Simulation of spatial systems with demographic noise

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The demographic (shot) noise in population dynamics scales with the square root of the population size. This process is very important, as it yields an absorbing state at zero field, but simulating it, especially on spatial domains, is a nontrivial task. Here, we analyze two similar methods that were suggested for simulating the corresponding Langevin equation, one by Pechenik and Levine and the other by Dornic, Chaté, and Muñoz (DCM). These methods are based on operator-splitting techniques and the essential difference between them lies in which terms are bundled together in the splitting process. Both these methods are first order in the time step so one may expect that their performance will be similar. We find, surprisingly, that when simulating the stochastic Ginzburg-Landau equation with two deterministic metastable states, the DCM method exhibits two anomalous behaviors. First, the stochastic stall point moves away from its deterministic counterpart, the Maxwell point, when decreasing the noise. Second, the errors induced by the finite time step are larger by a significant factor (i.e., $>10\times$) in the DCM method. We show that both these behaviors are the result of a finite-time-step induced shift in the deterministic Maxwell point in the DCM method, due to the particular operator splitting employed. In light of these results, care must be exercised when computing quantities like phase-transition boundaries (as opposed to universal quantities such as critical exponents) in such stochastic spatial systems.

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I. INTRODUCTION

Systems with demographic noise play an important role in many different problems, including birth-death processes, ecology, population genetics, reaction-diffusion processes, infection models, etc. [1]. Demographic noise is a very important factor in the dynamics of these systems since the noise vanishes at zero abundance, rendering the inactive state an absorbing state. Technically, demographic stochasticity is a special kind of multiplicative noise, proportional to the square root of the fluctuating field. Thus, at small amplitudes, the noise is typically dominant. As a result, simulating the dynamics of a system with demographic stochasticity requires special care; it is crucial, for instance, that the noise not cause the amplitude to become negative [2], which would occur with a naive treatment of the noise.

These problems are especially severe in spatial systems, where many local populations are connected by diffusion of individuals. There have been numerous schemes proposed to efficiently simulate a spatially extended system. One approach is to simulate an individual-based model using a Gillespie-type algorithm [3], where the rate for each event is taken into account and the next event is picked from the complete set of possibilities. If the demographic noise arose, as is typical, from the original model having discrete individuals, then this is an exact treatment of the problem. However, for large spatial systems this procedure is prohibitively expensive, especially at large population levels. Having to pick the next event from the very large number of different processes that can occur, each with its own unique rate, slows down the calculation enormously [4]. Alternatively, one may still use an individual-based model but discretize the process in time, along the lines of how one simulates a deterministic partial differential equation. In this scheme [5], the events on different sites occur in parallel (the number of births, for example, at a given site, is given by a Poisson or binomial deviate, with the mean determined by the birth rate times the time step) and so this is much more efficient than Gillespie, especially at large density (small noise). This is still not ideally efficient, in that a different random deviate must be generated at every site for every different process. Furthermore, being essentially a Euler-type scheme, there are strong restrictions on how large the time step can be.

Aside from the individual-based approach, one can instead use a continuum approach, either Fokker-Planck or Langevin [6]. If the model under study is also a continuum equation, such as the stochastic Fisher equation [7], then an individual-based model approach discussed above is only an approximation. Otherwise, the continuum equation is a certain large-population limit of the original model. In this continuum approach, the strength of the demographic noise is given by a single parameter, the amplitude of the square-root noise. Here, one has to face squarely the difficulties inherent in the square-root type multiplicative noise. One noteworthy approach was that of Pechenik and Levine (PL) [8], who introduced an operator-splitting method, based on an analytic solution of the pure birth-death process. In a single time step of the PL simulations, the birth-death process is simulated exactly, followed by a deterministic update implementing the diffusion and reaction terms. This method has the possibility of being stable for all time step sizes. Using this method, PL were able to simulate the Fisher process at very small noise levels, thereby confirming the analytical predictions of Brunet and Derrida [7] for this limit.

