Rapid Communications

Collective $1/f$ fluctuation by pseudo-Casimir-invariants

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In this study, we propose a universal scenario explaining the $1/f$ fluctuation, including pink noises, in Hamiltonian dynamical systems with many degrees of freedom under long-range interaction. In the thermodynamic limit, the dynamics of such systems can be described by the Vlasov equation, which has an infinite number of Casimir invariants. In a finite system, they become pseudoinvariants, which yield quasistationary states. The dynamics then exhibit slow motion over them, up to the timescale where the pseudo-Casimir-invariants are effective. Such long-time correlation leads to $1/f$ fluctuations of collective variables, as is confirmed by direct numerical simulations. The universality of this collective $1/f$ fluctuation is demonstrated by taking a variety of Hamiltonians and changing the range of interaction and number of particles.

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Introduction. The $1/f$ fluctuation is ubiquitous in nature: Certain quantities that fluctuate in time exhibit a power spectrum of the form $1/f^{\nu}$, where f is the frequency and the exponent ν is typically in the range of $1/2 \lesssim \nu \lesssim 3/2$. The $1/f$ fluctuation, suggesting a long-time correlation, is observed in nature such as vacuum tubes [1], semiconductors [2], spin transport [3], oceans [4,5], quasars [6], solar wind [7,8], and proteins [9]. The $1/f$ fluctuation is also observed in model systems of water molecules [10,11], proteins [12], ferromagnetic bodies [13], and accretion disks [14]. (See, for instance, [15–19] for reviews.)

There have been multiple efforts to understand the mechanism of the $1/f$ fluctuation; however, a coherent explanation remains lacking. For instance, the superposition of Lorentzians (see [18,19], for instance) requires a certain distribution of multiple timescales, but the origin of multiple timescales must be explained. Our strategy here is to restrict our concern to Hamiltonian dynamical systems with many degrees of freedom and search for the possibility of $1/f$ spectra for collective variables, as observed in water molecules [10] and ferromagnetic bodies [13].

In Hamiltonian systems with a few degrees of freedom, the $1/f$ fluctuation has been observed and analyzed by the hierarchical structure of the phase space constructed by the Kolmogorov-Arnold-Moser tori and chaotic sea [20–26]. The hierarchical structure is analyzed by perturbation theory in two-degrees-of-freedom systems [27] and is also observed in a symplectic coupled map with four particles [28]. Nevertheless, such microscopic hierarchical structures exist only in a certain range for a system with a few degrees of freedom and is thus not generic in systems with many degrees of freedom. Hence, it remains to be elucidated how the $1/f$ fluctuation is generated in the collective motion of many-degree-of-freedom systems.

The aim of this Rapid Communication is to propose a universal scenario for the collective $1/f$ fluctuation in long-range Hamiltonian systems with many degrees of freedom. The target class of systems includes self-gravitating systems, plasmas, geophysical flows, trapped ions [29], among others [30–32]. In the thermodynamic limit with an infinite number of particles, the dynamics of such systems can be described by the Vlasov equation, the partial differential equation for the one-particle distribution function [33–35]. This equation is described by the distribution function on the one-particle phase space; therefore, the collective motion is naturally treated through the Vlasov equation.

An important feature of Vlasov dynamics is that they have an infinite number of Casimir invariants. The Casimir invariants are exact in the thermodynamic limit, but in a system with a finite number of particles, they become pseudoinvariants and fluctuate slowly with time. These pseudo-Casimir-invariants play the role of constraints up to a certain timescale, but they break down in the long timescale, as has also been observed recently in the two-step relaxation of fluctuation amplitude in thermal equilibrium [13]. Such slow motion is owing to the pseudo-Casimir-invariants and one may expect a long-time correlation in the dynamics of collective variables, i.e., the collective $1/f$ fluctuation. In the present Rapid Communication, we numerically demonstrate that this is indeed true and propose a general scenario for the collective $1/f$ fluctuation that persists up to the timescale where the constraint by pseudo-Casimir-invariants is effective.

