

**Fidelity susceptibility of the anisotropic XY model: The exact solution**Qiang Luo,<sup>1,\*</sup> Jize Zhao,<sup>2</sup> and Xiaoqun Wang<sup>3,4,†</sup><sup>1</sup>*Department of Physics, Renmin University of China, Beijing 100872, China*<sup>2</sup>*Center for Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, China*<sup>3</sup>*Key Laboratory of Artificial Structures and Quantum Control, Ministry of Education, School of Physics and Astronomy, Tsung-Dao Lee Institute, Shanghai Jiao Tong University, Shanghai 200240, China*<sup>4</sup>*Collaborative Innovation Center for Advanced Microstructures, Nanjing 210093, China*

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We derive several closed-form expressions for the fidelity susceptibility (FS) of the anisotropic XY model in the transverse field. The basic idea lies in a partial fraction expansion of the expression so that all the terms are related to a simple fraction or its derivative. The critical points of the model are reiterated by the FS, demonstrating its validity for characterizing the phase transitions. Moreover, the critical exponents  $\nu$  associated with the correlation length in both critical regions are successfully extracted by the standard finite-size scaling analysis.

DOI: [10.1103/PhysRevE.98.022106](https://doi.org/10.1103/PhysRevE.98.022106)**I. INTRODUCTION**

Quantum phase transitions (QPTs), which are driven solely by quantum fluctuations and are characterized by drastic changes in the ground state, are of great interest for the interpretation of widespread phenomena in physics [1,2]. Over the past few decades, numerous theoretical methods for detecting QPTs have been introduced regarding the aspect of quantum information sciences [3–9]. The ground-state fidelity  $F$  [10] is one of such methods. It measures the overlap between two wave functions of the same Hamiltonian but at different values of the control parameter  $\lambda$ . As a result, a notable change in the fidelity is expected to occur at the transition point  $\lambda_c$  even for a finite-size system. However, the fidelity is sometimes chaotic numerically in that it depends on the increment of the control parameter and it vanishes exponentially with an increase of the system size. Therefore, the fidelity susceptibility (FS)  $\chi_F$  [11–13], the derivation of the fidelity with respect to  $\lambda$ , is introduced to eliminate such drawbacks and turns out to be more powerful. Technically, the FS is nothing but the nontrivial leading quadratic term of the fidelity, so its divergence at the transition point is reminiscent of the singularity of the latter. The past 15 years have witnessed the explosive applications of fidelity and FS to the QPT of various strongly correlated systems [14–25], including the intricate Berezinskii-Kosterlitz-Thouless transition [26–28] and the unconventional topological phase transition [29–34].

Historically, the plausible evidence of the FS as a probe for quantum criticality is revealed by the similarity between the scaling behavior of the FS at the critical point and that of the second derivative of the ground-state energy, where the transverse-field Ising model (TFIM) is illustrated as an example [35,36]. Following standard arguments in the

scaling theory of a continuous QPT, one obtains that the FS per site in a  $d$ -dimensional system with length  $L$  scales as [37–39]

$$\chi_F(\lambda)/L^d \sim L^{2/\nu-d} f(|\lambda - \lambda_c|L^{1/\nu}), \quad (1)$$

where  $\nu$  is the critical exponent of the correlation length and  $f(\cdot)$  is a scaling function. It is inferred that near the critical point the scaling expression behaves as  $\chi_F(\lambda_c)/L^d \sim L^{k_1}$  ( $k_1 = 2/\nu - d$ ). Alternately, one may look at  $\chi_F$  slightly away from the critical point at the thermodynamic limit (TDL), where  $\chi_F(\lambda)/L^d \sim |\lambda - \lambda_c|^{k_2}$  ( $k_2 = d\nu - 2$ ). Consequently, the scaling ansatz in the system exhibiting logarithmic divergences requires that the absolute value of the ratio  $k_2/k_1$  is the critical exponent  $\nu$  [37–39]. The critical exponent  $\nu$  is usually calculated numerically because of the absence of analytical expression for FS. The main obstacle lies in that no algebraic technique is available to obtain its closed form at finite length. A breakthrough was made on the single-parameter TFIM by Damski and Rams in light of several elegant summation formulas [40–42]. This method, however, is difficult to follow and can hardly be generalized to a double-parameter XY model where a quadratic summation is explicitly involved [10,43]. Notably, after a proper symmetry analysis, we find that the partial fraction expansion method can be used to divide the quadratic term into two coupled linear terms. As a result, all the terms are related to a simple fraction or its derivative (see Appendixes A and B) that can be treated exactly in principle.

