

Design strategies for generalized synchronization

Shahrukh Chishti* and Ram Ramaswamy†

School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India



(Received 15 July 2018; published 24 September 2018)

We describe a general procedure to couple two dynamical systems so as to guide their joint dynamics onto a *specific* transversally stable invariant submanifold in the phase space. This method can thus be viewed as a means of constraining the dynamics, with the coupling functions providing the forces of constraint, which results in the coupled systems being in generalized synchronization. The required coupling functions are, however, not uniquely defined and can therefore be chosen in order to satisfy a desired design criterion.

DOI: [10.1103/PhysRevE.98.032217](https://doi.org/10.1103/PhysRevE.98.032217)

I. INTRODUCTION

In recent decades there has been a surge of interest in the study of synchronization [1,2], arguably one of the most striking temporal patterns that coupled oscillatory dynamical systems can exhibit. Although this is a subject that has been investigated extensively for nearly four centuries now [3], with the current developments in nonlinear science, the definitions and scenarios that come under rubric of synchronization have been extended considerably [1].

At the most general level of description, synchrony can be understood as the development of strong dynamical correlations between two dynamical systems [4,5]. This covers a range of behavior, from when the dynamics of the two subsystems is identical [1,2], namely, *complete* synchronization, to when there is a functional dependence between the variables of the two subsystems: this is termed *generalized* synchronization [6–8]. A variety of intermediate behaviors is possible, and accordingly, different forms of synchronization can be seen in systems that are coupled one-way or mutually, when the coupling is asymmetric, or when the coupling involves time delay [1,9,10]. Furthermore, the dynamics of the combined system in synchrony necessarily takes place in a lower dimensional subspace of the phase space: the functional relationship between the variables ensures that the dynamics lies in the so-called synchronization manifold [11]. In most situations the synchronized dynamics is typically not very different from that of the isolated systems since the coupling functions that are employed often vanish on the synchronization manifold.

Our aim in the present work is to design the coupling between two systems so as to reach a specific state of generalized synchronization, namely, one with a desired functional dependence of the variables of one system on the other. In other words, our objective is to determine a procedure so as to constrain the dynamics in the coupled system to a *specific* submanifold or algebraic variety [12] within the phase space. Synchronization is thus viewed as effectively being a control strategy whereby two systems mutually drive each other into a correlated dynamical state (or if the coupling is one-way, the

interaction is of the master-slave type and the slave dynamics becomes correlated to that of the master).

Generalized synchronization [6–8] (GS) has been extensively studied since the 1990s for a variety of different scenarios [13], complex coupling topologies [14], with time delay in the coupling [15], and so on. These studies have mainly adopted a passive viewpoint that can be summarized as follows: Given a specified form of the coupling, how and when does GS result and how can it be detected?

At the same time, there has been motivation to study a variety of linear and nonlinear coupling mechanisms [16] for reasons that include potential applications of chaos synchronization in secure communication or control [17–19]. The possibility of achieving a required state of synchronization by modifying the coupling function is therefore desirable from an engineering point of view and offers a level of flexibility and control. The role of coupling in modifying the resulting dynamics has been studied in detail, especially since the collective dynamics of coupled systems includes phenomena such as amplitude or oscillation death [20], phase flip [21], or induced multistability [22] in addition to synchronization. Indeed, as has been noted [16], the functional form of the coupling function can be an instrument of control. The possibility of achieving a required state of synchronization by modifying the coupling function is therefore desirable from an engineering point of view and offers a level of flexibility and control.

The control framework is particularly moot when the coupling between two systems is of the master-slave or skew-product form [23]. The dynamics of the drive or master system remains unaffected while that of the response or slave system correlates with that of the drive. Our additional objective in such a scenario is to determine whether there is an *optimal* coupling that can achieve a specific objective, in terms of the functional form of coupling rather than coupling strength [24]. Within the master-slave scenario, the autonomous flow dynamics of the enslaved system can often be completely destroyed [25], and this level of over-design may not be desirable. Accordingly, we present a general framework for bidirectional coupling that offers an inherently stable procedure for synchronization and aims to affect the inherent dynamics minimally.

*shahrukh.chishti@gmail.com

†ramaswamy@jnu.ac.in

There is overlap between the objectives of the present work and of synchronization engineering [26], the technique for controlling phase dynamics by tuning coupling functions in an ensemble of oscillators. Further, there is also overlap between the present work and the techniques used in projective synchronization [25,27–30], although our approach that is outlined in Sec. II is more general. Particular attention is paid to the question of stability, and it is shown how the coupling functions can be selected to ensure stability in directions that are transverse to the desired synchronization submanifold. We then present specific results for the case of two coupled chaotic Lorenz oscillators and show how the coupling can be arranged for specific choices of the synchronization submanifold in Sec. III. The paper concludes with a summary and discussion in Sec. IV. It should be noted that while the discussion and presentation in this paper is for two coupled systems, it is a simple matter to extend this to larger sets of coupled oscillators with different coupling topologies. Explicit consideration of such systems will be taken up in subsequent work.