Model. We consider the α-Hamiltonian mean-field (α-HMF) model [36], which is described by the Hamiltonian

$$H_{\alpha}(q,p) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \frac{1}{2N_{\alpha}} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{1-\cos(q_j - q_k)}{r_{jk}^{\alpha}}, \quad (1)$$

where $\alpha \geqslant 0$. This system represents XY spins, each of which is located at a one-dimensional lattice point with a unit lattice

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spacing. The variable q_j denotes the phase of the jth particle, and p_j is the conjugate momentum. A quantum version of this system can be experimentally realized through trapped ions [29]. Here, the spatial boundary condition is set to be periodic, and accordingly, the distance between the jth and kth particles is defined as $r_{jk} = \min\{\,|j-k|,\ N-|j-k|\,\}$ for $k \neq j$ and $r_{jk} = 1$ for $k = j$. The normalization factor N_α is introduced to ensure the extensivity of energy, as is defined by $N_\alpha = \sum_{k=0}^{N-1} 1/r_{jk}^\alpha$. By taking $\alpha = 0$ with $N_0 = N$, the α-HMF model is reduced to the Hamiltonian mean-field (HMF) model [37,38]. In the opposite limit, $\alpha \to \infty$, it can be reduced to the model with the nearest-neighbor couplings with $N_\infty = 3$. Hence, the dependence on the coupling ranging from long to short is investigated by varying the value of α.

Here, we investigate the order parameter defined by $\boldsymbol{M} = \sum_{j=0}^{N-1}(\cos q_j, \sin q_j)/N$. The α-HMF model shows the second-order phase transition at the specific energy $E_c = 3/4$, which corresponds to the critical temperature $T_c = 1/2$, irrespective of the value of α for $0 \leqslant \alpha < 1$ [39–41]. The boundary between the long and short range is given by $\alpha = 1$, beyond which the mean-field description is not valid. We introduce the scaling $x = j/N$ such that the domain of x is restricted to $x \in [-1/2, 1/2]$ because of the periodic boundary condition.

In the limit $N \to \infty$, the dynamics of the α-HMF model are described by the Vlasov equation [42]

$$\frac{\partial F}{\partial t} + \frac{\partial \mathscr{H}[F]}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial \mathscr{H}[F]}{\partial q}\frac{\partial F}{\partial p} = 0, \tag{2}$$

where $F(q,p,x,t)$ is the one-particle distribution function, $\mathscr{H}[F](q,p,x,t)$ is the one-particle Hamiltonian functional defined by

$$\mathscr{H}[F] = \frac{p^2}{2} + \mathscr{V}[F](q,x,t),$$
$$\mathscr{V}[F] = \frac{-1}{\kappa_\alpha}\int_{-1/2}^{1/2} dx' \int_{-\pi}^{\pi} dq' \int_{-\infty}^{\infty} dp' \times \frac{\cos(q-q')}{|x-x'|^\alpha} F(q',p',x',t), \tag{3}$$

and $\kappa_\alpha = \int_{-1/2}^{1/2}(1/|x|^\alpha)dx$. From the Poisson structure of the Vlasov equation, it is easy to show that a Casimir functional,

$$\mathscr{C}[F](t) = \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp\, c(F(q,p,x,t)) \tag{4}$$

is a constant of motion for any differentiable function c if $c(F) \to 0$ in $|p| \to \infty$ [43,44]. We stress that the validity of the Vlasov description is guaranteed for $\alpha < 1$, as the integral of κ_α does not converge for $\alpha \geqslant 1$ [45].

Numerical tests and results. The initial values of $\{(q_k, p_k)\}_{k=0}^{N-1}$ are randomly chosen from the one-particle distribution function in thermal equilibrium:

$$F_{\rm eq}(q,p;T,M) = A\exp[-(p^2/2 - M\cos q)/T], \tag{5}$$

where A is the normalization factor, and T is temperature. From the rotational symmetry of the system, the direction of the order parameter is set to $\boldsymbol{M} = (M, 0)$. The magnetization M is determined for a given T by solving the self-consistent equation $M = \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp\ F_{\rm eq}(q,p;T,M)\cos q$.

The temperature T is introduced to parametrize the set of thermal equilibria. All numerical simulations are performed by integrating the canonical equations of motion associated with the N-body Hamiltonian (1) [46] without thermal noise by using the fourth-order symplectic integrator [47] with the time step $\Delta t = 0.1$.

We recall that the Casimir invariants hold under conditions of (i) thermodynamic limit $N \to \infty$ and (ii) long-range interaction ($\alpha < 1$). For large but finite N, the Casimir invariants are no longer invariants; they fluctuate with time, thus becoming pseudoinvariants. To unveil the possible relationship between the pseudo-Casimir-invariants and $1/f$ fluctuation, we first numerically examine the N dependence and α dependence in the low- and high-energy sides of the α-HMF model, respectively.

