Activated dynamics: An intermediate model between the random energy model and the *p*-spin model

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To study the activated dynamics of mean-field glasses, which takes place on times of order $\exp(N)$, where N is the system size, we introduce a new model, the correlated random energy model (CREM), that allows for a smooth interpolation between the REM and the *p*-spin models. We study numerically and analytically the CREM in the intermediate regime between REM and *p*-spin. We fully characterize its energy landscape, which is like a golf course but, at variance with the REM, has metabasins (or holes) containing several configurations. We find that an effective description for the dynamics, in terms of traps, emerges, provided that one identifies metabasins in the CREM with configurations in the trap model.

I. INTRODUCTION

When approaching the glass transition, supercooled liquids undergo an extraordinary slowing down [1,2]. This phenomenon is argued by many to be related to some kind of activated dynamics [3].

Mean-field models of glasses have been particularly influential to study the glass transition: First, they capture the first decades of the slowing down by providing a microscopic realization of the so-called mode coupling transition (MCT) [4], as shown, e.g., by the *p*-spin spherical model which displays an MCT transition at a certain temperature T_d [5]. Second, they provide a basis for the random first-order transition theory [6], which is one of the most prominent approaches to understand the formation of glasses and provides a sound phenomenological explanation of activated dynamics of glass-forming liquids.

To construct a full first-principles microscopic theory of activated dynamics, a promising strategy consists of understanding first activated dynamics in mean-field models and then extending this approach to finite-dimensional systems. Pioneering results were obtained more than a decade ago by Crisanti and Ritort, who numerically studied mean-field models on times which diverge exponentially with the system size N (since barriers are extensive in mean-field models) [7,8]. Analytical results have been hampered by the nonperturbative character of the activated processes.

A successful workaround to this issue has been the introduction of the Trap model (TM) [9–11] (see also Ref. [12]). It

is a further simplification of usual mean-field glasses, which captures the main features of activation and is analytically tractable. The TM is based on a drastic simplification of the dynamics: The energy landscape is envisioned as a golf course characterized a large amount of separate minima at the bottom of which the system spends most of the time. Transitions between these local minima require reaching a high threshold energy $E_{\mathrm{th}} = 0$ with an Arrhenius rate of order $e^{(E-E_{\mathrm{th}})/k_B T}$ [13]. From E_{th} any part of the phase space can be attained with equal probability and in negligible time. This leads to simple and appealing description of the dynamics. For instance, the aging following a quench from high temperature can be precisely described in terms of an exploration of deeper and deeper minima: The system spends a substantial fraction of the time, t, elapsed after the quench, in the deepest minimum visited along the dynamics. This leads to a logarithmic decrease of the average energy, $E(t) \simeq -T \log(t)$ [10,11].

Even though traits of the TM were searched and found in simulations of realistic glass-formers more than a decade ago [14,15], only very recently it has been shown that in the limit of large-enough times and system sizes the activated dynamics of the simplest mean-field model of the glass transition, the random energy model (REM) [16], is fully TM-like [17,18], meaning that in the limit of large times and sizes, the REM dynamics can be predicted through the TM. On the analytical side, this result was obtained in Ref. [17] after almost two decades of effort on simplified versions of the REM [19–23]. On the numerical side [18] (see also Ref. [24]) this was done by extracting information on the basins through the time series of the energy, as proposed in Ref. [25].

The REM is thought to represent a limiting case ($p \to \infty$) of the *p*-spin model [16,26], which is one of the most studied

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mean-field models of glasses [1,2,5]. A natural question is therefore whether the TM-like description of the activated dynamics holds for this richer system as well. As a first step to solve this interesting but difficult issue, in this paper we introduce and study a variation of the REM, the correlated random energy model (CREM). In the CREM, a parameter α controls the amount of correlations between energy levels. By tuning the parameter α it is possible to interpolate smoothly and exactly between the REM and the p-spin model (as discussed below, the common lore that the p-spin model tends to the REM for $p \to \infty$ is problematic when studying activated dynamics, this is the reason why we introduce the CREM). We study the CREM in the intermediate regime, where energy levels are weakly correlated, and show that although the energy landscape structure is richer with respect to the REM, a description in terms of separate, well-defined traps can still be done.

