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Universal behavior of the full particle statistics of one-dimensional Coulomb gases with an arbitrary external potential

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We present a complete theory for the full particle statistics of the positions of bulk and extremal particles in a one-dimensional Coulomb gas (CG) with an arbitrary potential, in the typical and large deviations regimes. Typical fluctuations are described by a universal function which depends solely on the general properties of the external potential. The rate function controlling large deviations is, rather unexpectedly, *not* strictly convex and has a discontinuous third derivative around its minimum for both extremal *and* bulk particles. This implies, in turn, that the rate function cannot predict the anomalous scaling of the typical fluctuations with the system size for bulk particles, and it may indicate the existence of an intermediate phase in this case. Moreover, its asymptotic behavior for extremal particles differs from the predictions of the Tracy-Widom distribution. Thus many of the paradigmatic properties of the full particle statistics of Dyson log gases do *not* carry over into their one-dimensional counterparts, hence proving that one-dimensional CG belongs to a different universality class. Our analytical expressions are thoroughly compared with Monte Carlo simulations, showing excellent agreement.

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As universality is one of the pillars of modern theoretical physics, an important goal is to understand under which conditions universal properties do emerge in strongly correlated systems, together with their range of validity. In order to pursue an answer to this poignant question, one uses classical ensembles of random matrices as a mathematical laboratory to test possible ideas, as the joint probability density of their eigenvalues offers a correlated system which, moreover, is simple enough to be amenable to a thorough and rigorous mathematical treatment. This can quite generally be written as

$$P(\mathbf{x}) = C_N e^{-\frac{\beta}{2}(V(\mathbf{x}) - \sum_{i\neq j}\log(|x_i - x_j|))}, \tag{1}$$

with $\mathbf{x} = \{x_i\}_{i=1}^N$, C_N is a normalization constant, and $V(\mathbf{x})$ is a function depending on the particular ensemble [1,2]. Physically, Eq. (1) can be identified as the two-dimensional Coulomb interacting system N charged particles, constrained to move along a single direction, with an external potential $V(\mathbf{x})$, the so-called Dyson's log gas [3]. Using path integral methods, this Coulomb fluid picture has been used to study the asymptotic behavior of the statistics of extreme and bulk eigenvalues in several classical ensembles [4–19]. In particular, the statistics of extremal eigenvalues follows a universal behavior governed by the Tracy-Widom (TW) distribution [20,21], while the typical fluctuations of bulk eigenvalues scale logarithmically with the system size rather than linearly [22,23]. The main physical reason behind these, and other findings, has been well established by now and corresponds to abrupt changes, or phase transitions, on the different mechanisms governing the statistical fluctuations.

The validity of this ubiquitous statistical behavior has been further explored in other correlated systems inspired mainly in ensembles of random matrices by either considering noninvariant ensembles [24–26] or by probing correlated systems similar to that in Eq. (1) but with a different interparticle interaction. An important result on the latter was considered in Ref. [27], where it was shown that there is a discontinuity in the third derivative of the rate function describing the large deviations of the extremal particle in a Coulomb gas (CG) confined by an arbitrary external central potential for any dimension, thus pinpointing a universal third-order phase transition, according to the Ehrenfest criterion. Moreover, in Ref. [28] it has been shown that when the one-dimensional (1D) CG is subjected to an external harmonic potential, the statistics of the rightmost particle exhibits a different distribution from the TW around its typical value. Interestingly, this system corresponds to the so-called jellium model or the one-dimensional component plasma [29,30], with many important applications in colloidal suspensions and polyelectrolyte solutions [31–33], and has been studied on distinct scenarios as many relevant quantities can be calculated exactly [34–37]. These recent results make clear that the study of CG with different dimensionality may provide either a deeper understanding of their shared universal properties or give rise to new behaviors that contrast the traditional, celebrated ones of random matrix theory (RMT).