In the low-energy ordered phase ($T < T_c$), the relationship is examined through the N dependence by fixing the parameter α at $\alpha = 0$ (the HMF model) for simplicity. The power spectra of $M^2(t)$ are presented in Fig. 1(a) for several values of N,

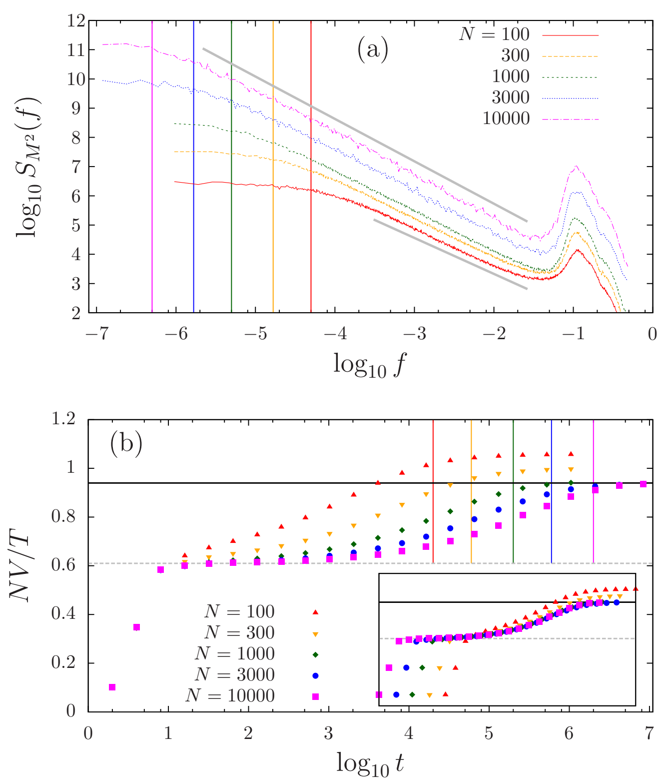

FIG. 1. N-dependence in the HMF model ($\alpha = 0$). $N = 100$ (300, red triangles), 300 (300, orange inverse triangles), 1000 (300, green diamonds), 3000 (50, blue circles) and 10000 (50, magenta squares). The number in parentheses represents the number of sample orbits over which the average is taken. $T = 0.45(<T_c)$. (a) Power spectra of $M^2(t)$. N increases from bottom to top. For graphical reasons, the vertical scales have been changed suitably. The vertical lines indicate the time-scale $1/\tau$ where $\tau = 200N$. The gray line segments guide the eyes for the slopes -1.45 (upper) and -1.23 (lower) obtained by the least-square method in the intervals of segments. (b) Temporal evolution of the scaled variance of magnetization M. The lower gray broken and upper black solid horizontal lines are, respectively, the theoretically predicted plateau level and thermal equilibrium level. N increases from top to bottom. The vertical lines represent $\tau = 200N$. (inset) The horizontal axis is $\log_{10}(t/N)$.

which exhibits $1/f$ fluctuations down to a certain frequency τ^{-1}. From the figure, this maximum timescale is estimated as $\tau \sim 200N$, suggesting that the $1/f$ fluctuation persists to small frequencies, in the larger N. The timescale $\tau \propto N$ is consistent with the fact that the collision term of the product of two $O(1/\sqrt{N})$ terms is added to the Vlasov equation (2) for finite N, to break the Casimir invariants. The power spectra of individual $\cos q_j(t)$ do not exhibit $1/f$ spectra [44], and the observed $1/f$ spectra are caused by collective motion.

To quantitatively reveal the timescale where the constraint by the pseudo-Casimir-invariants is effective, we also compute the time-dependent variance $V(t)$ defined by

$$V(t) = \sum_{n=0}^{n_0-1} \frac{1}{n_0} \left[\frac{1}{t} \sum_{k=1}^{t} \|\boldsymbol{M}(nt+k)\|^2 - \left(\frac{1}{t} \sum_{k=1}^{t} \|\boldsymbol{M}(nt+k)\| \right)^2 \right], \quad (6)$$

where $t = 2, 4, 8 \cdots$ and n_0 is a sufficiently large number. The variance V is scaled as NV/T, which represents the susceptibility at the thermal equilibrium level, according to the fluctuation-response relation [13]. The temporal evolution of the scaled variance is shown in Fig. 1(b) with indications of the timescale $\tau = 200N$ around which the variance reaches the asymptotic level.