Subsequently, Dornic, Chaté, and Muñoz [9] (DCM) proposed a different split-operator scheme. They introduced a much simpler method of generating the stochastic update, which allowed them to incorporate more terms of the Langevin equation in the exact stochastic update. These included the linear growth term as well as a first order in time approximation of the diffusive nearest-neighbor coupling. As is the PL scheme, the DCM scheme is overall first order in time, and thus one would expect the accuracy of the two methods to be comparable.

The reason for revisiting the two methods has its origin in an anomaly in the results presented in a recent study of Martin *et al.* [10] of the phase diagram of the stochastic Ginzburg-Landau (GL) equation generated using the DCM scheme. Deterministically, this system supports two stable fixed points, one at zero field and the other at some finite value. On spatial domains, one of these states invades the other, unless the external parameters (like, e.g., the stress affecting the population) are tuned to the stall (Maxwell) point. The numerical results of Martin *et al.* ([10], Fig. 2) indicated that in the stochastic system the active state invades the absorbing state even on the high-stress side of the deterministic Maxwell point, i.e., in the regime where (in the deterministic system) the inactive state is dominant. If true, this would be a very surprising feature since the demographic noise destabilizes the active state, but cannot touch the inactive state where its amplitude (which is proportional to the square root of the concentration) vanishes. Moreover, Martin *et al.* presented a renormalization group analytic calculation showing that, at each length scale, the effect of an increase in the noise is a reduction of the positive feedback; hence, it must increase the basin of attraction of the inactive phase.

In an attempt to understand this point better, we systematically decreased the time step Δt of the simulation, using the same DCM algorithm. We discovered that the location of the stochastic stall point converges linearly to a certain value on the correct (i.e., low-stress) side of the deterministic Maxwell point, namely, the stress parameter at the stochastic stall point is in the range where in the deterministic system the active phase invades. Thus, as $\Delta t \to 0$, the noise indeed renders the active phase less active. We then attempted to reproduce this effect using the PL scheme, and found that both methods converge linearly to the same critical stress, but here the stress parameter at the stochastic stall point is *always* on the low-stress side of the deterministic Maxwell point. Moreover, the convergence of the PL algorithm is faster, i.e., the size of the finite-Δt correction was over 10 times smaller. These surprising results call for a more systematic study of the size and sign of the finite-Δt corrections in the two methods. As pointed out by Lee, Kwon, and Park [11], the particular choice of how to split the operators may lead to a nontrivial shift of the effective values of the bare parameters for finite Δt. Here, we show that for the DCM operator-splitting scheme on spatial domains, this shift causes large finite-Δt corrections, moving the phase boundaries even in the absence of noise. The root of the problem, it turns out, is a mixing between the linear growth and diffusion terms, when the the local and the nonlocal parts of the diffusion operator are split in such a way that only one acts together with the growth. We thus conclude that in an efficient splitting scheme, one should not split the diffusion operator.

The plan of the paper, then, is as follows. We first review the PL and DCM algorithms, highlighting their similarities and differences. In Sec. III, we present our results for the model of Martin *et al.*, comparing the DCM and PL schemes for different Δt. We show the anomalous DCM behavior persists even in the deterministic limit. This allows an analytic calculation of the leading finite-Δt corrections in the deterministic limits of DCM and PL, presented in Sec. IV. We then summarize our conclusions and the implications for other numerical studies involving spatial demographic noise.

II. TWO METHODS: PL AND DCM

In this section, we review the details of the two methods, in order to fix notation and nomenclature. The PL scheme consists of breaking up the Langevin equation into two pieces, a stochastic balanced birth-death process yielding the demographic noise and a deterministic piece containing the rest of the dynamics. An elementary time step consists of a stochastic update of duration Δt, followed by a deterministic update of the same duration. The stochastic update constitutes an exact solution of the local equation

$$\frac{\partial \phi}{\partial t}(x,t) = \sigma\sqrt{\phi(x,t)}\eta; \quad \langle \eta(x,t)\rangle = 0;$$
$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t'). \tag{1}$$