II. CONSTRAINED DYNAMICS

Consider two independent systems $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ wherein the flows are specified by

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{F}_1(\mathbf{x}), \\ \dot{\mathbf{y}} &= \mathbf{F}_2(\mathbf{y}).\end{aligned}\quad (1)$$

This may be written compactly in terms of $\mathbf{X} \equiv [\mathbf{x} \ \mathbf{y}]^T \in \mathbb{R}^{m+n}$ as

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}), \quad (2)$$

with \mathbf{F} defined appropriately.

Regardless of the precise form of synchrony (namely complete, phase, lag, etc. [1]), one can identify a lower-dimensional subspace \mathcal{M} in the coupled system on which the synchronized dynamics is effectively confined: this is the synchronization manifold [11]. In principle, all specific types of synchronization can be seen as special cases of generalized synchronization, namely, to have the variables functionally dependent on each other [6],

$$\mathbf{x} = \Phi_x(\mathbf{y}) \text{ and } \mathbf{y} = \Phi_y(\mathbf{x}), \quad (3)$$

or equivalently through the relationship [31] $\Phi(\mathbf{x}, \mathbf{y}) = 0$, where the functional Φ typically specifies an algebraic variety and is defined by a set of $N < n + m$ functional relations or constraints, each of which is an algebraic relationship of the form

$$\phi_i(\mathbf{x}, \mathbf{y}) = 0, \quad (4)$$

all of which should be distinct and independent, and $\Phi = [\phi_1 \ \phi_2 \ \dots \ \phi_N]^T$.

Our objective is to couple the systems in order to achieve a specified state of GS, along with two additional related requirements in order that the synchronization is stable. The synchronization manifold must be an attractor of the dynamics, at least locally, and, furthermore, this subspace should be stable under transversal perturbations in order for synchrony to be observed. Our approach here is to incorporate these

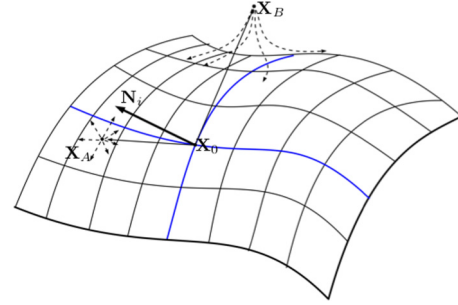


FIG. 1. Schematic of our general strategy. The surface represents the target synchronization manifold $\mathcal{M}(\Phi)$. The points \mathbf{X}_0 , \mathbf{X}_A are on $\mathcal{M}(\Phi)$, and also shown is \mathbf{N}_i , a normal to the surface. The coupling is designed to bring nearby trajectories (from \mathbf{X}_B , say) onto this submanifold in the phase space. Subsequently, the additional terms in the coupling keep the flow on $\mathcal{M}(\Phi)$ (as indicated at \mathbf{X}_A) and, furthermore, constrain it to this surface.

features in our design strategy. Since the variables of the two are functionally related to each other, the subsystems need to be coupled in a manner so as to ensure that trajectories in the system are asymptotically attracted to the submanifold $\mathcal{M}(\Phi) = \{\mathbf{X} \in \mathbb{R}^{m+n} : \Phi(\mathbf{X}) = 0\}$ or, equivalently, that

$$\lim_{t \rightarrow \infty} \phi_j(\mathbf{X}(t)) \rightarrow 0 \ \forall j, \text{ and therefore } \lim_{t \rightarrow \infty} \Phi(\mathbf{X}(t)) = 0. \quad (5)$$

In order that the submanifold specified by $\Phi(\mathbf{x}, \mathbf{y}) = 0$ is invariant, namely, that a trajectory that is in $\mathcal{M}(\Phi)$ remains in it, we require that the flow direction be orthogonal to the normal vectors at any point \mathbf{X} of Φ ; see Fig. 1. The normals are given by

$$\begin{aligned}\mathbf{N}_i(\mathbf{X}) &= \nabla_{\mathbf{x}} \phi_i(\mathbf{x}, \mathbf{y}) \Rightarrow \mathfrak{N} \equiv \nabla_{\mathbf{X}}^T \Phi(\mathbf{X}) \\ &= [\mathbf{N}_1 \ \mathbf{N}_2 \ \dots \ \mathbf{N}_N]^T,\end{aligned}\quad (6)$$

and thus we have the condition

$$\mathbf{N}_i^T(\mathbf{X})\mathbf{F}(\mathbf{X}) = 0 \quad \forall \mathbf{X} \in \mathcal{M}(\Phi) \Rightarrow \mathfrak{N}(\mathbf{X})\mathbf{F}(\mathbf{X}) = \mathbf{0}. \quad (7)$$

If the additive coupling terms for each of the flows are given by $\zeta_i(\mathbf{X})$, the coupled system is given as

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{F}_1(\mathbf{x}) + \zeta_1(\mathbf{X}), \\ \dot{\mathbf{y}} &= \mathbf{F}_2(\mathbf{y}) + \zeta_2(\mathbf{X}).\end{aligned}\quad (8)$$

It should be pointed out that the coupling functions ζ_i are defined only up to addition; one can include terms in the coupling, namely, $\zeta_i \rightarrow \zeta_i + \chi_i$, with the requirement that the χ_i should vanish on the synchronization manifold:

$$\chi_i(\mathbf{X}) = 0 \quad \forall \mathbf{X} \in \mathcal{M}(\Phi). \quad (9)$$

This does not affect the formalism, but the additional stabilizing terms $\chi \equiv [\chi_1(\mathbf{x}, \mathbf{y}) \ \chi_2(\mathbf{x}, \mathbf{y})]^T$ may be required in order to ensure that the manifold is made attracting. The variable coupling strength can also be included here.

