Management of matter-wave solitons in Bose-Einstein condensates with time-dependent atomic scattering length in a time-dependent parabolic complex potential

E. Kengne[1,2] and W. M. Liu[2]

[1]*Laboratory of Advanced Microsystems Engineering, Department of Computer Science and Engineering, University of Quebec at Outaouais, 101 St-Jean-Bosco, Succursale Hull, Gatineau(PQ) J8Y 3G5, Canada*

[2]*Laboratory of Condensed Matter Theory and Materials Computation, Institute of Physics, Chinese Academy of Sciences, No. 8 South-Three Street, ZhongGuanCun, Beijing 100190, China*

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In this paper, we consider a Gross-Pitaevskii (GP) equation with a time-dependent nonlinearity and a spatiotemporal complex linear term which describes the dynamics of matter-wave solitons in Bose-Einstein condensates (BECs) with time-dependent interatomic interactions in a parabolic potential in the presence of feeding or loss of atoms. We establish the integrability conditions under which analytical solutions describing the modulational instability and the propagation of both bright and dark solitary waves on a continuous wave background are obtained. The obtained integrability conditions also appear as the conditions under which the solitary waves of the BECs can be managed by controlling the functional gain or loss parameter. For specific BECs, the dynamics of bright and dark solitons are investigated analytically through the found exact solutions of the GP equation. Our results show that under the integrability conditions, the gain or loss parameter of the GP equation can be used to manage the motion of both bright and dark solitons. We show that for BECs with loss (gain) of atoms, the bright and dark solitons during their propagation have a compression (broadening) in their width. Furthermore, under a safe range of parameters and under the integrability conditions, it is possible to squeeze a bright soliton of BECs with loss of atoms into the assumed peak matter density, which can provide an experimental tool for investigating the range of validity of the 1D GP equation. Our results also reveal that under the conditions of the solitary wave management, neither the injection or the ejection of atoms from the condensate affects the soliton peak during its propagation.

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I. INTRODUCTION

Since the first experimental and theoretical investigations of Bose-Einstein condensates [1,2], a rapid development and extensive studies have been done in this field. In particular, extensive research works carried out in the fields on bright and/or dark matter-wave solitons and vortices, modulational instability (MI), coherent structures, and domain formation in BECs trapped in optical lattices have stimulated intensive studies of the nonlinear excitations of the atomic matter waves [3–11]. These different works realized experimentally and/or theoretically on BECs provide many opportunities for both exploring quantum phenomena on a macroscopic scale and developing concrete applications of BECs in the future. As we know, one of the most important aspects of the BEC is the dynamics of its solitary waves; indeed, the generation, dynamics, and management of a BEC solitary wave is important for a number of BEC applications [12]. Therefore, it is of great interest to develop mathematical techniques that allow us to investigate analytically and/or numerically the dynamics of matter-wave solitons in BECs.

It is well known that at absolute zero temperature, the properties of weakly interacting bosonic gases trapped in a potential are usually described by a distributed nonlinear Schrödinger (NLS) equation with a trap potential, frequently called, in the theory of BECs, the GP equation [2]. In this GP equation, the two-body interatomic interaction which corresponds to the cubic nonlinear term of the NLS equation has been reported to be generally the dominant term [13]. In the physically important case of the cigar-shaped BECs in presence of either the feeding or the loss of atoms, it is reasonable to reduce the dissipative GP equation into a dissipative 1D NLS equation [14–17],

$$i\frac{\partial\psi}{\partial t}-\frac{\alpha_0}{2}\frac{\partial^2\psi}{\partial x^2}+g|\psi|^2\psi-kx^2\psi=i\gamma\psi. \qquad (1)$$

Here, α_0 is a real (negative in the context of BECs) constant and coefficients $g=g(t)$, $k=k(t)$, and $\gamma=\gamma(t)$ are all real functions of time t. In Eq. (1), time t and spatial coordinate x are, respectively, measured in units $1/\omega_\perp$ and $a_\perp$, where $a_\perp=\sqrt{\hbar/(m\omega_\perp)}$ and $a_0=\sqrt{\hbar/[m\omega_0(t)]}$ are linear oscillator lengths in the transverse and cigar-axis directions, respectively; $\omega_\perp$ which in our study is considered as a constant is the radial oscillation frequency and $\omega_0=\omega_0(t)$ is the axial oscillation frequency. m is the atomic mass and the wave function ψ is measured in units of $1/\sqrt{2\pi a_B a_\perp}$, a_B being the Bohr radius. The nonlinearity functional parameter $g(t)=-\frac{2a_s(t)}{a_B}$ is the time-dependent scattering length, which is positive for repulsive interatomic interactions and negative for attractive interatomic interactions, the potential parameter $k(t)=\pm\frac{\omega_0^2(t)}{2\omega_\perp^2}$ measures the strength of the magnetic trap and may be positive (confining potential) or negative (repulsive

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potential). $k(t)$ is typically fixed in current experiments, but adiabatic changes in the strength of the trap are experimentally feasible; hence, we examine the more general time-dependent case. The time-dependent dissipative parameter $\gamma(t)$ relates the feeding ($\gamma > 0$) or loss ($\gamma < 0$) of atoms in the condensate.

In some special cases, the dynamics of matter-wave solitons in BECs described by the GP Eq. (1) has been investigated. Kengne and Talla [16] investigated the dynamics of of a bright soliton using Eq. (1) with constant coefficients when $\alpha_0 = 2$, $g = 2g_0$ (g_0 being a real constant), $k(t) = \lambda^2/4$ (λ being a real constant), and $\gamma = -\lambda/2$; their results showed that, under a safe range of parameters, the bright soliton can be compressed into very high local matter densities by increasing the absolute value of the atomic scattering length or feeding parameter. Biao *et al.* [14] used Eq. (1) with $\alpha_0 = -2$, $k(t) = \lambda^2/4$ (λ being a real constant), and γ constant, to investigate the soliton properties in BECs with time-dependent atomic scattering length in an expulsive parabolic and complex potential; with the help of exact solitary wave solutions of Eq. (1), they demonstrated that the lifetime of both a bright and dark solitons in BECs can be extended by reducing both the ratio of the axial oscillation frequency to radial oscillation frequency and the loss of atoms. In the presence of the three-body interatomic interactions and a linear term in the potential term (gravitational field) of Eq. (1), Mohamadou *et al.* [7] studied the modulational instability of BECs. They pointed out the effects of the gravitational field, as well as of the parameter related to the feeding or loss of atoms in the condensate, on the instability growth rate. Their study also showed that the gravitational field changes the condensate trail of the soliton trains during the propagation. In the case where the feeding or loss of atoms in the condensates is neglected (i.e., when the dissipative term $\gamma(t)\psi$ of Eq. (1) is absent), Lei *et al.* [12] established the integrable condition of Eq. (1) and presented its exact analytical solution which describes the modulational instability and the propagation of a bright solitary wave on a continuous wave (cw) background.

In this work, we investigate through Eq. (1) the dynamics of matter-wave solitons in BECs with time-dependent atomic scattering length in a parabolic external potential in the presence of gain or loss of atoms. The main objective of this work is to give an appropriate answer to the following question: How to manage the motion of one-solitary waves in BECs by either injecting atoms into the condensate or ejecting atoms from the condensate? We start by establishing an integrability criterion of Eq. (1). Then, we find, under the integrability criterion, analytical solitary wave solutions that are used to give some thorough analysis of dynamics of solitons in BECs under consideration. The rest of our work is organized as follows. In Sec. II, we establish the integrability criterion for Eq. (1) and reduce Eq. (1) to the standard cubic nonlinear Schrödinger equation by employing a modified lens-type transformation; then, under the integrability criterion, we will show that the analytical solitary wave solutions Eq. (1) on a nonvanishing cw background for equation can be expressed in only terms of the dissipative parameter $\gamma(t)$. The dynamics of solitary waves for specific BECs are discussed analytically in Sec. III. The main work is summarized in Sec. IV.

II. INTEGRABILITY CONDITION AND ANALYTICAL SOLITARY WAVE SOLUTIONS

In this section, we establish an integrability condition of Eq. (1) under which its analytical solitary wave solutions are then presented. Here, we also give some properties of found analytical solutions.

A. Modified lens-type transformation and integrability condition

Theorem. For Eq. (1) to be integrable, it is sufficient that the real constant α_0 and the three time-dependent parameters $g(t)$, $k(t)$, and $\gamma(t)$ should satisfy the differential conditions

$$\frac{d\gamma}{dt} + 2\gamma^2 - \alpha_0 k = 0 \quad \text{and} \quad \frac{1}{g}\frac{dg}{dt} + 4\gamma = 0. \tag{2}$$

Proof. To prove this theorem, we introduce the modified lens-type transformation [18],

$$\psi(x,t) = \frac{1}{\sqrt{\ell}} u(X,T) \exp\left[\eta - i\frac{\gamma}{\alpha_0}x^2\right], \tag{3}$$

where $\eta = \eta(t)$, $T = T(t)$, and $\ell = \ell(t) > 0$ are three real functions of time t, and $X = x/\ell(t)$ is a real function of x and t. We impose to $\eta(t)$, $T(t)$, and $\ell(t)$ to satisfy the ordinary differential system

$$\frac{dT}{dt} - \frac{1}{\ell^2} = 0, \tag{4a}$$

$$\frac{1}{\ell}\frac{d\ell}{dt} - 2\gamma = 0, \tag{4b}$$

$$\frac{d\eta}{dt} - \frac{1}{2\ell}\frac{d\ell}{dt} = 0. \tag{4c}$$

Equation (4a) is introduced to preserve the scaling. Following Mohamadou *et al.* [7], we impose to $T(t)$ to be a continuous function t varying from zero to plus infinity. Inserting ansatz Eq. (3) into Eq. (1) and using system Eqs. (4a)–(4c), we find under conditions Eqs. (2) that Eq. (1) in terms of rescaled variables X and T is converted to the following NLS equation:

$$i\frac{\partial u}{\partial T} - \frac{\alpha_0}{2}\frac{\partial^2 u}{\partial X^2} + g(t)\ell(t)\exp\left[2\eta(t)\right]|u|^2 u = 0. \tag{5}$$

Now, we solve Eqs. (4a), (4b), (4c), and the second equation in conditions Eqs. (2) in terms of the feeding or loss parameter $\gamma(t)$ and obtain

$$T(t) = \frac{1}{\ell_0^2}\int_0^t \exp\left[-4\int_0^y \gamma(\tau)d\tau\right]dy, \tag{6a}$$

$$\ell(t) = \ell_0 \exp\left[2\int_0^t \gamma(\tau)d\tau\right], \tag{6b}$$

$$\eta(t) = \int_0^t \gamma(\tau)d\tau + \frac{1}{2}\ln|\eta_0|, \tag{6c}$$

$$g(t) = g_0 \exp\left[-4\int_0^t \gamma(\tau)d\tau\right], \tag{6d}$$

where $\ell_0 = \ell(0)$, $g_0 = g(0)$, and $\frac{1}{2}\ln|\eta_0| = \eta(0)$. Inserting Eqs. (6a)–(6d) into Eq. (5) and asking to integration constants

ℓ_0, g_0, and η_0 to satisfy condition $g_0\ell_0|\eta_0| = sign(g_0) = p^2$ yield the standard cubic NLS equation,

$$i\frac{\partial u}{\partial T} - \frac{\alpha_0}{2}\frac{\partial^2 u}{\partial X^2} + p^2|u|^2u = 0, \quad (7)$$

which is an integrable equation, and the theorem is proven.

Main result. As the result of the above theory, it is seen that (res1)It follows from Eqs. (6a)–(6d) that each of $X(x,t)$ and $T(t)$ is expressed in only the terms of the gain or loss parameter $\gamma(t)$;

(res2) Under the integrable condition Eqs. (2) and ansatz Eq. (3) with time-dependent function Eqs. (6a)–(6d), the solution for Eq. (1) can be expressed in only terms of the feeding or loss parameter $\gamma(t)$ as follows:

$$\psi(x,t) = \sqrt{\frac{|\eta_0|}{\ell_0}}\,u\left(\frac{1}{\ell_0\exp\left[2\int_0^t\gamma(\tau)d\tau\right]}x, \frac{1}{\ell_0^2}\int_0^t \exp\left[-4\int_0^y\gamma(\tau)d\tau\right]dy\right)\exp\left[-i\frac{\gamma(t)}{\alpha_0}x^2\right]. \quad (8)$$

We have thus obtained that under the integrability conditions Eqs. (2), the solutions $\psi(x,t)$ of form Eq. (3) of the GP Eq. (1) is expressed in only terms of the gain or loss parameter $\gamma(t)$, meaning that under conditions Eqs. (2), parameter $\gamma(t)$ can be used to manage the motion of any solitary wave described by Eq. (8). Thus, we conclude that conditions Eqs. (2) are the conditions [on the equations parameters $g(t)$, $k(t)$, and $\gamma(t)$] under which the motion of one-solitary waves of BECs described by the GP Eq. (1) will be managed by controlling the functional parameter $\gamma(t)$ of gain or loss of atoms of the condensates. Henceforth, Eqs. (2) will be referred to as the integrability conditions [when talking about the integrability of the GP Eq. (1)] or the "*conditions of the solitary wave management*" (when talking about the BECs solitons).