The locality of the stochastic dynamics for ϕ means that every discrete site can be updated independently. The Probability Density Function (PDF) of the updated $\tilde{\phi}$ at a given site, given its initial value ϕ, is given by

$$P(\tilde{\phi}) = \delta(\tilde{\phi})e^{-2\phi/(\sigma^2\Delta t)} + \frac{2e^{-2(\phi+\tilde{\phi})/(\sigma^2\Delta t)}}{\sigma^2\Delta t}\sqrt{\frac{\phi}{\tilde{\phi}}}\,\mathrm{I}_1\left(\frac{4\sqrt{\phi\tilde{\phi}}}{\sigma^2\Delta t}\right), \tag{2}$$

where I_1 is a Bessel function. PL implemented this update via a look-up table and interpolation for the inverse cumulative distribution function for various initial values of ϕ.

As opposed to this algorithm for the stochastic update employed by PL, DCM aptly noted that it can be more conveniently accomplished via first generating a Poisson deviate Q_i with mean $\lambda\phi$, where

$$\lambda = \frac{2}{\sigma^2\Delta t}, \tag{3}$$

and then generating a gamma deviate R with shape parameter Q and scale unity. The new $\tilde{\phi}$ is then R/λ_i. This new $\tilde{\phi}$ was shown to obey precisely the PDF (2). This is the first aspect of the DCM work and, being an exact reformulation, has no effect on the results. As this trick is both more efficient and simpler to implement, and as it does not change the results at all, we incorporate it into the PL method in the following without further comment.

Following this stochastic updating of $\phi(x)$, the rest of the PL dynamics (diffusion, linear, and nonlinear interactions) is simulated deterministically for $\phi(x)$ for duration Δt and then the stochastic step is taken again. There are many possible schemes to implement the deterministic update, with the minimalistic requirement being that it be first order accurate in time. An additional desideratum is that it be stable for all Δt, so as to not to constrain the size of the time step. In their original work, PL used an Euler scheme for the deterministic update.

In practice, to ensure stability, we implement a diffusion step using an alternating-direction implicit (ADI) [12] update, followed by a Runge-Kutta update of the reaction terms. We verified that similar results are obtained by other deterministic update schemes as long as Δt is small enough to guarantee stability.

We now turn to a description of the DCM method. In the context of their method of performing the stochastic update discussed above, DCM additionally noted that this method could be generalized to include arbitrary constant source and linear growth terms. They chose to take advantage of this to incorporate the linear growth term present in the Langevin equation, as well as an approximate first order in time approximation of the diffusion term. DCM decomposed the second-difference diffusion operator $D(\phi_{i+1} - 2\phi_i + \phi_{i-1})/2(\Delta x)^2$, into two pieces, one a local decay term $-2D\phi_i/(\Delta x)^2$ and the second a source term $D(\phi_{i+1} + \phi_{i-1})/2(\Delta x)^2$ which is approximated to be constant during the interval $(t_0, t_0 + \Delta t)$. Denoting the linear growth rate of ϕ by α, this gives rise to a different Langevin equation for the "noise step":

$$\frac{\partial \phi}{\partial t}(x,t) = \left[\alpha - \frac{2D}{(\Delta x)^2}\right]\phi(x,t) + S(x) + \sigma\sqrt{\phi(x,t)}\eta;$$

$$S(x) = \frac{D}{(\Delta x)^2}[\phi(x + \Delta x, t_0) + \phi(x - \Delta x, t_0)]. \tag{4}$$

This (approximate) local Langevin equation is also exactly solvable, and again involves generating a Poisson variate Q at each site with mean $\lambda\phi\ \exp(\nu\Delta t)$,where now

$$\lambda = \frac{2\nu}{\sigma^2[\exp(\nu\Delta t) - 1]}; \quad \nu = \alpha - 2D/(\Delta x)^2, \tag{5}$$

followed by a gamma deviate, R, with shape parameter $Q + 2S/\sigma^2$. The new $\tilde{\phi}$ is then again $R\lambda$. The remaining nonlinear dynamics is then implemented deterministically for an interval Δt, similar to the deterministic update in PL.