To show that many universal features of the CG are indeed sensitive to the physical dimensions of the system, we present here the complete solution of the full particle statistics of the 1D case with an arbitrary external potential, obtaining exact expressions for both the typical or large fluctuations regimes. To be specific, we consider a Hamiltonian of the form

$$\mathcal{H}(\mathbf{x}) = N^2 \sum_{i=1}^{N} v(x_i) - N\alpha \sum_{i<j}^{1,N} |x_i - x_j|, \tag{2}$$

where the choice of the powers of N ensures that we have a nontrivial contribution in the thermodynamic limit. Clearly, to have a confined configuration, $v(x)$ must be a convex function, but an upper bound on α might also be required to guarantee that $v(x)$ dominates over the electrostatic repulsion and an equilibrium particle density $\rho_{\rm eq}(x)$ is attained [27,28,38]. As the Hamiltonian in Eq. (2) is invariant under the permutation of particles, we will henceforth assume that $x_{\min} \equiv x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N \equiv x_{\max}$. Then the optimal position of the ith particle x_i^* is thus given by [39]

$$v'(x_i^*) = \frac{\alpha}{N}(2i - N - 1), \quad i = 1, \ldots, N, \tag{3}$$

thus in the thermodynamic limit $\rho_{\rm eq}(x)$ has a natural domain $x \in [x_-, x_+]$, with $v'(x_\pm) = \pm\alpha$. Depending on the external potential, a restriction on the values of α may be necessary to obtain a physical solution [39]. In addition, when $v(x)$ is of class C^3, we have that for the Hamiltonian of Eq. (2), the typical fluctuations regime correspond to deviations of order $O(N^{-1})$, while large deviations are of order $O(N^0)$ [27,28,38,39].

To obtain the cumulative distribution function (CDF) of the typical fluctuations of both the extremal particles and bulk particles (x_K, $1 < K < N$) around their average positions, $\langle x_i \rangle = x_i^*$, $i = 1, \ldots, N$, we write the probability Prob$[x_1 < x_2 < \cdots < x_i < w] = Z_c(w; N)/Z_c(\infty; N)$, for a fixed but arbitrary particle indexed according to $i = cN$ and with [28,39]

$$Z_c(w;N) = N! \int_{-\infty}^{w} dx_i \prod_{j=1}^{i-1} \int_{-\infty}^{x_{j+1}} dx_j \times \prod_{j=i+1}^{N} \int_{x_{j-1}}^{\infty} dx_j e^{-\mathcal{H}(\mathbf{x})}. \tag{4}$$

Notice that, since $x_1 < x_2 < \cdots x_N$, the absolute value in Eq. (2) does not play any role and we can perform a second-order Taylor expansion $\mathcal{H}(\mathbf{x})$ around the minimum $x_i^\star$, since all the remaining terms of the expansion are at least of order $O(N^{-1})$ and consequently vanish in the thermodynamic limit. Defining $W_i \equiv N u_i (w - x_i^*)$, $\epsilon_j \equiv N u_j (x_j - x_j^*)$, $\Delta_j^{(\pm)} = \pm N u_j (x_{j\pm1}^* - x_j^*)$, $y_j^{(\pm)} = \frac{u_{j\mp1}}{u_j}\epsilon_j \pm \Delta_{j\mp1}^{(\pm)}$, and $u_i = \sqrt{v''(x_i^*)}$, the last integral can be approximated as

$$Z_c(w;N) \approx \frac{N!\, e^{-\mathcal{H}(\mathbf{x}^*)}}{N^N \prod_{j=1}^{N} u_j} \int_{-\infty}^{W_i} d\epsilon_i e^{-\frac{1}{2}\epsilon_i^2} \times \left(\prod_{j=1}^{i-1} \int_{-\infty}^{y_{j+1}^{(+)}} d\epsilon_j e^{-\frac{1}{2}\epsilon_j^2} \right) \left(\prod_{j=i+1}^{N} \int_{y_{j-1}^{(-)}}^{\infty} d\epsilon_j e^{-\frac{1}{2}\epsilon_j^2} \right). \tag{5}$$

Note that the first (second) multiple integral inside the parentheses in Eq. (5) is proportional to the CDF of the extremal particle, being smaller (greater) than x_i, but for a smaller system of size $i-1$ $(N-i)$. This suggests that the fluctuations of the bulk particles can be described in terms of the CDF of the extremal ones, i.e., $c = 1$ and $c = 0$ [39]. To shorten notation let us write $F_c(W_i(w); N) \equiv Z_c(w; N)/Z_c(\infty; N)$. Then the statistics of $x_{\max}$, whose CDF is $F_1(W; N)$, obeys the following forward differential equation in the thermodynamic limit [28,34,39]

$$\frac{dF_1(W)}{dW} = A_1\, e^{-W^2/2}\, F_1\left(W + \frac{2\alpha}{u_+}\right), \tag{6}$$

where $u_\pm = \sqrt{v''(x_\pm^*)}$ and A_1 is a constant which is fixed upon imposing boundary conditions $F_1(W) \to 1$ as $W \to \infty$, and $F_1(W) \to 0$ as $W \to -\infty$. An entirely analogous analysis can be made for the statistics of the leftmost particle, corresponding to F_0, for which the resulting probability density function (PDF) is determined by a delayed differential equation.

