Scaling properties of monolayer graphene away from the Dirac point

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The statistical properties of the carrier density profile of graphene in the ground state in the presence of particle-particle interaction and random charged impurity in zero gate voltage has been recently obtained by Najafi *et al.* [Phys. Rev. E **95**, 032112 (2017)]. The nonzero chemical potential (μ) in gated graphene has nontrivial effects on electron-hole puddles, since it generates mass in the Dirac action and destroys the scaling behaviors of the effective Thomas-Fermi-Dirac theory. We provide detailed analysis on the resulting spatially inhomogeneous system in the framework of the Thomas-Fermi-Dirac theory for the Gaussian (white noise) disorder potential. We show that the chemical potential in this system as a random surface destroys the self-similarity, and also the charge field is non-Gaussian. We find that the two-body correlation functions are factorized to two terms: a pure function of the chemical potential and a pure function of the distance. The spatial dependence of these correlation functions is double logarithmic, e.g., the two-point density correlation behaves like $D_2(r,\mu) \propto \mu^2 \exp[-(-a_D \ln \ln r^{\beta_D})^{\alpha_D}] (\alpha_D = 1.82, \beta_D = 0.263, and a_D = 0.955)$. The Fourier power spectrum function also behaves like $\ln[S(q)] = -\beta_S^{-a_S}(\ln q)^{a_S} + 2 \ln \mu (a_S = 3.0 \pm 0.1 and \beta_S = 2.08 \pm 0.03)$ in contrast to the ordinary Gaussian rough surfaces for which $a_S = 1$ and $\beta_S = \frac{1}{2}(1 + \alpha)^{-1}$ (α being the roughness exponent). The geometrical properties are, however, similar to the ungated ($\mu = 0$) case, with the exponents that are reported in the text.

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I. INTRODUCTION

The transport parameters of the monolayer graphene scale with the average density of carriers which itself is tuned by the gate potential [1]. By tuning the chemical potential, a graphene system meets various phases which are characterized by the carrier density, such as the electron-hole puddles (EHPs) which arise in low densities that is the phase with some strong density inhomogeneities with fluctuations much larger than the average density. The explanation of this phase needs the consideration of the joint effect of extrinsic disorder and interaction [1-3], and needs a method that treats these two in a same footing [1,4]. EHPs are believed to be responsible for the observed minimum conductivity of graphene [5], i.e., in low densities (around the Dirac point) a complex network of small random puddles with semimetal character forms, through which the electrons hop and percolate and contribute to the conductivity. The configuration of these puddles depends on the chemical potential (which determines the system density) and the (extrinsic) charged impurity configuration in the sample. It has been proposed that such an inhomogeneity dominates the graphene physics at low ($\leq 10^{12}$ cm⁻²) carrier densities [5] for which the self-consistent Thomas-Fermi-Dirac (TFD) theory was employed to simulate the graphene charge profile on the SiO₂ substrate. The examples that experimentally confirmed EHPs are [4,6–15].

Many other studies on the unusual properties of graphene are still based on idealized models which neglect the effect of disorder and particle-particle interactions. The Thomas-Fermi-Dirac (TFD) theory as a simplified model which treats interaction and disorder in the same footing can be very helpful in such phenomena which need some statistical analysis. This theory has been properly developed for the graphene systems in the low density limit [5,16,17]. Using this theory and the Schramm-Lowewner evolution technique it has been proposed that the isocharge lines of EHPs at the Dirac point are conformal invariant and some critical exponents were obtained [16]. This, along with the observation of Herrmann according to whom the contour lines in the graphene membranes are conformal invariant [18], shows that the Dirac point is very special.

A substantial feature of the experiments on graphene near the Dirac point is the formation of large (spanning) clusters of negative or positive charge densities. The presence of the spanning cluster in a system may be the fingerprint of the scale invariance and the self-affinity of the system. This leads to some scaling behaviors which are expected to present in the scale-free systems [19–22]. The presence of carrier charge self-similarity is an important question in graphene. At the zero chemical potential, it has been shown that the graphene is very different from the ordinary 2D electron gas, for which the charge fluctuations are maximal and some critical exponents are obtained [16,23]. By analyzing the contour lines of the system at the Dirac point it has been shown that the fractal dimension of the corresponding random loops is $D_f(\mu = 0) =$ 1.38 ± 0.02 . In all of these studies the graphene system has been mapped to the random rough surfaces (which are scale

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invariant in the zero chemical potential), in which the charge density of the graphene sheet plays the role of the height of the rough surface and the contours play the role of the isoheight lines.