III. STOCHASTIC GINZBURG-LANDAU EQUATION

As discussed in the Introduction, the origins of this study lay in some anomalies in the results presented in the paper of Martin *et al.*, obtained simulating a model via the DCM prescription. The model studied by Martin *et al.* is described by the two-dimensional Langevin equation

$$\frac{\partial \phi}{\partial t} = D\nabla^2\phi + \alpha\phi + \beta\phi^2 - \gamma\phi^3 + \sigma\sqrt{\phi}\eta, \tag{6}$$

where, as usual, η is a unit-strength zero-mean white noise. Let us first review the features of the deterministic dynamics, $\sigma = 0$. α controls the degree of stress in this model, with the stress increasing as α is lowered. If α is positive (low stress), the state $\phi = 0$ is unstable and there is only one stable state at $\bar{\phi} = (\beta + \sqrt{\beta^2 + 4\alpha\gamma})/(2\gamma)$. When $\alpha < 0$ (high stress), the zero state is stable, but the active state invades as long as α is above the Maxwell (stall) point, $\alpha_{\rm MP} = -2\beta^2/9\gamma$. If $\alpha < \alpha_{\rm MP}$, the inactive phase invades the active one. Finally, below $\alpha_T = -\beta^2/4\gamma$, the active phase loses its stability and the only stable solution is at $\phi = 0$. α_T is thus the tipping point below which the deterministic active state collapses (i.e., the local deterministic dynamics does not support an active state).

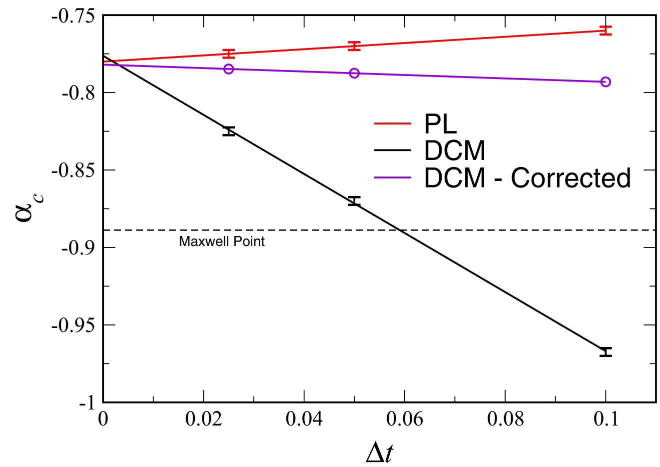

FIG. 1. The location α_s of the stochastic stall point for the active-inactive phase boundary for the two-dimensional GL model as a function of Δt, with $D = 1$, $\beta = 2$, $\gamma = 1$, $\sigma^2 = 1$. Also shown (dotted line) is the value of the deterministic Maxwell point, which is crossed by the DCM results as Δt is varied. The system size is $2^7 \times 2^7$. The diffusion step of the PL method was done using ADI.

Martin *et al.* simulated this system using the DCM algorithm with $\Delta t = 0.1$ [13] for the parameters $D = 1$, $\beta = 2$, $\gamma = 1$, $\sigma^2 = 1$, for various values of α. Tracking the motion of the phase boundary in a system initially divided lengthwise between the active and inactive phases, they found a transition between active phase invasion and inactive phase invasion at a value of $\alpha_s = -0.9640$. We denote this transition point as the stochastic stall point α_s. As noted above, this is *below* the deterministic Maxwell point for these parameters of $\alpha_{\rm MP} = -0.889$. To examine the sensitivity of the measured α_s to the time step Δt, we repeated the DCM simulations for a variety of time steps, the results of which are presented in Fig. 1. We reproduced the result of Martin *et al.* for $\Delta t = 0.1$, and found that as Δt is decreased, α_s increases, crossing the Maxwell point, and linearly converging to a value of $\alpha_s = -0.775$. Thus, the DCM calculation of α_s using $\Delta t = 0.1$ is off by 28%.

It was then natural to inquire whether the PL simulations show the same anomalies. The results of this calculation for the Martin parameters, labeled PL, are shown in Fig. 1 alongside the DCM results. The results converge linearly as $\Delta t \to 0$ to the same limiting value as the DCM simulations. That said, we see that the PL value of α_s is above $\alpha_{\rm MP}$ for all Δt's examined. Thus, the finite Δt error was of the opposite sign of the DCM error, and was a factor of 11 smaller in magnitude.