The typical fluctuations for bulk particles are obtained by choosing $K = cN$ and considering the thermodynamic limit while c remains finite. The corresponding CDF, denoted $F_c(W)$, obeys the following forward and delayed differential equation [39],

$$\frac{dF_c(W)}{dW} = A_c\, e^{-W^2/2}\, F_1\left(W + \frac{2\alpha}{u_c}\right) F_0\left(W - \frac{2\alpha}{u_c}\right), \tag{7}$$

where $u_c = \sqrt{v''(x)}$ with x such that $c = \int_{x_-}^{x} \rho_{\rm eq}(y)dy$, and A_c can be obtained by requiring $F_c(W)$ to be normalized.

Equations (6) and (7) are the first of our main results, for they provide a complete description of the full set of particles, now indexed according to c. Second, they show that the joint contribution of the confining potential $v(x)$ as well as the electrostatic interaction is captured succinctly by the constants $\frac{2\alpha}{u_\pm}$ and $\frac{2\alpha}{u_c}$. This means that whenever two ensembles share the same expression of $v''(x)$, then their typical statistical properties are the same and, therefore, described by the *universal function* $F_c(W)$. It is fairly straightforward to show [39] that the asymptotic behavior of $F_1'(W)$ is given by

$$F_1'(W) \sim \begin{cases} \exp(-W^2/2), & W \to \infty, \\ \exp\left(-\frac{u_+}{12\alpha}|W|^3\right), & W \to -\infty. \end{cases} \tag{8}$$

Notice that the right tail, $W \to \infty$, is rather different from the case of Dyson log gases governed by the asymptotic TW PDF that decays as $e^{-\frac{3}{2}W^{3/2}}$ [15]. Similarly, the asymptotic behavior of the PDF for bulk particles turns out to be

$$F_c'(W) \sim \exp\left(-\frac{u_c}{12\alpha}|W|^3\right), \quad W \to \pm\infty. \tag{9}$$