II. MODELS

A. The random energy model

In the REM [16], there are N binary variables $s_i = \pm 1$ called spins. Each configuration of the N spins (also called state) is assigned a random energy from a Gaussian distribution of mean 0 and variance N. The energies of different states are independent. Each state has N neighbors, corresponding to the flipping of a single spin. Typically, i.e., with probability 1 in the large-N limit, the energies of the neighbors are included in the interval $(-\sqrt{2N \log(N)}, \sqrt{2N \log(N)})$, which corresponds to zero intensive energy [18]. The majority of the states are also at zero intensive energy. This leads to a landscape like a golf course where to escape an energy minimum the system typically must climb up to $E_{\text{th}}^{\text{REM}} = -\sqrt{2N \log(N)}$, and configurations at low energy E act like trap with a lifetime of the order $e^{(E_{\text{th}} - E)/k_B T}$. It was recently shown that in large-enough systems and on exponentially large time scales, the equilibrium and aging dynamics of the REM can be effectively described through the TM [17,18] (see also the previous work in Refs. [19–21,24,27]).

B. The p-spin model

The Hamiltonian of the p-spin model contains p-body interactions between N Ising spins $s_i = \pm 1$ and reads [16]:

$$\mathcal{H} = -\sum_{1 \leqslant i_1 < i_2 < \ldots < i_p \leqslant N} J_{i_1, i_2, \ldots, i_p} s_{i_1} s_{i_2} \ldots s_{i_p}, \qquad (1)$$

where the couplings $J_{i_1, i_2, \ldots, i_p}$ are extracted from a random Gaussian distribution with mean zero and variance $\frac{p!}{2N^{p-1}}$.

In the p-spin model, as in many other mean-field glassy systems [1], there exists a threshold energy E_{th} below which dynamics become activated and exponentially slow (in the system size)[28,29]. At variance with the REM, this threshold energy is extensive and negative [30].

For any p of order 1 and in the large-N limit, the correlation matrix between two generic configurations $\vec{x} = [s_1^{(x)}, s_2^{(x)}, \ldots, s_N^{(x)}]$ and $\vec{y} = [s_1^{(y)}, s_2^{(y)}, \ldots, s_N^{(y)}]$ reads [16]

$$\overline{E(\vec{x}) E(\vec{y})} = N q(\vec{x}, \vec{y})^p, \qquad (2)$$

where $\overline{(\ldots)}$ is the average over different instances of the couplings and $q(\vec{x}, \vec{y}) = \frac{1}{N} \sum_i^N s_i^{(x)} s_i^{(y)}$ is the overlap between $\vec{x}$ and $\vec{y}$. Thus, contrary to the REM, now energies are correlated. Note that Eq. (2) is valid at leading order in N (subleading corrections have been neglected). The REM can be *formally* recovered in the $p \to \infty$ limit. In fact, if one could take $p \to \infty$ before $N \to \infty$ then Eq. (2) would lead to uncorrelated energies for different configurations. This limit, however, does not make sense since $p = N$ at most [31]. This is not an issue for thermodynamics, which was indeed shown to converge to that of the REM even for $N \to \infty$ first and $p \to \infty$ later [16,26]; it is instead an issue for activated dynamics, since uncorrelated energies are a key ingredient for the analysis of the dynamics of the REM. For this reason we consider below a different model that allows us to interpolate continuously between the p-spin model with $p \sim \mathcal{O}(1)$ and the REM.