The remainder of the paper is organized as follows. In Sec. II we introduce the anisotropic XY model in the transverse field and give the expressions for the FS. In Sec. III the closed form of the expressions are presented in detail and the critical exponent  $\nu$  is calculated analytically by the scaling ansatz. Section IV is devoted to a summary.

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## II. MODEL

We calculate analytically the FS of the one-dimensional spin- $\frac{1}{2}$  anisotropic  $XY$  model in the transverse field [44]

$$\hat{H} = - \sum_{n=1}^{\mathcal{N}} \left( \frac{1+\gamma}{2} \sigma_n^x \sigma_{n+1}^x + \frac{1-\gamma}{2} \sigma_n^y \sigma_{n+1}^y + h \sigma_n^z \right), \quad (2)$$

where  $\sigma_n^\alpha$  ( $\alpha = x, y, z$ ) is the  $\alpha$  component of the Pauli operator acting on site  $n$ ,  $\gamma$  is the anisotropy parameter at the  $xy$  plane, and  $h$  is the external field in the  $z$  direction. The number of sites  $\mathcal{N} = 2N$  is assumed to be even for brevity and both symbols  $N$  and  $\mathcal{N}$  are used throughout the paper. Here a periodic boundary condition ( $\sigma_{\mathcal{N}+1} = \sigma_1$ ) is imposed so that Eq. (2) can be diagonalized exactly via the Jordan-Wigner transformation. The physical ground state depends strongly on the proper choice of momentum quantization, which results in a positive- or negative-parity sector [45,46]. The ground state could be determined by a competition between the vacuum states of the two parity sectors at given parameters [45]. Interestingly, the energy gap, obtained from the two states, shows a rather anomalous behavior [46]. For even  $\mathcal{N}$ , it is argued that the physical ground state lies in the positive sector at least outside the disordered circle  $h^2 + \gamma^2 = 1$ . In the circle, the energy gap vanishes rapidly near  $\gamma = 0$ . So for simplicity we will only consider the momentum quantization in the positive sector hereafter. The single-particle energies are given by  $\Lambda_k = \sqrt{\epsilon_k^2 + \gamma^2 \sin^2 k}$ , where  $\epsilon_k = \cos k - h$ ,  $k = (2n-1)\pi/\mathcal{N}$  ( $n = 1, 2, \dots, N$ ) [45,46]. With  $\cos \theta_k = \epsilon_k/\Lambda_k$  in mind, we have the analytic expressions for the FS as  $\chi^{(q)}(h, \gamma) = \frac{1}{4} \sum_{n=1}^{\mathcal{N}} (\partial \theta_k / \partial q)^2$  with  $q = h$  and  $\gamma$  [10]. Consequently, the explicit expressions for the FS with respect to the two kinds of QPTs (see below) are [10]

$$\chi^{(h)}(h, \gamma) = \frac{1}{4} \sum_{k>0} \frac{\gamma^2 \sin^2 k}{[(\cos k - h)^2 + \gamma^2 \sin^2 k]^2} \quad (3)$$

and

$$\chi^{(\gamma)}(h, \gamma) = \frac{1}{4} \sum_{k>0} \frac{\sin^2 k (\cos k - h)^2}{[(\cos k - h)^2 + \gamma^2 \sin^2 k]^2}. \quad (4)$$

Actually, the FS defined above is none other than the diagonal element of a more general concept, the quantum metric tensor [47]. The expression for the remaining off-diagonal element is shown in Appendix C.

As shown in Fig. 1, the model has a richer phase diagram when compared with the TFIM, a special case of Eq. (2) at  $\gamma = 1$ . There are four different phases in the  $(h, \gamma)$  plane, which are separated by the lines  $h = \pm 1$  and by the segment  $|h| < 1, \gamma = 0$ . The corresponding QPTs are referred to as the Ising transition and the anisotropy transition, respectively. The Ising critical lines are the boundaries between ferromagnetic phases and paramagnetic phases, whereas the anisotropy transition separates the ferromagnets with spins in the  $x$  and  $y$  directions. Both Ising and anisotropy transitions share the same critical exponent  $\nu = 1$  [48,49], and we intend to calculate it analytically according to expressions (3) and (4) in the following section. There are two fascinating curves in Fig. 1. The circular curve ( $h^2 + \gamma^2 = 1$ , dashed pink line) separates the regions with oscillatory and nonoscillatory correlations