Since the two methods, while built on the same general principles, have such widely different errors, we proceeded to inquire as to the underlying mechanism behind this effect, and how general the phenomenon is. A key insight into this problem is obtained by examining the dependence on the noise strength σ. When we lower σ (at fixed $\Delta t = 0.1$) in the PL framework, the value of α_s decreases, approaching $\alpha_{\rm MP} = -0.889$ from above. This is as expected, in that the noise should favor the inactive state. Lowering σ within the DCM scheme also lowers α_s, but in this case, it is moving yet further from $\alpha_{\rm MP}$, as seen in Fig. 2. Lowering σ further towards zero, α_s is seen to continue to decrease, approaching

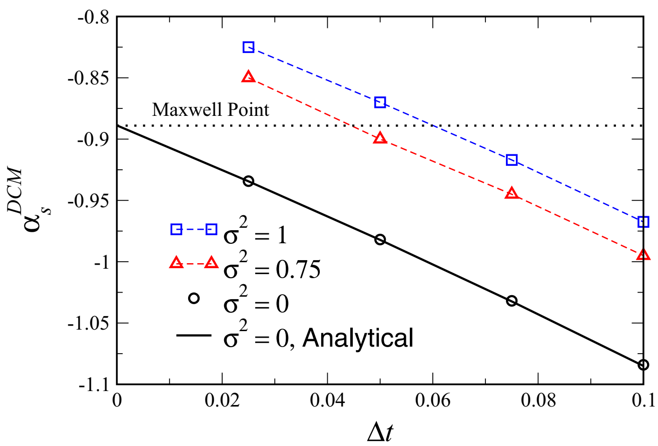

FIG. 2. The location of the stochastic stall point α_s for the DCM scheme as a function of Δt for $\sigma^2 = 1$, 3/4, and 0, the latter being $\alpha_{\rm MP}^{\rm DCM}(\Delta t)$. $D = 1$, $\beta = 2$, $\gamma = 1$. The finite σ^2 results are seen to be approaching the deterministic result for the corresponding Δt. Also shown is the analytical approximation from Eq. (10), as well as the true Maxwell point, $\alpha_{\rm MP}$, for reference.

some limiting value below $\alpha_{\rm MP}$ as $\sigma \to 0$. This leads us to conclude that even in the deterministic model, when done with the DCM operator breakup, there is a Δt dependent shift in the value of the deterministic Maxwell point, so that we have that $\alpha_{\rm MP}^{\rm DCM} = \alpha_{\rm MP}^{\rm DCM}(\Delta t)$. We have verified this directly by simulating the deterministic version of the DCM algorithm, first treating the quadratic and cubic reaction terms together via Runge-Kutta, and then doing a combined update of the linear term and the approximate diffusion term. This latter update is

$$\phi_i = \frac{S}{\nu}(e^{\nu\Delta t} - 1) + \phi_i^0 e^{\nu\Delta t}, \tag{7}$$

where $S = D/(\Delta x)^2 \sum_{j\in nn} \phi_j^0$ and $\nu = \alpha - 2Dd/(\Delta x)^2$, where d is the spatial dimensionality. The results of this calculation are presented in Fig. 2. We see that indeed the deterministic MP in the DCM scheme has acquired a Δt dependent shift toward more negative values. Thus, it is not that α_s is on the high-stress side of $\alpha_{\rm MP}^{\rm DCM}$, it is simply that, as we have seen, for the DCM algorithm, $\alpha_{\rm MP}^{\rm DCM}(\Delta t)$ itself has moved to due to the finite value of Δt, an effect not present in PL.

IV. DETERMINISTIC MAXWELL POINT IN DCM

We have seen that the finite Δt shift in the stochastic stall velocity, α_s, is, in DCM, essentially the same in sign and magnitude as the shift in the deterministic Maxwell point $\alpha_{\rm MP}^{\rm DCM}(\Delta t)$. This suggests that this shift has nothing to do with the noise, and its origin comes from the operator-splitting procedure itself.

