Steady thermocapillary migration of a droplet in a uniform temperature gradient combined with a radiation energy source at large Marangoni numbers

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Steady thermocapillary droplet migration in a uniform temperature gradient combined with a radiation energy source at large Reynolds and Marangoni numbers is studied. To reach a terminal quasisteady process, the magnitude of the radiation energy source is required to preserve the conservative integral thermal flux across the surface. Under a quasisteady state assumption, an analytical result for the steady thermocapillary migration of a droplet at large Reynolds and Marangoni numbers is derived by using the method of matched asymptotic expansions. It is shown that the thermocapillary droplet migration speed increases as the Marangoni number increases, while a radiation energy source with a sine square dependence is provided.

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I. INTRODUCTION

A droplet in an external fluid or on a solid substrate can be driven by body forces generated in gravitational, electric, magnetic, and ultrasonic fields [1]. Even in the absence of body forces, a variable surface tension along the interface can also drive the droplet migration in an external fluid or solid substrate. Thermocapillary migration of a droplet in a microgravity environment is a very interesting topic in both fundamental hydrodynamic theory and engineering applications [2]. Young, Goldstein, and Block [3] carried out an initial study on the thermocapillary migration of a droplet in a uniform temperature gradient in the limits of zero Reynolds (Re) and Marangoni (Ma) numbers (YGB model). Subramanian [4] proposed the quasisteady state assumption and obtained analytical results with high-order expansions at small Ma numbers. The thermocapillary droplet migration processes at small Ma numbers are understood very well in a series of theoretical analyses, numerical simulations, and experimental investigations [5,6]. However, the physical behaviors at large Ma numbers appear rather complicated due to the momentum and energy transfer through the interface of two-phase fluids. Meanwhile, to perform a feasible numerical simulation of thermocapillary migration of a droplet at large Ma numbers is still a challenge due to very thin thermal boundaries [$O(\mathrm{Ma}^{-1/2})$] and very long migration times [$O(\mathrm{Ma})$]. Under the assumption of a quasisteady state, Balasubramaniam and Subramanian reported [7] that the migration speed of a droplet increases as the Ma number increases, in qualitative agreement with a corresponding numerical simulation [8]. The experimental investigation carried out by Hadland *et al.* [9] and Xie *et al.* [10] showed that the droplet migration speed decreases as the Ma number increases, which qualitatively disagrees with the above theoretical and numerical results. Wu and Hu [11] and Wu [12] identified a nonconservative integral thermal flux across the surface in steady thermocapillary droplet migration at large Ma (Re) numbers, which indicates that thermocapillary droplet migration at large Ma (Re) numbers is an unsteady process. To preserve a conservative integral thermal flux across the surface, two methods, i.e., adding a thermal source inside the droplet or at the surface, were also suggested. With a thermal source added inside the droplet, an analytical result of steady thermocapillary migration of the droplet at large Ma (Re) numbers was determined [13]. Therefore, thermocapillary droplet migration at large Ma numbers remains a topic to be studied with respect to its physical mechanism.

In the above studies, the variable surface tension exerted on the interface of two phases is generated by adding a nonuniform temperature field. On the other hand, a radiative heating contrary to the direction of movement, which provides a thermal source at the surface through absorption, can also form a variable surface tension exerted on the interface. Oliver and Dewitt [14] first analyzed thermocapillary migration of a droplet caused by thermal radiation in a microgravity environment in the zero Re and Ma number limits. Rednikov and Ryzzantsev [15] independently derived similar results and determined the deformation of the droplet. Shen [16] and Khodadadi and Zhang [17] numerically studied the effects of thermocapillary convection on the melting of droplets for a short duration and uniform heat pulses under zero gravity conditions at large Ma numbers, respectively. Lopez *et al.* [18] experimentally observed thermocapillary migration of a droplet caused by laser beam heating due to the absorption of laser radiation in making a strongly nonhomogeneous distribution of temperature inside the droplet as well as at its surface.

In this paper, when a radiation energy source contrary to the direction of movement is placed to preserve the conservative integral thermal flux across the surface, thermocapillary droplet migration at large Re and Ma numbers can thus reach a quasisteady process. Steady thermocapillary droplet migration in a uniform temperature gradient combined with a radiation energy source at large Re and Ma numbers is studied. In comparing with the previous method to preserve

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a conservative integral thermal flux across the surface [13], the current method, i.e., placing the radiation energy source on the outside of the droplet, is easier to carry out in a real space experiment. In principle, the previous method adds a thermal source in the energy equation within the droplet, but the current method adds a heat flux at the interface of the droplet. The paper is organized as follows. In Sec. II, the magnitude of the radiation energy source is required to preserve the conservative integral thermal flux across the surface. An analytical result for steady thermocapillary droplet migration at large Re and Ma numbers is determined in Sec. III. Finally, in Sec. IV, conclusions and discussions are given.