There is increasing numerical and experimental evidence that the isoheight lines in the random fluctuating fields in (2+1)-rough surfaces are scale invariant. The size distribution in these systems is characterized by a few scaling functions and scaling exponents [24]. The graphene system, as a 2 + 1random system, can be mapped to the rough surfaces [16,23]. By analyzing the contour lines of the electron-hole density, the authors showed that ungated graphene ($\mu = 0$) is a non-Gaussian self-similar system. In the present paper we test numerically the Thomas-Fermi-Dirac theory for the finite- μ graphene system and investigate the local and geometrical properties of the random electronic charge. The finite chemical potential breaks apparently the scale invariance of the system. We show that the distribution functions of electrons and holes become different for finite μ s and the two-point functions are factorized to two different pure functions: one a function of μ and the other a function of the spatial scale. The numerical fits of these functions show that their dependence is double logarithmic with the distance. We also investigate the geometrical properties of the system and show that when the surface is cut in the vicinity of the mean density, the geometrical properties of the system are similar to the ungated graphene.

The paper is organized as follows. In the next section we will introduce the TFD model for the graphene. In Sec. III we will fix the notation and introduce different scaling behaviors and scaling exponents corresponding to the contour loop ensembles (CLE). In Sec. IV we will numerically measure the proposed scaling exponents for the disorder potential and the carrier density in graphene. In the final section, we summarize the obtained results and our conclusions.

II. GROUND STATE OF GRAPHENE

The carrier density in graphene \bar{n} is controlled by the gate voltage V_g , i.e., $\bar{n} = \kappa_S V_g / 4\pi t$, in which κ_S is the substrate dielectric constant and t is its thickness. Additionally, the transport functions of graphene (like the conductivity) depend on $x \equiv \bar{n}/n_i$ in which n_i is the impurity density. In ordinary densities, the conductivity is a linear function of x and for very low x's it reaches a minimum of order $\sigma \sim e^2/h$, which is linked to the formation of EHP's. The pattern of charge density of this system has been extensively investigated for mono- and bilayer graphene [4,10,12] (close to the Dirac point) which reveals the formation of EHPs. These experiments support the theoretical predictions of Adam *et al.* [25] and Hwang *et al.* of large scale electron density fluctuations [1].

The linear dependence of the conductivity on carrier density in graphene sheets [1,26] and also the fact that the spatial pattern of EHPs is not correlated with the topography of the graphene sheets [27] are two important evidences that the remote Coulomb impurities are the dominant disorder source in most graphene samples. The disorder in addition to being the main source of the scattering has an additional effect: it locally shifts the Dirac point. This means that, even at the zero gate voltage, the Fermi energy is moved to the positive or the negative values with respect to the charge neutrality (Dirac) point. This causes some valleys and mountains of density throughout the sample, which are investigated in this paper for various rates of chemical potentials.

The case of relevance in the present paper is a slow (spatial) varying charge density system in which the use of the Thomas-Fermi-Dirac theory is permissible, i.e., $|\nabla_r n(\mathbf{r})/n(\mathbf{r})| \ll k_F(\mathbf{r})$ in which $k_F(\mathbf{r})$ is the Fermi wave number at position \mathbf{r} . The form of the energy functional of the graphene in this limit and for low densities has been obtained [17]. By minimizing the energy functional of the system in the local density approximation one yields the following equation for the local density of the system [28]:

$$\operatorname{sgn}(n)\sqrt{|\pi n|} + \frac{r_s}{2} \int d^2 \mathbf{r}' \frac{n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + r_s V_{xc}[n] + r_s V_D(\mathbf{r}) - \frac{\mu}{\hbar v_F} = 0, \qquad (1)$$