Lee *et al.* [11] have presented a scheme that allows one to compare two splittings of the local linear (growth and diffusion outflow) terms, including the effect of noise. Their scheme provides the renormalized values of the reaction-diffusion constants and noise amplitude of one splitting procedure in terms of those of the other one and the time step Δt. Since we are interested in the shift in the deterministic Maxwell point from its exact value induced by the DCM splitting, it is simpler to obtain it directly by considering performing a DCM update on a spatially uniform state ϕ^0.

Obviously, diffusion should play no role in this case. However, implementing the combined diffusion and linear reaction step, Eq. (7) here yields

$$\phi^1 = \phi^0\left[\frac{2D\nu}{\nu(\Delta x)^2}(e^{\nu\Delta t} - 1) + e^{\nu\Delta t}\right] \tag{8}$$

which is clearly D dependent and not equal to the exact result $e^{\alpha\Delta t}\phi^0$. Expanding for small Δt, we get

$$\phi^1 = \phi^0\left[1 + \alpha\Delta t + (\alpha^2 - 2dD\alpha)\frac{(\Delta t)^2}{2}\right]. \tag{9}$$

In principle, we need to consider also the implementation of the separate update of the nonlinear reaction terms as well. However, as we shall see, this latter update does not introduce a further shift of the Maxwell point, and we postpone it for the moment. Just considering the linear or diffusion update, we see that for $\alpha < 0$, the updated value ϕ^1 is larger than it should be by an amount $dD\alpha\Delta t^2\phi^0$. A convenient way to think of this, along the lines of Lee *et al.*, is that there is an effective $\alpha_{\rm eff}(\Delta t) \approx \alpha(1 - dD\Delta t)$. This can be shown to agree exactly with the shift of α between the PL and DCM splittings, using an extension of the Lee *et al.* results (due to the incorporation of the nonlocal diffusional source term in the deterministic update in PL, which is not considered in their work), as the shift is completely absent in the PL splitting, the diffusion there having no effect on a constant state. This effective shift in α induces a shift in the Maxwell point, so that the DCM Maxwell point is at $\alpha_{\rm eff} = -2\beta^2/(9\gamma)$, or $\alpha_{\rm MP}^{\rm DCM} \approx -2\beta^2/9c/(1 - Dd\Delta t)$. Since we are working at $D = 1$, $d = 2$, and $\Delta t = 0.1$, this amounts to a 20% increase in the magnitude of $\alpha_{\rm MP}$, consistent with our numerics.

Moreover, we can extend this concept of $\alpha_{\rm eff}$ to all values of Δt, which gives that $\alpha_{\rm MP}^{\rm DCM}(\Delta t)$ is given implicitly by

$$e^{-2\beta^2\Delta t/9\gamma} = \left[\frac{2Dd}{\nu(\Delta x)^2}(e^{\nu\Delta t} - 1) + e^{\nu\Delta t}\right];$$
$$\nu = \alpha_{\rm MP}^{\rm DCM}(\Delta t) - 2Dd/(\Delta x)^2. \tag{10}$$

This is indicated as well in Fig. 2, showing excellent agreement. It should be noted that the tipping point is also similarly shifted in the DCM scheme, so that it remains safely on the high-stress side of the Maxwell point, as it physically reasonable.

To check that the treatment of the nonlinear terms does not induce a further shift, we note that using a second-order update of the nonlinear terms $\mathcal{N}(\phi) = \beta\phi^2 - \gamma\phi^3$ yields a final value of ϕ of

$$\begin{aligned}\phi^2 &= \phi^1 + \Delta t\mathcal{N}(\phi^1) + \frac{(\Delta t)^2}{2}\mathcal{N}(\phi^1)\mathcal{N}'(\phi^1)\\ &\approx \phi^1 + \Delta t\mathcal{N}(\phi^1) + \frac{(\Delta t)^2}{2}\mathcal{N}(\phi^0)\mathcal{N}'(\phi^0)\\ &\approx \phi^0 + \Delta t[\alpha\phi^0 + \mathcal{N}(\phi^0)] + \frac{(\Delta t)^2}{2}[(\alpha^2 - 2dD\alpha)\phi^0\\ &\quad + \mathcal{N}'(\phi_0)(2\alpha\phi^0) + \mathcal{N}(\phi)\mathcal{N}'(\phi^0)].\end{aligned} \tag{11}$$