II. PROBLEM FORMULATION

Consider the thermocapillary migration of a spherical droplet of radius R_0, density $\gamma\rho$, dynamic viscosity $\alpha\mu$, thermal conductivity βk, and thermal diffusivity $\lambda\kappa$ in a continuous phase fluid of infinite extent with density ρ, dynamic viscosity μ, thermal conductivity k, and thermal diffusivity κ under a uniform temperature gradient G in the direction of movement and an inhomogeneous radiation energy source S contrary to the direction of movement. It is assumed that the continuous phase fluid is transparent and that the radiation is absorbed totally on the droplet surface. The rate of change of the interfacial tension between the droplet and the continuous phase fluid with temperature is denoted by σ_T. Unsteady energy equations for the continuous phase and the fluid in the droplet in a laboratory coordinate system denoted by a bar are written as follows,

$$\begin{aligned}\frac{\partial \bar{T}}{\partial t} + \bar{\mathbf{v}}\bar{\nabla}\bar{T} &= \kappa\bar{\Delta}\bar{T},\\ \frac{\partial \bar{T}'}{\partial t} + \bar{\mathbf{v}}'\bar{\nabla}\bar{T}' &= \lambda\kappa\bar{\Delta}\bar{T}',\end{aligned} \tag{1}$$

where $\bar{\mathbf{v}}$ and $\bar{T}$ are velocity and temperature, and a prime denotes quantities in the droplet. The solutions of Eqs. (1) have to satisfy the boundary conditions at infinity,

$$\bar{T} \to T_0 + G\bar{z}, \tag{2}$$

where T_0 is the undisturbed temperature of the continuous phase and the boundary conditions at the interface $\bar{\mathbf{r}}_b$ of the two-phase fluids,

$$\begin{aligned}\bar{T}(\bar{\mathbf{r}}_b,t) &= \bar{T}'(\bar{\mathbf{r}}_b,t),\\ \frac{\partial \bar{T}}{\partial n}(\bar{\mathbf{r}}_b,t) + S &= \beta\frac{\partial \bar{T}'}{\partial n}(\bar{\mathbf{r}}_b,t).\end{aligned} \tag{3}$$

Under the quasisteady state assumptions, steady axisymmetric energy equations nondimensionalized by taking the radius of the droplet R_0, the YGB model velocity $v_o = -\sigma_T G R_0/\mu$, and GR_0 as reference quantities to make the coordinates, velocity, and temperature dimensionless, can be written in a spherical coordinate system (r,θ) moving with a droplet velocity V_∞ as follows,

$$\begin{aligned}1 + u\frac{\partial T}{\partial r} + \frac{v}{r}\frac{\partial T}{\partial \theta} &= \epsilon^2\Delta T,\\ 1 + u'\frac{\partial T'}{\partial r} + \frac{v'}{r}\frac{\partial T'}{\partial \theta} &= \lambda\epsilon^2\Delta T',\end{aligned} \tag{4}$$

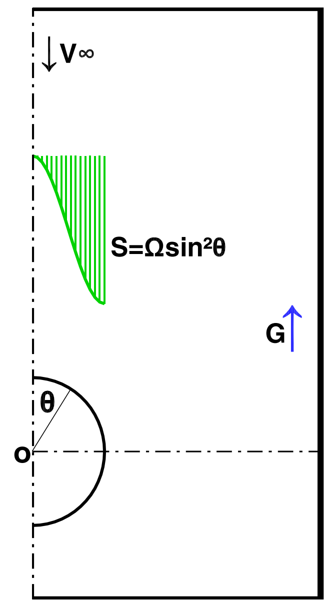

FIG. 1. A schematic diagram of thermocapillary droplet migration under the combined actions of a temperature gradient G and a radiation energy source $S = \Omega\sin^2\theta$ in an axisymmetric spherical coordinate system moving with a droplet velocity V_∞.

where the small parameter ϵ and Ma number are defined, respectively, as

$$\epsilon = \frac{1}{\sqrt{\mathrm{Ma}\, V_\infty}} \tag{5}$$

and

$$\mathrm{Ma} = \frac{v_0 R_0}{\kappa}. \tag{6}$$

The boundary conditions (2) and (3) are rewritten, respectively, as

$$T \to r\cos\theta, \quad \text{as } r \to \infty, \tag{7}$$

and at the interface of two-phase fluids,

$$\begin{aligned}T(1,\theta) &= T'(1,\theta),\\ \frac{\partial T}{\partial r}(1,\theta) + \Omega\sin^2\theta\cos\theta &= \beta\frac{\partial T'}{\partial r}(1,\theta), \quad 0 \leqslant \theta \leqslant \pi/2,\\ \frac{\partial T}{\partial r}(1,\theta) &= \beta\frac{\partial T'}{\partial r}(1,\theta), \quad \pi/2 < \theta \leqslant \pi.\end{aligned} \tag{8}$$

The inhomogeneous radiation energy source nondimensionalized by the reference quantity kG is assumed as $S = \Omega\sin^2\theta$. Its contribution to the interface thermal flux $S\cos\theta$ is zero at $\theta = \pi/2$, which reveals that the upper and lower interface thermal boundary conditions in Eq. (8) are continuous. A schematic diagram of thermocapillary droplet migration in a coordinate system moving with a droplet speed V_∞ is shown in Fig. 1.