which should be solved self-consistently. In this equation v_F is the Fermi velocity, $r_s \equiv e^2/\hbar v_F \kappa_S$ is the dimensionless interaction coupling constant, μ is the chemical potential, and $g = g_s g_v = 4$ is the total spin and valley degeneracy. The exchange-correlation potential is [17]

$$V_{xc} = \frac{1}{4} [1 - gr_s \zeta(gr_s)] \operatorname{sgn}(n) \sqrt{\pi |n|} \ln\left(\frac{4q_c}{\sqrt{4\pi |n|}}\right) \quad (2)$$

in which q_c is the momentum cutoff and $\zeta(y) = \frac{1}{2} \int_0^\infty \frac{dx}{(1+x^2)^2(\sqrt{1+x^2}+\pi y/8)}$. For the graphene on the SiO₂ substrate, $\kappa_S \simeq 2.5$, so that $r_s \simeq 0.8$, $d \simeq 1$ nm, and $q_c = 1/a_0$, where a_0 is the graphene lattice constant $a_0 \simeq 0.246$ nm corresponding to the energy cutoff $E_c \simeq 3$ eV.

For zero chemical potential $\mu = 0$, the charged impurities are not screened in graphene [17]. For nonzero μ s, however, the screening effects become important and cannot be neglected. It has been shown that in the $q \rightarrow 0$ limit the Thomas-Fermi dielectric function becomes $\epsilon_{\text{TF}} \equiv \epsilon_{\text{RPA}}(q \rightarrow 0) = 1 + \frac{q_{\text{TF}}}{q}$ in which $q_{\text{TF}} = g_s g_v e^2 / \kappa_S v_F$ [29]. Within this approximation, the potential of a charged impurity located at a distance *d* from the substrate is $\tilde{v}(q) = \frac{2\pi e^2}{\kappa} \frac{e^{-qd}}{q+q_{\text{TF}}}$, which gives rise to the following form for the real space [30]:

$$V(\mathbf{r}) = \frac{e^2}{\kappa_s} \frac{e^{-q_{\rm TF}r}}{\sqrt{r^2 + d^2}}$$
(3)

in which $r \equiv |\mathbf{r}|$. Therefore, the potential in Eq. (1) is of the Yukawa type in the finite μ s. Therefore, the remote Coulomb disorder potential is calculated by the relation

$$V_D(r) = \int d^2 r' \frac{\rho(\mathbf{r}') e^{-q_{\rm TF}|\mathbf{r}-\mathbf{r}'|}}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2 + d^2}}$$
(4)

in which $\rho(r)$ is the charged impurity density and *d* is the distance between substrate and the graphene sheet.

In this paper we consider the disorder to be white noise with Gaussian distribution $\langle \rho(\mathbf{r}) \rangle = 0$ and $\langle \rho(\mathbf{r}) \rho(\mathbf{r}') \rangle =$ $(n_i d)^2 \delta^2(\mathbf{r} - \mathbf{r}')$. It is notable that, in the $\mu = 0$ limit, due to the pure 1/r dependence of the Hartree and disorder terms, the convergence of the equation is slow. Let us first concentrate on the scaling properties of this equation for $\mu = 0$ (for which $q_{\text{TF}} \rightarrow 0$) and ignore V_{xc} . By applying the scale transformation $\mathbf{r} \rightarrow \lambda \mathbf{r}$, we see that the equation remains unchanged if we transform $n(\mathbf{r}) \rightarrow n(\lambda \mathbf{r}) = \lambda^{-2} n(\mathbf{r})$, which is compatible with the spatial dimension of $n(\mathbf{r})$ [note that $V_D V_D(\lambda \mathbf{r}) =$ $\lambda^{-1}V_D(\mathbf{r})$]. This guarantees the self-affinity of the graphene when viewed as a random rough surface. This scale invariance in two dimensions leads to power-law behaviors with some critical exponents [24], and may also lead to conformal invariance of the system. V_{xc} breaks this scale invariance, since it transforms as $V_{xc}(\mathbf{r}) \rightarrow V_{xc}(\lambda \mathbf{r}) = \lambda^{-1}(V_{xc} - \beta \operatorname{sgn}(n)\sqrt{\pi |n|} \ln \lambda)$ in which $\beta \equiv \frac{1}{4} [1 - gr_s \zeta(gr_s)]$. Therefore, after applying the scale transformation, one reaches the same equation as Eq. (1)with logarithmically corrected coefficients [16]. Therefore, the scale invariance survive marginally in the infrared limit even in the presence of V_{xc} . This symmetry implies that the correlation functions show power-law behaviors, but further details of the system need exact or numerical solutions. The nonzero μ s break apparently the scale invariance of the system. It has two main effects: it controls \bar{n} and the density fluctuations, and also it breaks the scale invariance. This breaking is along with some scaling functions which are studied in Sec. IV.