From Fig. 1(b), we can understand that the constraint by the pseudo-Casimir-invariants is effective up to the timescale τ: The small collision term is negligible in the short-time regime and the fluctuation is restricted in the initial Casimir level set, which is an iso-Casimir surface in function space, up to the end of the plateau [48]. The plateau level is theoretically predicted using linear response theory for the Vlasov dynamics [49,50] under the conservation of the Casimir invariants [51]. With time, the collision term becomes non-negligible, altering the Casimir invariants and moving to another level set. The state is trapped on the new level set (although the next plateau is invisible in the figure because of the logarithmic axis). The change in level sets continues and the cumulative variance (6) thus slowly increases, whereas the suppression by the pseudo-Casimir-invariants remains effective. We underline that the plateau is not perfectly flat, where a perfect plateau suggests that the Casimir invariants are exact as in the limit $N \to \infty$. The arrival to the asymptotic level at τ suggests that the state has traveled over the possible level sets and the constraint by the pseudo-Casimir-invariants is no longer effective for timescales larger than τ [52].

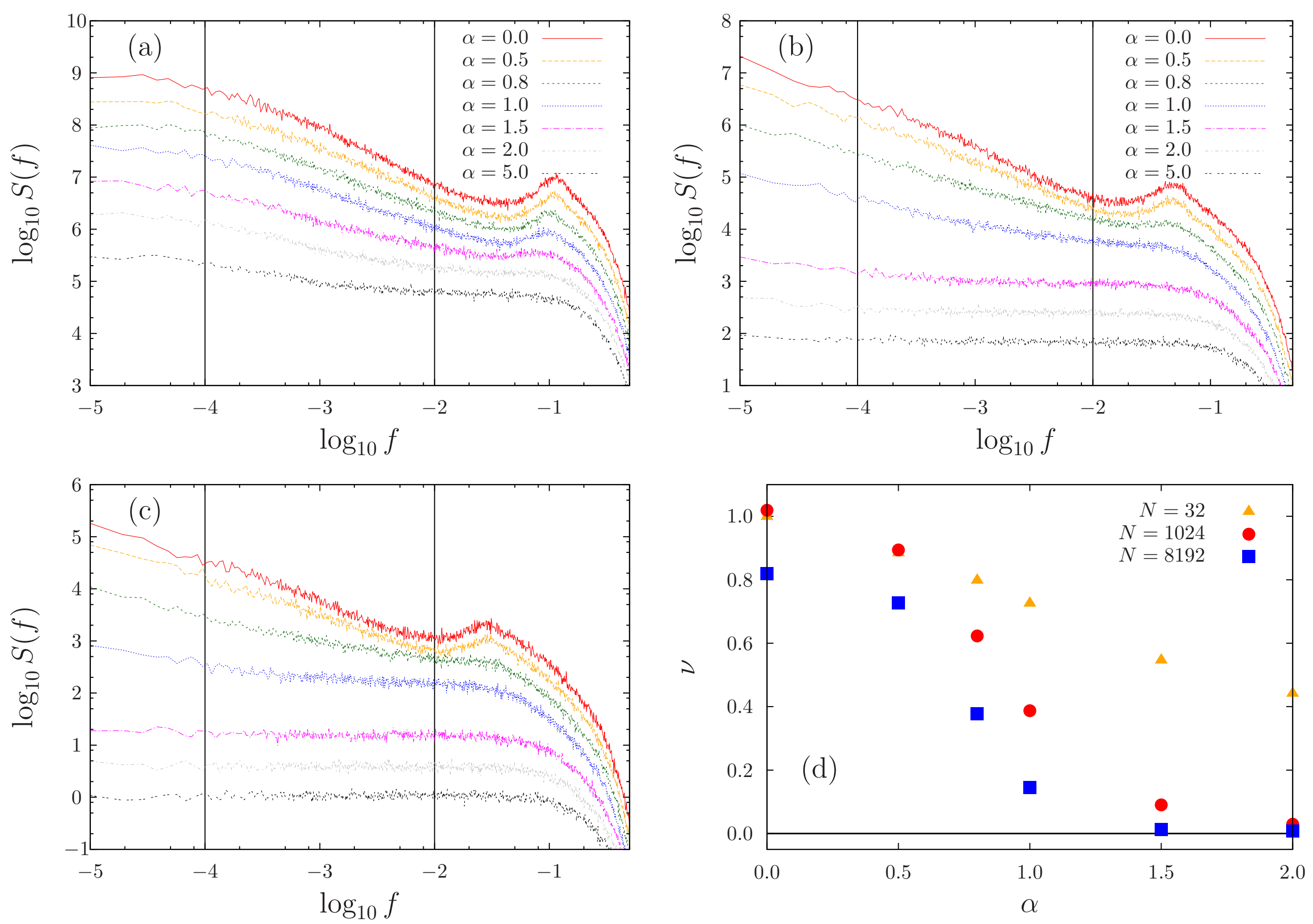

FIG. 2. Power spectra in the α-HMF model. $T = 0.6$. (a) $N = 32\,(100)$. (b) $N = 1024\,(100)$. (c) $N = 8192\,(60)$. The number in parentheses represents the number of sample orbits. $\alpha = 0,\ 0.5,\ 0.8,\ 1,\ 1.5,\ 2$, and 5 from top to bottom. For graphical reasons, the vertical scale has been modified suitably. (d) α dependence of the exponent ν (minus of the slope), which is computed by the least-square method between the two vertical lines of the panels (a)–(c).