For large Re numbers ($\mathrm{Re} = \frac{v_0 R_0}{\nu}$), the velocity fields of the continuous phase and the fluid within the droplet can be

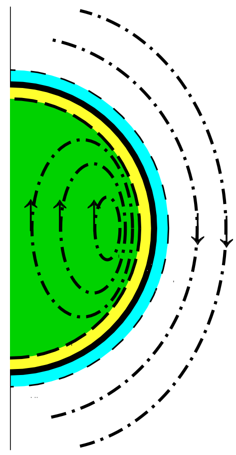

FIG. 2. A schematic diagram of potential flows and boundary layer flows of thermocapillary droplet migration at large Re numbers. Solid line: The interface of the droplet. Dashed/long-dashed lines: The interface between potential flow (white/green zone) and boundary layer flow (blue/yellow zone) in the continuous flow/within the droplet. Dashed-dotted lines: Streamlines of the potential flows inside and outside the droplet.

described by potential flows and boundary layer flows, as shown in Fig. 2. The scaled potential flow fields around a fluid sphere,

$$u = -\cos\theta\left(1-\frac{1}{r^3}\right),\quad v = \sin\theta\left(1+\frac{1}{2r^3}\right), \tag{9}$$

and

$$u' = \frac{3}{2}\cos\theta(1-r^2),\quad v' = -\frac{3}{2}\sin\theta(1-2r^2), \tag{10}$$

are taken as those in the continuous phase and within the droplet, respectively [19,20]. It is noticed that the potential flow fields (9) and (10) for large Re numbers may be obtained from the general solutions for small Re numbers by setting $D_n = 0$, $n \geqslant 3$ [21,22]. For large Ma numbers, the temperature field at infinity in Eq. (7) is further expressed as [11]

$$T \approx r\cos\theta - \frac{1}{2r^2}\cos\theta + o(1). \tag{11}$$

Integrating Eqs. (4) in the continuous phase domain ($r \in [1, r_\infty], \theta \in [0, \pi]$) with the boundary condition (11) and within the droplet region ($r \in [0, 1], \theta \in [0, \pi]$), respectively, we obtain

$$\int_0^\pi \frac{\partial T}{\partial r}(1,\theta)\sin\theta d\theta + \int_0^{\pi/2} \Omega \sin^3\theta\cos\theta d\theta = -\frac{1}{3\epsilon^2} + \frac{\Omega}{4} \tag{12}$$

and

$$\int_0^\pi \frac{\partial T'}{\partial r}(1,\theta)\sin\theta d\theta = \frac{2}{3\lambda\epsilon^2}. \tag{13}$$

From Eqs. (12) and (13), we have

$$\beta\int_0^\pi \frac{\partial T'}{\partial r}(1,\theta)\sin\theta d\theta - \int_0^\pi \frac{\partial T}{\partial r}(1,\theta)\sin\theta d\theta - \int_0^{\pi/2}\Omega\sin^3\theta\cos\theta d\theta = \frac{1}{3\epsilon^2}\left(1+\frac{2\beta}{\lambda}\right) - \frac{\Omega}{4}. \tag{14}$$

For large Ma numbers and finite V_∞, Eqs. (12) and (13) should satisfy the thermal flux boundary condition (8), i.e., the right-hand side of Eq. (14) will be zero. So, we have

$$\Omega = \frac{4}{3\epsilon^2}\left(1+\frac{2\beta}{\lambda}\right) = \frac{4}{3}\left(1+\frac{2\beta}{\lambda}\right)V_\infty \mathrm{Ma}, \tag{15}$$

which preserves the conservative integral thermal flux across the surface. In following, we will focus on the steady thermocapillary migration of a droplet in a uniform temperature gradient G combined with an external thermal radiation source S and determine the dependence of the migration speed on large Ma numbers.

III. ANALYSIS AND RESULTS

A. Outer temperature field in the continuous phase

By using an outer expansion for the scaled temperature field in the continuous phase,

$$T = T_0 + \epsilon T_1 + o(\epsilon), \tag{16}$$

the energy equation for the outer temperature field in its leading order can be obtained from Eqs. (4) as follows,

$$1 + u\frac{\partial T_0}{\partial r} + \frac{v}{r}\frac{\partial T_0}{\partial\theta} = 0. \tag{17}$$