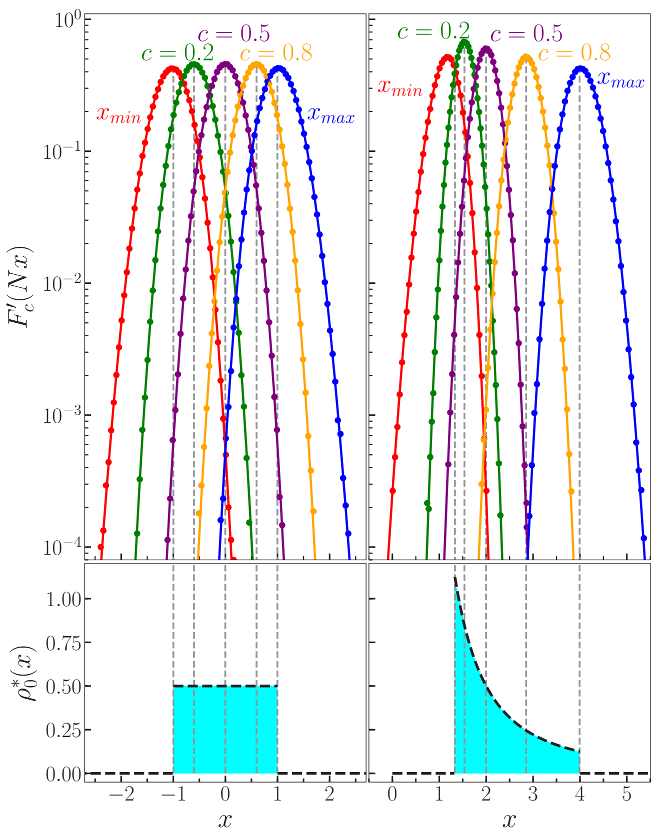

FIG. 1. Top panels: Comparison of the PDFs $\frac{dF_c(W)}{dw}$ obtained by solving numerically Eqs. (6) and (7) for different values of c (solid curves) and MC simulations (markers) considering two potentials. Left panel: $v(x) = \frac{x^2}{2}$ that corresponds to the classical Gaussian ensemble. Right panel: $v(x) = \frac{1}{2}(x - a\log x)$, with $a > 1$, inspired by the Wishart-Laguerre ensemble. A factor N in the argument of F_c has been included to magnify the size of the fluctuations. Bottom panels: Density of the 1D CGs using the same potentials, comparing the MC samples (cyan histogram) with the analytic expression for $\rho_{\rm eq}(x)$ of Eq. (13) (black dashed curve). The dashed vertical lines indicate the position for which the typical fluctuations were calculated. We used 10^6 MC steps for a system of $N = 500$ particles.

As typically large fluctuations are expected to match atypical small ones, Eq. (9) indicates that the rate function is not strictly convex and therefore will be unsuitable to describe the Gaussian-like behavior found in the case of Dyson log gases [8,16,19,22,23]. Moreover, we will see that the rate function has a third-order discontinuity. These results show that the 1D CG belongs to a different universality class than the one determined by the TW distribution. To conclude our analysis of the typical fluctuations regime, we present in Fig. 1 the results obtained by solving numerically the differential equations above and a comparison with Monte Carlo (MC) simulations using the Hamiltonian of Eq. (2) for two paradigmatic potentials of RMT.

To study the large deviations regime for which $|w - x_i| = O(1)$, for any $i = 1, \dots, N$ we use the Coulomb fluid method [4–19] to compute $\varrho(c,w) \equiv \text{Prob}[\mathcal{C}_w = c]$, with $\mathcal{C}_w = \frac{1}{N}\sum_{j=1}^N \Theta(w - x_j)$, which corresponds to the probability that *exactly* cN particles have positions smaller than w. We start by writing

$$\varrho(c,w) = \frac{1}{\Omega_0}\int d\mathbf{x}\, p(\mathbf{x})\delta\left(c - \frac{1}{N}\sum_{i=1}^N \Theta(w - x_i)\right), \quad (10)$$

with $p(\mathbf{x}) = \frac{1}{\Omega_0}e^{-\mathcal{H}(\mathbf{x})}$. This can be written as the following path integral (and two integrals over variables μ and ν) $\varrho(c,w) = \frac{1}{\Omega_0}\int D[\rho,\mu,\nu]e^{-N^3\mathcal{S}[\rho,\mu,\nu]}$ with $\mathcal{S}$ being the action [39],

$$\begin{aligned}\mathcal{S}[\rho,\mu,\nu] = &\int dx\rho(x)v(x) - \frac{\alpha}{2}\int dx dx'|x - x'|\rho(x)\rho(x')\\ &- \mu\left(1 - \int dx\rho(x)\right)\\ &- \nu\left(c - \int dx\Theta(w - x)\rho(x)\right).\end{aligned} \quad (11)$$

Here, μ and ν are Lagrange multipliers to enforce normalization in the density ρ and that a fraction c of particles are to the left of w, respectively. Similarly, the normalization constant can be written as $\Omega_0 = \int D[\rho_0,\mu_0]e^{-N^3\mathcal{S}_0[\rho_0,\mu_0]}$ and corresponds, in turn, to a CG without a wall. In the thermodynamic limit both expressions can be evaluated by the saddle-point method obtaining $\varrho(c,w) \sim e^{-N^3\psi(c,w)}$, where

$$\psi(c,w) = \mathcal{S}[\rho^*,\mu^*,\nu^*] - \mathcal{S}_0[\rho_0^*,\mu_0^*] \quad (12)$$

is the rate function. Here, $\rho^*(x)$ corresponds physically to the equilibrium particle density of a system constrained to have a fraction of particles c to the left of w, while $\rho_0^*(x)$ is the unconstrained equilibrium particle density. As noted in Refs. [16,19], the rate function $\psi(c,w)$ has a dual role, for it describes the large deviations of $\mathcal{C}_w$, when w is taken as a parameter or, conversely, the statistics of the ith particle when ψ is viewed as a function of w.

Noteworthy, the stationarity conditions of $\mathcal{S}$ yield an integral equation that can be solved *exactly* for *any* external potential $v(x)$. The solution for the unconstrained system is [39]

$$\rho_0^*(x) = \frac{v''(x)}{2\alpha}\mathbb{I}[x_- \leqslant x \leqslant x_+], \quad (13)$$

where $\mathbb{I}[A]$ is an indicator function, whose value is 1 whenever condition A is true. The bottom panels of Fig. 1 show a comparison of this formula with unconstrained MC samplings.