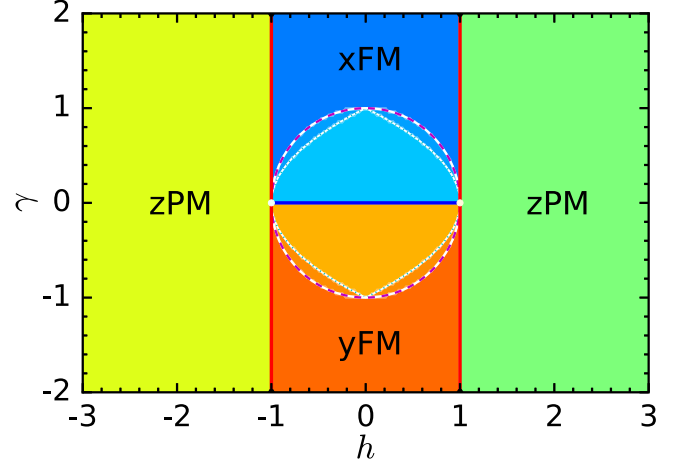


FIG. 1. Ground-state phase diagram of the anisotropic  $XY$  model in the transverse field at the  $(h, \gamma)$  plane. There are paramagnetic phases with opposing orientation along the  $z$  direction when  $|h| > 1$  and ferromagnetic phases when  $|h| < 1$ . There are two fascinating curves in the ferromagnetic phases (see the main text) [44].

asymptotic behaviors, while the parabolic curve ( $\gamma^2 \pm h = 1$ , dotted cyan line) is the boundary between commensurate and incommensurate phases [50]. It is worth mentioning that the critical exponents at the multicritical points obey a different universality [51] and is beyond the scope of the present paper.

## III. EXACT SOLUTIONS

### A. Ising transition

From Eq. (3) we can rewrite the FS for the  $XY$  model with respect to the external magnetic field  $h$  as

$$\chi^{(h)}(h, \gamma) = \frac{\gamma^2}{4(1-\gamma^2)^2} \sum_{n=1}^{\mathcal{N}} \frac{1-c_n^2}{[c_n^2 - \frac{2h}{1-\gamma^2}c_n + \frac{h^2+\gamma^2}{1-\gamma^2}]^2}, \quad (5)$$

where  $c_n = \cos \frac{(2n-1)\pi}{\mathcal{N}}$ . Hereafter,  $|\gamma| < 1$  is assumed to avoid possible ambiguity and we note that our main results remain unchanged for arbitrary  $\gamma$ . For example, our results are still valid for  $|\gamma| = 1$  since the FS is continuous when crossing the lines. The summation in Eq. (5) is not easy to handle directly due to the existence of the quadratic terms. To eliminate them, we here employ a factorization method. It can be found that when  $h^2 + \gamma^2 > 1$  we can factorize  $\phi(t) = t^2 - \frac{2h}{1-\gamma^2}t + \frac{h^2+\gamma^2}{1-\gamma^2} = (t - \lambda_+)(t - \lambda_-)$ , where  $\lambda_{\nu}$ 's ( $\nu = \pm$ ) are the real roots of the equation  $\phi(t) = 0$  and

$$\lambda_{\nu} = \frac{h + \nu|\gamma|\sqrt{h^2 + \gamma^2 - 1}}{1 - \gamma^2}. \quad (6)$$

The signs of the two roots  $\lambda_{\nu}$  are the same and are consistent with the sign of the field  $h$ . In addition, their absolute values are both larger than 1 so long as  $|h| \neq 1$  (otherwise the smaller one equals 1). When  $h^2 + \gamma^2 < 1$ , however, the roots are complex and the imaginary unit  $i = \sqrt{-1}$  should be involved. By virtue of the partial fraction expansion of the expression

$\frac{1-c_n^2}{(c_n-\lambda_+)^2(c_n-\lambda_-)^2}$ , Eq. (5) is recast into the symmetric form

$$\chi^{(h)}(h, \gamma) = \frac{1}{16(h^2 + \gamma^2 - 1)} \sum_{v=\pm} \mathcal{F}_v^{(h)}, \quad (7)$$

where

$$\mathcal{F}_v^{(h)} = (\lambda_v^2 - 1)L'(\lambda_v) - \frac{2(\lambda_v\lambda_{\bar{v}} - 1)}{(\lambda_v - \lambda_{\bar{v}})}L(\lambda_v), \quad (8)$$