C. The correlated random energy model

We consider a variant of the REM with correlated energiesCREM [32]. In the CREM, there are N spins $s_i = \pm 1$, ($i = 1, \ldots, N$), so there are 2^N different states. As in the REM, each state is assigned a random Gaussian energy of mean 0 and variance 1. However, in the CREM the energies of two configurations $\vec{x}$ and $\vec{y}$ are not independent. Their covariance is

$$\overline{E(\vec{x}) E(\vec{y})} = N q(\vec{x}, \vec{y})^{\alpha N}, \qquad (3)$$

where $\alpha \in [0, \infty)$ is a parameter. Contrary to (2), the equation above is strictly valid for any α and N. The largest covariance is obtained for nearest-neighboring configurations $\vec{x}$ and $\vec{x}'$. Since they differ by one spin flip only, in the large-N limit one finds $\overline{E(\vec{x}) E(\vec{x}')} = N e^{-2\alpha}$. The parameter α allows for a smooth interpolation between the REM and the p-spin model. In fact, for $\alpha = \frac{p}{N}$, Eq. (3) reduces to Eq. (2), and the p-spin model is recovered. Whereas if α diverges with N, then the energies become independent variables at large N, as in the REM. In the following, to study activated dynamics in an intermediate case between REM and p spin, we focus on the regime α of the order of 1.

The study of the CREM can be easily implemented numerically, since its computational complexity does not increase with p. As a matter of fact, by going to Fourier space on the hypercube the energies of the CREM become independent, so one can generate them as Gaussian independent random variables in Fourier space and then antitransform them back to real space (see Appendix A).

III. A GOLF COURSE WITH STRUCTURE IN THE HOLES

For α of order 1 and very large N, the correlation matrix reads

$$\mathcal{Q}_{xy} \equiv \overline{E(\vec{x}) E(\vec{y})} = N e^{-2\alpha r_{xy}}, \qquad (4)$$

where $r_{xy} = \frac{1}{2} \sum_{i=1}^N |s_i^{(x)} - s_i^{(y)}|$, the number of spins that are different between $\vec{x}$ and $\vec{y}$, indicates the *distance* between the two configurations. The exponential decay of $\mathcal{Q}_{xy}$ with the distance determines a correlation length $\xi = \frac{1}{2\alpha}$ for the typical size of correlated domains on the hypercube. Note that ξ is a distance in phase space.

Since the energy distribution is Gaussian, one can easily obtain, given a configuration $\vec{x}$ with energy E_x, the conditional probability of a configuration $\vec{y}$ at distance r from it:

$$P(E_y|E_x) = \frac{1}{\sqrt{2\pi N(1-\rho^{2r})}} e^{-\frac{(E_y-\rho^r E_x)^2}{2N(1-\rho^{2r})}} , \quad (5)$$

where we defined $\rho = e^{-2\alpha}$. From Eq. (5) one can get the expectation of the energy E_y conditioned to E_x, $\mathbb{E}[E_y|E_x] = \rho^r E_x$. Thus, we find that for $\alpha = p/N \to 0$, the usual p-spin case, the energy landscape a finite number of steps away from a given configuration is flat, with every energy almost equal to E_x, consistently with an infinite correlation length $\xi \to \infty$. When α diverges, the REM case, $\mathbb{E}[E_y|E_x] = 0$ and E_y is independent from E_x, consistently with a vanishing correlation length. For $\alpha \sim 1$, one has instead an intermediate situation in which the energies of neighboring configurations are typically higher but do not reach directly zero intensive energy. As we will show, the energy landscape is still golf-course-like but has gained some more structure with respect to the REM.

A. Complexity of "critical points" and threshold energy

In analogy with calculations on the spherical p-spin model [33–36], and with other systems displaying complex energy landscape [37–39], we focus on the discrete counterparts of critical points in continuous systems. Given a configuration $\vec{x}$, we call it "a critical point of order k" if exactly k of its neighbors have lower energy. For example, local minima correspond to $k = 0$. In the case of large funnel-like basins, the analogous of a saddle of order 1 connecting them would be a configuration with $k = 2$. Note, however, that even though two local minima must be connected by a configuration with $k \geqslant 2$, it is not true that every configuration with $k \geqslant 2$ connects different minima [40]. The choice of considering only nearest-neighbor configurations to define "critical points" makes sense for the CREM and the REM where the energy changes substantially by one spin flip.