From a physical perspective, we expect the constrained density to be similar to $\rho_0^*(x)$ since by placing a barrier at w there will only be a *local* density rise in its vicinity. This is a consequence of the regularity of the 1D interparticle potential when particles overlap, in contrast with its 2D counterpart. Hence we have that once the wall is present, there will be an accumulation of particles next it, whose magnitude depends on the fraction of the particles "pushed" by it, compared to the one of the unconstrained system. The latter one, denoted as $c^*(w)$, is easily calculated by integrating $\rho_0^*(x)$ up to w. When $w \in [x_-,x_+]$ we have $c^*(w) = \frac{v'(w)+\alpha}{2\alpha}$ and the constrained equilibrium density results into [39]

$$\begin{aligned}\rho^*(x) = &\frac{v''(x)}{2\alpha}(\mathbb{I}[x_- \leqslant x \leqslant a] + \mathbb{I}[b < x \leqslant x_+])\\ &+ |c - c^*(w)|\delta(w - x).\end{aligned} \quad (14)$$

Here, the several parameters involved in Eq. (14) are defined as follows: x_0 is such that $v'(x_0) = \alpha(2c-1)$; when $c > c^*(w)$ we must take $a = w$ and $b = x_0$, while for $c < c^*(w)$ we have instead that $a = x_0$ and $b = w$. For $c = c^*(w)$, the wall becomes ineffective, and therefore $x_0 = w$, recovering the unconstrained solution $\rho_0^*(x)$. Finally, for $w < x_-$ ($w > x_+$) we have that $c^* = 0$ ($c^* = 1$) and similar expressions for $\rho^*(x)$ apply. The fact that the resulting equilibrium density has, in general, an infinitely sharp peak at w as well as a noncompact and bounded support resembles some well-known results for the spectral densities of RMT [4–19]. However, in those systems the effect of introducing the wall significantly modifies the unconstrained density and obtaining an analytical expression for $\rho^*(x)$ is only possible in a few exemplary cases. Surprisingly, this is not longer true for 1D CG, where we have found the equilibrium density for any convex potential and any fraction of particles to the left of w. It is important to mention that so long as $v(x)$ is strong enough to dominate as $|x| \to \infty$, the equilibrium densities of Eqs. (13) and (14) are the unique minimizers of the corresponding actions [38], while the convexity of the potential assures that they are non-negative functions.

Putting these results together and evaluating Eq. (12) yields a rather simple expression for the rate function,

$$\psi(c,w) = \frac{|c - c^*(w)|v(w)}{2} - \int_a^b dx\, \frac{v''(x)v(x)}{4\alpha} - \frac{\mu^* - \mu_0^* + \nu^* c}{2}, \tag{15}$$

whenever the wall is inside the natural support $w \in [x_-, x_+]$ (analogous expressions for $w \notin [x_-, x_+]$ together with explicit formulas are given in Ref. [39]). Figure 2 shows a comparison of the analytical value of $\psi(c,w)$ and MC estimations of the rate function for the same two potentials used above. Importantly, through Eq. (15) we can recover straightforwardly the results of Refs. [27,28] for the rate functions $\phi_M^{(\pm)}$ and $\phi_m^{(\pm)}$ controlling the left and right deviations of the rightmost and leftmost particles (see Ref. [39] for details). The left panels' insets of Fig. 2 show the comparison of the rate functions of extremal particles with MC simulations, while the ones in the right panels depict a histogram obtained by MC sampling and the analytical expression for $\rho^*(x)$ according to Eq. (14). In all cases, the agreement is outstanding. Thus Eq. (15) provides a general and exact expression for the rate function of one-dimensional CGs and it constitutes the main result of this second part.