with  $\bar{v}$  the complementary component of the branch  $v$  ( $\bar{v}v = -1$ ). In Eq. (8), the complex analytic function  $L(\alpha) = \sum_{n=1}^N \frac{1}{c_n + \alpha}$  ( $\alpha \in \mathbb{C}$ ) is introduced and the odevity of it and its derivative are utilized as well. The highlight of the paper is that the closed form of  $L(\alpha)$  is discovered, making it possible to have exact expressions for FS. The mathematical details are omitted here and are presented in Appendix A; the relation between  $L'(\alpha)$  and  $L(\alpha)$  is shown in Appendix B. According to the formula presented in Appendix A, it is useful to have  $g_v = \lambda_v + \sqrt{\lambda_v^2 - 1}$  so that

$$g_v = \frac{h + vp_v\sqrt{h^2 + \gamma^2 - 1}}{1 - p_v|\gamma|}, \quad (9)$$

where  $p_v = p_v(h)$  is a piecewise sign function of the field  $h$  and branch  $v$ ,

$$p_v = \begin{cases} v, & |h| > 1 \\ \text{sgn}(h), & |h| < 1. \end{cases} \quad (10)$$

The relevant correlation length near the critical point reads  $\xi = 1/|\ln g_+|$  [44], indicating that it is the positive branch of  $\mathcal{F}_v^{(h)}$  ( $v = 1$ ) that gives rise to the divergence behavior of the FS in the TDL. The explicit expression of  $\mathcal{F}_v^{(h)}$  is calculated analytically as

$$\mathcal{F}_v^{(h)} = \frac{\mathcal{N}^2 g_v^{\mathcal{N}}}{(g_v^{\mathcal{N}} + 1)^2} + \frac{\mathcal{N} C_v g_v^{\mathcal{N}} - 1}{2 g_v^{\mathcal{N}} + 1}, \quad (11)$$

with the expression

$$C_v = \frac{p_v}{h^2 - 1} \left[ \frac{v\gamma^2 h}{\sqrt{h^2 + \gamma^2 - 1}} - \frac{h^2 + \gamma^2 - 1}{|\gamma|} \right]. \quad (12)$$

We note here the detailed derivations are presented in the Supplemental Material (SM) [52].

We now consider the FS at special values. When  $\gamma = 1$ , the XY model is reduced to the TFIM and the FS has the form

$$\begin{aligned} \chi^{(h)}(h, 1) &= \frac{\mathcal{N}}{16h^2} \\ &\times \left[ \frac{\mathcal{N}h^{\mathcal{N}}}{(h^{\mathcal{N}} + 1)^2} + \frac{1}{2} \left( \frac{(h^2 + 1)(h^{\mathcal{N}} - 1)}{(h^2 - 1)(h^{\mathcal{N}} + 1)} - 1 \right) \right], \end{aligned} \quad (13)$$

which agrees with the result by finite sums of hyperbolic functions [40]. When  $h = h_c$ , i.e.,  $|h| = 1$ , the FS is

$$\begin{aligned} \chi^{(h)}(h_c, \gamma) &= \frac{\mathcal{N}^2 - \frac{3-\gamma^2}{2\gamma}\mathcal{N}}{32\gamma^2} \\ &+ \frac{\mathcal{N} \left[ \left( \mathcal{N} + \frac{3-\gamma^2}{2\gamma} \right) \left( \frac{1+\gamma}{1-\gamma} \right)^{\mathcal{N}} + \frac{3-\gamma^2}{2\gamma} \right]}{16\gamma^2 \left[ \left( \frac{1+\gamma}{1-\gamma} \right)^{\mathcal{N}} + 1 \right]^2}. \end{aligned} \quad (14)$$

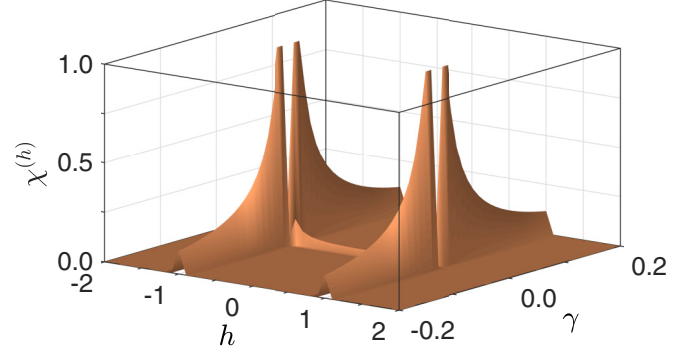


FIG. 2. Normalized fidelity susceptibility  $\chi^{(h)}$  with respect to  $h$ . It has a sharp extremum at the critical lines  $|h| = 1$  where continuous QPTs with  $v = 1$  occur.