The probability that a configuration with negative energy E be a "critical point of order k" is

$$P_k(E) = \binom{N}{k} \left\{ \frac{\text{erfc}\,[B(E)]}{2} \right\}^k \left\{ 1 - \frac{\text{erfc}\,[B(E)]}{2} \right\}^{N-k}, \quad (6)$$

where $B(E) = -E\sqrt{\frac{(1-\rho)}{2N(1+\rho)}}$ and $\text{erfc}(x)$ is the complementary error function. As shown in Appendix B, this equation can be easily established by realizing that after conditioning on the value of the energy E of a given configuration, the energies of all its nearest neighbors are independent. For large N, one can use Eq. (6) to calculate the entropy of "critical points of order k," which we call *complexity* Σ_k in analogy with the one of the spherical p spin [33–36]. To leading order one gets (see Appendix B 2)

$$\frac{\Sigma_k(E)}{N} = \log(2) - \frac{E^2}{2N^2}\left(1 + k\frac{1-\rho}{1+\rho}\right). \quad (7)$$

We plot Σ_k for several k in Fig. 1 to stress its qualitative similarity with the complexity in the p-spin model (with the already-mentioned caveat on the analogy with saddle points).

Even though at present, nonmininum configurations are exponentially fewer than minima for any $E < 0$, indicating
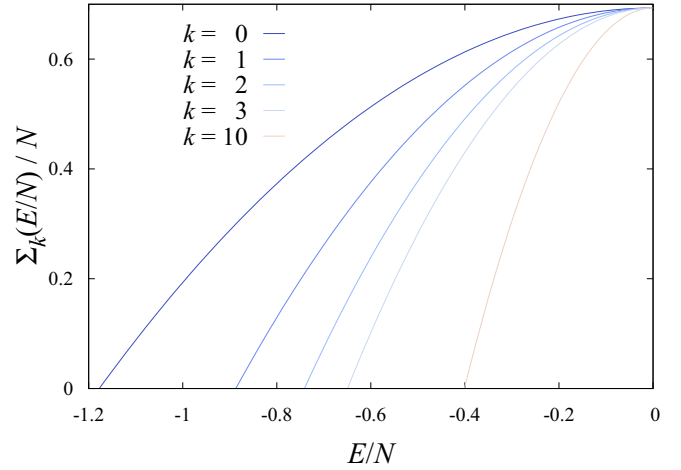

FIG. 1. Complexity curves Σ_ks in the CREM for $\alpha = 1$.

that the intensive threshold energy is zero, as in the REM but differently from the p spin [1,18]. This also implies that for any $E < 0$, to leading order in N, the entropy is equal to the complexity of the minima: $S(E) = \Sigma_0(E)$. Since $S(E)$ is the same as in the REM [16], the thermodynamics is the same in both models (even though the Σ_ks for $k > 0$ are different).

To calculate more precisely the threshold energy, we say that a configuration of energy E is at E_{th} if the lowest-lying among its neighbors has energy $E_{\min}^{(\text{neigh})}(E) = E$. The quantity $E_{\min}^{(\text{neigh})}(E)$ is found by combining Eq. (5) with common results from extreme statistics (see Appendix B 3), resulting in

$$E_{\text{th}} = -\sqrt{2\frac{1+\rho}{1-\rho}N\log(N)} = \sqrt{\frac{1+\rho}{1-\rho}}\,E_{\text{th}}^{\text{REM}}. \quad (8)$$

As expected from the "complexity" calculation, even though correlations have the effect of lowering the threshold energy with respect to the REM, E_{th} is still intensively zero for large N.