B. Applicability of the results to real experiments

Here, we give some examples of either the potential (through its strength k) or s-wave scattering length (through the nonlinearity parameter g) leading to the applicability of the above theory. We focus ourselves to BECs with (a) the time-independent harmonic potential; (b) the temporal periodic modulation of the s-wave scattering; and (c) a specific time-varying potential. Because many authors either use $2a(t)$ instead of $g(t)$ as the nonlinear parameter or use different letters to defined the strength k of the harmonic potential of the GP Eq. (1), we will, when using the results of some authors, introduce if necessary, some numbers or constant, that allow us to have the same GP equation as that used by these authors.

1. BECs with time-independent harmonic potential

As the first example, let us consider the time-independent harmonic potential that was used by Khaykovich *et al.* [4] in the creation of bright BEC solitons. In that experiment, $\omega = 2\pi i \times 70$ Hz, $\omega_\perp = 2\pi \times 710$ Hz, so $k = -2\kappa^2$ ($\kappa \approx 0.05$).

(i) By letting $\lambda = \pm 2\sqrt{2}\kappa$, we obtained $k = -\lambda^2/4$. Taking $g(t) = 2g_0\exp(\lambda t)$, we obtain the special form of $g(t)$ and $k(t)$ [that is, $g(t) = 2g_0\exp(\lambda t)$ and $k = -\lambda^2/4$] used by Liang *et al.* [5]. With these $g(t)$ and k, the conditions of the solitary wave management Eqs. (2) are satisfied for $\gamma = -\lambda/4$ and $\alpha_0 = -1/2$.

(ii) Also, the special form of $k(t) = -\lambda^2/4$ used by Biao *et al.* [14] with constant feeding or loss term also satisfies the conditions of the solitary wave management for $\gamma = \pm\frac{\sqrt{-2\alpha_0}}{4}\lambda$ and $g(t) = g_0\exp[\mp\sqrt{-2\alpha_0}\lambda t]$ with any negative α_0.

2. BECs with a temporal periodic modulation of the s-wave scattering length

As the second example, we consider the temporal periodic modulation of the s-wave scattering length. The oscillation of the scattering length with frequency $\omega \approx \omega_\perp$ is studied by Abdullaev *et al.* [19] in a different context. According to Refs. [17], we consider the nonlinearity parameter of the form $g(t) = 1 + m_0\sin[\omega t]$ for $0 < m_0 < 1$. It is important to note that for this nonlinearity parameter, we have $g_0 = g(0) = 1$. This avoids the condensate from the collapse if the oscillating term $m_0\sin[\omega t]$ of the nonlinearity parameter were absent [20]. For such a nonlinearity parameter, the conditions of the solitary wave management Eqs. (2) are satisfied for $\gamma(t) = -\frac{m_0\omega}{4}\frac{\cos[\omega t]}{1+m_0\sin[\omega t]}$ and $k(t) = \frac{m_0\omega^2}{8\alpha_0}(\frac{2m_0+2\sin[m_0 t]+m_0\cos^2[\omega t]}{(1+m_0\sin[\omega t])^2})$. Therefore, $\int_0^t\gamma(\tau)d\tau = -\frac{1}{4}\ln(1+m_0\sin[\omega t])$ and $T(t) = \frac{1}{\ell_0^2}[\frac{m_0}{\omega} + t - \frac{m_0}{\omega}\cos[\omega t]]$.

The interaction and the confinement potential can be changed independently using an optical trap and magnetic-field-induced Feshbach resonance. To compress the soliton in a controllable manner, Xue [21] gradually increased the interaction and simultaneously turn off the longitudinal confinement potential. In this purpose, he considered a nonlinearity parameter of the form $g(t) = 1 + a\tanh[\Omega t]$, where Ω is the normalized longitudinal confinement frequency at the initial stage. In our study, we follow Xue [21] and consider the nonlinearity parameter of the form $|g(t)| = 1 + \tanh[\Omega t]$. By taking the strength k of the potential as $k(t) = -\frac{\Omega^2}{8\alpha_0}(\tanh^2[\Omega t] + 2\tanh[\Omega t] - 3)$, the integrability condition Eqs. (2) will be satisfied for $\gamma(t) = \frac{\Omega}{4}(\tanh[\Omega t] - 1)$. It is important to note that $|g(t\to+\infty)| = 2$, while $k(t\to+\infty) = 0$, meaning that the longitudinal confinement potential gradually turns off, while the interaction gradually turns on. Therefore, this example can be used both in the stabilization of a soliton state and for the compression of the soliton in a controllable manner [17,21].

3. BECs with the specific time-varying potential $k(t) = a(t+t_0)^{-2}$

Now, let us consider BECs with the specific time-varying potential with strength $k(t) = a(t+t_0)^{-2}$, which has been suggested as being of interest in quite different setups such as the study of explosion-implosion dualities for the quintic (critical) GP equation [22]. Here, $a \neq 0$ is any real constant, and t_0 is an arbitrary constant whose sign is related to the sign of $\gamma(t)$; $t_0\gamma(t) > 0$ and which essentially determines the width of the trap at time $t = 0$ [4]; notice that $t_0 < 0$ describes a BEC in a shrinking trap, while the case corresponds to a broadening condensate. This type of time-varying potential has been used by Theocharis *et al.* [4] when investigating the MI of BECs with a specific time-varying potential. With this strength k of potential, the conditions of the solitary wave management Eqs. (2) will be satisfied under the condition $1 + 8\alpha_0 a \geqslant 0$ if we choose $\gamma(t) = b(t + t_0)^{-1} \neq 0$ and $g(t) = g_0|t_0|^{4b}|t + t_0|^{-4b}$, where $b = (1 \pm \sqrt{1 + 8\alpha_0 a})/4$. Inserting the expression of $\gamma(t)$ into Eqs. (6a)–(6c) yields

$$T(t) = \begin{cases} \ell_0^{-2}|t_0| \ln |(t + t_0)/t_0|, & \text{If} \quad 4b = 1, \\ \frac{t_0}{\ell_0^2(1-4b)}\left[\left(\frac{t_0}{t+t_0}\right)^{4b-1} - 1\right], & \text{if} \quad 1 - 4b \neq 0, \end{cases}$$

$$\ell(t) = \ell_0|(t + t_0)/t_0|^{2b}, \quad \eta(t) = \ln \sqrt{|\eta_0|} + b \ln \left|\frac{t + t_0}{t_0}\right|.$$

C. Analytical solitary wave solutions under the integrable condition

Under the integrable condition Eqs. (2), we can now use the standard cubic NLS Eq. (7) to present analytical bright and dark solitary wave solutions of Eq. (1). For this purpose, we follow Kengne and Talla [16] and apply the Darboux transformation on the Standard NLS Eq. (7) with the so-called seed solution,

$$u_c(X,T) = A_c \exp\left[i\left(k_c X + \frac{\alpha_0 k_c^2 + 2p^2 A_c^2}{2}\right)T\right], \tag{9a}$$

associated with the cw solution $\psi_c(x,t) = A_c\sqrt{\frac{|\eta_0|}{\ell_0}} \exp\left[i\left(-\frac{\gamma(t)}{\alpha_0}x^2 + k_c X(x,t) + \frac{\alpha_0 k_c^2 + 2p^2 A_c^2}{2}T(t)\right)\right]$ of the GP Eq. (1). Here, $X(x,t) = \ell_0^{-1}x \exp\left[-2\int_0^t \gamma(\tau)d\tau\right]$, $T(t) = \ell_0^{-2}\int_0^t \exp\left[-4\int_0^y \gamma(\tau)d\tau\right]dy$, k_c and A_c being two arbitrary real numbers. We then obtain

$$u(X,T) = \left[A_c + A_s\frac{D\cosh[\xi_b] + \cos[\theta] + i\{\alpha\sinh[\xi_b] + \beta\sin[\theta]\}}{\cosh[\xi_b] + D\cos[\theta]}\right]\exp[i\varphi_b] \tag{9b}$$

for $p^2 = 1$, and

$$u(X,T) = [A_c + i A_s \tanh[\xi_d]]\exp[i\varphi_d] \tag{9c}$$

for $p^2 = -1$, where

$$\xi_b = \xi_b(X,T) = A_s\left[\frac{\beta}{\sqrt{-\alpha_0}}X - \sqrt{2}\left(\rho + k_0\beta - \frac{2\rho A_c^2}{A_s^2 + 2\rho^2}\right)T\right],$$

$$\xi_d = \xi_d(X,T) = \pm\frac{X}{\sqrt{-2\alpha_0}} + A_s(A_c + \sqrt{2}k_0)T, \quad \theta = \theta(X,T) = \rho\sqrt{-\frac{2}{\alpha_0}}X - \frac{4\rho k_0 + (2\rho^2 - A_s^2)\beta}{2}T,$$

$$\varphi_b = \varphi_b(X,T) = k_0\sqrt{-\frac{2}{\alpha_0}}X + (A_c^2 - k_0^2)T, \quad \varphi_d = \varphi_d(X,T) = k_0\sqrt{-\frac{2}{\alpha_0}}X - (A_s^2 + A_c^2 + k_0^2)T,$$

$$D = -\frac{2A_c A_s}{A_s^2 + 2\rho^2}, \quad \alpha = -\frac{2\sqrt{2}A_c\rho}{A_s^2 + 2\rho^2}, \quad \beta^2 = \frac{A_s^2 - 4A_c^2 + 2\rho^2}{A_s^2 + 2\rho^2}, \quad k_0 = \frac{\sqrt{-\frac{\alpha_0}{2}}A_c^2 k_c + A_s^2 k_s}{A_c^2 + A_s^2}.$$

Here, A_c, A_s, k_c, k_s, and ρ are arbitrary real constants with $(|A_c| + |A_c|)(A_s^2 + 2\rho^2) > 0$.

D. Analysis of solutions Eqs. (9b) and (9c)

In what follows, we analyze two main situations associated with the solution parameters A_c and A_s, namely, the situation $A_c A_s = 0$ with $|A_c| + |A_s| > 0$ and the situation $A_c A_s \neq 0$. In what follows, we mainly focus ourselves to the bright solitary wave solution Eq. (9b).

1. Situation $A_c A_s = 0$ with $|A_c| + |A_s|\rangle 0$

It is important to notice the following important properties of solitary wave solutions Eqs. (9b) and (9c). On the one hand, when $A_s = 0$, they reduce to the cw solution Eq. (9a). On the other hand, when $A_c = 0$, they reduce respectively the well-known bright and dark soliton solutions

$$u(X,T) = \frac{A_s}{\cosh\left[A_s\left(\frac{1}{\sqrt{-\alpha_0}}X - \sqrt{2}(k_0 \pm \rho)T\right)\right]} \exp\left[i\Theta_b\right] \tag{10a}$$

and

$$u(X,T) = i A_s \tanh\left[\pm\frac{X}{\sqrt{-2\alpha_0}} + \sqrt{2}A_s k_0 T\right] \exp\left[i\Theta_d\right], \tag{10b}$$

where $\Theta_b = (k_0 \pm \rho)\sqrt{-\frac{2}{\alpha_0}}X - (k_0^2 \pm \frac{4\rho k_0 + (2\rho^2 - A_s^2)\beta}{2})T$ and $\Theta_d = k\sqrt{-\frac{2}{\alpha_0}}X - (A_s^2 + k_0^2)T$. Therefore, the analytical solutions Eqs. (9b) and (9c) represent respectively a bright and a dark solitary wave embedded on a cw background. Thus, with the use of Eq. (8), we can obtain both the bright and the dark solitary wave solutions which propagates on a cw background for Eq. (1) when the integrability conditions Eqs. (2) are satisfied.