The first term of Eq. (14) is the leading term, while the second term is either exponential decay or linearly increased with the system size  $\mathcal{N}$ , depending on the sign of  $\gamma$ . In the finite-size system, the maximal value of the FS  $\chi^{(h)}(h = h_m, \gamma)$  is slightly larger than that of  $\chi^{(h)}(h = h_c, \gamma)$ , but with the quadratic term of  $\mathcal{N}$  unchanged. Therefore,  $\chi^{(h)}(h_m, \gamma) \simeq \chi^{(h)}(h_c, \gamma) \simeq \frac{\mathcal{N}}{32\gamma^2} (\mathcal{N} - \frac{3-\gamma^2}{2|\gamma|})$ . Considering only the leading term, we have

$$\ln \left( \frac{\chi^{(h)}(h_m, \gamma)}{\mathcal{N}} \right) = \ln \mathcal{N} - \ln(32\gamma^2). \quad (15)$$

In the TDL, the exact expression for the FS can be calculated by the residue theorem and turns out to be [53]

$$\frac{\bar{\chi}^{(h)}(h, \gamma)}{\mathcal{N}} = \frac{1}{16} \times \begin{cases} \frac{1}{|\gamma|(1-h^2)}, & |h| < 1 \\ \frac{|h|\gamma^2}{(h^2-1)(h^2+\gamma^2-1)^{3/2}}, & |h| > 1 \end{cases} \quad (16)$$

and the scaling behavior around the critical points is

$$\ln \left( \frac{\bar{\chi}^{(h)}(h, \gamma)}{\mathcal{N}} \right) = -\ln |h - h_c| - \ln(32\gamma). \quad (17)$$

It can be noticed that the absolute value of the prefactors of Eqs. (15) and (17) are equal, indicating that the critical component  $v = 1$  according to the ansatz (1). The normalized FS  $\bar{\chi}^{(h)}(h, \gamma)$  with respect to  $h$  in the TDL is shown in Fig. 2. It is found that the FS has a sharp extremum at  $|h| = 1$  where a continuous QPT occurs. This result demonstrates convincingly that FS can be used to characterize the quantum critical behavior.

## B. Anisotropy transition

Calculation for the FS with respect to the anisotropy transition is similar to the Ising transition discussed above. To begin with, we should rewrite the expression (4) as

$$\chi^{(\gamma)}(h, \gamma) = \frac{1}{4(1-\gamma^2)^2} \sum_{n=1}^N \frac{(1-c_n^2)(c_n-h)^2}{(c_n-\lambda_+)^2(c_n-\lambda_-)^2}, \quad (18)$$

where the  $\lambda'_v$ s ( $v = \pm$ ) are defined in Eq. (6). Once the partial fraction expansion of the expression  $\frac{(1-c_n^2)(c_n-h)^2}{(c_n-\lambda_+)^2(c_n-\lambda_-)^2}$  is obtained, the FS is readily split into two symmetric forms, each of which could be calculated according to the method shown

in Sec. III A. So we neglect the tedious details and present the final result

$$\chi^{(\gamma)}(h, \gamma) = \frac{1}{16(h^2 + \gamma^2 - 1)} \sum_{\nu=\pm} \mathcal{F}_\nu^{(\gamma)} - \frac{\mathcal{N}}{8(1 - \gamma^2)^2}, \quad (19)$$

where

$$\mathcal{F}_\nu^{(\gamma)} = \frac{(g_\nu^2 - 1)^2}{4g_\nu^2} \left[ \frac{\mathcal{N}^2 g_\nu^{\mathcal{N}}}{(g_\nu^{\mathcal{N}} + 1)^2} + \mathcal{N} \left( \frac{C_\nu}{2} + \frac{p_\nu}{|\gamma|} \right) \frac{g_\nu^{\mathcal{N}} - 1}{g_\nu^{\mathcal{N}} + 1} \right]. \quad (20)$$