B. Size and structure of metabasins

We now focus on metabasins, defined as sets of configurations connected by paths that do not overcome E_{th}. The threshold energy can be used to define a typical linear size d_{th} of the metabasins, as the typical minimum number of spin flips required, starting from a local energy minimum, to reach E_{th}. In the REM case, one needs just a spin flip, i.e., the typical metabasin consists of a single configuration. This is no longer the case for the CREM. Since every configuration has N neighbors, as long as $d_{\text{th}} \ll N$, a rough estimate of the phase-space volume Ω_b of a basin is $\Omega_b \sim N^{d_{\text{th}}}$. According to Eq. (5), given an energy minimum $\vec{x}$ at energy $E_x = E$, configurations $\vec{y}$ at distance r from it have in average an energy $E_y = \rho^r E_x$. The linear size $d_{\text{th}}(E_x)$ is therefore found by imposing that E_y reach the threshold level: $\rho^{d_{\text{th}}(E_x)}E_x = E_{\text{th}}$. The resulting linear metabasin size is therefore:

$$d_{\text{th}}(E) = \frac{1}{2\alpha}\log\left(\frac{E}{E_{\text{th}}}\right). \quad (9)$$

From this equation one sees that deep configurations (meaning with negative intensive energy) are in metabasins made of multiple configurations and that the metabasins become larger and larger as α decreases [41].

To obtain further information on the structure of the metabasins, we focus on the distance $d_{\mathrm{sad}}(E)$ between a local minimum at energy E and the closest configuration of order $k \geqslant 2$. In a continuous system, this second definition would correspond to the distance between a local minimum and the nearest saddles Since to leading order there are N^d configurations at distance d, the typical distance d_{sad} from the minimum at which one finds a saddle is obtained by imposing $N^{d_{\mathrm{sad}}(E)} P_k[\rho^{d_{\mathrm{sad}}(E)} E] \sim 1$. As we previously showed [Eq. (7)], the saddles of order $k = 2$ are overwhelmingly more common that those of higher order, and thus one can impose the simpler condition $N^{d_{\mathrm{sad}}(E)} P_2[\rho^{d_{\mathrm{sad}}(E)} E] \sim 1$, which to leading order yields $d_{\mathrm{sad}} \sim \frac{1}{2\alpha} \log(\frac{E}{E_{\mathrm{th}}})$. This shows that one has to climb up to E_{th} before finding a "saddle," thus implying that a metabasin contains several configurations but no additional higher local minima. This is in agreement with Eq. (5), which implies that energy typically increases by going further away from a given low-energy configuration.

C. Trap dynamics

The previous analysis shows that when $\alpha \sim 1$ the energy landscape of the CREM is more complex than the one of the REM. Nevertheless, the golf-course structure of the landscape still holds with the additional characteristic that the holes actually contain a large number of configurations. Since to escape from a hole (or metabasin) the system has to climb up to a zero intensive threshold energy, as in the REM, an effective description in terms of trap dynamics should hold if the motion at $E > E_{\mathrm{th}}$ is rapid and decorrelates quickly as it happens in the REM. The difference with the REM is that one needs to coarse grain the energy landscape: The counterpart of configurations of the trap model are metabasins in the CREM. As a check of Trap dynamics, we have studied the energy as a function of time for quenches at different temperatures. The resulting curves are given in Fig. 2 for $\alpha = 1$ and agree with the behavior $E(t) \simeq -T \log(t)$. The linear logarithmic decrease combined with the prefactor equal to the temperature are a strong indication that a description in terms of Trap dynamics holds [10,18,24], as recalled in the Introduction (the final plateau for $T = 0.75$ is a finite-size effect due to the fact that the system is small enough to eventually equilibrate). We did not study aging functions or distributions of trapping times since these observables present strong finite-size and finite-time effects, which makes the comparison with the trap model problematic [42] even in the REM where it is rigorously known that trap predictions hold [17,18].