The requirement of invariance for such a system on the submanifold gives

$$\mathfrak{N}(\mathbf{F} + \zeta) = \mathbf{0} \Rightarrow \mathfrak{N}\zeta = -\mathfrak{N}\mathbf{F}, \quad (10)$$

where $\zeta \equiv [\zeta_1(\mathbf{x}, \mathbf{y}) \ \zeta_2(\mathbf{x}, \mathbf{y})]^T$.

For the synchronization submanifold to be transversally stable, perturbations should decay so that the trajectory returns

to $\mathcal{M}(\Phi)$. Consider a local perturbation about a point \mathbf{X}_0 on $\mathcal{M}(\Phi)$ as $\xi(t) = \mathbf{X}(t) - \mathbf{X}_0$. Linearizing Eq. (2) gives

$$\dot{\xi} = \mathbf{F}(\mathbf{X}_0) + \nabla_{\mathbf{x}}^T \mathbf{F}(\mathbf{X}_0) \xi + \text{higher order terms.} \quad (11)$$

Expanding the perturbation in the normal and tangent subspaces, namely, as $\xi = \sum_i \alpha_i \mathbf{N}_i(\mathbf{X}) + \sum_i \beta_i \mathbf{T}_i(\mathbf{X})$ about \mathbf{X}_0 , we get

$$\dot{\xi} = \mathbf{F}(\mathbf{X}_0) + \nabla_{\mathbf{x}}^T \mathbf{F}(\mathbf{X}_0) \left[\sum_i \alpha_i \mathbf{N}_i(\mathbf{X}_0) + \sum_i \beta_i \mathbf{T}_i(\mathbf{X}_0) \right].$$

For transverse stability it is necessary that the perturbation should decay along the normal directions,

$$\dot{\xi} \cdot \alpha_i \mathbf{N}_i < 0 \quad \forall i = 1, 2, \dots, N.$$

The tangential component vanishes since the flow on the submanifold is always along the surface as already ensured by Eq. (7):

$$\left[\mathbf{F}(\mathbf{X}_0) + \nabla_{\mathbf{x}}^T \mathbf{F}(\mathbf{X}_0) \sum_i \beta_i \mathbf{T}_i(\mathbf{X}_0) \right] \cdot \alpha_i \mathbf{N}_i = 0.$$

Hence the stability criterion is equivalent to the requirement that $\dot{\xi}$ decays along the normal directions:

$$\nabla_{\mathbf{x}}^T \mathbf{F}(\mathbf{X}_0) \sum_i \alpha_i \mathbf{N}_i \cdot \alpha_i \mathbf{N}_i < 0; \quad \forall i = 1, 2, \dots, N. \quad (12)$$

This can be ensured by requiring that along the trajectory the real parts of all eigenvalues of the Jacobian remain nonpositive. At every point \mathbf{X} on the submanifold, if the eigenvalues $\{\lambda_i\}$ of the Jacobian matrix given by

$$\mathfrak{J} = \mathfrak{N} \nabla_{\mathbf{x}}^T \mathbf{F}(\mathbf{X}) \mathfrak{N}^T \quad (13)$$

satisfy the condition

$$\forall \mathbf{X} \in \mathcal{M}(\Phi), \lambda_i < 0 \quad \forall i, \quad (14)$$

stability is ensured. More generally, one may require that the transverse Lyapunov exponents should be negative: this is the condition that is employed in projective synchronization [14,27].

Equations (7), (10), and (14), along with the constraints specifying $\mathcal{M}(\Phi)$ represent our strategic equations for achieving the desired synchronization between the two systems given by Eq. (1). The stabilizing functions χ can be included in the dynamics so as to make the manifold transversally stable in the following manner. Defining

$$\chi \equiv \epsilon \Phi(\mathbf{X}), \quad (15)$$

where ϵ is a matrix which can be chosen suitably (and need not be constant), one obtains

$$\nabla_{\mathbf{x}}^T \chi = \epsilon \nabla_{\mathbf{x}}^T \Phi(\mathbf{X}) + \nabla_{\mathbf{x}}^T \epsilon \Phi(\mathbf{X}). \quad (16)$$

Note that the second term on the right vanishes on the synchronization manifold, and therefore

$$\nabla_{\mathbf{x}}^T \chi = \epsilon \mathfrak{N}. \quad (17)$$

By varying the coupling strength (which is included in ϵ) there is complete control on making the synchronization manifold attractive, and this is applicable in all cases of generalized synchronization. Note that the systems of equations are not overdetermined, and in fact there is considerable flexibility in

the choice of both χ and ζ . This will be explicitly illustrated in the examples we discuss below.