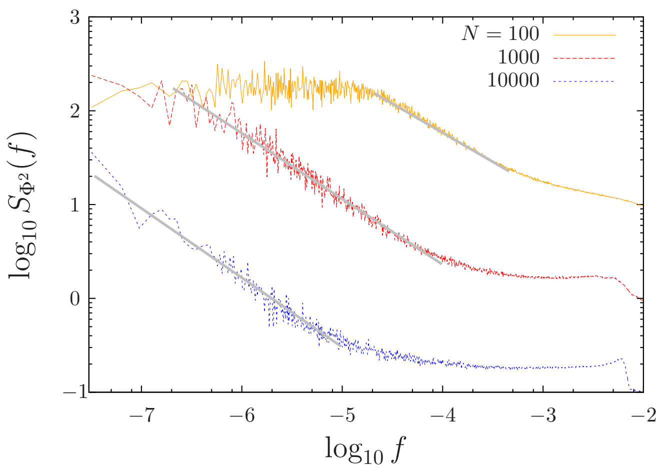

FIG. 3. Power spectra of $\mathbf{\Phi}^2(t)$ in the globally coupled FPUT model. The initial state is in thermal equilibrium with $T = 1$. $N = 100$ (top orange), 1000 (middle red), and 10 000 (bottom blue), where the vertical scales of the last two lines have been multiplied by 10 and 100, respectively, for graphical reasons. Each curve is the averages over 20 samples. The time step of simulations is $\Delta t = 0.01$. The initial interval $t \in [0, 10^4]$ has been removed to avoid transience [55]. The gray line segments are obtained from the least-square method in the intervals of segments. The slopes are -0.62, -0.70, and -0.74 from top to bottom.

In the high-energy disordered phase ($T > T_c$), linear response theory predicts that the variance on a Casimir level set agrees with the thermal equilibrium level [49,53], and, accordingly, no two-step relaxation of $V(t)$ appears [44]. In contrast, the disordered phase exists for any value of α. We, therefore, change the strategy and investigate the dependence of the range of interaction α by choosing the initial condition from thermal equilibrium (5) by setting $M = 0$ for $T = 0.6 (> T_c)$. The power spectra of $M^2(t)$ for $T = 0.6$ are presented in Fig. 2 for $N = 32$, 1024, and 8192. As we expected, the $1/f$ spectrum tends to disappear as α increases, i.e., as the interaction range becomes shorter. Moreover, this tendency is enhanced by increasing N and the exponent ν goes to 0 in $\alpha > 1$ as reported in Fig. 2(d). Note that the collective motion is again necessary for observing the $1/f$ spectrum [44]. At the low-energy end, a similar figure to Fig. 2 has the same tendency but does not coincide exactly [44].