As it can be explicitly verified from Eq. (15), the rate function of the 1D CG has the noticeable feature that its first two partial derivatives vanish at c^* and w^*, where this latter quantity is obtained by inverting the relation defining c^*. This is in stark contrast with the analogous result in RMT, where the second partial derivatives are different from zero, meaning that the fluctuations of x_i around w^* ($\mathcal{C}_w$ around c^*) are of Gaussian type [22,23]. Instead, in the 1D case we found that the *third* derivatives correspond to the first nonvanishing term in the expansion of $\psi(c,w)$ around c^* and w^*. In fact, we have that

$$\Pr(x_i = w) \sim \exp\left(-\frac{N^3(v''(w^*))^2}{12\alpha}|w - w^*|^3\right), \tag{16}$$

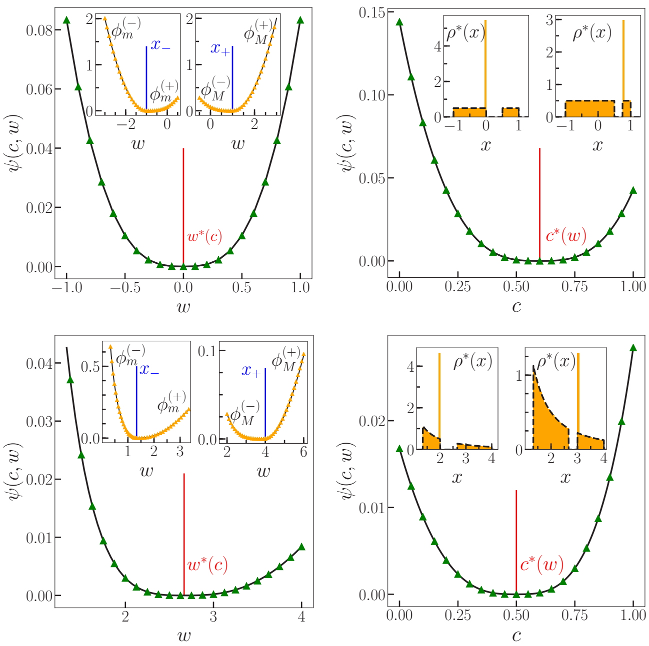

FIG. 2. Left panels: Comparison between MC simulations (triangular markers) of $\psi(c,w)$ and the exact expression of Eq. (15) (solid curves) as a function of w. The upper panel shows the results using the harmonic potential with $\alpha = 1$ and $c = 0.5$, while the lower one corresponds to the Wishart-Laguerre potential with $\alpha = 0.25$, $a = 2$, and $c = 0.75$. The right (left) inset shows $\phi_M(w)$ [$\phi_m(w)$], the rate function for $x_{\max}$ ($x_{\min}$). The MC estimations of $\phi_M^{(-)}(w)$ and $\phi_m^{(+)}(w)$ have been scaled as N^3, while the other rate functions were scaled as N^2. Right panels: $\psi(c,w)$ as a function of c, fixing $w = 0.2$ for the harmonic potential (upper panel) and $w = 2$ for the Wishart-Laguerre one (lower panel). The insets compare MC samples of the constrained density $\rho^*(x)$ (orange histograms) with the exact formula (14) (blue dashed curve), using $c = 0.75$ in all the cases.

which is straightforward to verify that it matches exactly with the asymptotic expansion of $F'_{c=i\,N}(W)$ of Eq. (9). This last expression implies that the rate function is *not* analytical around its minimum, which is flatter than a quadratic one because of the vanishing second derivatives. We thus end up with the unusual case of having a rate function that is not *strictly* convex nor analytic near its minimum, once again differing from the features of Dyson log gases. This is not a minor difference indeed, for it is known [40,41] that a rate function that is not strictly convex cannot be extended to the regime of typical fluctuations as in the Dyson log-gas case [19]. In other words, the 1D CG follows a *weak* large deviation principle, for the rate function cannot be expanded to match smoothly the typical fluctuations regime [40]. A similar behavior has been found in the two-dimensional Ising system [40,42,43] as well as for a drifted Brownian motion [44]. In Ref. [39] we provide further evidence that $\psi(c,w)$ does not provide the correct description in the regime of typical fluctuations. This unusual behavior on the statistical properties of bulk particles may indicate the existence of an intermediate regime, as it was recently found in the statistical properties of extremal eigenvalues in the Ginibre ensemble [45]. As a final remark, our results showed that the rate function for bulk particles in 1D

CG exhibits a discontinuity in its third derivative and, while analogous results for extremal particles seem to indicate the presence of a phase transition, this feature does not necessarily carry over for bulk particles as fluctuations behave in the same way at each side of the optimal value x^*, as can be observed in Eqs. (9) and (16). Thus a phase transition for bulk particles, if any, must lie in another explanation.

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