IV. DISCUSSION AND CONCLUSION

A central question in the study of glasses is the nature of activated dynamics. The dynamics of mean-field models on time scales diverging with N provides a useful and interesting paradigm. Yet, although some results are known [7,8,43], the behavior of simple models such as the one with p-spin interactions or variants of it has not been fully elucidated.
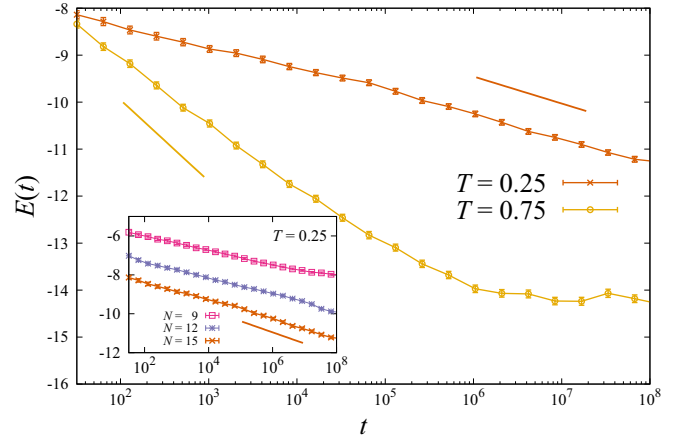

FIG. 2. Energy as a function of time in the CREM in the aging regime induced by a quench from $T = \infty$ to $T = 0.25$ (and 0.75) for $\alpha = 1$, $N = 15$. Curves are averaged along trajectories of 3×10^8 time steps on 2250 (and 900) different instances of the disorder. The straight lines correspond to a slope $-T$. Especially at high temperatures, where the equilibrium energy $\langle E(T) \rangle$ is higher, finite-size effects on the slope are expected [18] due to the convergence to $\langle E(T) \rangle$. The inset portrays the finite-size dependence of $E(t)$ for $T = 0.25$.

Only the dynamics of the REM was completely worked out. The new model we introduced and studied in this work, the correlated REM, provides a way to bridge the gap between these systems. We studied it in an intermediate regime between REM and p spin and found that its energy landscape and its dynamics are traplike provided that one identifies metabasins with single configurations of the trap model. The next crucial step is therefore to understand to what extent trap dynamics hold in the p-spin case, too. A likely possibility is that during the activated dynamics the system does not have to climb up to E_{th}, i.e., that a more complex structure of activated paths arises in the p-spin case. Nevertheless, some features of TM-like dynamics, such as partial equilibration at all energies above the one reached at time t during aging [24], could still effectively hold. Another aspect worth future analysis is the effective temperature description of the aging dynamics [44], which has been found to hold also in the activated regime of mean-field systems [45] but it is known to be violated in trap models [46,47]. To address these and other questions, further studies of the CREM dynamics can provide very valuable insights.

APPENDIX A: GENERATING THE ENERGIES OF THE CREM IN FOURIER SPACE

In this section we show how to generate the energy levels of the CREM by passing through Fourier space, where the energy levels are independent.

The overlap between two configurations $\vec{x}$ and $\vec{y}$ is $q(\vec{x}, \vec{y}) = \frac{1}{N} \sum_i^N s_i^{(x)} s_i^{(y)}$, where $s_i^{(x)} = \pm 1$ are the spins of configuration $\vec{x}$. The spins can be rewritten as $s_i^{(x)} = (-1)^{x_i}$, with $x_i = 0, 1$. This way, the overlap becomes $q(\vec{x}, \vec{y}) = \frac{1}{N} \sum_i^N (-1)^{x_i + y_i}$. The x_i $(i = 1, \dots, N)$ are coordinates of a hypercube of size $L = 2$, with periodic boundary conditions, so $-x_i = x_i$. As a consequence, the overlap can be written as a difference of the degrees of freedom,

$$q(\vec{x}, \vec{y}) = \frac{1}{N} \sum_i^N (-1)^{x_i - y_i} = q(\vec{x} - \vec{y}) \equiv q(\vec{z}), \quad \text{(A1)}$$

where we defined $\vec{z} = \vec{x} - \vec{y}$.

We now want to show that random energies $E(\vec{x})$, with the correlations defined by Eq. 3, are independent in Fourier space and calculate their wave-vector-dependent variance $\overline{E(\vec{k})E(-\vec{k})}$. Therefore, to generate the full set of energies of a sample, it is enough to generate the $E(\vec{k})$ as independent Gaussian random variables with variance $\overline{E(\vec{k})E(-\vec{k})}$ and then take the antitransform to have the energies in real space,

$$E(\vec{x}) = \frac{1}{2^N} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} E(\vec{k}), \quad \text{(A2)}$$

where the wave vectors take the form $\vec{k} = \frac{2\pi}{L}(n_1, \dots, n_N)$, and $n_i = 0, 1$, so, since $L = 2$, the antitransform can be simplified to $E(\vec{x}) = 2^{-N} \sum_{\vec{k}} (-1)^{\vec{n}\cdot\vec{x}} E(\vec{k})$.