Here  $p_\nu$ ,  $g_\nu$ , and  $C_\nu$  are as defined earlier. Again, the detailed derivations are presented in the SM [52]. It is well known that the anisotropy transition occurs when  $|h| < 1$ , and in this segment  $\chi^{(\gamma)}(h, \gamma)$  reaches its maximum exactly at  $\gamma = 0$ . Precisely, we have

$$\chi^{(\gamma)}(h, 0) = -\frac{\mathcal{N}}{8} + \frac{1}{4} \left[ \frac{\mathcal{N}^2 g^{\mathcal{N}}}{(g^{\mathcal{N}} + 1)^2} + \frac{\mathcal{N}h}{2i\sqrt{1-h^2}} \frac{g^{\mathcal{N}} - 1}{g^{\mathcal{N}} + 1} \right], \quad (21)$$

where  $g = h + i\sqrt{1-h^2}$ . Physically,  $\chi^{(\gamma)}(h, 0)$  is a real expression with a vanishing imaginary part. This can be seen clearly by parametrizing  $h = \cos \theta$  and  $\theta \in (0, \pi)$ ,

$$\chi^{(\gamma)}(h, 0) = \frac{\mathcal{N}}{4} \left[ \frac{\mathcal{N}}{\cos^2[N \cos^{-1}(h)]} + \frac{\tan[N \cos^{-1}(h)]}{\tan[\cos^{-1}(h)]} - 1 \right]. \quad (22)$$

It can be proved that the ratio of the second term to the first one in the large square brackets is an oscillating function and is bounded. So the second term can be neglected in that it contributes the leading term to a prefactor at most. The third term also does not need to be considered for large enough system size  $\mathcal{N}$ . In light of the relation  $\chi^{(\gamma)}(h, \gamma_m) = \chi^{(\gamma)}(h, \gamma_c) \simeq \frac{\mathcal{N}^2}{16 \cos^2[N \cos^{-1}(h)/2]}$ , we obtain that

$$\ln \left( \frac{\chi^{(\gamma)}(h, 0)}{\mathcal{N}} \right) = \ln \mathcal{N} - \ln \left[ 16 \cos^2 \frac{\mathcal{N} \cos^{-1}(h)}{2} \right]. \quad (23)$$

Similarly, the exact expression for the FS is [53]

$$\frac{\bar{\chi}^{(\gamma)}(h, \gamma)}{\mathcal{N}} = \frac{1}{16 |\gamma| (1 + |\gamma|)^2}, \quad |h| < 1 \quad (24)$$

in the TDL, indicating that the scaling behavior around the critical points is

$$\ln \left( \frac{\bar{\chi}^{(\gamma)}(h, \gamma)}{\mathcal{N}} \right) = -\ln |\gamma| - 4 \ln 2. \quad (25)$$

Similarly, we are safe to conclude from Eqs. (23) and (25) that the critical exponent  $\nu = 1$  according to the ansatz (1). In addition, the normalized FS  $\chi^{(\gamma)}(h, \gamma)$  with respect to  $\gamma$  in the TDL is shown in Fig. 3. This result also demonstrates convincingly that FS can be used to characterize the quantum critical behavior.

#### IV. CONCLUSION

We have derived several closed-form expressions for the fidelity susceptibility of the anisotropic XY model in the

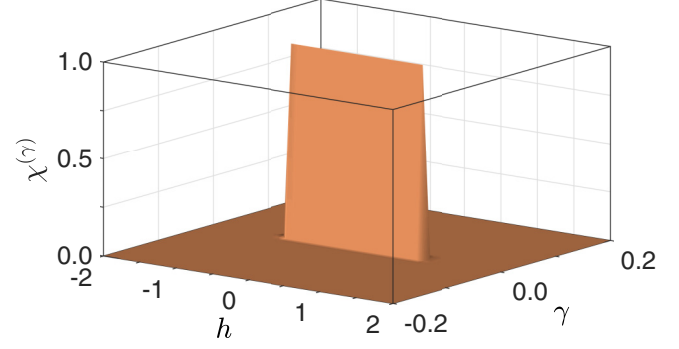


FIG. 3. Normalized fidelity susceptibility  $\chi^{(\gamma)}$  with respect to  $\gamma$ . It has a sharp extremum at the segment  $|h| < 1$  and  $\gamma = 0$  where continuous QPTs with  $\nu = 1$  occur.

transverse field after a symmetry analysis. The FS for the special case  $\gamma = 1$  can be recovered and is consistent with the results obtained by Damski and Rams in light of several elegant summation formulas [40–42] of the transverse-field Ising model. Our method is easy to follow and promises to be useful to other exactly solvable models, such as the XY model with bond-alternated interaction [54] or a staggered field [55]. The correlation length critical exponent  $\nu = 1$  is calculated analytically according to the standard finite-size scaling ansatz.