III. APPLICATIONS

For simplicity, we present applications of the present procedure for constraining the dynamics in a system of coupled chaotic Lorenz oscillators [32]. Each of the subsystems is a flow in \mathbb{R}^3 :

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), & \dot{y}_1 &= \sigma'(y_2 - y_1), \\ \dot{x}_2 &= x_1(\rho - x_3) - x_2, & \dot{y}_2 &= y_1(\rho' - y_3) - y_2, \\ \dot{x}_3 &= x_1 x_2 - \beta x_3, & \dot{y}_3 &= y_1 y_2 - \beta' y_3, \end{aligned} \quad (18)$$

σ, ρ , and β (and their primed counterparts) being parameters. Writing this compactly as

$$\dot{\mathbf{X}} = \mathbf{F}_0(\mathbf{X}), \quad (19)$$

one can see that the Jacobian of the intrinsic flow of the Lorenz systems is

$$\nabla_{\mathbf{x}}^T \mathbf{F}_0 = \begin{bmatrix} -\sigma & \sigma & 0 & 0 & 0 & 0 \\ \rho - x_3 & -1 & -x_1 & 0 & 0 & 0 \\ x_2 & x_1 & -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma' & \sigma' & 0 \\ 0 & 0 & 0 & \rho' - y_3 & -1 & -y_1 \\ 0 & 0 & 0 & y_2 & y_1 & -\beta' \end{bmatrix}. \quad (20)$$

We take the Lorenz systems to be identical, $\sigma = \sigma', \rho = \rho', \beta = \beta'$ in the three examples of designed synchrony presented below.

A. Modified projective synchronization

The first example we consider is the case of modified projective synchronization [25], where we take as the target submanifold the three-dimensional subspace within the six-dimensional phase space given by the conditions $y_1 = ax_1$, $y_2 = bx_2$, and $y_3 = cx_3$, which can be written as $\mathbf{y} = \mathfrak{S}\mathbf{x}$ with

$$\mathfrak{S} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Expressing the constraints simply as $(y_1 - ax_1) = 0$, $(y_2 - bx_2) = 0$, $(y_3 - cx_3) = 0$ effectively describes the constraint as $\Phi(\mathbf{X}) = \mathbf{y} - \mathfrak{S}\mathbf{x}$. Defining \mathfrak{N} as a matrix of dimension 3×6 ,

$$\mathfrak{N} = [\mathfrak{S} \quad -\mathbf{1}],$$

where $\mathbf{1}$ is a unit matrix of order 3, a straightforward calculation gives

$$\mathfrak{N}\zeta = \begin{bmatrix} (b-a)\sigma x_2 \\ (a-b)\rho x_1 - acx_1 x_3 + bx_1 x_3 \\ abx_1 x_2 - cx_1 x_2 \end{bmatrix}. \quad (21)$$

There are several possible algebraic solutions for ζ_i that will satisfy the constraint of keeping the dynamics on the invariant submanifold $\mathcal{M}(\Phi)$. Here we choose the following

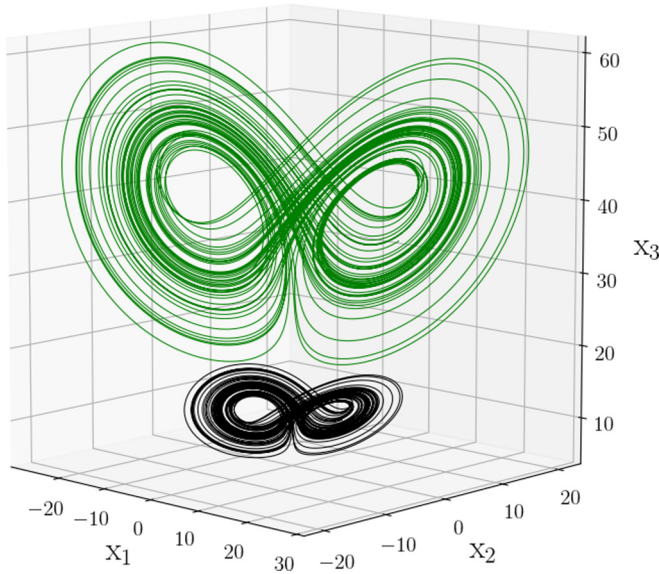


FIG. 2. Projective synchronization in bidirectionally coupled Lorenz systems for $a = 2.6$, $b = 1.8$, $c = 3.6$. (These values are arbitrary: any scale factors can be achieved with this form of the coupling and the desired a , b , and c .) The \mathbf{x} dynamics is shown in black, while the \mathbf{y} dynamics is in blue. The axes are labeled only by the \mathbf{x} variables.