Finally, we investigate whether the existence of phase transition in the α-HMF model is essential to the $1/f$ fluctuation. To this end, we introduce a globally coupled Fermi-Pasta-Ulam-Tsingou (FPUT) system [54]:

$$H_{\rm FPUT} = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \varphi(q_j - q_k),$$
$$\varphi(q) = \frac{q^2}{2} + \frac{q^4}{4}, \qquad (7)$$

which does not exhibit the phase transition. The collective observables that are naively expected, such as the variance of q or the potential energy, do not exhibit the $1/f$ fluctuation [44]. In contrast, the power spectrum of the collective variable $\mathbf{\Phi}^2(t)$ exhibits $1/f$ fluctuation as shown in Fig. 3, where $\mathbf{\Phi} = \sum_j (\cos\phi_j, \sin\phi_j)/N$ and ϕ is defined on the (q, p) plane as $(q, p) = (r\cos\phi, r\sin\phi)$ with $r = \sqrt{q^2 + p^2}$. To understand this result, we note that on one Casimir level set, fluctuations should occur on each iso-action curve in the leading order [51], where the action variable is associated with the one-particle Hamiltonian for the thermal equilibrium state. Traveling over level sets gives rise to fluctuations over iso-action curves, whereas the angle variable, conjugate with the action variable, captures the long-time correlation [44]. The variable ϕ here corresponds to the angle variable [56]. Therefore, $\mathbf{\Phi}$ is a suitable collective variable to extract the hidden $1/f$ fluctuation arising from the pseudo-Casimir-invariants.

Summary and discussions. In this study, we investigated the origin of the collective $1/f$ fluctuation in many-degree-of-freedom Hamiltonian systems under long-range interaction. Such systems have an infinite number of Casimir invariants in the limit $N \to \infty$, which, in a finite system, constrains the motion as pseudoinvariants and leads to the slow motion of collective variables. We then propose a simple but universal scenario: The collective $1/f$ fluctuation appears in the timescale up to which the constraint by the pseudo-Casimir-invariants is effective. This scenario has been successfully verified numerically by investigating the α-HMF model, FPUT model, and the dependence of the power spectra on the number of elements and the interaction range.

As the Casimir invariants are based on the distribution function and the average of an observable over it gives a collective variable, it is natural that the $1/f$ fluctuation is observed only for collective variables. For local variables such as the position of each particle, such long-term fluctuations are not observed. The collective variable of concern is given by an order parameter, as the average of the angle variable, corresponding to the slow change of the level of pseudo-Casimir-invariants. In the globally coupled FPUT model, the $1/f$ fluctuation may be hidden and is extracted by setting suitable observables with the aid of the proposed scenario.

To confirm the importance of the Casimir invariants (that are based on the Poisson structure of the Vlasov equation), we also examined the Kuramoto model [58], as a non-Hamiltonian and dissipative version of the HMF model, and confirmed that the system does not exhibit $1/f$ fluctuation [44,59]. The superposition of Lorentzian spectra is also unsuitable to explain the observed $1/f$ fluctuation by considering the timescales determined by the Landau damping modes [44].

It is important to examine the universality of our result. Although we considered the thermal equilibrium states of simple models here, the existence of Casimir invariants does not depend on consideration of reference states or the details of interaction potentials in long-range Hamiltonian systems, and the slow dynamics by pseudo-Casimir-invariants will generally be applied to the so-called quasistationary states [30], for instance. As there exists a variety of examples that can be effectively described by long-range Hamiltonian systems, such as plasmas, free electron laser [30,60–63], water molecules [64], and trapped ions [29,65–68], collective $1/f^\nu$ fluctuation will be experimentally verifiable. In real systems, the Poisson structure providing the Casimir invariants will be disturbed by dissipation and randomness. It will be important to examine the robustness of our scenario under such perturbations. Finally, the theory to predict the exponent ν in the $1/f^\nu$ spectrum must be formulated based on the slow motion of pseudo-Casimir-invariants.

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[43] In our study, F rapidly goes to zero in $|p| \to \infty$ and $c(F) = F^n (n = 1, 2, \dots)$ make Casimir invariants, for instance.

[44] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.98.020201 for a proof of the invariance of Casimir invariants and additional numerical results.

[45] We note that the divergence of κ_α can be avoided by introducing, for instance, long-range-like coupling constants in the interval $x \in [-\epsilon, \epsilon]$ with small $\epsilon > 0$. However, the scaling $x = j/N$ says that the particle at $x = 0$ interacts with an infinite number of particles in a long-range manner, and the short-range nature disappears accordingly. The case $\alpha = 1$ is included in the long-range coupling, but we do not discuss this boundary case.

[46] For $N = 2^n$, $n \in \mathbb{N}$, the interaction coefficient matrix, $R = [1/(N_\alpha r_{jk}^\alpha)]$, is diagonalized by the fast Fourier transform (FFT) matrix. By using this diagonalization and the FFT algorithm, we can reduce the numerical cost of the α-HMF model from $O(N^2)$ to $O(N \log N)$. This trick helps to perform long-time evolution.

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