This is to be compared to the exact second-order result

$$\phi^2 = \phi^0 + \Delta t[\alpha\phi^0 + \mathcal{N}(\phi^0)] + \frac{(\Delta t)^2}{2}[\alpha\phi^0 + \mathcal{N}(\phi^0)][\alpha + \mathcal{N}'(\phi^0)]$$
$$\approx \phi^0 + \Delta t[\alpha\phi^0 + \mathcal{N}(\phi^0)] + \frac{(\Delta t)^2}{2}[\alpha^2\phi^0 + \alpha\phi^0\mathcal{N}'(\phi_0) + \alpha\mathcal{N}(\phi^0) + \mathcal{N}(\phi^0)\mathcal{N}'(\phi^0)]. \tag{12}$$

We see that in addition to the extra term $-dD\alpha\phi^0(\Delta t)^2$ noted above, there is an additional term arising from the splitting off of the linear reaction term from the nonlinear ones, equal to $(\Delta t)^2\alpha/2[\phi^0\mathcal{N}'(\phi^0) - \mathcal{N}(\phi^0)]$. The additional terms in ϕ^2 can be considered to this order as arising from additional reaction terms in the differential equation

$$\Delta R = (\Delta t)\left\{-dD\alpha\phi + \frac{\alpha}{2}[\phi\mathcal{N}'(\phi) - \mathcal{N}(\phi)]\right\}. \tag{13}$$

These additional reaction terms cause a shift in ϕ_{eq}, the equilibrium value of ϕ, even for $D = 0$. For example, at the (true) Maxwell point $\alpha = \alpha_{\rm MP} = -2\beta^2/(9\gamma)$, the $D = 0$ shift amounts to a shift in ϕ_{eq} of $2\beta^3(\Delta t)/(27\gamma^2)$. To calculate the shift in the Maxwell point itself, we note that the added reaction terms ΔR give rise to an additional effective potential ΔV, where $\Delta R = -\Delta V'$, of

$$\Delta V = \frac{\Delta t}{2}[dD\alpha\phi^2 - \alpha\phi\mathcal{N}(\phi) - 2\alpha V_{\mathcal{N}}(\phi)]. \tag{14}$$

Here, $V_{\mathcal{N}}$ is the effective potential due to the nonlinear reaction terms. The Maxwell point is the value of α at which the effective potential $V(\phi_{eq}) = 0$. The shift in ϕ_{eq} does not contribute to a shift in the value of the effective potential to leading order since the derivative of V at an equilibrium point vanishes. At the unshifted equilibrium ϕ^0_{eq},

$$\alpha\phi^0_{eq} = -\mathcal{N}\left(\phi^0_{eq}\right),$$
$$\frac{\alpha\left(\phi^0_{eq}\right)^2}{2} = V_{\mathcal{N}}. \tag{15}$$

Substituting these in Eq. (14) leads to the cancellation of the change in $V(\phi_{eq})$ due to the nonlinear reaction terms, so the entire shift in the Maxwell point is indeed due to the particular mixing of the linear growth and diffusion terms in the DCM scheme.

It should be noted that, if instead of treating the nonlinear reaction terms separately in the DCM scheme, they were incorporated as constant source terms, akin to the treatment of the off-site diffusion terms, there would be no shift in the Maxwell point at all. In this case, there is only one update step which handles all the terms, and so the total update for a constant ϕ^0 would be

$$\phi^1 = \frac{1}{\nu}\left[\frac{2D}{(\Delta x)^2}\phi^0 + \mathcal{N}(\phi^0)\right](e^{\nu\Delta t} - 1) + e^{\nu\Delta t}\phi^0$$
$$\approx \phi^0 + (\Delta t)[\alpha\phi^0 + \mathcal{N}(\phi^0)] + \frac{\nu(\Delta t)^2}{2}[\alpha\phi^0 + \mathcal{N}(\phi^0)]. \tag{16}$$