bidirectional coupling:

$$\zeta_1 = \begin{bmatrix} \sigma y_2/a \\ (\rho y_1 - y_1 y_3)/b \\ y_1 y_2/c \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} \sigma a x_2 \\ \rho b x_1 - b x_1 x_3 \\ c x_1 x_2 \end{bmatrix}.$$

With this choice of ζ , it happens that the eigenvalues of the corresponding Jacobian matrix (13), namely,

$$\tilde{\mathfrak{J}} = \begin{bmatrix} -\sigma a^2 - \sigma & 0 & 0 \\ 0 & -b^2 - 1 & 0 \\ 0 & 0 & -c^2 \beta - \beta \end{bmatrix}, \quad (22)$$

are by construction all negative, additional stabilizing functions are superfluous. The fact that this matrix is diagonal indicates that at each point any perturbation will decay along the normals, naturally ensuring the stability of the synchronization manifold. Incidentally, this solution is valid for all real values of a , b , and c , and a particular realization is illustrated in Fig. 2.

Owing to the bidirectional coupling, though, while the subsystems are synchronized, the original flow has been partly modified since the coupling does not vanish on the synchronization manifold. This can be avoided by choosing a master-slave form for the coupling. Given the form of the matrix \mathfrak{N} , master-slave or one-way coupling can be ensured, for example, by making ζ_1 the null vector. One choice for ζ_2 that has been made earlier [14] is

$$\zeta_2 = \begin{bmatrix} (\sigma b/a - 1)y_2 - (\sigma - 1)y_1 \\ -x_1 x_3/b + \rho(a/b - 1)y_1 + y_1 y_3 \\ x_1 x_2/c - y_1 y_2 \end{bmatrix}, \quad (23)$$

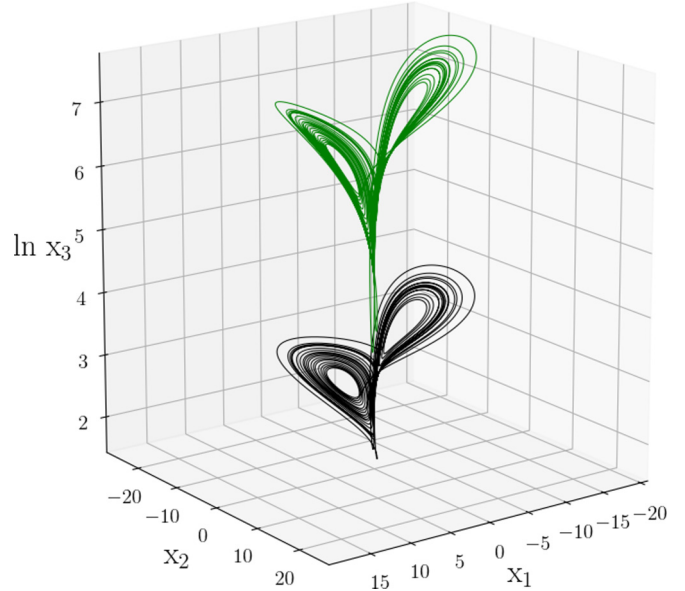


FIG. 3. Master-slave coupling in two Lorenz systems achieves the desired functional relationship, $y_3 = x_3^2$, between the master and slave subsystems. The form of the coupling is, however, quite complex. As in Fig. 2, the axes are labeled by the \mathbf{x} variables whose trajectory is shown in black, while the \mathbf{y} dynamics is in green. Note the logarithmic scale on the vertical axis.

while a simpler possibility is clearly [33]

$$\zeta_2 = \begin{bmatrix} \sigma(a - b)x_2 \\ (b - a)\rho x_1 + (ac - b)x_1 x_3 \\ (c - ab)x_1 x_2 \end{bmatrix}. \quad (24)$$

B. Nonlinear scaling

As a second example, consider the choice $y_1 = x_1$, $y_2 = x_2$, $y_3 = x_3^2$ for the equations of constraint in the coupled Lorenz system. In matrix notation this can be written as $\mathbf{y} = \mathfrak{S}(\mathbf{x})\mathbf{x}$ where \mathfrak{S} is not constant but depends on the position along the trajectory. The matrix of vectors that are normal to the constraint surface is

$$\mathfrak{N} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2x_3 & 0 & 0 & 1 \end{bmatrix}.$$

In order to keep the fidelity of the Lorenz dynamics, we choose to couple the two oscillators in the master-slave configuration with ζ_1 being the null vector. This gives, following the procedure outlined above [see Eqs. (10) and (21)], the highly nonintuitive coupling

$$\zeta_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \zeta_2 = \begin{bmatrix} 0 \\ -x_3 y_1 + x_1 x_3^2 \\ -x_1 x_2 + 2x_1 x_2 x_3 - \beta x_3^2 \end{bmatrix}$$

that (see Fig. 3) ensures that the dynamics of the original master Lorenz oscillator is maintained, and that of the slave Lorenz has the desired relationship, namely, $y_3 = x_3^2$. The stability matrix for this arrangement is not constant and is