III. GRAPHENE AS A ROUGH SURFACE, SCALING PROPERTIES OF CLE

In the above arguments we mentioned that n(x,y) is the important field in our analysis by which the energy of the system is obtained. Characterizing this field is very important in distinguishing the local phases and the phase separation pattern. To this end we analyze the local and global properties of the system. The first analysis on the contour loop ensemble (CLE) of this system has been done in [16] in which Eq. (1)was solved for $\mu = 0$. For the scale-invariant rough surfaces the height field (here the density) behaves like $n(\lambda \mathbf{r}) \stackrel{d}{=} \lambda^{\alpha} n(\mathbf{r})$ in which α is the *roughness* exponent or the *Hurst* exponent and λ is a scaling factor and the symbol $\stackrel{d}{=}$ means the equality of the distributions. For self-similar surfaces ($\mu = 0$ in our case) the density-density correlation function $D_2(r) \equiv$ $\langle [n(\mathbf{r} + \mathbf{r_0}) - n(\mathbf{r_0})]^2 \rangle$ behaves like $|\mathbf{r}|^{2\alpha_l}$ and also the total variance $W(L) \equiv \langle [n(\mathbf{r}) - \bar{n}]^2 \rangle_L$ behaves like $L^{2\alpha_g}$, where the parameter α_l (α_g) is called the local (global) roughness exponent, $\bar{n} = \langle n(\mathbf{r}) \rangle_L$; $\langle \dots \rangle_L$ means that the average is taken over \mathbf{r} in a box of size L. Self-affine surfaces are monofractals just if $\alpha_g = \alpha_l = \alpha$ [31]. For these systems the Fourier power spectrum [the second moment of n_q , \equiv the Fourier component of $n(\mathbf{r})$ scales like

$$S_{\mathbf{q}} \equiv \langle |n_{\mathbf{q}}|^2 \rangle_{\mu=0} \sim |\mathbf{q}|^{-2(1+\alpha)}$$
(5)

and also the distribution function of the density P(n) is of the Gaussian form $\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{n^2}{2\sigma^2}}$, where σ is the standard deviation. It has been shown that $\alpha_l^{\mu=0} = 0.35 \pm 0.03$ and $\alpha_g^{\mu=0} = 0.38 \pm 0.03$ [16]. Another quantity whose moment distribution should be Gaussian is the local curvature which is defined (at position **r** and at scale *b*) as $C_b(\mathbf{r}) = \sum_{m=1}^{M} [n(\mathbf{r} + b\mathbf{e}_m) - n(\mathbf{r})]$, in which the offset directions $\{\mathbf{e}_1, \dots, \mathbf{e}_M\}$ are a fixed set of vectors whose sum is zero, i.e., $\sum_{m=1}^{M} \mathbf{e}_m = 0$. If the rough surface is Gaussian and the first and all odd moments of C_b manifestly vanish since the random field has up or down

symmetry $n(\mathbf{r}) \longleftrightarrow -n(\mathbf{r})$. Additionally, for Gaussian random fields we have $\frac{\langle C_b^{\lambda} \rangle}{\langle C_b^{2} \rangle^2} = 3$. The self-similarity can also be addressed in geometrical

The self-similarity can also be addressed in geometrical quantities like the fractal dimension of loops and also the exponents of the distribution function of gyration radius and loop length. One can extract the loops from the isodensity (contour) lines of the profile $n(\mathbf{r})$ at the level set $n(\mathbf{r}) = n_0$ from which some nonintersecting loops result which come in many shapes and sizes. For scale-invariant random surfaces these geometrical objects show various power-law behaviors. The exponent of the distribution functions of loop lengths l [P(l)] and the gyration radius of loops r [P(r)] are of special importance. The fractal dimension of loops for ungated graphene has numerically determined $D_f^{\mu=0} = 1.38 \pm 0.02$ and the exponent of the distribution for loop length $\tau_l^{\mu=0} = 2.30 \pm 0.02$ [16].