If we write Eq. (8) for soliton solutions Eqs. (10a) and (10b), let us say, for example, for solution Eq. (10a), we obtain the following bright soliton solution for the GP Eq. (1) under the integrability conditions Eqs. (2),

$$\psi(x,t) = \frac{A_s \exp\left[i\Theta(x,t)\right]}{\cosh\left[A_s\left(\frac{\exp[-2\int_0^t \gamma(\tau)d\tau]}{\sqrt{-\ell_0\alpha_0|\eta_0|}}x - \frac{\sqrt{2}(k_0\pm\rho)}{\ell_0\sqrt{\ell_0|\eta_0|}}\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]}\right)\right]}, \tag{11}$$

where

$$\Theta(x,t) = -\frac{\gamma(t)}{\alpha_0}x^2 + \sqrt{-\frac{2}{\alpha_0}}\frac{(k_0 \pm \rho)}{\ell_0 \exp\left[2\int_0^t \gamma(\tau)d\tau\right]}x \quad - \frac{1}{\ell_0^2}\left(k_0^2 \pm \frac{4\rho k_0 + \left(2\rho^2 - A_s^2\right)\beta}{2}\right)\int_0^t \exp\left[-4\int_0^y \gamma(\tau)d\tau\right]dy + \Theta_0,$$

Θ_0, ρ, A_s, and k being four arbitrary real constant. It follows from the bright soliton solution Eq. (11) that: (i) the width of the bright solitary wave is inversely proportional to $\exp\left[-2\int_0^t \gamma(\tau)d\tau\right]/\sqrt{-\ell_0\alpha_0}$. So the total number of atoms of the BEC under consideration is $\int_{-\infty}^{+\infty}|\psi(x,t)|^2dx = 2|A_s|\sqrt{-\alpha_0|\eta_0|\ell_0}\exp\left[2\int_0^t \gamma(\tau)d\tau\right]$, which increases or decreases when the feeding or loss of atoms in the condensate is taken into consideration. Thus, Eq. (11) can be used to describe the compression (enlargement) of bright solitons during its propagation when the loss (feeding) of atoms in the condensates is taken into account. (ii) The center of bright soliton is $\zeta(t) = x = \frac{\sqrt{-2\alpha_0}(k_0\pm\rho)}{\ell_0}\exp\left[2\int_0^t \gamma(\tau)d\tau\right]\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]}$ and satisfies the following equation:

$$\frac{d^2\zeta}{dt^2} - \left(2\frac{d\gamma}{dt} + 4\gamma^2 + \exp\left[-8\int_0^t \gamma(\tau)d\tau\right]\right)\zeta = 0. \tag{12}$$

Equation (12) means that the center of mass of the macroscopic wave packet behaves like a classical particle, and allows us to manipulate the motion of bright solitons in BEC systems by controlling the functional parameter $\gamma(t)$ of feeding or loss of atoms of the condensates.

2. Situation $A_c A_s \neq 0$

For given A_c and A_s such that $A_c A_s \neq 0$, only one of the following two important situations associated with solitary wave solutions holds, either $A_s^2 - 4A_c^2 > 0$ or $A_s^2 - 4A_c^2 < 0$. In what follows, we associated with these two situations the condition $\alpha\beta = 0$ with $|\alpha| + |\beta| > 0$ (situation $\alpha = \beta = 0$ is not associated with any solitary wave solution). Indeed, if $\alpha = 0$ leads to $\rho = 0$ and $\beta^2 = \frac{A_s^2 - 4A_c^2}{A_s^2}$, which makes sense only if $A_s^2 - 4A_c^2 > 0$. Condition $\beta = 0$ leads to condition $A_s^2 - 4A_c^2 + 2\rho^2 = 0$, which is possible only if $A_s^2 - 4A_c^2 < 0$. As we will see below, either $\xi(x,t)$ or $\theta(x,t)$ has no x-dependent when $\alpha\beta = 0$. It is important to point out that the cw background will affect the total number of atoms in the condensate. Also, when $A_c << A_s$, the cw background is small within the existence of a bright soliton. Considering the dynamics of either the bright or the dark soliton in the cw background, the length $2\mathcal{L}$ of the spatial background must be very large compared to the scale of the soliton.

a. Situation when $\alpha = \rho = 0$. When $\alpha = \rho = 0$, we have $A_s^2 - 4A_c^2 > 0$ and the bright solitary wave solution of Eq. (1) reads

$$\psi(x,t) = \sqrt{\frac{|\eta_0|}{\ell_0}}\left[A_c + A_s\frac{-2A_c\cosh\left[\xi\right] + A_s\cos\left[\theta\right] + i\sqrt{A_s^2 - 4A_c^2}\sin\left[\theta\right]}{A_s\cosh\left[\xi\right] - 2A_c\cos\left[\theta\right]}\right]\exp\left[i\varphi\right], \tag{13}$$

where

$$\xi(x,t) = \sqrt{A_s^2 - 4A_c^2}\left\{\frac{\exp\left[-2\int_0^t \gamma(\tau)d\tau\right]}{\sqrt{-\alpha_0}\ell_0}x - \frac{\sqrt{2}k_0}{\ell_0^2}\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]}\right\},$$

$$\theta(x,t) = \frac{A_s\sqrt{A_s^2 - 4A_c^2}}{2\ell_0^2}\int_0^t \exp\left[-4\int_0^y \gamma(\tau)d\tau\right]dy,$$

$$\varphi(x,t) = -\frac{\gamma(t)}{\alpha_0}x^2 + \frac{k_0\sqrt{-\frac{2}{\alpha_0}}}{\ell_0\exp\left[2\int_0^t \gamma(\tau)d\tau\right]}x + \frac{A_c^2 - k_0^2}{\ell_0^2}\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]} + T_0,$$

with $k_0 = (\sqrt{-\frac{\alpha_0}{2}}A_c^2k_c + A_s^2k_s)/(A_c^2 + A_s^2)$. From the analysis of Eq. (13), it is seen that the center of the bright solitary wave embedded on a cw background in the situation $\alpha = \rho = 0$ also satisfies Eq. (12). From the expressions of $\theta(x,t)$ and $\xi(x,t)$, it is seen that the solution Eq. (13) oscillates and is aperiodic in the temporal coordinate. Because $\theta(x,t)$ has no x-dependence, the solitary wave solutions will be nonoscillatory in the x direction.

Solving equation $d|\psi(x,t)|^2/dx = 0$ leads to $\sinh[\xi] = 0$ and $\cosh[\xi] = \frac{A_s}{A_c}\frac{1}{\cos[\theta]} - \frac{2A_c}{A_s}\cos[\theta]$. This shows that the intensity (i.e., the peak matter density) of bright solitary wave given by Eq. (13) arrives at its maximum and minimum,

$$|\psi|^2_{\max} = \frac{|\eta_0|}{\ell_0}\frac{\left[\left(A_s^2 - 2A_c^2\right)\cos[\theta] - A_sA_c\right]^2 + A_s^2\left(A_s^2 - 4A_c^2\right)\sin^2[\theta]}{(A_s - 2A_c\cos[\theta])^2},$$

$$|\psi|^2_{\min} = \frac{|\eta_0|}{\ell_0}\frac{\left(A_s^2\cos[\theta] - A_s^2\cos^{-1}[\theta]\right)^2 + A_s^2\left(A_s^2 - 4A_c^2\right)\sin^2[\theta]}{\left(\frac{A_s^2}{A_c}\cos^{-1}[\theta] - 4A_c\cos[\theta]\right)^2}. \tag{14a}$$

respectively. Therefore the bright solitary wave Eq. (13) can only be squeezed into the assumed peak matter density between the minimum and maximum values. It is important to note that condition $\sinh[\xi] = 0$ for which the solitary wave intensity arrives at its maximum is the same as $\xi = 0$ leading to the center of bright solitary wave.

Now, let us evaluate the exact number of atoms in the bright soliton against the background described Eq. (13) within $[-L,\ L]$. For simplicity, we let $|\eta_0|/\ell_0 = 1$ (from condition $\ell_0 g_0|\eta_0| = p^2 = 1$, this means that $\ell_0 = 1/\sqrt{g_0}$) and find

$$\int_{-L}^{L}(|\psi|^2 - |\psi_c|^2)dx = 2\sqrt{-\frac{\alpha_0\left(A_s^2 - 4A_c^2\right)}{g_0}}\exp\left[2\int_0^t \gamma(\tau)d\tau\right]\frac{2A_s\sinh L}{2A_s\cosh L - 4A_c\cos\theta}. \tag{14b}$$

From Eq. (14b) we obtain the total number $N_0(t)$ of atoms of the bright solitary wave against the background described Eq. (13) as follows:

$$N_0(t) = \lim_{L\to+\infty}\int_{-L}^{L}(|\psi|^2 - |\psi_c|^2)dx = 2\sqrt{-\frac{\alpha_0\left(A_s^2 - 4A_c^2\right)}{g_0}}\exp\left[2\int_0^t \gamma(\tau)d\tau\right], \tag{14c}$$

which depends on only α_0, g_0, and γ and proves the stability of the bright solitary wave in the background described by Eq. (13). Equation (14c) reveals that under the conditions of the solitary wave management Eqs. (2), the bright solitary wave is stable against the variation of the both the scattering length and the potential magnitude. As one can see from Eq. (14c), the condensate may gain (loss) atoms for positive (negative) $\gamma(t)$. This means that during the compression (associating with $\gamma(t) < 0$) or the broadening (associating with $\gamma(t) > 0$) of the bright solitary wave described by Eq. (13), the number of atoms in the bright solitary wave either decreases or increases, respectively.

Finally, using the equality $|\psi - \psi_c|^2 = |\psi|^2 - |\psi_c|^2 + 2A_cA_s\frac{-2A_c\cosh[\xi]+A_s\cos[\theta]}{A_s\cosh[\xi]-2A_c\cos[\theta]}$, we have evaluated the number of atoms in both the bright solitary wave and background within $[-L,\ L]$ under the condition of $\psi(\pm L,t) \neq 0$ and $|\eta_0|/\ell_0 = 1$. We then find $\int_{-L}^{L}|\psi - \psi_c|^2dx = N_0(t)(\frac{2A_s\sinh L}{2A_s\cosh L-4A_c\cos\theta} + \frac{4A_c\cos[\theta]}{\sqrt{A_s^2-4A_c^2\cos^2[\theta]}} \times \arctan[\frac{A_s+2A_c\cos[\theta]}{\sqrt{A_s^2-4A_c^2\cos^2[\theta]}}\tanh[\frac{L}{2}]])$. Tending L to $+\infty$, we find the total number of atomic exchange between the bright solitary wave and the cw background as

$$\chi(t) = \lim_{L\to+\infty}\int_{-L}^{L}|\psi - \psi_c|^2dx = N_0(t)\left\{1 + \frac{4A_c\cos[\theta]}{\sqrt{A_s^2 - 4A_c^2\cos^2[\theta]}}\arctan\left[\frac{A_s + 2A_c\cos[\theta]}{\sqrt{A_s^2 - 4A_c^2\cos^2[\theta]}}\right]\right\}, \tag{14d}$$

where $N_0(t)$ is given by Eq. (14c). Equation (14d) reveals that a time-periodic atomic exchange is formed between the bright solitary wave and the nonzero-background. In the case of a zero-background, i.e., $A_c = 0$, from Eq. (14d), $\chi(t)$ is time-aperiodic, increases for $\gamma(t) > 0$ and decreases for $\gamma(t) < 0$. It is important to note that when $A_c = 0$, all the atoms are available only in the bright solitary wave; this means that there is no exchange of atoms between the solitary wave and zero-background. Because of the factor $\exp[2\int_0^t \gamma(\tau)d\tau]$ in the expression of $N_0(t)$ as shown in Eq. (14c), we conclude that the exchange of atoms continuously decreases (increases) for negative (positive) $\gamma(t)$, and cannot be controlled by applying the external magnetic field via Feshbach resonance. In the context of optical physic, the quantity $\chi(t)$ denotes the energy exchange between the attenuated pulse and the cw background.