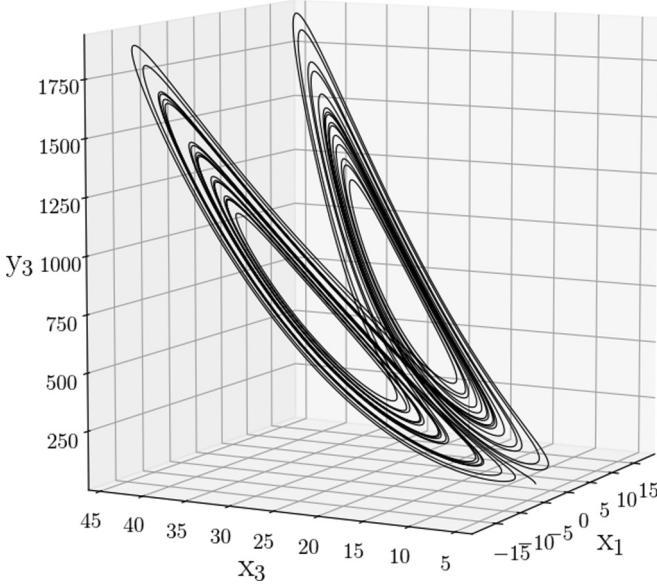


FIG. 4. Trajectory on the synchronization submanifold. The parabolic dependence of y_3 on x_3 is quite evident.

given by

$$\mathfrak{J} = \begin{bmatrix} -2\sigma & \sigma & 0 \\ 2(\rho - x_3 - x_3^2) & -2 & -4x_1x_3^2 - x_1 \\ 2x_3x_2 + x_2 + 4\beta x_3^2 - 4x_1x_2x_3 & 2x_1 & -4\beta x_1x_2x_3 - \beta \end{bmatrix}.$$

In the given range of parameter values the eigenvalues of the above matrix are all negative, and thus it is not required that additional stabilizing forces be added. A view of the synchronization manifold is provided in Fig. 4.

C. Crossed signals

The concept of constraining systems to a desired submanifold can be extended beyond the conventional sense of synchronization that has been discussed so far. In the final example presented here, we require that the coupling “flips” two of the variables, namely, the constraint function is

$$\Phi(\mathbf{X}) = \begin{bmatrix} x_1 - y_2 \\ x_2 - y_1 \end{bmatrix} = 0, \quad (25)$$

which is a two-dimensional submanifold, since no constraint has been set on the variable y_3 of the slave. The procedures to be followed are standard, and choosing master-slave coupling

$$\mathfrak{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathfrak{s}_2 = \begin{bmatrix} \sigma(y_1 - x_1) + (\rho - z_1)x_1 - y_1 \\ \sigma(y_1 - x_1) - (\rho - z_2)y_1 + x_1 \\ 0 \end{bmatrix},$$

we find the stability matrix involves the variable z_2 :

$$\mathfrak{J} = 2 \begin{bmatrix} -1 & (\rho - z_2) \\ \sigma & -\sigma \end{bmatrix}. \quad (26)$$

Since the eigenvalues are not necessarily negative (as required) an additional stabilizer function \mathfrak{s}_2 is needed in order to ensure the stability of the submanifold in the range of the

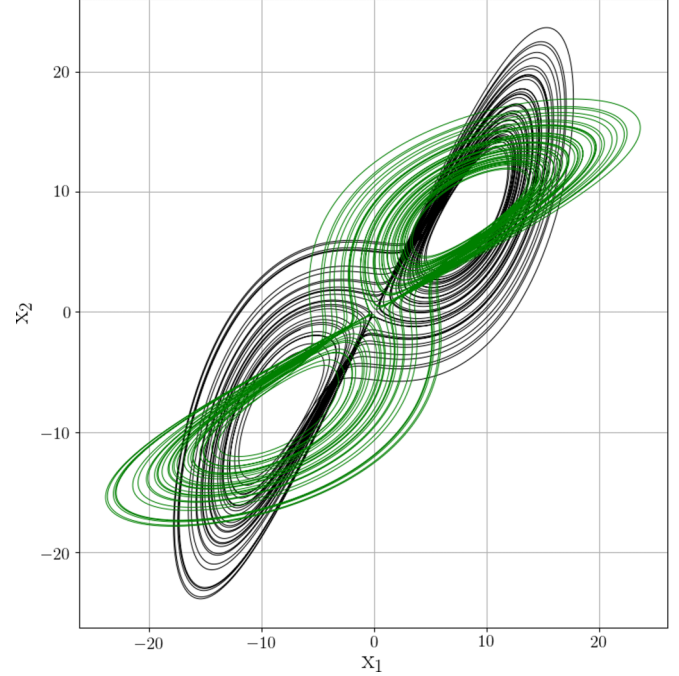


FIG. 5. Flipped synchrony, with $x_1 = y_2$ and $y_1 = x_2$, which automatically constrains x_3 to be identical to y_3 . Trajectories are shown in projection, with the axes being labeled by the \mathbf{x} variables whose trajectory is shown in black, while the \mathbf{y} dynamics is in green.

attractor. Choosing the variable coupling strength

$$\mathfrak{x}_2 = \begin{bmatrix} 0 \\ (\rho - y_3)(x_2 - y_1) \\ 0 \end{bmatrix} \quad (27)$$

effectively destroys a part of the original Lorenz system; we consider the alternate additional linear function with static coupling that gives negative eigenvalues

$$\mathfrak{x}_2 = \begin{bmatrix} \sigma(x_1 - y_2) \\ 0 \\ 0 \end{bmatrix}. \quad (28)$$

The efficacy of the procedure is apparent; see Fig. 5.