As stated in the Introduction for nonzero μ s the scale invariance is broken and n(x, y) (as a random field) is not self-similar. The characterization of this symmetry-breaking is important in determining the system transport, e.g., the density-density correlation function which is important in determining the dielectric function. In the next section we address the deformation of various functions in the nonzero chemical potential limit.

IV. NUMERICAL RESULTS AND DISCUSSION

In this section we present the numerical results. We have simulated the model on the L = 200 square lattice. We have also examined the code on the systems with lower sizes to control the finite size effects, i.e., L = 50 and L = 100, and extracted some exponents. Just like Ref. [16], the results do not change with system size and we have put the results of the largest size in the paper, i.e., L = 200. Finding a stable solution of Eq. (1) took about 20 min for a 3.2 GHz CPU. Therefore, it took about two months (per CPU) to generate 4×10^3 independent samples for each μ and system size. This section has been divided into three subsections. In the next subsection we present the results for the local quantities, i.e., the two-body (density-density) correlation function $D_2(r,\mu)$, the total variance $W_L(L,\mu)$, and the Fourier power spectrum S_a . In Sec. IV B the higher moments are analyzed. The fractal dimension of loops and the distribution functions of the geometrical observables are calculated in Sec. IVC.

A. Local exponents

The chemical potential tunes the average density. When $\mu = 0$ this average is zero, showing that we are right at the Dirac point. For nonzero values, the Fermi surface moves above or under the Dirac point and the system acquires nonzero average density. This can be seen in Fig. 1 in which the distribution of *n* has been shown. It is known that this function is non-Gaussian for graphene [16] that is evident in this figure. This figure reveals that the logarithm of this function is linear in *n* with two nonequal (μ -dependent) slopes. For small μ values this function behaves like the following relation:

$$P_{\mu}(n) \propto \begin{cases} \exp\left[-a_{R}(n-n_{0}(\mu))\right], & n \ge n_{0}(\mu), \\ \exp\left[a_{L}(n-n_{0}(\mu))\right], & n < n_{0}(\mu), \end{cases}$$
(6)



FIG. 1. (a) Semilog plot of the distribution function of density P(n). Left inset: $\epsilon_{Pn}(\mu)$, which is defined as the (absolute value of the) difference between the slopes of two sides. Right inset: the position of the peak of the distribution function $n_0(\mu)$. (b) The distribution function of the roughness function $P(W_2)$ for the boxes of size l = 100. Inset: the position of the peak in terms of μ .

in which a_R and a_L are the mentioned slopes (that are equal only for $\mu = 0$) and $n_0(\mu)$ is the density at which the distribution function shows a peak $[n_0(\mu = 0) = 0]$ and the function becomes singular. This function differs from the result of Ref. [16]:

$$P_n = A \exp\left[-\zeta \left(\operatorname{sgn}(n)\sqrt{\pi |n|} - \frac{\mu}{\hbar S v_F}\right)\right],\tag{7}$$