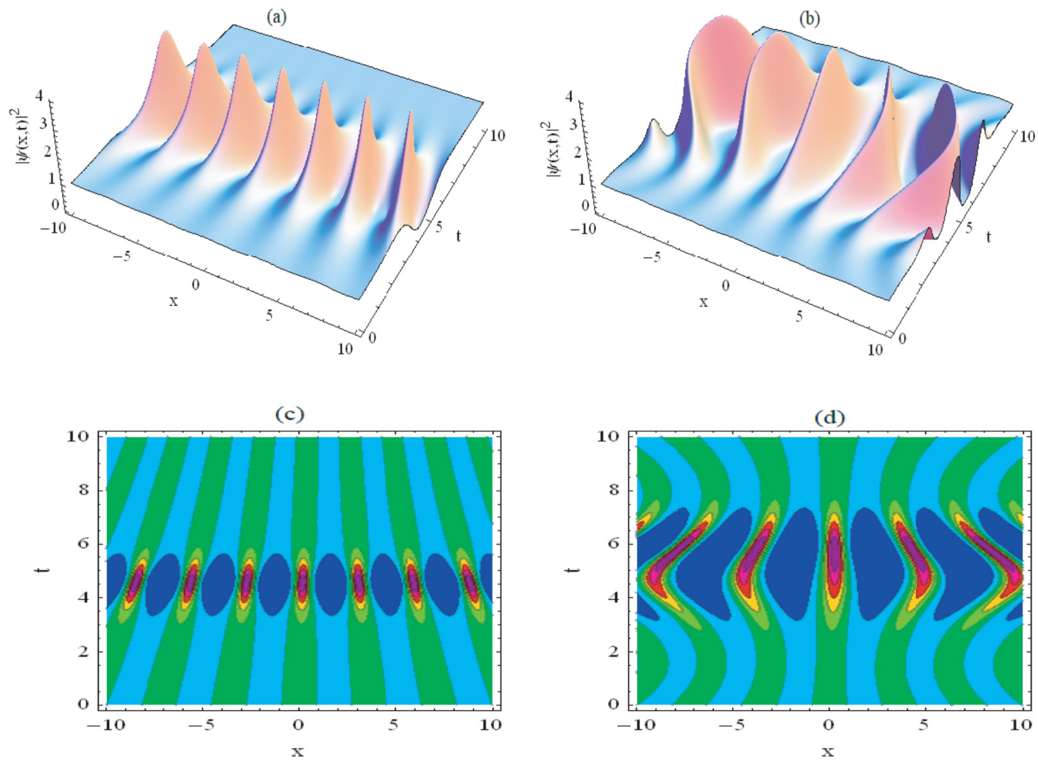

FIG. 1. The evolution of modulational instability given by solution Eq. (15) with the parameters $A_c = A_s = 1$, $k_c = 0.03$, $k_s = 0.04$, $\alpha_0 = -1$, $|\eta_0| = \ell_0 = 1$. (a, c) $g(t) = \exp[2\lambda t]$, $\gamma = -\frac{\lambda}{2}$, $k = \frac{\lambda^2}{2\alpha_0}$, $\lambda = 0.05$, $g_0 = 1$. (b, d) $g(t) = 1 + m_0 \sin[\omega t]$, $\gamma(t) = -\frac{m_0\omega}{4}\frac{\cos[\omega t]}{1+m_0\sin[\omega t]}$, $k(t) = \frac{m_0\omega^2}{8\alpha_0}\left(\frac{2m_0+2\sin[m_0 t]+m_0\cos^2[\omega t]}{(1+m_0\sin[\omega t])^2}\right)$, $\omega = 1$, $m_0 = 0.4$.

3. Situation when $\beta = 0$

In the present situation, we must have $\rho \neq 0$ and $A_s^2 - 4A_c^2 < 0$. Using Eqs. (8) and (9b), we can rewrite a bright solitary wave solution of Eq. (1) under conditions Eqs. (2) as follows:

$$\psi(x,t) = \sqrt{\frac{|\eta_0|}{\ell_0}}\left[A_c + A_s\frac{-A_s\cosh[\xi] + 2A_c\cos[\theta] + i\sqrt{4A_c^2 - A_s^2}\sinh[\xi]}{2A_c\cosh[\xi] - A_s\cos[\theta]}\right]\exp[i\varphi], \tag{15}$$

where

$$\xi(x,t) = \frac{A_s\sqrt{4A_c^2 - A_s^2}}{2\ell_0^2}\int_0^t \exp\left[-4\int_0^y \gamma(\tau)d\tau\right]dy,$$

$$\theta(x,t) = \sqrt{4A_c^2 - A_s^2}\left(\frac{1}{\ell_0\sqrt{-\alpha_0}\exp\left[2\int_0^t \gamma(\tau)d\tau\right]}x - \frac{\sqrt{2}k_0}{\ell_0^2}\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]}\right),$$

$$\varphi(x,t) = -\frac{\gamma(t)}{\alpha_0}x^2 + \frac{k_0\sqrt{-\frac{2}{\alpha_0}}}{\ell_0\exp\left[2\int_0^t \gamma(\tau)d\tau\right]}x + \frac{(A_c^2 - k_0^2)}{\ell_0^2}\int_0^t \frac{dy}{\exp\left[4\int_0^y \gamma(\tau)d\tau\right]},$$

with $k_0 = \sqrt{-\frac{\alpha_0}{2}}A_c^2 k_c + A_s^2 k_s/(A_c^2 + A_s^2)$. As we can see from the expressions of $\xi(x,t)$ and $\theta(x,t)$, $\xi(x,t)$ is $x-$independent and solution Eq. (15) is periodic with the period $\widetilde{T} = 2\pi\ell_0\sqrt{\frac{\alpha_0}{A_s^2 - 4A_c^2}}\exp[2\int_0^t \gamma(\tau)d\tau]$ in the spatial coordinate x and aperiodic in the temporal coordinate t (remember that in the context of BECs, $\alpha_0 < 0$). It is aperiodic in temporal coordinate. Due to the presence $\int_0^t \gamma(\tau)d\tau$ in the expression of $\widetilde{T}$, period $\widetilde{T}$ is not a constant, but when $|\gamma(t)| << 1$ and t is very small, $\widetilde{T}$ is very close to $2\pi\ell_0\sqrt{\frac{\alpha_0}{A_s^2 - 4A_c^2}}$. As Lei Wu *et al.* [12] pointed out, solution Eq. (15) can be used to describe the MI process of a nonlinear matter wave governed by the GP Eq. (1). Indeed, we can see from figure 1 that the cw background becomes unstable. Therefore, in the case of $A_s^2 - 4A_c^2 < 0$, a small perturbation for Eq. (9b) may lead to the modulation instability. It is also seen from plots of Fig. 1 that with a strong cw background, a train of bright solitons is excited. We can then conclude that an important condition for exciting a train of bright solitons is that the background be strong enough.

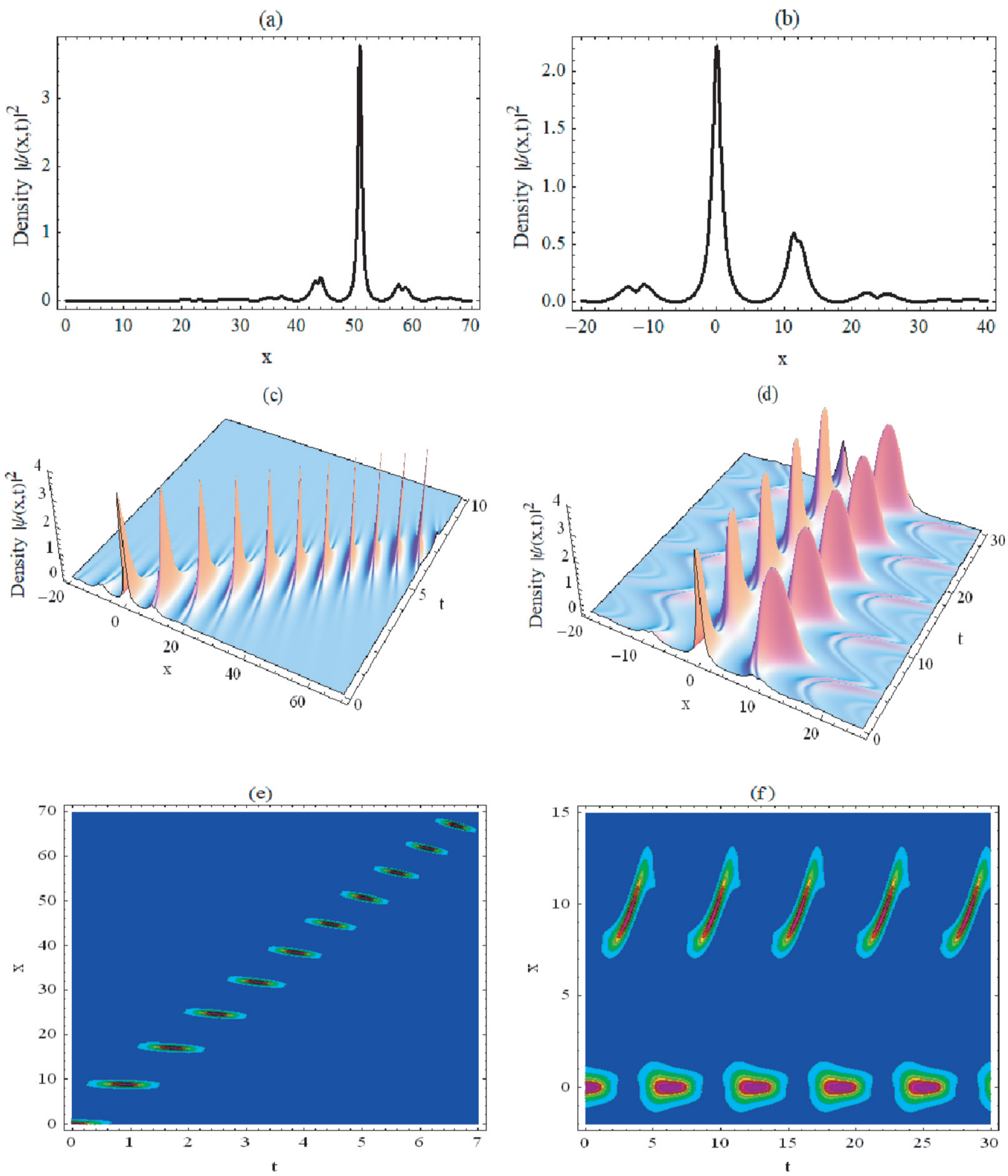

FIG. 2. (a, b) Spatial profiles at time $t = 5$, (c, d) the evolution plots, and (e, f) the contour plots of the density $|\psi(x,t)|^2$ for the case when α and β in solution Eq. (9b) satisfy condition $\alpha\beta \neq 0$. The data used to generate plots (a), (c), and (e) and plots (b), (d), and (f) are the same as for plots (a) and (b) in Fig. 1, respectively. Here, we used $\rho = 0.02$ with a positive β.

4. Situation when $\alpha\beta \neq 0$

In the situation when $\alpha\beta \neq 0$, each of functions $\xi(x,t)$ and $\theta(x,\tau)$ depends on both x and t. Therefore, the bright solitary wave solution Eq. (9b) will be a combination of both nonoscillatory and oscillatory with respect with x but with a localized envelope which has a density profile that is similar to that of gap solitons. As we can see from the plots of Figs. 2, the situation $\alpha\beta \neq 0$ may lead to the formation of multisoliton.

III. ANALYTICAL STUDY OF DYNAMICS OF ONE-SOLITARY WAVES OF BECS IN PRESENCE OF FEEDING OR LOSS OF ATOMS

In this section, we apply the results of the previous section to study analytically the dynamics of BECs in presence of feeding or loss of atoms. By using the above properties of solitary wave solutions Eqs. (9b) and (9c), we show that the manipulation of one of or either the three parameters g, γ, and k can be used to manipulate the motion of bright and/or dark solitary waves in BEC systems. For simplicity, we assume, to avoid the MI phenomenon for the bright solitary waves, $\alpha = \rho = 0$ leading to the situation when the power of the cw background A_c^2 is less than the quarter of soliton's peak power A_s^2, that is, when $A_s^2 - 4A_c^2 > 0$. An analytical bright solitary wave solution of the GP Eq. (1) under the integrability condition Eqs. (2) is then given by Eq. (13).