By using the coordinate transformation from (r,θ) to (ψ,θ) in solving Eq. (17), its solution can be written as

$$T_0(r,\theta) = G(\psi) - \int \frac{2r^4}{2r^3+1}\frac{d\theta}{\sin\theta}, \tag{18}$$

where $G(\psi)$ is a function of ψ (the stream function in the continuous phase). Following Refs. [7,13], the solution near $r = 1$ is simplified as

$$\begin{aligned} T_0(r,\theta) = {} & \left(1+\frac{\pi}{6\sqrt{3}}-\frac{1}{6}\ln 432\right) - \frac{1}{18}\left(\frac{\pi}{\sqrt{3}}+\ln 432\right) \\ & \times\left(r^2-\frac{1}{r}\right)\sin^2\theta + \frac{1}{3}\ln\left(r^2-\frac{1}{r}\right) \\ & + \frac{2}{3}\ln(1+\cos\theta) + \frac{2}{9}\left(r^2-\frac{1}{r}\right)\cos\theta \\ & + \frac{1}{9}\left(r^2-\frac{1}{r}\right)\ln\left(r^2-\frac{1}{r}\right)\sin^2\theta \\ & + \frac{2}{9}\left(r^2-\frac{1}{r}\right)\sin^2\theta\ln(1+\cos\theta). \end{aligned} \tag{19}$$

In the boundary layer approximation near the interface, the convective transport term must be of the same order as the conduction term. By using the scaled radial coordinate in the boundary layer,

$$x = \frac{r-1}{\epsilon}, \tag{20}$$

the temperature field near the interface can be expressed as

$$t(x,\theta) = 1 + \frac{\pi}{6\sqrt{3}} - \frac{1}{6}\ln 48 + \frac{2}{3}\ln\left(\frac{1+\cos\theta}{\sin\theta}\right) + \frac{1}{3}x\sin^2\theta\epsilon\ln\epsilon + o(\epsilon\ln\epsilon). \quad (21)$$

B. Outer temperature field within the droplet

By using the outer expansion for the scaled temperature field within the droplet in Eqs. (4),

$$T' = \frac{1}{\epsilon^2}T'_{-2} + \frac{1}{\epsilon}T'_{-1} + T'_0 + o(1), \quad (22)$$

the equation in its leading order can be written as

$$u'\frac{\partial T'_{-2}}{\partial r} + \frac{v'}{r}\frac{\partial T'_{-2}}{\partial\theta} = 0. \quad (23)$$

Its solution is

$$T'_{-2} = F_0(\psi'), \quad (24)$$

where $\psi' = \frac{3}{4}\sin^2\theta(r^4 - r^2)$ is the stream function within the droplet. The unknown function $F_0(\psi')$ can be obtained from the following equation for the temperature field T'_0 in its second order,

$$1 + u'\frac{\partial T'_0}{\partial r} + \frac{v'}{r}\frac{\partial T'_0}{\partial\theta} = \lambda\Delta F_0. \quad (25)$$

Following Ref. [7], in solving Eq. (25), the coordinate transformation from (r,θ) to (m,q) is applied in the form of

$$m = -\frac{16}{3}\psi', \quad q = \frac{r^4\cos^4\theta}{2r^2-1}, \quad (26)$$

where m and q denote the streamlines and their orthogonal lines, respectively. The solution of Eq. (25) is thus written as

$$T'_{-2}(r,\theta) = F_0 = \frac{1}{\lambda}\left[B' - \frac{1}{16}m + \frac{3}{256}\left(3\ln 2 - 1\frac{3}{4}\right)m^2 - \frac{3}{512}m^2\ln m\right] + o(m^2\ln m), \quad (27)$$

where B' is an unknown constant. Similarly, by using the scaled radial coordinate in the boundary layer,

$$x' = \frac{1-r}{\sqrt{\lambda}\epsilon}, \quad (28)$$

the temperature field near the interface can be expressed as follows,

$$t'(x',\theta) = -\frac{1}{2\sqrt{\lambda}}x'\sin^2\theta\frac{1}{\epsilon} - \frac{3}{8}x'^2\sin^4\theta\ln\epsilon + o(\ln\epsilon). \quad (29)$$

C. Inner temperature fields in the leading order

By using inner expansions for the continuous phase and the fluid in the droplet,

$$t(x,\theta) = t_{-1}\frac{1}{\epsilon} + t_{l0}\ln\epsilon + o(\ln\epsilon), \quad (30)$$

$$t'(x',\theta) = t'_{-1}\frac{1}{\epsilon} + t'_{l0}\ln\epsilon + o(\ln\epsilon), \quad (31)$$

and the inner variables given in Eqs. (20) and (28), the scaled energy equations for the inner temperature fields in the leading order can be written as follows,

$$-3x\cos\theta\frac{\partial t_{-1}}{\partial x} + \frac{3}{2}\sin\theta\frac{\partial t_{-1}}{\partial\theta} = \frac{\partial^2 t_{-1}}{\partial x^2}, \quad (32)$$

$$-3x'\cos\theta\frac{\partial t'_{-1}}{\partial x'} + \frac{3}{2}\sin\theta\frac{\partial t'_{-1}}{\partial\theta} = \frac{\partial^2 t'_{-1}}{\partial x'^2}. \quad (33)$$