in which $\zeta \equiv \frac{4 \ln(L/a)}{dn_i^2 r_s} \bar{n}$, A is a normalization constant, and S is the area of the sample. This equation has been calculated for low densities and low interactions and disorder strengths. In fact the form (6) is not true for very low densities, and log P(n) varies with \sqrt{n} in accordance with Eq. (7). We find from this figure that the asymmetry parameter $\epsilon_{Pn}(\mu) \equiv$ $|a_L(\mu) - a_R(\mu)|$ increases with μ in a power-law fashion $\epsilon_{Pn} \sim \mu^2$, and $n_0(\mu)$ behaves like $\mu \ln \mu$ as depicted in Fig. 1 (the left and right panels, respectively). The asymmetry of $P_{\mu}(n)$ shows that the dynamics of electrons and holes in gated graphene is not the same as expected. Also it is seen that the graphs become wider for larger μ values showing that the density fluctuations (which are proportional to the system compressibility κ_{μ}) increase by increasing μ . Figure 1(b) shows the distribution of the total variance of density $P(W_2)$ (or roughness function in the rough surface language) for boxes with linear size l = 100. We see that the peak of the distribution (which is here equal to the average value \overline{W}_2) scales with the square of the chemical potential, i.e., $\bar{W}_2 \sim \mu^2$. This reveals that the density fluctuations increase with μ is compatible with the widening of P(n) shown above. To see this, we have plotted $(\delta n)^2 \equiv \langle n^2 \rangle - \langle n \rangle^2$ in the left inset in this figure. It is evident that \overline{W}_2 and $(\delta n)^2$ have the same behaviors. The widening of the distribution function can be inferred from Eq. (7) in which large μ value is compensated by large density fluctuations.

Now let us consider the two point correlation functions which have been analyzed in Fig. 2, i.e., $D_2(r,\mu)$, $W_2(L,\mu)$, and $S_q(\mu)$. Interestingly, we have observed that all multipoint functions considered in this paper are factorized to two parts: one part depending only on μ and the other a pure function of the other variable, e.g., $D_2(r,\mu) = \frac{f_D(\mu)}{\lambda(r)}$ [see the right hand inset of Fig. 2(a)]. For this function, our analysis reveals that $f(\mu) \sim \mu^2$ and $\lambda(r)$ is best fitted by [the left inset of Fig. 2(a)]

$$\lambda(r) = \exp[(-a_D \ln \ln r^{\beta_D})^{\alpha_D}] \tag{8}$$

in which $\alpha_D = 1.82 \pm 0.01$, $\beta_D = 0.26 \pm 0.01$, and $a_D = 0.96 \pm 0.01$. This function along with the relevant quantities have been sketched in Fig. 2(b) in terms of μ . The quadratic form of $f(\mu)$ has been shown in the insets. Therefore, the two point function D_2 is not power law in r; instead it behaves double logarithmically. This behavior is independent of μ , which appears as a multiplicative constant, i.e., $f(\mu)$.

The same behavior is seen for $W_2(L)$ [in which *L* is the box in which the variance (roughness) has been calculated; see Sec. III] as is evident in Figs. 2(c) (in terms of *L*) and 2(d) (in terms of μ). In these figures $\lambda(L)$ is best fitted by the same relation as Eq. (8), with the parameters $\alpha_W = 1.35 \pm 0.01$, $\beta_W = 0.26 \pm 0.01$, and $a_W = 1.375 \pm 0.005$. The fact that D_2 and W_2 show the same *r* and μ dependences is not surprising since the role of the spatial extent of the boxes in which the total variance is calculated is the same as the role of *r* in the densitydensity correlation function. The example is the equality of the global and local roughness exponents in the scale-invariant rough surfaces.

A completely different behavior is seen for S_q . Our observations reveal that this function shows the following form:

$$\ln S_q = -(\beta_S^{-1} \ln q)^{a_S} + \ln f_S(\mu)$$
(9)

in which $a_S = 3.0 \pm 0.1$ and $\beta_S = 2.1 \pm 0.03$. Equivalently one finds that $S_q = \frac{f_S(\mu)}{\lambda(q)}$ in which $\lambda(q) = \exp[(\ln(q)/\beta_S)^{a_S}]$. This relation shows that $\ln S_q$ for the graphene varies with the third power of $\ln q$. This should be compared with the same expression for the scale-invariant rough surface:

$$\ln S_q = -2(1+\alpha)\ln q + \text{const.} \tag{10}$$



FIG. 2. $D_2(r,\mu) = \frac{f_D(\mu)}{\lambda(r)}$ in terms of (a) r and (b) μ . The fitting is done for $\lambda(r) = \exp\left[(-a_D \ln \ln r^{\beta_D})^{\alpha_D}\right]$, with the parameters $\alpha_D = 1.82$, $\beta_D = 0.263, a_D = 0.955$, and $f_D(\mu) \sim \mu^2$. $W_2(r,\mu) = \frac{f_W(\mu)}{\lambda(L)}$ in terms of (c) L and (d) μ . The fitting is done for $\lambda(L) = \exp\left[(-a_W \ln \ln L^{\beta_W})^{\alpha_W}\right]$ with the parameters $\alpha_W = 1.35 \pm 0.01$, $\beta_W = 0.26 \pm 0.01$, $a_W = 1.375 \pm 0.005$, and $f_W(\mu) \sim \mu^2$. $S_q = \frac{f_q(\mu)}{\lambda(q)}$ in terms of (e) q and (f) μ with $\lambda(q) \exp\left[(\ln(q)/\beta_S)^{\alpha_S}\right]$, $a_S = 3.0 \pm 0.1$, $\beta_S = 2.1 \pm 0.03$, and $f_S(\mu) \sim \mu^2$.

Therefore, apart from the proportionality constant, the main difference of the S_q of graphene and the ordinary rough surfaces is that the logarithm of the former depends on the third power of ln q, whereas the latter is liner. The graphs for S_q have

been shown in Figs. 2(e) and 2(f). It is evident from Fig. 2(f) that $f_S(\mu) \sim \mu^2$ just like the functions D_2 and W_2 . This is not surprising, since S_q is related to the Fourier transformation of D_2 and consequently with the same μ dependence.



FIG. 3. $C_2(b,\mu)$ in terms of (a) *b* and (b) μ , for which the fitting is done for $D_2(b,\mu) = \frac{f_C(\mu)}{\lambda(b)}$. The fitting parameters are $\lambda(b) = \exp\left[(-a_C \ln \ln r^{\beta_C})^{\alpha_C}\right]$, $\alpha_C = 1.35 \pm 0.02$, $\beta_C = 0.25 \pm 0.01$, $a_C = 1.36 \pm 0.01$, and $f_C(\mu) \sim \mu^2$. The same analysis for $C_4(b,\mu)$ in terms of (c) *b* and (d) μ , with the parameters $\alpha_C = 1.85 \pm 0.02$, $\beta_C = 0.26 \pm 0.01$, $a_C = 1.86 \pm 0.01$, and $f_C(\mu) \sim \mu^3$.

The same features have been observed for $C_2(b)$ and $C_4(b)$ in which $C_n(b) \equiv \langle C_b^n \rangle$. These quantities have been shown in Fig. 3, in which α , β , and a have been reported for each case separately. The dependence on b is just like Eq. (8) (with rreplaced by b). Also the dependence of $f_C(\mu)$ is power law with the exponent 2 [see Fig. 3(b)]. This exponent is 3 for C_4 . This demonstrates that the distribution of C_2 and C_4 is not Gaussian and shows again that their dependence on b is double logarithmic.

B. Higher order moments, non-Gaussian surface

An important check for the systems which are mapped to the rough surfaces is the Gaussian and non-Gaussian behaviors. It is known that graphene is a non-Gaussian rough surface, even at the Dirac point [16]. However, the exact characterization of this non-Gaussian rough surface needs some critical investigation on other variables, like $C_2(b,\mu) = \langle C_b^2 \rangle$ [see the definition of $C_b(\mathbf{r})$ in Sec. III], and the higher moments of density, especially the odd powers like $D_3(r,\mu) = \langle [n(\mathbf{r}_0 + \mathbf{r})n(\mathbf{r}_0)]^3 \rangle$ and $C_3(b,\mu) = \langle (C_b)^3 \rangle$, etc., which are expected to be zero for

scale-invariant symmetric rough surfaces [24]. Figure 4 shows these functions.

We see from Fig. 4(a) that C_3 does not vanish, and increases with *b*. Also we know that $\frac{C_2^2}{C_4}$ should be constant (for all *b*'s equal to $\frac{1}{3}$) for a scale-invariant rough surface. Figure 4(b) shows that it is not the case for the gated graphene, and this function has a nontrivial increasing behavior in terms of *b*. All of these show that the system is non-Gaussian.