We focus ourselves to some special cases when either the strength $g(t)$ of the two-body interatomic interactions or the potential strength $k(t)$ or both $g(t)$ and $k(t)$ are know. In what follows, we take some examples to demonstrate the dynamics of bright and/or dark solitary waves in 1D BEC systems with different kinds of either scattering length and harmonic

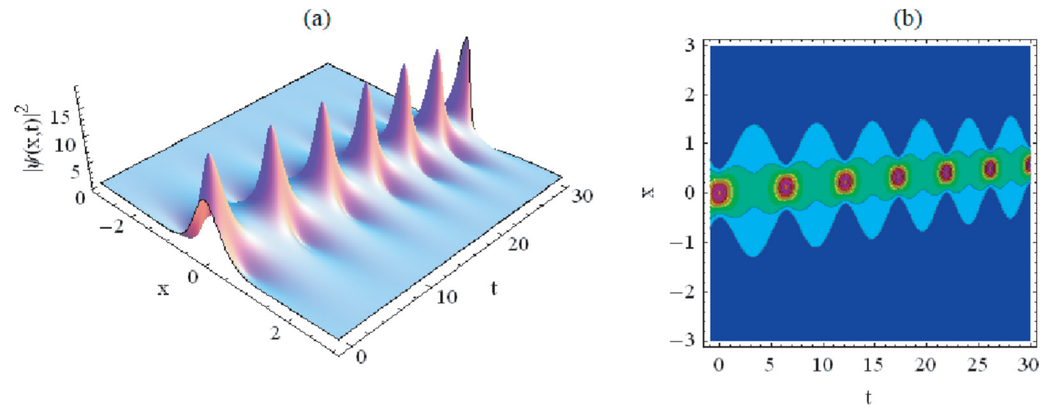

FIG. 3. The dynamics of a bright solitary wave in the expulsive parabolic potential given by Eq. (16). Different plots are generated with parameters $A_c = 1.12$, $A_s = 3.2$, $|g_0| = 0.25$, $k_c = 0.03$, $k_s = 0.04$, and $\lambda = 0.02$.

trapping potential, or scattering length, or harmonic trapping potential.

To argue the applicability of our theory, all the examples used here are taken from Sec. II B. To have fewer peaks so that the structure of the curves can be seen more clearly, we will sometimes in our examples, some parameters with values slightly different from those used in real experiments.

A. Special case of BECs with $g(t) = \pm 2|g_0| \exp[\lambda t]$ and $k(t) = -\frac{1}{4}\lambda^2$

In this example, we follow Liang *et al.* [5] and consider the GP Eq. (1) with parameters $g(t) = \pm 2|g_0| \exp[\lambda t]$ and $k(t) = -\frac{1}{4}\lambda^2$. Solving the integrability conditions Eqs. (2) in γ and α_0 yields $\gamma = -\lambda/4$ and $\alpha_0 = -1/2$. Therefore, $T(t) = \frac{1}{\lambda \ell_0^2}(\exp[\lambda t] - 1)$ and $X(x,t) = \frac{\exp[\frac{\lambda}{2}t]}{\ell_0}x$. For $T(t)$ to vary from zero to plus infinity, we must have $\lambda > 0$, meaning that $\gamma < 0$ and corresponds to the loss of atoms in the condensate. In this example, the potential strength k and the loss parameter γ are constant. In our study, we restrict ourselves to a safe range of parameters used by Liang *et al.* [5]; for this range of parameters, the system described by Eq. (1) becomes effectively 1D, i.e., the energy of two body interactions is much less than the kinetic energy in the transverse direction (Brazhnyi *et al.* in Ref. [16]). For the above set of equation parameters, solution Eq. (13) for $g(t) = 2|g_0| \exp[\lambda t]$ becomes, for $|\eta_0|\ell_0^{-1} = 1$,

$$\psi(x,t) = \left[A_c + A_s \frac{-2A_c \cosh[\xi] + A_s \cos[\theta] + i\sqrt{A_s^2 - 4A_c^2}\sin[\theta]}{A_s \cosh[\xi] - 2A_c \cos[\theta]}\right] \exp[i\varphi(x,t)],$$
$$\xi(x,t) = \sqrt{2|g_0|\left(A_s^2 - 4A_c^2\right)}\left\{\exp\left[\frac{\lambda}{2}t\right]x - \frac{\sqrt{|g_0|}k_0}{\lambda}(\exp[\lambda t] - 1)\right\},$$
$$\theta(x,t) = \frac{|g_0|A_s\sqrt{A_s^2 - 4A_c^2}}{2\lambda}(\exp[\lambda t] - 1), \quad \varphi(x,t) = -\frac{\lambda}{2}x^2 + 2k_0\sqrt{|g_0|}\exp\left[\frac{\lambda}{2}t\right]x + \frac{|g_0|\left(A_c^2 - k_0^2\right)}{\lambda}(\exp[\lambda t] - 1), \tag{16}$$

where $k_0 = \frac{A_c^2 k_c + 2A_s^2 k_s}{2(A_c^2 + A_s^2)}$.

The dark solitary wave solution Eq. (9c) becomes

$$\psi(x,t) = [A_c + iA_s \tanh[\xi(x,t)]] \exp[i\varphi(x,t)], \quad \xi(x,t) = \pm\sqrt{|g_0|}\exp\left[\frac{1}{2}\lambda t\right]x + \frac{A_s(A_c + \sqrt{2}k_0)}{|g_0|\lambda}(\exp[\lambda t] - 1) + T_0,$$
$$\varphi(x,t) = -\frac{\lambda}{2}x^2 + 2k_0\sqrt{|g_0|}\exp\left[\frac{\lambda}{2}t\right]x - \frac{\left(A_s^2 + A_c^2 + k_0^2\right)}{|g_0|\lambda}(\exp[\lambda t] - 1), \tag{17}$$

where $k_0 = \frac{A_c^2 k_c + 2A_s^2 k_s}{2(A_c^2 + A_s^2)}$.

1. Dynamics of bright solitary wave described by Eq. (16)

We start with the study of the dynamics of bright soliton described by Eq. (16). In the case of the absence of feeding or loss of atoms in the condensate with the above equation parameters $g(t)$ and $k(t)$, Liang *et al.* [5] reported that with the increasing absolute value of the scattering length (from our computation, this corresponds to $\gamma = -\lambda/4 < 0$, i.e., to the loss of atoms in the condensate), the bright soliton has an increase in the peaking value and a compression in the width. As we can see from Eq. (9b) as well as from Eq. (16), the increasing absolute value of the scattering length preserves the soliton peak. For a better understanding, we use Eq. (16) to show in Fig. 3 the dynamics of the Feshbach resonance managed bright soliton in the expulsive parabolic potential in

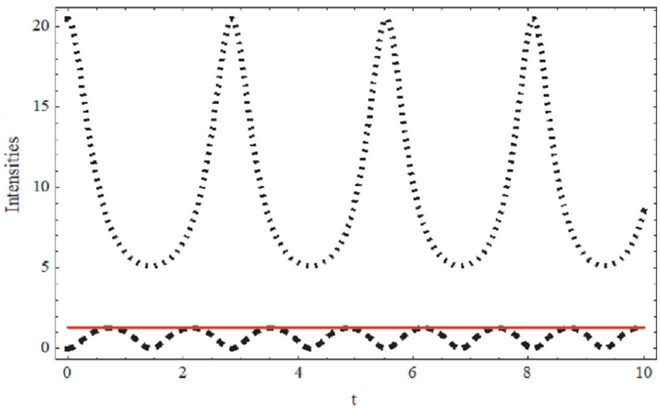

FIG. 4. The evolution plots of the maximal intensity $|\psi|^2_{\max}$ (dotted lines), the minimal intensity $|\psi|^2_{\min}$ (dashed lines), and the background intensity $|\psi_c|^2$ (solid lines) for BECs with loss of atoms. To generate these plots, we have used the parameters $A_c = 0.8$, $A_s = 2.4$, $g_0 = 0.25$, $k_c = 0.03$, $k_s = 0.04$, $T_0 = 0$, and $\lambda = 0.02$.

the presence the loss of atoms. As one can clearly see from Fig. 3(a), the bright solitary wave preserves its peak value when the loss of atoms is taken into consideration. The contour plot Fig. 3(b) shows that in the presence of loss of atoms, the solitary wave has a compression in its width during its propagation. From the above computation, the loss of atoms correspond to an increasing $g(t)$ as function of time t. It is important to notice, as we can see from Fig. 3, that in an expulsive parabolic potential, the bright solitary wave is set into motion and propagates in the longitudinal direction, instead of oscillating as in an attractive parabolic potential.

With the help of Eq. (14a), we show in Fig. 4 the evolution plots of the maximal and minimal intensities given by $|\psi|^2_{\max}$ (dotted lines) and $|\psi|^2_{\min}$ (dashed lines), and the background intensity $|\psi_c|^2$ (solid (red) lines) for $\lambda = 0.02$. Plot of Fig. 4 reveal that the bright soliton can only be squeezed into the assumed peak matter density between the minimum and maximum values.

With the help of Eq. (14d), we depict in Fig. 5 the atomic exchange between the bright solitary wave and background. If the amplitude of the $A_c = 0$, there is no exchange of atoms between the solitary wave and cw background, that is, all the atoms are available only in the soliton. Figure 5(a) clearly indicates that the atoms does not undergo exchange from the solitary wave to cw background; due to the loss of atoms, $\chi(t) = N_0(t)|_{A_c=0}$ decreases as the solitary wave propagates. In other words, we can also say that the bright solitary wave only appears and the background disappears. However in the case of nonzero background, the physical situation is entirely different as follows. The exchange of atoms between the bright solitary and the cw background becomes quicker as the bright solitary wave propagates. As we can see from different Figs. 5(b) and 5(c), a slow-fast-slow process of atomic exchange is performed between the bright solitary wave and the cw background, but the whole trend of the atomic exchange between the bright soliton and the cw background decreases.

2. Propagation of dark solitary wave described by Eq. (17)

Now, we use expressions Eq. (17) to investigate analytically the dynamics of dark soliton in the BEC under consideration when the conditions of the solitary wave management Eqs. (2) are satisfied. Because $\xi(x,t)$ in Eq. (17) depends on both x and t, the solution Eq. (17) should be a time-dependent dark soliton embedded on a cw background with intensities $|\psi_c(x,t)|^2 = A_c^2$. To investigate the stability of the dark soliton against the loss or gain of atoms in the condensate, we have

$$\int_{-\infty}^{+\infty}(|\psi(x,t)|^2 - |\psi(\pm\infty,t)|)dx = \mp\frac{2A_s^2}{\sqrt{|g_0|}}\exp\left[-\frac{1}{2}\lambda t\right], \tag{18}$$

which is the exact number of the atoms in the dark soliton against the background described by Eq. (17). This shows that during the propagation of the dark soliton described by Eq. (17), the number of atoms in the dark soliton decreases (increases) when the loss (gain) of atoms is taken into consideration. It follows from Eq. (18) that the dark soliton associated with sign "+" in Eq. (17) leads to a negative "number of atoms."

We show in Fig. 6 the evolution of the density $|\psi(x,t)|^2$ of the dark soliton described by Eq. (17) with sign "−." As we can see from Fig. 6(a), the dark soliton Eq. (17) during its propagation maintains its shape to the background intensity. The evolution contour Fig. 6(b) shows that during its propagation, the dark soliton in BECs has a compression in the width.

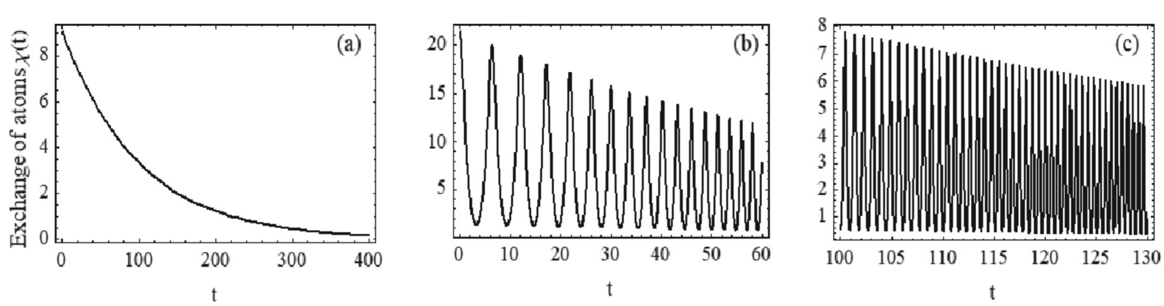

FIG. 5. The atomic exchange between the bright solitons and the background given by Eq. (14d) for BECs with $g(t) = 2|g_0|\exp[\lambda t]$ and $k(t) = -\frac{1}{4}\lambda^2$. The range of time is (a) $t \in [0, 400]$, (b) $t \in [0, 60]$, and (d) $t \in [100, 130]$. Plot (a) corresponds to a zero cw background, while plots (b) and (c) are obtained are associated with a nonzero cw background. The parameters are given as follows: $A_c = 1.12$, $A_s = 3.2$, $g_0 = 0.25$, $k_c = 0.03$, $k_s = 0.04$, and $\lambda = 0.02$.