The boundary conditions are

$$\begin{aligned} t_{-1}(0,\theta) &= t'_{-1}(0,\theta),\\ \delta\frac{\partial t_{-1}}{\partial x}(0,\theta) + \omega\delta\sin^2\theta\cos\theta &= -\frac{\partial t'_{-1}}{\partial x'}(0,\theta),\ 0\leqslant\theta\leqslant\pi/2,\\ \delta\frac{\partial t_{-1}}{\partial x}(0,\theta) &= -\frac{\partial t'_{-1}}{\partial x'}(0,\theta),\ \pi/2<\theta\leqslant\pi,\\ t_{-1}(x\to\infty,\theta) &\to 0,\\ t'_{-1}(x'\to\infty,\theta) &\to B - \frac{1}{2\sqrt{\lambda}}x'\sin^2\theta, \end{aligned} \quad (34)$$

where $\delta = \sqrt{\lambda}/\beta$ and $\omega = \Omega\epsilon^2 = \frac{2}{3}(1+\frac{2\beta}{\lambda})$. We transform the independent variables from $[(x,x'),\theta]$ to $[(\eta,\eta'),\xi]$ and the functions from (t_{-1},t'_{-1}) to (f_0,f'_0) as

$$(\eta,\eta') = \left(\frac{3}{2}x\sin^2\theta, \frac{3}{2}x'\sin^2\theta\right),$$
$$\xi = \frac{1}{2}(2-3\cos\theta+\cos^3\theta) = \frac{1}{2}(2+\cos\theta)(1-\cos\theta)^2, \quad (35)$$

and

$$f_0(\eta,\xi) = t_{-1}(x,\theta),$$
$$f'_0(\eta',\xi) = t'_{-1}(x',\theta) - B + \frac{1}{2\sqrt{\lambda}}x'\sin^2\theta. \quad (36)$$

The corresponding energy equations for f_0, f'_0 and the boundary conditions can be written as follows,

$$\frac{\partial f_0}{\partial\xi} = \frac{\partial^2 f_0}{\partial\eta^2},$$
$$\frac{\partial f'_0}{\partial\xi} = \frac{\partial^2 f'_0}{\partial\eta'^2}, \quad (37)$$

and

$$\begin{aligned} &f_0(0,\xi) = f'_0(0,\xi) + B,\\ &\delta\frac{\partial f_0}{\partial\eta}(0,\xi) = -\frac{\partial f'_0}{\partial\eta'}(0,\xi) + \frac{1}{3\sqrt{\lambda}} + \Phi(\xi),\ 0\leqslant\xi\leqslant 1,\\ &\delta\frac{\partial f_0}{\partial\eta}(0,\xi) = -\frac{\partial f'_0}{\partial\eta'}(0,\xi) + \frac{1}{3\sqrt{\lambda}},\quad 1<\xi\leqslant 2,\\ &f_0(\eta\to\infty,\xi) = 0,\\ &f'_0(\eta'\to\infty,\xi) = 0, \end{aligned} \quad (38)$$

where $\Phi(\xi) = -\frac{2\omega\delta}{3}\cos\theta = -\frac{2\omega\delta}{3}(\cos\frac{\phi}{3} - \sqrt{3}\sin\frac{\phi}{3})$, and $\phi = \arccos(1-\xi)$ in Shengjin's formula [23]. To solve

Eqs. (37), the initial conditions are provided below,

$$\begin{aligned} f_0(\eta,0) &= 0,\\ f_0'(\eta',0) &= f_0'[\eta',\xi(\pi)] = f_0'(\eta',2) = g_0(\eta'),\\ g_0(\eta'\to\infty) &\to 0. \end{aligned} \tag{39}$$

Following the methods given by Carslaw and Jaeger [24] and Harper and Moore [19], the solutions of Eqs. (37) for the continuous phase and the fluid in the droplet can be respectively determined as

$$\begin{aligned} f_0(\eta,\xi) &= \frac{1}{1+\delta}\left\{-\frac{1}{\sqrt{\pi}}\int_0^{\xi}\Phi(\xi-\tau)\exp\left(-\frac{\eta^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}} + \left(B+\frac{\eta}{3\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right)\right.\\ &\quad\left.+\frac{1}{\sqrt{\pi\xi}}\int_0^{\infty}g_0(\eta^*)\exp\left[-\frac{(\eta+\eta^*)^2}{4\xi}\right]d\eta^*\right\}\quad(0\leqslant\xi\leqslant 1),\\ f_0(\eta,\xi) &= \frac{1}{1+\delta}\left\{-\frac{1}{\sqrt{\pi}}\int_0^{\xi}\Phi(\xi-\tau)\exp\left(-\frac{\eta^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}} + \frac{1}{\sqrt{\pi}}\int_0^{\xi-1}\Phi(\xi-1-\tau)\exp\left(-\frac{\eta^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}}\right.\\ &\quad\left.+\left(B+\frac{\eta}{3\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta}{2\sqrt{\xi}}\right)+\frac{1}{\sqrt{\pi\xi}}\int_0^{\infty}g_0(\eta^*)\exp\left[-\frac{(\eta+\eta^*)^2}{4\xi}\right]d\eta^*\right\}\quad(1<\xi\leqslant 2), \end{aligned} \tag{40}$$