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#### APPENDIX A: CLOSED-FORM EXPRESSION

*Lemma 1.* For any nonzero (complex) variable  $\alpha$ , the closed-form of the summation

$$L(\alpha) = \sum_{n=1}^{\mathcal{N}} \frac{1}{\alpha + c_n}, \quad c_n = \cos \frac{(2n-1)\pi}{\mathcal{N}} \quad (A1)$$

can be expressed as

$$L(\alpha) = \begin{cases} \frac{\mathcal{N}}{\sqrt{\alpha^2-1}} \frac{\beta^{\mathcal{N}-1}}{\beta^{\mathcal{N}+1}}, & |\alpha| > 1 \\ \text{sgn}(\alpha) \mathcal{N}^2, & |\alpha| \rightarrow 1 \\ \frac{\mathcal{N}}{\sqrt{1-\alpha^2}} \tan[N \cos^{-1}(\alpha)], & |\alpha| < 1, \end{cases} \quad (A2)$$

where  $\mathcal{N} = 2N$ ,  $\beta = \alpha + \sqrt{\alpha^2 - 1}$ , and  $\text{sgn}(\cdot)$  is the sign function.

*Proof.* The function  $L(\alpha)$  is an odd function with respect to the variable  $\alpha$ , which can be verified immediately by noting the symmetry relation  $c_n = -c_{N+1-n}$ . The monotonic behavior of the function depends highly on  $\alpha$  and we will consider the case  $|\alpha| > 1$  first.

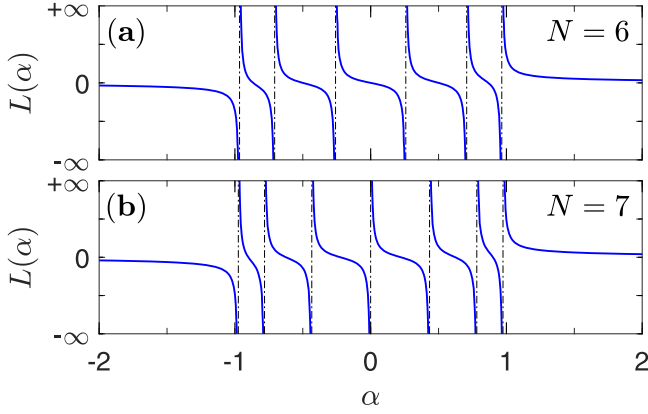


FIG. 4. Illustration of the function  $L(\alpha)$  for (a) the even case  $N = 6$  and (b) the odd case  $N = 7$ . In both cases, the solid blue lines are the curve for  $L(\alpha)$ , while the dash-dotted black lines represent the discontinuous points.

To begin with, let  $\zeta = e^{(\pi/N)i}$  be the primitive  $2N$ th root of unity and

$$\Omega = \{\zeta^{2n-1} \mid -N < n \leq N\} = \{\zeta^{\pm(2n-1)} \mid 1 \leq n \leq N\}$$

be the set of roots for the polynomial  $z^N + 1 = 0$ . Therefore, for any allowed integer  $n$ ,  $c_n = \frac{\omega + \omega^{-1}}{2}$  with  $\omega = \omega_n = \zeta^{2n-1}$ . Employing the logarithmic derivative with respect to  $z$  of the equality

$$\prod_{\omega \in \Omega} (z + \omega) = \prod_{\omega \in \Omega} (z - \omega) = z^N + 1,$$

we then arrive at the important summation identity

$$\sum_{\omega \in \Omega} \frac{z}{\omega + z} = N \frac{z^N}{z^N + 1}. \quad (\text{A3})$$

Actually, it is natural for us to define the new variable  $\beta = \alpha + \sqrt{\alpha^2 - 1}$  so that  $\alpha = \frac{\beta + \beta^{-1}}{2}$ . In light of Eq. (A3) we have

$$\begin{aligned} L(\alpha) &= \frac{1}{2} \sum_{n=-N+1}^N \frac{1}{\alpha + c_n} = \frac{1}{2} \sum_{\omega \in \Omega} \frac{1}{\frac{\beta + \beta^{-1}}{2} + \frac{\omega + \omega^{-1}}{2}} \\ &= \frac{1}{\beta - \beta^{-1}} \sum_{\omega \in \Omega} \left( \frac{\beta}{\omega + \beta} - \frac{\beta^{-1}}{\omega + \beta^{-1}} \right) \\ &= \frac{N}{\sqrt{\alpha^2 - 1}} \frac{\beta^N - 1}{\beta^N + 1}. \end{aligned} \quad (\text{A4})$$

If the hyperbolic functions are involved, Eq. (A4) is equivalent to

$$L(\alpha) = \frac{N \operatorname{sgn}(\alpha)}{\sqrt{\alpha^2 - 1}} \tanh[N \cosh^{-1}(\alpha)]. \quad (\text{A5})$$