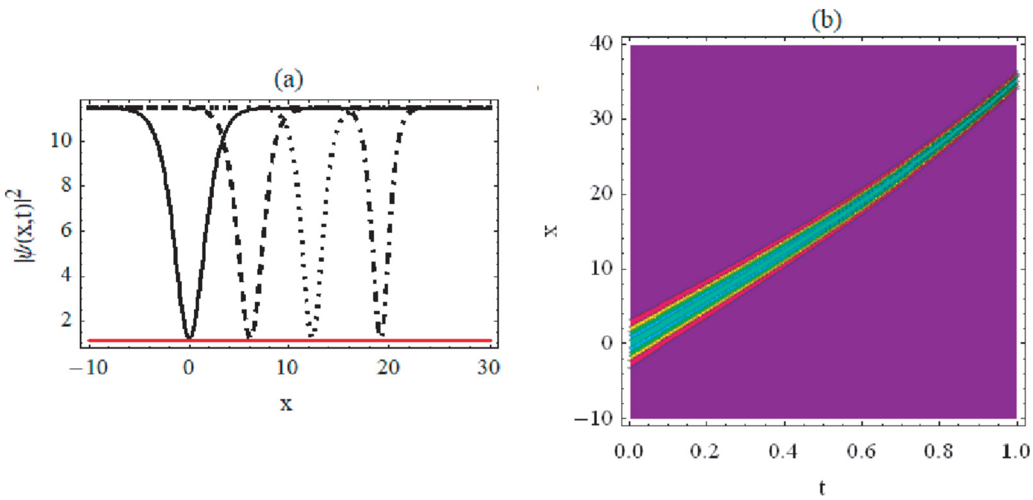

FIG. 6. The evolution plots of $|\psi(x,t)|^2$ associated with the dark soliton described by Eq. (17) with sign "−." (a) The profile of dark soliton at different times: $t = 0$ (solid line), $t = 0.2$ (dashed line), $t = 0.4$ (dotted line), and $t = 0.6$ (dot-dashed line); (b) the evolution contour plot of density $|\psi(x,t)|^2$ of dark soliton. The horizontal (red) line in panel (a) shows the density of the cw background. The parameters used to generate these plots are $g_0 = 0.25$, $A_c = 1.12$, $A_s = 3.2$, $k_c = 0.03$, $k_s = 0.04$, and $\lambda = 2$.

B. Special case of BECs with $g(t) = 1 + m_0 \sin[\omega t]$ for $0 < m_0 < 1$

In this section, we consider the special of BECs with the temporal periodic modulation of the s-wave scattering length [17], and the nonlinearity parameter takes the form $g(t) = 1 + m_0 \sin[\omega t]$ for $0 < m_0 < 1$; here, we consider that $g_0 = 1$. Without loss of generality, we focus ourselves on the bright soliton described by Eq. (13). For simplicity, we take $|\eta_0|\ell_0^{-1} = 1$, leading to $\ell_0 = 1/\sqrt{g_0} = 1$; moreover, we take $\alpha_0 = -1$. With the above form of $g(t)$, the expressions for $k(t)$ and $\gamma(t)$ are given in Sec. II B. It follows from the expression $\gamma(t) = -\frac{m_0\omega}{4}\frac{\cos[\omega t]}{1+m_0\sin[\omega t]}$ that the parameter γ of gain or loss of atoms in the condensate, as well as the nonlinearity parameter $g(t)$, is a time-periodic function with period $T_p = 2\pi/\omega$; moreover, $\gamma(t)$ will have alternate signs as one can see from Fig. 7(a). Therefore, during the propagation of the BEc soliton, the condensate will alternatively loss and gain atoms; this behavior is well seen in Fig. 7(c). Figure 7(c) reveals that in each interval of negative $\gamma(t)$ [positive $\gamma(t)$], the number $N_0(t)$ of atoms in the bright solitary wave against the cw background given by Eq. (14c), i.e.,

$$N_0(t) = 2\sqrt{\frac{A_s^2 - 4A_c^2}{1 + m_0 \sin[\omega t]}}, \tag{19a}$$

decreases (increases). It follows from Eq. (14d) that the number $\chi(t)$ of atoms in both the bright soliton and cw background is

$$\chi(t) = N_0(t)\left\{1 + \frac{4A_c\cos[\theta]}{\sqrt{A_s^2 - 4A_c^2\cos^2[\theta]}}\arctan \times\left[\frac{A_s + 2A_c\cos[\theta]}{\sqrt{A_s^2 - 4A_c^2\cos^2[\theta]}}\right]\right\} \tag{19b}$$

with $\theta(x,t) = \frac{A_s m_0\sqrt{A_s^2-4A_c^2}}{2\omega}(1 - \cos[\omega t])$. Equation (19b) displays that a time-periodic atomic exchange with period T_p is formed between the solitary wave and the background. Figure 7(e) shows the exchange of atoms between the solitary wave and the background within two periods of time, i.e., for $t \in [0, 2T_p]$. This plot shows that a slow-fast-slow-fast process of atomic exchange is periodically performed between the solitary wave and the background. From Figs. 7(b), 7(d) and 7(f), we can see the bright solitary wave behaves like a breather soliton propagating on a cw background, which corresponds, as we can see from Fig. 7(e), to the atom exchanging between the bright solitary wave and the cw background. The conclusion can be made that the atomic exchange between the solitary wave and the background keeps the bright solitary wave dynamically stable against the variation of the nonlinearity, as demonstrated by Liang *et al.* [5]. From Fig. 7(d), we can see that during its propagation, the bright solitary wave on each time interval of negative (positive) parameter $\gamma(t)$ has a compression (broadening) it the width; therefore, a compression -broadening-compression-broadening-compression process of the bright solitary wave is periodically performed during the bright soliton propagation. Figure 7(f) shows the evolution plots of the maximal (solid line) and minimal (dashed line) intensities given by Eq. (14a), and the background intensity $|\psi_c|^2 = A_c^2$ (horizontal red line). Plots of Fig. 7(f) mean that the bright solitary wave can only be squeezed into the assumed peak matter density between the minimum and maximum values.

C. Special case of BECs with $k(t) = -\frac{3}{(t+t_0)^2}$, with $t_0 \neq 0$ (see Theocharis *et al.* in Ref. [4])

In this special case, we use either the integrability conditions Eqs. (2) or Sec. IIB 3 to obtain $\gamma(t) = -\frac{1}{t+t_0}$, $g(t) = g_0(\frac{t+t_0}{t_0})^4$, and $\alpha_0 = -1$. To avoid the singularity, parameter t_0 must be a positive real number; this real parameter t_0 essentially determines the width of the trap at time $t = 0$ according to Eq. (1). For simplicity, we take $|\eta_0|\ell_0 = 1$, implying that $\ell_0 = 1/\sqrt{g_0}$.

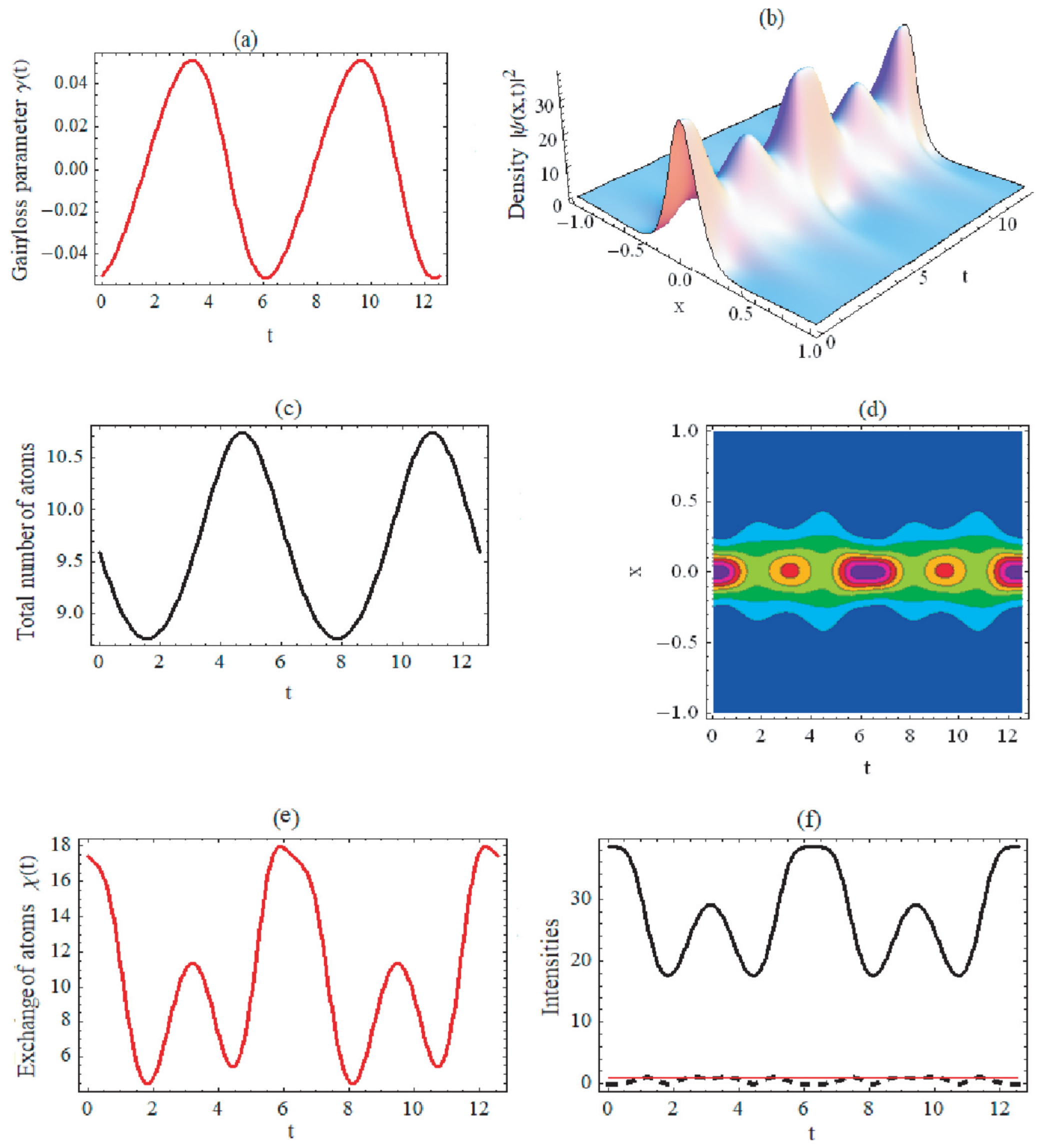

FIG. 7. Plots of different scenarios for a BEC with the temporal periodic modulation of the s-wave scattering length [17] with the nonlinearity parameter $g(t) = 1 + m_0 \sin[\omega t]$, $0 < m_0 < 1$. (a) Time evolution of the gain/loss parameter $\gamma(t) = -\frac{m_0\omega}{4}\frac{\cos[\omega t]}{1+m_0\sin[\omega t]}$; (b) the evolution plot of the density $|\psi(x,t)|^2$ of the bright soliton described by Eq. (13); (c) plot of the total number of atoms $N_0(t)$ defined by Eq. (19a); (d) the evolution contour plot of the density $|\psi(x,t)|^2$; (e) time-periodic atomic exchange between the bright soliton and the background given by Eq. (14d); (f) the evolution plots of $|\psi|^2_{\max}$ (solid line) and $|\psi|^2_{\min}$ (dashed line) described by Eq. (14a). The horizontal (red) line is the background intensities $|\psi_c|^2 = A_c^2$. The range of time is two periods, i.e., $t \in [0,\ 4\pi/\omega]$. The parameters used in these plots are $A_c = 1$, $A_s = 5.2$, $k_c = 0.03$, $k_s = 0.04$, $T_0 = 0$, $\omega = 1$, and $m_0 = 0.2$.