and

$$\begin{aligned} f_0'(\eta',\xi) &= \frac{\delta}{1+\delta}\left\{-\frac{1}{\sqrt{\pi}\delta}\int_0^{\xi}\Phi(\xi-\tau)\exp\left(-\frac{\eta'^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}} - \left(B-\frac{\eta'}{3\delta\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta'}{2\sqrt{\xi}}\right)\right\}\\ &\quad+\frac{1}{2\sqrt{\pi\xi}}\int_0^{\infty}g_0(\eta^*)\left\{\exp\left[-\frac{(\eta'-\eta^*)^2}{4\xi}\right]+\frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta^*)^2}{4\xi}\right]\right\}d\eta^*\quad(0\leqslant\xi\leqslant 1),\\ f_0'(\eta',\xi) &= \frac{\delta}{1+\delta}\left\{-\frac{1}{\sqrt{\pi}\delta}\int_0^{\xi}\Phi(\xi-\tau)\exp\left(-\frac{\eta'^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}}\right.\\ &\quad\left.+\frac{1}{\sqrt{\pi}\delta}\int_0^{\xi-1}\Phi(\xi-1-\tau)\exp\left(-\frac{\eta'^2}{4\tau}\right)\frac{d\tau}{\tau^{1/2}} - \left(B-\frac{\eta'}{3\delta\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta'}{2\sqrt{\xi}}\right)\right\}\\ &\quad+\frac{1}{2\sqrt{\pi\xi}}\int_0^{\infty}g_0(\eta^*)\left\{\exp\left[-\frac{(\eta'-\eta^*)^2}{4\xi}\right]+\frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta^*)^2}{4\xi}\right]\right\}d\eta^*\quad(1<\xi\leqslant 2). \end{aligned} \tag{41}$$

D. Steady migration velocity of the droplet

Due to the zero net force acting on the droplet at the flow direction, the migration speed of the droplet can be obtained as

$$\begin{aligned} V_\infty &= -\frac{1}{2(2+3\alpha)}\int_0^{\pi}\sin^2\theta\frac{\partial t}{\partial\theta}(0,\theta)d\theta\\ &= \frac{1}{2+3\alpha}\int_0^{\pi}\sin\theta\cos\theta t(0,\theta)d\theta. \end{aligned} \tag{42}$$

When the inner expansion in the temperature field (30) is truncated at the $o(\ln\epsilon)$ order, we rewrite Eq. (42) as

$$V_\infty = \frac{1}{2+3\alpha}\int_0^{\pi}\sin\theta\cos\theta\left[t_{-1}(0,\theta)\frac{1}{\epsilon}+t_{l0}(0,\theta)\ln\epsilon\right]d\theta. \tag{43}$$

Since $\epsilon = 1/\sqrt{\mathrm{Ma}\,V_\infty}$, the migration speed of the droplet is evaluated as

$$V_\infty \approx a_1^2\,\mathrm{Ma} - 2a_{l0}\ln\mathrm{Ma} + a_0, \tag{44}$$

where

$$a_1 = \frac{1}{2+3\alpha}\int_0^{\pi}\sin\theta\cos\theta t_{-1}(0,\theta)d\theta \tag{45}$$

and

$$a_{l0} = \frac{1}{2+3\alpha}\int_0^{\pi}\sin\theta\cos\theta t_{l0}(0,\theta)d\theta. \tag{46}$$

From Eqs. (40), we obtain the inner temperature field in its leading order for the continuous phase near the surface of the droplet,

$$\begin{aligned} t_{-1}(0,\theta) = f_0(0,\xi) &= \frac{1}{1+\delta}\left[-\frac{1}{\sqrt{\pi}}\int_0^{\xi}\Phi(\xi-\tau)\frac{d\tau}{\tau^{1/2}} + B + \frac{1}{\sqrt{\pi\xi}}\int_0^{\infty} g_0(\eta^*)\exp\left(-\frac{\eta^{*2}}{4\xi}\right)d\eta^*\right] \\ &= \frac{1}{1+\delta}\left[-\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\xi}}\Phi(\xi-s^2)ds + B + \frac{2}{\sqrt{\pi}}\int_0^{\infty} g_0(2\xi^{1/2}\zeta)\exp(-\zeta^2)d\zeta\right], \quad 0\leqslant\theta\leqslant\pi/2, \\ t_{-1}(0,\theta) = f_0(0,\xi) & \\ &= \frac{1}{1+\delta}\left[-\frac{1}{\sqrt{\pi}}\int_0^{\xi}\Phi(\xi-\tau)\frac{d\tau}{\tau^{1/2}} + \frac{1}{\sqrt{\pi}}\int_0^{\xi-1}\Phi(\xi-1-\tau)\frac{d\tau}{\tau^{1/2}} + B + \frac{1}{\sqrt{\pi\xi}}\int_0^{\infty} g_0(\eta^*)\exp\left(-\frac{\eta^{*2}}{4\xi}\right)d\eta^*\right] \\ &= \frac{1}{1+\delta}\left[-\frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\xi}}\Phi(\xi-s^2)ds + \frac{1}{\sqrt{\pi}}\int_0^{\sqrt{\xi-1}}\Phi(\xi-1-s^2)ds + B \right. \\ &\quad \left. + \frac{2}{\sqrt{\pi}}\int_0^{\infty} g_0(2\xi^{1/2}\zeta)\exp(-\zeta^2)d\zeta\right], \quad \pi/2<\theta\leqslant\pi. \end{aligned} \tag{47}$$