IV. DISCUSSION AND SUMMARY

Generalized synchronization has been studied extensively in various settings, which include a range of topologies such as complex networks [34], with or without time delay [15], Hölder continuity [14], the Lipschitz condition [35–37], etc. The condition of GS can be verified through analytical as well as numerical approaches such as the auxiliary system method [14,23,38], normal hyperbolicity [39], development of error systems, and Lyapunov function stability [8].

It is natural to consider *any* form of synchronization between two coupled dynamical systems in terms of invariant manifolds [11]. Additionally if the synchronization manifold is k -hyperbolic [40], then the synchronization will be stable. In the present paper, we implement a set of procedures in an *active* approach to achieve these objectives, by first specifying the desired synchronization manifold and then reverse

engineering the required coupling function to guarantee k -hyperbolicity. Our approach is constructive in the sense that we first introduce effective forces of constraints that keep the dynamics on the manifold, and then introduce transverse stabilization so that the desired synchronization manifold (or here, more properly, an algebraic variety) is also the attractor of the dynamics. This geometric approach addresses the two conditions, Eqs. (7) and (14), independently. There is considerable flexibility in how the dynamics can be stabilized on the desired synchronization manifold, and there is also considerable choice in the coupling functions that will ensure that the manifold is attracting.

The present methodology has been illustrated in a system of two coupled chaotic Lorenz oscillators, where a number of different objectives were specified. The variables of the two oscillators were required to depend on each other in a specified way and both linear and nonlinear dependences were considered. Synchronization forces the dynamics onto the desired low-dimensional submanifold, but given the latitude with which the coupling can be chosen, the dynamics of the original systems may or may not be retained. In the usual forms of coupling that have been studied in the literature, the coupling is often selected so as to vanish on the synchronization manifold. Here this cannot always be guaranteed, and in extreme cases, the dynamics in the coupled system can be quite different from that in the original. It is well known, for instance that within the master-slave scenario, the coupling can seriously alter the intrinsic flow of the slave system [25]. At the same time coupling functions that are *over*-designed may not have a simple action, and additional stabilizing terms (that vanish on the synchronization manifold) may be needed to achieve stability. The fact that the solution of the residual

flow equation is not unique offers flexibility in the choice of coupling, starting with whether one desires unidirectional or bidirectional interactions. There will thus be several coupling options that are inherently stable for synchronization.

Our procedure complements the synchronization engineering [26] wherein a systematic design linear and nonlinear feedback (possibly with time delay) is implemented in order to achieve a number of different collective states such as cluster synchronization, desynchronization, and chimeric dynamics. Control strategies for achieving different collective dynamical states are clearly of importance not just in the engineering context [26] but in biology as well [41]. In the examples we have considered here, the constraints are separable, but clearly an important extension of the present methodology is to the case of nonseparable constraints. Further, in the examples presented here both the uncoupled flows are identical; this is not a limitation, and the formalism goes through quite easily for nonidentical systems as well as to extended systems, namely, ensembles of oscillators with complex coupling topologies. However, not all constraints are possible to implement: there appear to be intrinsically *nonsynchronizable* conditions that depend on the details of the systems [42], and it will be important to examine the limitations of the present approach in this context.

ACKNOWLEDGMENT

R.R. acknowledges the support of the the SERB, Department of Science and Technology, India (Grant No. SR/S2/JCB/2008), through the award of the J. C. Bose Fellowship.