In what follows, we focus our attention on the dynamics of bright solitary wave described by Eq. (13), i.e., the bright solitary waves described by equation

$$\psi(x,t) = \left\{A_c + A_s\frac{-2A_c\cosh[A_s\beta(X-\sqrt{2}k_0T)] + A_s\cos\left[\frac{A_s^2\beta}{2}T\right] + iA_s\beta\sin\left[\frac{A_s^2\beta}{2}T\right]}{A_s\cosh[A_s\beta(X-\sqrt{2}k_0T)] - 2A_c\cos\left[\frac{A_s^2\beta}{2}T\right]}\right\}$$
$$\times\exp\left\{i\left[\sqrt{2}k_0X + \left(A_c^2-k_0^2\right)T - \frac{1}{(t+t_0)}x^2\right]\right\},\quad X = \sqrt{g_0}\left(\frac{t+t_0}{t_0}\right)^2x,\ T = \frac{g_0t_0}{5}\left[\left(\frac{t+t_0}{t_0}\right)^5 - 1\right],\qquad(20)$$

where $k_0 = \frac{A_c^2k_c/\sqrt{2}+A_s^2k_s}{A_c^2+A_s^2}$, $\beta^2 = \frac{A_s^2-4A_c^2}{A_s^2}$, and k_c and k_s are real constants. Using Eqs. (14c) and (14d), the total number $N_0(t)$ of atoms in the bright solitary wave against the background and the total number $\chi(t)$ of atoms in both the bright solitary wave and

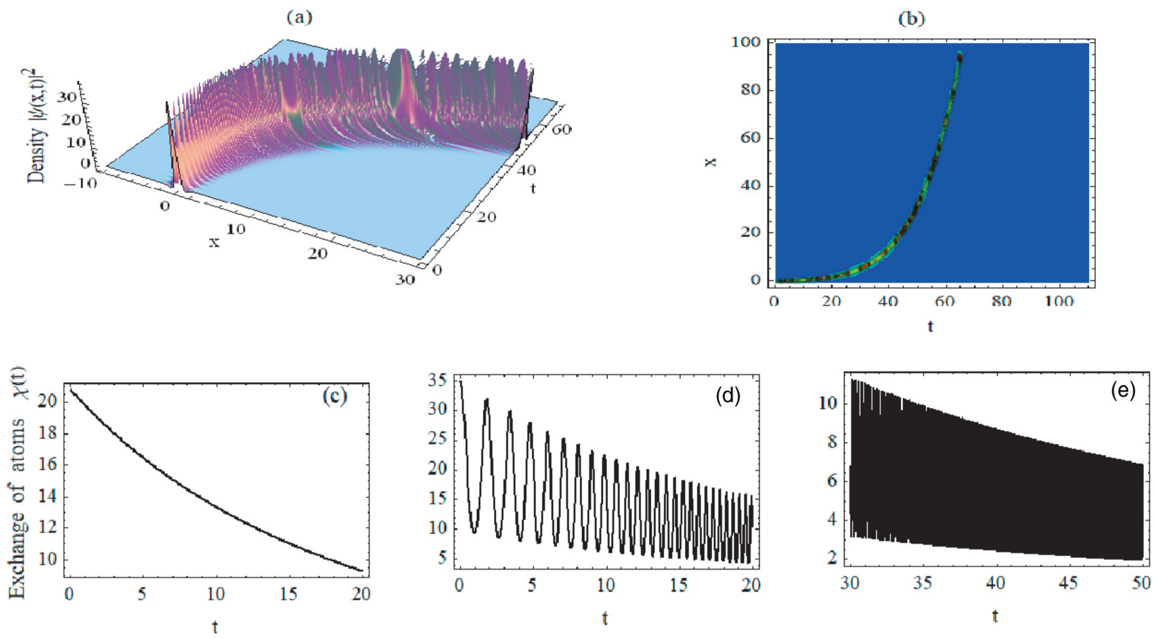

FIG. 8. The evolution of bright solitary waves on cw background given by Eq. (20) with $t_0 = 40$ for parameters $A_c = 1$, $A_s = 5.2$, $g_0 = 0.25$, $k_c = 0.03$, and $k_s = 0.04$. Plots (a) and (b) show respectively the evolution and the contour plots of density $|\psi(x,t)|^2$; plots (d) and (e) show the time-aperiodic atomic exchange between the bright soliton and the cw background in the range of time $t \in [0,\ 20]$ and $t \in [30,\ 50]$, respectively. Panel (c) plots the atomic exchange in the situation of zero background.

the cw background are, respectively,

$$N_0(t) = 2\sqrt{\frac{A_s^2 - 4A_c^2}{g_0}}\left(\frac{t_0}{t+t_0}\right)^2, \tag{21a}$$

$$\chi(t) = N_0(t)\left(1 + \frac{4A_c \cos\left[\frac{A_s^2\beta}{2}T\right]}{\sqrt{A_s^2 - 4A_c^2\cos^2\left[\frac{A_s^2\beta}{2}T\right]}} \arctan\left[\frac{A_s + 2A_c\cos\left[\frac{A_s^2\beta}{2}T\right]}{\sqrt{A_s^2 - 4A_c^2\cos^2\left[\frac{A_s^2\beta}{2}T\right]}}\right]\right). \tag{21b}$$

It is seen from Eq. (21a) that the total number of atoms decreases as the bright soliton propagates. Also, Eq. (21b) displays that a time-aperiodic atomic exchange is formed between the bright soliton and the cw background. In Fig. 8 we show the evolution density plot [Fig. 8(a)], the evolution contour plot [Fig. 8(b)], and the time-aperiodic atomic exchange between the bright soliton and the cw background [Figs. 8(c)–8(e)]. As we can see from plot 8(a), the bright soliton preserves its peak during its propagation. Figure 8(b) reveals that during its propagation, the bright soliton has compression process. As shown in Fig. 8(c), in the case of zero background, there will not be the exchange of atoms. However, in the case of nonzero background as we can see from Figs. 8(d) and 8(e), a slow-fast-slow process of atomic exchange is performed between the bright soliton and the background, but the whole trend of the atomic exchange between the bright soliton and the background decreases.

D. Special case of BECs with $k(t) = \frac{1}{16(t+t_0)^2}$, with $t_0 \neq 0$ (see Theocharis *et al.* in Ref. [4])

Here, we follow Theocharis *et al.* [4] and consider a BEC with time-dependent atomic scattering length in an attractive parabolic potential. From the conditions of the solitary wave management we obtain $\gamma(t) = \frac{1}{4(t+t_0)}$, $g(t) = g_0|\frac{t_0}{t+t_0}|$, and -2, g_0 being an arbitrary real number. Here, t_0 is a real constant whose sign is related to the sign of $\gamma(t) : t_0\gamma(t) > 0$ and which essentially determines the "width" of the trap at time $t = 0$ according to $k(t)$. Moreover, $t_0 < 0$ corresponds to the case of loss of atoms and describes a BEC in a shrinking trap, while the case $t_0 > 0$ is associated with the gain of atoms and corresponds to a broadening condensate (see Theocharis *et al.* in Ref. [4]). Computing $X(x,t)$ and $T(t)$ yields $X(x,t) = \ell_0^{-1}\sqrt{|\frac{t_0}{t+t_0}|}x$ and $T(t) = t_0\ell_0^{-2}\ln|\frac{t+t_0}{t_0}|$. In the case of negative t_0, X and T are defined for $t \in [0,\ -t_0[$; here, $T(0) = 0$ and $T(t \to -t_0) = +\infty$. Moreover, $\gamma(t) = \frac{1}{4(t+t_0)} < 0$ on $[0,\ -t_0[$.

1. *Case of BEC with loss of atoms:* $t_0 < 0$

Without loss of generality, we consider in this case only the situation when $g(t) > 0$ on $[0,\ -t_0[$, and for simplicity, we consider $|\eta_0|\ell_0^{-1} = 1$ so that the bright soliton solution Eq. (9b)

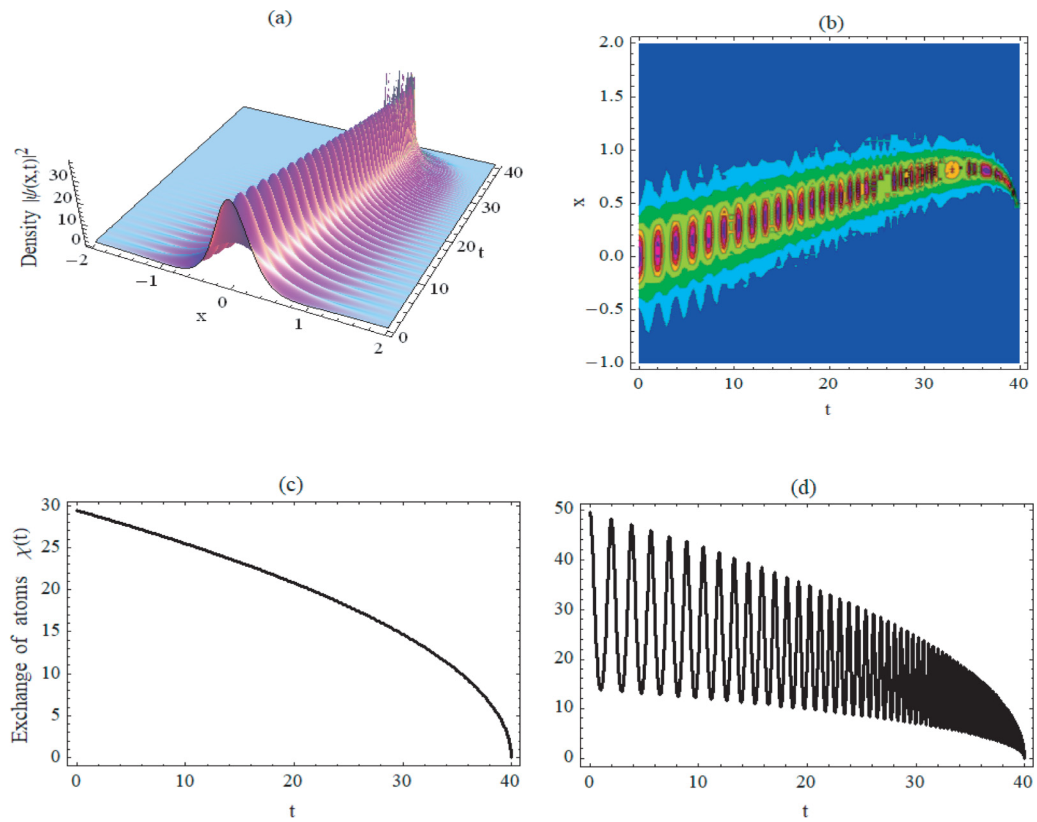

FIG. 9. The evolution plot (a) and the contour plot (b) of density $|\psi(x,t)|^2$ of the bright soliton described by Eq. (22) with positive β and $t_0 = -40$. (c, d) The time-aperiodic atomic exchange $\chi(t)$ between the bright solitons and the cw background. (c) shows the number of atoms in the absence of the cw background, i.e., when $A_c = 0$. The parameters used in these plots are $A_c = 1$, $A_s = 5.2$, $k_c = 0.03$, $g_0 = 0.25$, and $k_s = 0.04$.

under conditions $\alpha = \rho = 0$ becomes

$$\psi(x,t) = \left\{ A_c + A_s \frac{-2A_c \cosh\left[\sqrt{2}\beta A_s\left[\frac{1}{2}X - k_0T\right]\right] + A_s \cos\left[\frac{A_s^2\beta}{2}T\right] + i A_s\beta \sin\left[\frac{A_s^2\beta}{2}T\right]}{A_s \cosh\left[\sqrt{2}\beta A_s\left[\frac{1}{2}X - k_0T\right]\right] - 2A_c \cos\left[\frac{A_s^2\beta}{2}T\right]} \right\}$$
$$\times \exp\left\{ i\left[\frac{x^2}{8(t+t_0)} + k_0X + \left(A_c^2 - k_0^2\right)T\right]\right\},$$
$$X = x\sqrt{g_0\frac{t_0}{t+t_0}}, \quad T(t) = g_0t_0 \ln\frac{t+t_0}{t_0}, \quad k_0 = \frac{A_c^2k_c + A_s^2k_s}{A_c^2 + A_s^2}, \quad \beta^2 = \frac{A_s^2 - 4A_c^2}{A_s^2}, \tag{22}$$

where A_c, A_s, k_c, and k_s are arbitrary real constants, and $t \in [0, -t_0[$. From Eqs. (14c) and (14d) we obtain the number $N_0(t)$ of atoms and the quantity $\chi(t)$ that counts the number of atoms in both the bright soliton and background as $N_0(t) = 2\sqrt{\frac{2(A_s^2-4A_c^2)}{g_0}}\sqrt{|\frac{t+t_0}{t_0}|}$ and $\chi(t) = N_0(t)(1 + \frac{4A_c\cos[\theta]}{\sqrt{A_s^2-4A_c^2\cos^2[\theta]}} \arctan[\frac{A_s+2A_c\cos[\theta]}{\sqrt{A_s^2-4A_c^2\cos^2[\theta]}}])$, where $\theta(x,t) = \frac{g_0A_s^2\beta}{2}\ln|\frac{t+t_0}{t_0}|$. The expression for $N_0(t)$ shows that the number of atoms decreases in time interval $[0, -t_0[$.