Substituting Eq. (47) into Eq. (45), we obtain

$$\begin{aligned} a_1 = &-\frac{1}{\sqrt{\pi}(2+3\alpha)(1+\delta)}\left\{\int_0^{\pi}\sin\theta\cos\theta\left[\int_0^{\sqrt{\xi}}\Phi(\xi-s^2)ds\right]d\theta - \int_{\pi/2}^{\pi}\sin\theta\cos\theta\left[\int_0^{\sqrt{\xi-1}}\Phi(\xi-1-s^2)ds\right]d\theta\right\} \\ &+ \frac{2}{\sqrt{\pi}(2+3\alpha)(1+\delta)}\int_0^{\pi}\sin\theta\cos\theta\left[\int_0^{\infty} g_0(2\xi^{1/2}\zeta)\exp(-\zeta^2)d\zeta\right]d\theta. \end{aligned} \tag{48}$$

To determine the function g_0 in Eq. (48), we use the boundary condition within the droplet at the front and rear stagnation points in Eq. (39),

$$\begin{aligned} g_0(\eta') = &\frac{\delta}{1+\delta}\left\{-\frac{1}{\sqrt{\pi}\delta}\int_0^{\sqrt{2}}\Phi(2-s^2)\exp\left(-\frac{\eta'^2}{4s^2}\right)ds + \frac{1}{\sqrt{\pi}\delta}\int_0^{1}\Phi(1-s^2)\exp\left(-\frac{\eta'^2}{4s^2}\right)ds - \left(B-\frac{\eta'}{2\delta\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta'}{2\sqrt{2}}\right)\right\} \\ &+ \frac{1}{2\sqrt{2\pi}}\int_0^{\infty} g_0(\eta^*)\left\{\exp\left[-\frac{(\eta'-\eta^*)^2}{8}\right] + \frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta^*)^2}{8}\right]\right\}d\eta^*. \end{aligned} \tag{49}$$

The integral of the fourth term on the right-hand side of Eq. (49) is approximated as

$$\int_0^{\infty} g_0(\eta^*)h(\eta',\eta^*)d\eta^* = \int_0^{\eta_l^*} g_0(\eta^*)h(\eta',\eta^*)d\eta^* + g_0(\eta_l^*)\int_{\eta_l^*}^{\infty} h(\eta',\eta^*)d\eta^*. \tag{50}$$

Then, Eq. (49) is evaluated in a linear system of equations,

$$\begin{aligned} &g_0(\eta') - \frac{1}{4\sqrt{2\pi}}g_0(\eta_1^*)\left\{\exp\left[-\frac{(\eta'-\eta_1^*)^2}{8}\right] + \frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta_1^*)^2}{8}\right]\right\}\Delta\eta^* \\ &-\frac{1}{4\sqrt{2\pi}}g_0(\eta_{N+1}^*)\left\{\exp\left[-\frac{(\eta'-\eta_{N+1}^*)^2}{8}\right] + \frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta_{N+1}^*)^2}{8}\right]\right\}\Delta\eta^* \\ &-\frac{1}{2\sqrt{2\pi}}\sum_{j=2}^{N} g_0(\eta_j^*)\left\{\exp\left[-\frac{(\eta'-\eta_j^*)^2}{8}\right] + \frac{1-\delta}{1+\delta}\exp\left[-\frac{(\eta'+\eta_j^*)^2}{8}\right]\right\}\Delta\eta^* \\ &-\frac{1}{2}g_0(\eta_{N+1}^*)\left[\mathrm{erfc}\left(\frac{\eta_{N+1}^*-\eta'}{2\sqrt{2}}\right) + \frac{1-\delta}{1+\delta}\mathrm{erfc}\left(\frac{\eta_{N+1}^*+\eta'}{2\sqrt{2}}\right)\right] \\ &= \frac{\delta}{1+\delta}\left[-\frac{1}{\sqrt{\pi}\delta}\int_0^{\sqrt{2}}\Phi(2-s^2)\exp\left(-\frac{\eta'^2}{4s^2}\right)ds + \frac{1}{\sqrt{\pi}\delta}\int_0^{1}\Phi(1-s^2)\exp\left(-\frac{\eta'^2}{4s^2}\right)ds - \left(B-\frac{\eta'}{2\delta\sqrt{\lambda}}\right)\mathrm{erfc}\left(\frac{\eta'}{2\sqrt{2}}\right)\right], \end{aligned} \tag{51}$$

where $\eta_{N+1}^* = \eta_l^*$ and $\Delta\eta^* = \eta_l^*/N$. The physical coefficients used in space experiments [13] with the uniform temperature gradient $G = 12$ K/cm for the continuous phase of Fluorinert FC-75 (3M Corporation) and the droplet of 5-cSt silicone oil at $T = 333$ K are adopted to yield $\alpha = 0.342$, $\beta = 0.571$, and $\lambda = 0.299$. A typical value for η_l^* is chosen as 3. Using