-
- [1] A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Science* (Cambridge University Press, Cambridge, 2001).
- [2] S. H. Strogatz, *Sync: How Order Emerges from Chaos in the Universe, Nature, and Daily Life* (Hyperion, New York, 2004).
- [3] See M. Bennett, M. F. Schatz, H. Rockwood, and K. Wiesenfeld, *Proc. Roy. Soc. Math. Phys. Eng. Sci.* **458**, 563 (2002) for a discussion of Huygens' observations and their modern reexamination.
- [4] R. Brown and L. Kocarev, *Chaos* **10**, 344 (2000).
- [5] X.-S. Yang, *Chaos, Solitons Fractals* **11**, 1365 (2000).
- [6] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, *Radiophys. Quantum Electron.* **29**, 795 (1986).
- [7] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, *Phys. Rev. E* **51**, 980 (1995); H. D. I. Abarbanel, N. F. Rulkov, and M. M. Sushchik, *ibid.* **53**, 4528 (1996).
- [8] L. Kocarev and U. Parlitz, *Phys. Rev. Lett.* **76**, 1816 (1996).
- [9] L.-Y. Cao and Y.-C. Lai, *Phys. Rev. E* **58**, 382 (1998).
- [10] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, *Phys. Rev. Lett.* **78**, 4193 (1997); M. Rosenblum, A. Pikovsky, and J. Kurths, in *Stochastic Dynamics*, edited by L. Schimansky-Geier and T. Pöschel (Springer-Verlag, Berlin, 1997), pp. 232–244.
- [11] K. Josić, *Phys. Rev. Lett.* **80**, 3053 (1998); *Nonlinearity* **13**, 1321 (2000).
- [12] R. Hartshorne, *Algebraic Geometry* (Springer-Verlag, New York, 1977).
- [13] Z. Zheng, X. Wang, and M. C. Cross, *Phys. Rev. E* **65**, 056211 (2002).
- [14] L. Guo and Z. Xu, *Chaos* **18**, 033134 (2008).
- [15] M. Zhan, X. Wang, X. Gong, G. W. Wei, and C.-H. Lai, *Phys. Rev. E* **68**, 036208 (2003).
- [16] T. Stankovski, T. Pereira, P. V. E. McClintock, and A. Stefanovska, *Rev. Mod. Phys.* **89**, 045001 (2017).
- [17] C. Y. Chee and D. Xu, *IEE Proc. G Circ. Dev. Syst.* **153**, 357 (2006).
- [18] G. Chen and X. Dong, *Chaos Synchronization* (World Scientific, Singapore, 2012), pp. 537–618.
- [19] C. J. Stam and B. W. van Dijk, *Physica D* **163**, 236 (2002).
- [20] G. Saxena, A. Prasad, and R. Ramaswamy, *Phys. Rep.* **521**, 205 (2012).
- [21] A. Prasad, *Phys. Rev. E* **72**, 056204 (2005); A. Prasad, J. Kurths, S. K. Dana, and R. Ramaswamy, *ibid.* **74**, 035204(R) (2006).
- [22] See, e.g., H. Sakaguchi and Y. Kuramoto, *Prog. Theor. Phys.* **76**, 576 (1986), as well as the extensive bibliography in Ref. [16].
- [23] L. M. Pecora and T. L. Carroll, *Phys. Rev. Lett.* **64**, 821 (1990).
- [24] Optimality cannot be unequivocally defined in such a context: there is no obvious “cost function”; therefore measures of sufficient generality cannot be devised in a simple manner. The coupling terms that are introduced do not have a clear interpretation as a constraint force in all cases; these concerns will be addressed in our future work.

- [25] G.-H. Li, *Chaos, Solitons Fractals* **32**, 1786 (2007).
- [26] I. Z. Kiss, C. G. Rusin, H. Kori, and J. L. Hudson, *Science* **316**, 1886 (2007); I. Z. Kiss, *Curr. Opin. Chem. Eng.* **21**, 1 (2018).
- [27] M. A. Khan and S. Poria, *Pramana J. Phys.* **81**, 395 (2013).
- [28] R. Mainieri and J. Rehacek, *Phys. Rev. Lett.* **82**, 3042 (1999).
- [29] L. Wang, B. Zhen, and J. Xu, *Sci. World* **2013**, 568194 (2013).
- [30] Y. Chen and X. Li, *Int. J. Mod. Phys. C* **18**, 883 (2007).
- [31] B. R. Hunt, E. Ott, and J. A. Yorke, *Phys. Rev. E* **55**, 4029 (1997).
- [32] E. Lorenz, *J. Atmos. Sci.* **20**, 130 (1963); C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors* (Dover, New York, 2005).
- [33] I. Grosu, E. Padmanaban, P. K. Roy, and S. K. Dana, *Phys. Rev. Lett.* **100**, 234102 (2008).
- [34] H. Liu, J. Chen, J. Lu, and M. Cao, *Physica A* **389**, 1759 (2010).
- [35] W. He and J. Cao, *Chaos* **19**, 013118 (2009).
- [36] J. Yan and C. Li, *Chaos, Solitons Fractals* **26**, 1119 (2005).
- [37] W. K. Wong, B. Zhen, J. Xu, and Z. Wang, *Chaos* **22**, 033146 (2012).
- [38] L. Kocarev, U. Parlitz, and R. Brown, *Phys. Rev. E* **61**, 3716 (2000).
- [39] S. S. Yang and C. K. Duan, *Chaos, Solitons Fractals* **9**, 1703 (1998).
- [40] S. Wiggins, *Normally Hyperbolic Invariant Manifolds in Dynamical Systems* (Springer-Verlag, New York, 1994).
- [41] See, e.g., D. Peak, J. D. West, S. M. Messinger, and K. A. Mott, *Proc. Natl. Acad. Sci. U. S. A.* **101**, 918 (2004); P. Mehta and T. Gregor, *Curr. Opin. Genet. Dev.* **20**, 574 (2010); M. Nitzan, J. Casadiego, and M. Timme, *Sci. Adv.* **3**, e1600396 (2017) for some representative articles that address collective dynamics in different aspects of biology.
- [42] S. Chishti and R. Ramaswamy (unpublished).