The additional reaction term in this case is

$$\Delta R = \frac{(\Delta t)}{2}\left\{-\frac{2D}{(\Delta x)^2}[\alpha\phi + \mathcal{N}(\phi)] - \alpha\phi\mathcal{N}'(\phi) - \mathcal{N}(\phi)\mathcal{N}'(\phi)\right\}. \tag{17}$$

Here, there is no shift in ϕ_{eq} since ΔR is proportional to the original reaction term $\alpha\phi + \mathcal{N}(\phi)$. The shift in the effective potential reads as

$$\Delta V = \frac{(\Delta t)}{2}\left\{\frac{2D}{(\Delta x)^2}\left[\frac{\alpha\phi^2}{2} + V_{\mathcal{N}}(\phi)\right] + \alpha[\phi\mathcal{N}(\phi) + V_{\mathcal{N}}(\phi)] + \frac{\mathcal{N}^2(\phi)}{2}\right\}. \tag{18}$$

Using Eqs. (15), we see that $\Delta V(\phi^0_{eq}) = 0$, and so here there is no first-order shift to the Maxwell point, either, as in the PL approach. We have verified this point by direct numerical simulation as well (not shown).

It is intriguing to ask what effect presenting the DCM stochastic results in terms of $\alpha_{\rm eff}$ instead of α has. This is presented in Fig. 1 as the curve DCM corrected. We see that the remaining finite-Δt effects are roughly the same size as those of the PL method, and so the use of $\alpha_{\rm eff}$ serves as a simple ex post facto method of correcting the DCM algorithm.

Thus, we have seen that the joining together of the approximate diffusion operator and the linear term in the DCM scheme leads to the introduction of an $\alpha_{\rm eff}$. Any scheme, in particular the PL scheme, where the diffusion has no effect on a uniform state would not suffer from this problem.

V. DISCUSSION

We have seen that the DCM method captures the correct universal behavior and is exact in the $\Delta t \to 0$ limit. The apparent anomalies observed are due to the unexpected finite-Δt shift in the location of the deterministic Maxwell point, and so of the stochastic stall point as well. This is why in the Martin *et al.* simulation the stochastic stall point appeared to move away from the deterministic Maxwell point with decreasing noise, which is physically unacceptable,

To verify this specific point in the two-dimensional (2D) scenario considered in Martin *et al.*, we have measured the location of the stochastic stall point as a function of Δt for the set of parameters singled out by Martin *et al.*, where the stochastic transition was on the high-stress side of the deterministic Maxwell (stall) point. As seen in Fig. 1, and consistent with what we have seen above, the DCM transition point has a strong dependence of Δt and indeed crosses over to the correct side of the deterministic Maxwell point in the $\Delta t \to 0$ limit. The PL transition point has a very much weaker dependence on Δt and only slightly overestimates the effect of stochasticity. We are currently engaged in a detailed study of the phase diagram of the two-dimensional system using the more quantitatively (and qualitatively) reliable PL method.

We note in passing that there is no similar shift in the deterministic phase transition point of the stochastic Fisher equation ($\beta < 0, \gamma = 0$) since this occurs at $\alpha = 0$, and the shift in α is multiplicative and so vanishes at $\alpha = 0$. We also note that when measuring the finite Δt corrections for the stochastic Fisher equation, we failed to obtain the better than 1% accuracy observed by DCM for the α of the stochastic transition with their reported $\Delta t = 0.25$; instead, we obtained this accuracy only for a much smaller time step $\Delta t = 0.025$.

On its face, incorporating the diffusion into the stochastic term could be expected to improve the accuracy since one expects that the more terms one can handle analytically the better. However, the particular breakup employed in the DCM approach leads to a mixing of the linear decay with the diffusion and weakens the decay, thus favoring the active state. The PL method and a modified DCM method where the diffusion terms are incorporated in the deterministic update do not suffer this defect.

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