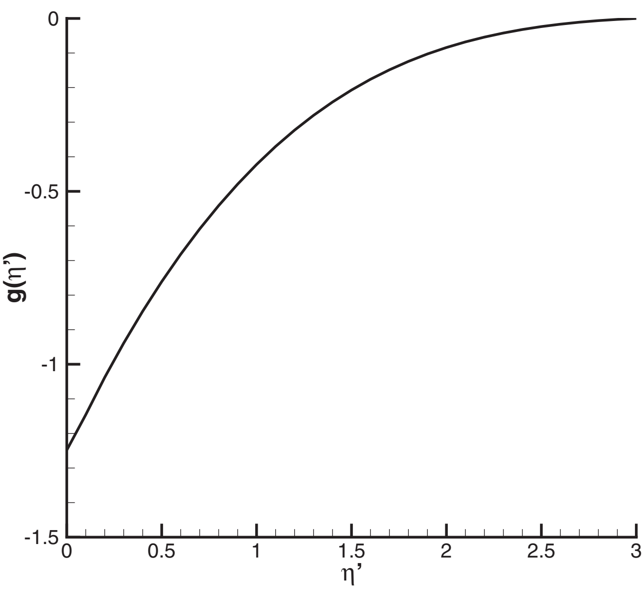

FIG. 3. Function g_0 vs η' determined from Eq. (51).

the trial and error method to satisfy the above approximation, we determine the unknown constant $B = 1.419$ and obtain the dependence of g_0 on η' as shown in Fig. 3. From Eq. (48), we can determine the root mean square of the leading-order term of the migration speed as

$$a_1 = 4.354 \times 10^{-2}. \tag{52}$$

Although equations and boundary conditions describing the second-order term of the migration speed can be obtained, we are unable to find an analytical result for t_{l0} in Eq. (46). Under the truncation after the leading-order term in Eq. (44), we obtain the migration speed of the droplet,

$$V_\infty \approx 1.896 \times 10^{-3}\,\mathrm{Ma}, \tag{53}$$

which indicates that the thermocapillary droplet migration speed increases as the Ma number increases. Using the migration speed V_∞, Eq. (15) is rewritten as

$$\Omega = \frac{2}{3}\left(1 + \frac{2\beta}{\lambda}\right)V_\infty\,\mathrm{Ma} \approx 6.098 \times 10^{-3}\,\mathrm{Ma}^2. \tag{54}$$

Therefore, to reach steady thermocapillary droplet migration in the space experiment at large Ma numbers [13], an external radiation energy source,

$$S = \Omega \sin^2\theta \approx 6.098 \times 10^{-3}\,\mathrm{Ma}^2 \sin^2\theta, \tag{55}$$

contrary to the direction of movement, should be provided.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, steady thermocapillary droplet migration in a uniform temperature gradient combined with a radiation energy source at large Re and Ma numbers is studied. The magnitude of the radiation energy source with the sine square dependence is determined to preserve the conservative integral thermal flux across the surface. Under the assumption of a quasisteady state, we have determined an analytical result for the steady thermocapillary migration of droplets at large Re and Ma numbers. The result shows that the thermocapillary droplet migration speed increases as the Ma number increases.

In general, when the droplet in a uniform temperature gradient moves upward, the thermal energy is not only transferred into the droplet from the top surface but also out of the droplet from the bottom surface. Meanwhile, the thermal flux across the surface is balanced. For large Ma numbers, once thermocapillary droplet migration reaches a quasisteady state, the relation of the nonconservative integral thermal flux across the surface will be required [11,12]. To satisfy the challenge, a thermal source at the surface through absorption from an external radiation energy source is provided for the system to make a balance of the integral thermal flux across the surface. The thermal source at the surface can bring more heat to the droplet, while the heat transfer in the system due to the thermal conduction across and around the droplet is weaker than that due to the thermal convection around the droplet at large Ma numbers. The thermocapillary migration of a droplet in the uniform temperature gradient combined with a radiation energy source at large Ma numbers can thus arrive at a quasisteady state process.

To perform a real space experiment to confirm the above theoretical analysis of the steady thermocapillary migration of a droplet, laser beam heating technology may be one of the possible physical means to provide an external radiation energy source contrary to the droplet movement direction.

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