Since the discrete Fourier transform of the energies reads

$$E(\vec{k}) = \sum_{\vec{x}} e^{-i\vec{k}\cdot\vec{x}} E(\vec{x}), \quad \text{(A3)}$$

the energy correlation matrix in Fourier space is

$$\overline{E(\vec{k})E(\vec{k}')} = \sum_{\vec{x}, \vec{y}} e^{-i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} \overline{E(\vec{x})E(\vec{y})}, \quad \text{(A4)}$$

$$= N \sum_{\vec{x}, \vec{y}} e^{-i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{y}} q(\vec{x} - \vec{y})^{\alpha N}. \quad \text{(A5)}$$

We can now define $\vec{u} = (\vec{x} + \vec{y})/2$ and $\vec{v} = (\vec{x} - \vec{y})/2$, so that

$$= N 2^N \left[2^{-N} \sum_{\vec{v}} e^{i\vec{v}\cdot(\vec{k}' - \vec{k})} \right] \sum_{\vec{u}} e^{-i(\vec{k}' + \vec{k})\cdot\vec{u}} q(2u)^{\alpha N}, \quad \text{(A6)}$$

where the term in square brackets is a representation of the Kronecker delta, $\delta_{\vec{k}\vec{k}'}$, so there is no correlation for any $\vec{k} \neq \vec{k}'$.

Consequently, the variance can be written as

$$\overline{E(\vec{k})E(-\vec{k})} = N 2^N \sum_{\vec{z}} e^{-i\vec{z}\cdot\vec{k}} q(\vec{z})^{\alpha N}, \quad \text{(A7)}$$

which can be simplified to a form that is easily implemented numerically,

$$\overline{E(\vec{n})E(-\vec{n})} = N 2^N \sum_{\vec{z}} (-1)^{\sum_i z_i n_i} \left[\frac{1}{N} \sum_{i=1}^N (-1)^{z_i} \right]^{\alpha N}. \quad \text{(A8)}$$

APPENDIX B: DETAILS OF THE CALCULATIONS

In this section we calculate explicitly some of the relations written in the main paper.

1. Order of a configuration

The order of a configuration $\vec{x}$ is the number of neighboring configurations with energy $E^{(\text{neigh})} < E$. Since each configuration has N neighbors, the probability of a configuration of energy E having order k reduces to the probability that it have exactly k neighbors with lower energy, and $N - k$ neighbors with $E^{(\text{neigh})} > E$.

This amounts to taking the joint distribution of $N + 1$ energies: the one of state $\vec{x}$, and its N neighbors. Calling $(E_0, \dots, E_N)$ the vector representing these $N + 1$ energies, the joint distribution can be written as

$$[(E_0, \dots, E_N)]$$

$$= \sqrt{\frac{\det(\mathcal{Q})}{(2\pi)^{N+1}}} \exp \left[(E_0, \dots, E_N) \mathcal{Q}^{-1} \begin{pmatrix} E_0 \\ \cdot \\ \cdot \\ \cdot \\ E_N \end{pmatrix} \right]. \quad \text{(B1)}$$

$\mathcal{Q}$ has a diagonal band structure, and its inverse $\mathcal{Q}^{-1}$ is tridiagonal [48]. As a consequence, its only nonzero nondiagonal elements are those relating each neighbor to $\vec{x}$. This means that once the energy $E \equiv E_0$ of the configuration $\vec{x}$ is fixed, all the neighbors are mutually independent.