The exact expression for the case  $|\alpha| < 1$  can be obtained from the former in the spirit of analytic continuation or simply by the substitution  $\beta = \alpha + i\sqrt{1 - \alpha^2}$ . In this situation we have

$$L(\alpha) = \frac{N}{\sqrt{1 - \alpha^2}} \tan[N \cos^{-1}(\alpha)]. \quad (\text{A6})$$

It should be noted that the point  $\alpha = 0$  is out of our consideration in general since whether this point make sense or not depends on the oddity of  $N$ . It is zero when  $N$  is even and diverging for the odd case (see Fig. 4). More generally, there are  $N$  discontinuous points in the range of  $|\alpha| < 1$ , namely,  $\alpha_{\text{disc}} = \cos \frac{2m+1}{N} \pi$ ,  $m = 0, 1, 2, \dots, N-1$ . Therefore, the function  $L(\alpha)$  is continuous at the points  $|\alpha| = 1$  and its values thereof are equal to  $\operatorname{sgn}(\alpha)N^2$ .

Altogether, we finish the full processes of the proof. We end this Appendix with Fig. 4, which presents the curvature of function  $L(\alpha)$  for even and odd  $N$ . ■

## APPENDIX B: DERIVATIVE RELATION

*Lemma 2.* For any nonzero (complex)  $\alpha$ , the first derivative of the function  $L(\alpha)$  defined in Eq. (A1) is

$$\begin{aligned} L'(\alpha) &= \frac{\partial L(\alpha)}{\partial \alpha} \\ &= \begin{cases} \frac{4\beta^2}{(\beta^2-1)^2} \left[ \frac{N^2 \beta^N}{(\beta^N+1)^2} - \frac{N}{2} \frac{(\beta^2+1)(\beta^N-1)}{(\beta^2-1)(\beta^N+1)} \right], & |\alpha| > 1 \\ -\operatorname{sgn}(\alpha) \frac{N^2(16N^2+9)}{24}, & |\alpha| \rightarrow 1 \\ -\frac{L(\alpha)}{\sqrt{1-\alpha^2}} \left[ \frac{N}{\sin(N \cos^{-1}(\alpha))} - \frac{\alpha}{\sqrt{1-\alpha^2}} \right], & |\alpha| < 1, \end{cases} \end{aligned} \quad (\text{B1})$$

where  $N = 2N$ ,  $\beta = \alpha + \sqrt{\alpha^2 - 1}$ , and  $\operatorname{sgn}(\cdot)$  is the sign function.

## APPENDIX C: QUANTUM METRIC TENSOR

The quantum metric tensor [47] is a concept stemming from differential geometry and information theory. It describes the absolute value of the overlap amplitude between neighboring ground states. Therefore, like the fidelity susceptibility, the metric also plays a vital role in understanding the quantum phase transition.

For the anisotropic  $XY$  model [see Eq. (2)] in the  $(h, \gamma)$  parameter space, the quantum metric tensor is defined as  $g_{\sigma\sigma} = \frac{1}{4} \sum_{n=1}^N (\partial\theta_k/\partial\mu^\sigma)(\partial\theta_k/\partial\mu^\sigma)$ , where  $\mu^{1,2} = h, \gamma$  [47]. So the solely off-diagonal element of the tensor is

$$\chi^{(h\gamma)}(h, \gamma) = \frac{\gamma}{4} \sum_{k>0} \frac{\sin^2 k(\cos k - h)}{[(\cos k - h)^2 + \gamma^2 \sin^2 k]^2}. \quad (\text{C1})$$

The method for the calculation of Eq. (C1) has been explained in the main text. The detail is omitted here and is presented in the SM [52]; the result is

$$\chi^{(h\gamma)}(h, \gamma) = \frac{\operatorname{sgn}(\gamma)}{16(h^2 + \gamma^2 - 1)} \sum_{v=\pm} \mathcal{F}_v^{(h\gamma)}, \quad (\text{C2})$$

where

$$\mathcal{F}_v^{(h\gamma)} = \frac{g_v^2 - 1}{2p_v g_v} \left[ \frac{N^2 g_v^N}{(g_v^N + 1)^2} + \frac{N}{2} \left( \frac{p_v}{|\gamma|} + C_v \right) \frac{g_v^N - 1}{g_v^N + 1} \right]. \quad (\text{C3})$$

Also, the  $g_v$ ,  $p_v$ , and  $C_v$  are defined in Eqs. (9), (10), and (12), respectively.



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