For a better understanding of the dynamics of the BEC in a shrinking trap (i.e., when $t_0 < 0$), we show in Fig. 9(a) the evolution and Fig. 9(b) the contour plots of $|\psi(x,t)|^2$ which display the dynamics of the Feshbach resonance managed bright soliton in the attractive parabolic potential with loss of atoms. As we can see from Fig. 9(a), the bright soliton peak oscillates between minimum and maximum values. From the contour plot [Fig. 9(b)], it can be seen that, a compression-broadening-compression process in the bright soliton width is performed, but in whole, the bright soliton has a compression in its width. As shown in Fig. 9(c), in the case of zero cw background, there will not be the exchange of atoms. However, in the presence of a cw background as shown in Fig. 9(d), a slow-fast-slow process of atomic exchange is performed between the bright soliton and the cw background, but the whole trend of the atomic exchange between the bright soliton and the background decreases.

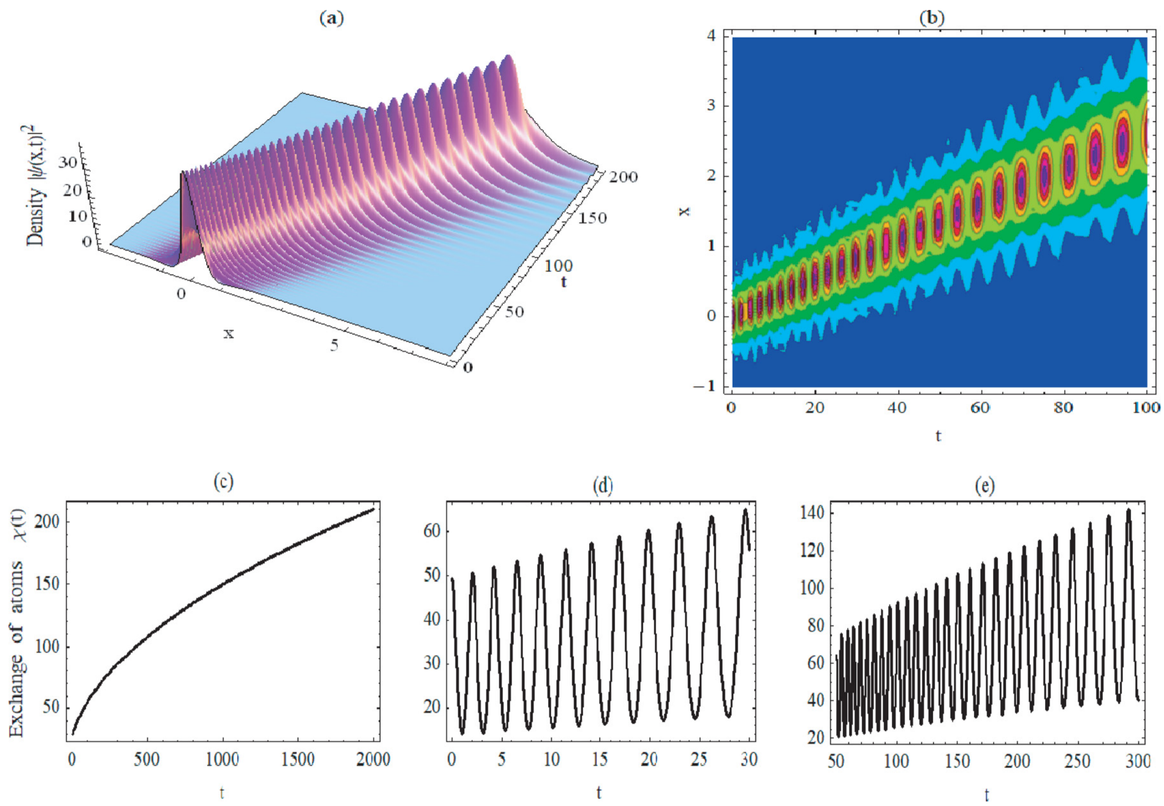

FIG. 10. The dynamics (a) and the evolution plot (b) of the bright soliton described by Eq. (22) with positive β and $t_0 = 40$. (c–e) The time-aperiodic atomic exchange $\chi(t)$ between the bright solitons and the cw background. The range of time is (c) $t \in [0, 200]$ for the case of zero cw background, (d) $t \in [0, 30]$ and $t \in [50, 300]$. The parameters used in these plots are the same as in Fig. 9.

2. *Case of BEC with gain of atoms:* $t_0 > 0$

Now we consider the case of BEC with gain of atoms. Here, we will distinguish two cases, the case $g_0 > 0$ leading to bright solitons and the case $g_0 < 0$ which leads to dark solitons.

a. Propagation of bright solitons of a BEC with gain of atoms. In the case of BECs with attractive interatomic interactions, i.e., $g_0 > 0$, the bright solitons [Fig. (9b)] under conditions $\alpha = \rho = 0$ is given by Eq. (22) with $t \in [0, +\infty[$. For a better understanding of the broadening condensate (i.e., with $t_0 > 0$), we have depicted in Figs. 9(a) and 9(d) the density $|\psi(x,t)|^2$ and the time-aperiodic atomic exchange $\chi(t)$ between the bright soliton and the cw background. It is seen from Fig. 10(a) that during its propagation, the bright soliton peak oscillates and is preserved. Figure 10(b) shows that with the increasing of the gain parameter $\gamma(t)$, a compression-broadening-compression process in the bright soliton width is performed, but in whole, the bright soliton has a broadening in its width. From Figs. 10(a) and 10(b), we can see that the bright soliton behaves like a breather soliton propagating on a cw background, which corresponds to the atom exchanging between the bright soliton and the cw background, and this exchange keeps the bright soliton dynamically stable (see Liang *et al.* in Ref. [5]) against the variation of the gain parameter $\gamma(t)$. These features can be well seen in Figs. 10(c)–10(e). From Fig. 10(c), it is seen that in the case of zero cw background, there will not be the exchange of atoms. From Figs. 10(d) and 10(e), it is seen that for BEC with gain of atoms, a slow-fast-slow process of atomic exchange is performed between the bright soliton and the cw background, but the whole trend of the atomic exchange between the bright soliton and the cw background increases.

a. Dynamics of dark solitons of a BEC with gain of atoms. We now focus ourselves to the study of BEC with repulsive interatomic interactions, i.e., $g_0 < 0$. In this situation, Eq. (9b) leads to the following dark soliton solution of Eq. (1):

$$\psi_\pm(x,t) = \left\{A_c + iA_s \tanh\left[\pm\frac{1}{2}X(x,t) + A_s(A_c+\sqrt{2}k_0)T(t)\right]\right\}\exp\left\{i\left[\frac{1}{8(t+t_0)}x^2 + k_0X(x,t) - \left(A_s^2+A_c^2+k_0^2\right)T(x,t)\right]\right\},$$

$$X(x,t) = \sqrt{-g_0\frac{t_0}{t+t_0}}\,x,\quad T(t) = -g_0t_0\ln\frac{t+t_0}{t_0},\quad k_0 = \frac{A_c^2k_c + A_s^2k_s}{A_c^2+A_s^2}. \tag{23}$$

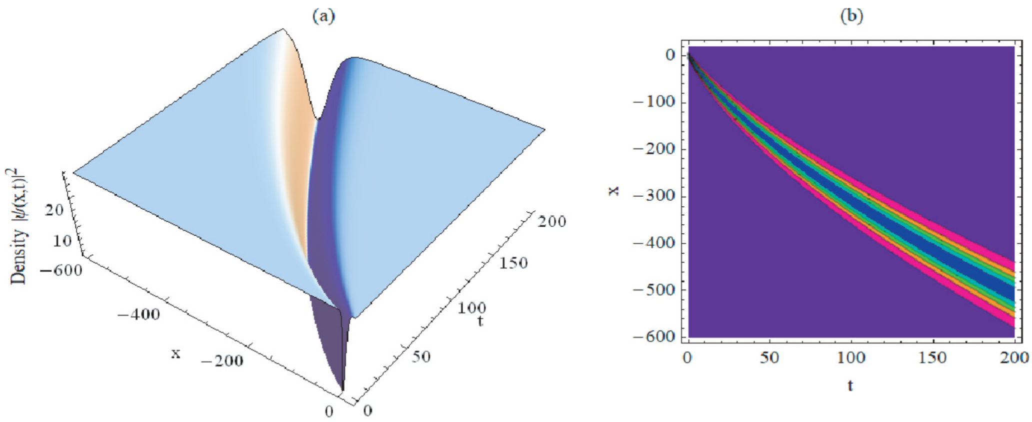

FIG. 11. The evolution (a) and the contour (b) plots of the density $|\psi(x,t)|^2$ of the dark soliton embedded on a cw background described by Eq. (23) with sign "+" for $t_0 = 2$. The parameters used in these plots are $A_c = 1$, $A_s = 5.2$, $k_c = 0.03$, $k_s = 0.04$, and $g_0 = -0.25$.

Here, A_c, A_s, k_c, and k_s are arbitrary real constants, and $t \in [0, -t_0[$; for simplicity, we have taken $|\eta_0|\ell_0^{-1} = 1$. From Eq. (24), we can obtain the intensities of the cw background corresponding to $A_s = 0$ as $|\psi_c|^2 = A_c^2$. Proceeding as in the case of the bright soliton, we obtain the exact number of the atoms in the bright soliton against the cw background described Eq. (23) as follows:

$$\int_{-\infty}^{+\infty} (|\psi_\pm|^2 - |\psi_\pm(\pm\infty,t)|^2)dx = \pm 2A_s^2\sqrt{-\frac{t+t_0}{g_0 t_0}}.$$

This indicates that during the propagation of the dark soliton on a cw background, the number of atoms in the dark soliton increases [and is "negative" if we take the sign "−" in Eq. (23)]. As shown plots of Fig. 11, independently on the gain parameter $\gamma(t)$, the dark soliton remains its intensity on the cw background and can propagate a longer distance. Plots of Fig. 11 also reveal that during its propagation, the dark soliton has a broadening in the width.

IV. CONCLUSION

In summary, we have considered a GP equation with dissipative term which describes the dynamics of matter-wave solitons in BECs with the time-dependent interatomic interaction in a time-dependent parabolic potential with gain or loss of atoms. The integrability conditions of the GP equation under consideration is established. Analytical solutions that describe the modulational instability and the propagation of bright and dark solitary waves on a cw background are obtained and expressed in terms of only the dissipative parameter. The obtained integrability conditions Eqs. (2) also appear as the conditions under which the solitary waves of the BECs can be managed by controlling the functional gain or loss parameter $\gamma(t)$. For specific BECs, the dynamics and the trains of bright and dark solitons are analyzed thoroughly. We have showed that under the found conditions of the solitary wave management, the gain or loss parameter can be used to manage the motion of both the bright and dark solitons. Our results also reveal that the train of bright solitons may be excited with a strong enough cw background. We have obtained that when our integrability criterion is satisfied, bright and dark solitons in BECs with gain of atoms during their propagation have a broadening in their width. In contrast, for BECs with loss of atoms satisfying the conditions of the solitary wave management, both bright and dark solitons during their propagation have a compression in their width; this possibility of compressing the solitons of BECs could provide an experimental tool for investigating the range of validity of the 1D GP equation. Also, we have showed that neither the injection nor the ejection of atoms does not affect the soliton peaks of BEC for which the integrability criterion is fulfilled. It is also shown that under the conditions of the solitary wave management, the number of atoms in the bright soliton of BECs with loss of atoms keeps dynamic stability: the exchange of atoms between the bright soliton and the cw background becomes quicker when decreasing the loss parameter $\gamma(t)$.

ACKNOWLEDGMENTS

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