Therefore, the probability of a state being of order k takes the binomial form

$$P_k(E) = \binom{N}{k} P[E^{(\text{neigh})} > E]^{N-k} P[E^{(\text{neigh})} < E]^k. \quad \text{(B2)}$$

As shown in the main Eq. (5), given a configuration $\vec{x}$ with energy E_x, the conditional probability of a configuration $\vec{y}$ at distance r from it is

$$P(E_y|E_x) = \frac{1}{\sqrt{2\pi N(1 - \rho^{2r})}} e^{-\frac{(E_y - \rho^r E_x)^2}{2N(1 - \rho^{2r})}}. \quad \text{(B3)}$$

Through Eq. (B3), Eq. (B2) can be rewritten as

$$P_k(E) = \binom{N}{k} \left[\frac{1}{\sqrt{2\pi N(1 - \rho^2)}} \int_E^\infty e^{-\frac{(E' - \rho E)^2}{2N(1 - \rho^2)}} dE' \right]^{N-k}$$

$$\times \left[\frac{1}{\sqrt{2\pi N(1 - \rho^2)}} \int_{-\infty}^E e^{-\frac{(E' - \rho E)^2}{2N(1 - \rho^2)}} dE' \right]^k. \quad \text{(B4)}$$

Through a variable change, the first integral can be rewritten as $\frac{1}{2}\,\mathrm{erfc}[B(E)]$, with

$$B = -E\sqrt{\frac{(1-\rho)}{2N(1+\rho)}},\tag{B5}$$

whereas the second one is equal to $\frac{1}{2}\,\mathrm{erfc}[-B(E)] = 1 - \frac{1}{2}\,\mathrm{erfc}[B(E)]$. Consequently,

$$P_k(E) = \binom{N}{k}\left\{\frac{\mathrm{erfc}\,[B(E)]}{2}\right\}^k\left\{1 - \frac{\mathrm{erfc}\,[B(E)]}{2}\right\}^{N-k}.\tag{B6}$$

2. Complexity

We now calculate the intensive (i.e., for $E \sim -N$) complexity, which is defined as

$$\Sigma_k = \log[2^N P(E) P_k(E)],\tag{B7}$$

where $P(E\frac{1}{\sqrt{2\pi N}}e^{-\frac{E^2}{2N}}$ is the distribution of the energies defining the model.

We are interested in finite k, with diverging N. In this limit, the binomial coefficient reduces to N^k. Also, since we focus on intensive energies, the term $B(E)$ in Eq. (B6) is large, and one can make an asymptotic expansion of the complementary error function. To first order, for large x, $\mathrm{erfc}(x) \simeq \frac{e^{-x^2}}{\sqrt{\pi}x}$, so,

keeping only the dominant order, one has

$$\Sigma_k(E) = N\log(2) - \frac{E^2}{2N} - kB^2,\tag{B8}$$

$$= N\log(2) - \frac{E^2}{2N}\left(1 - k\frac{1-\rho}{1+\rho}\right).\tag{B9}$$

3. Threshold energy

A configuration at E_th typically has its lowest neighbor at its same energy:

$$E_\mathrm{min}^{(\mathrm{neigh})}(E) = E_\mathrm{th}.\tag{B10}$$

In fact, this means that if $E < E_\mathrm{th}$, a configuration is typically a minimum, whereas for $E > E_\mathrm{th}$ it typically is not.

As shown in Eq. (B3), given a configuration at energy E, its neighbors' energies follow a Gaussian distribution centered in ρE, with variance $\sigma^2 = N(1-\rho^2)$.

The typical minimum of N Gaussians of variance σ^2 and mean μ is positioned at $\mu - \sqrt{2\sigma^2\log(N)}$ (finite-size corrections to this result are of order $\log[\log(N)]$, see, e.g., Ref. [49]). Therefore, the lowest neighbor of a configuration with energy E has energy

$$E_\mathrm{min}^{(\mathrm{neigh})}(E) = \rho E - \sqrt{2(1-\rho^2)N\log(N)}.\tag{B11}$$

By solving condition (B10) for E_th, one obtains

$$E_\mathrm{th} = -\sqrt{2\frac{1+\rho}{1-\rho}N\log(N)}.\tag{B12}$$

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