C. Geometrical quantities

The scale invariance of the critical systems dictates some scaling relations between the geometrical quantities of the system. The determination of the exponents of these global quantities, for example, the various fractal dimensions of the system, helps in determining their universality class of the model in hand. All of the analyses presented in the previous section are in terms of the local variable $n(\mathbf{r})$. There is, however, a nonlocal point of view in such problems, i.e., the isoheight (contour) lines of the profile $n(\mathbf{r})$ at the level set $n(\mathbf{r}) = n_0$, which also show the scaling properties. When we cut the



FIG. 4. Non-Gaussian parameters (a) $C_3(b,\mu)$ and (b) $[C_2(b,\mu)]^2/C_4(b,\mu)$. The latter changes with the second power of μ which confirms that the system is non-Gaussian.

self-affine surface n(x, y) some nonintersecting loops result which come in many shapes and sizes [24,32]. We choose 10 different n_0 between the maximum and the minimum densities from which a CLE is obtained. Some samples have been shown in Fig. 5 for $\mu = 10^{-3}$, 5×10^{-3} , 10^{-2} , 5×10^{-2} , 0.1, and 0.2. The different colors in each figure show connected clusters



FIG. 5. Charge pattern of the systems with chemical potentials: (a) $\mu = 0.001$, (b) $\mu = 0.005$, (c) $\mu = 0.01$, (d) $\mu = 0.05$, (e) $\mu = 0.1$, and (f) $\mu = 0.2$. The different colors show the connected components.



FIG. 6. Log-log plot of (a) *l*-*r* and (b) *S*-*r* with the exponents $\gamma_{lr} = 1.42 \pm 0.02$ and $\gamma_{Sr} = 1.8 \pm 0.1$. (c) The distribution function of the area inside the cluster *a* (inset: the distribution function of the loop length *l*). (d) The distribution function of the cluster mass *S* (upper inset: the distribution function of the mass gyration radius *R*; lower inset: the distribution function of the loop gyration radius *r*). The corresponding exponents are $\tau_a = 1.82 \pm 0.06$, $\tau_l = 2.28 \pm 0.07$, $\tau_s = 1.55 \pm 0.03$, $\tau_R = 1.90 \pm 0.06$, and $\tau_r = 2.43 \pm 0.05$.

each of which has its own gyration radius r, (exterior) loop length l, mass (S), and area inside (a) (this is the total area inside the loop). For the self-affine systems, these geometrical objects are scale invariant and show various power-law behaviors, e.g., their size distribution is characterized by a few power law relations and scaling exponents. These quantities (in the thermodynamic limit) scale with each other in the form $y \sim x^{\gamma_{xy}}$ and the distribution functions of them behave like $p(x) \sim x^{-\tau_x}$ in which x, y = l, r, S, a. The scaling theory of CLEs of self-affine Gaussian fields was introduced in Ref. [32] and developed in Refs. [24,33]. When a charge density pattern is obtained, we extract the contour lines by 10 different cuts with the same spacing between maximum and minimum values. The Hoshen-Kopelman [34] algorithm has been employed for identifying the clusters in the lattice. It is notable that for each L = 200 sample (for a given μ) $\sim 5 \times 10^2$ loops were obtained. Since we have generated 4×10^3 samples for each μ , about 2×10^6 loops have been generated for each μ .

We have calculated the fractal dimensions γ_{lr} and γ_{Sr} as seen in Figs. 6(a) and 6(b). Interestingly, we have observed that these exponents are μ independent. The numerical values

of these universal quantities are $\gamma_{lr} = 1.42 \pm 0.02$ and $\gamma_{Sr} =$ 1.80 ± 0.05 . The latter changes behavior (more precisely becomes nearly constant) for the scales larger than $r_0 \sim 30$, which is finite size effect. It is notable that for a space-filling cluster $\gamma_{Sr}^{\text{space filling}} = 2$. The difference between the obtained γ_{Sr} and $\gamma_{Sr}^{\text{space filling}}$ shows that there are some hallows (the regions with different densities) inside the clusters. The exponent γ_{lr} is not significantly different from the case $\mu = 0$ [16]. The exponents of the distribution function of the geometrical observables have also been presented in Figs. 6(c) and 6(d). We see that $\tau_a = 1.82 \pm 0.06$, $\tau_l = 2.28 \pm 0.07$, $\tau_S = 1.55 \pm$ 0.03, $\tau_R = 1.90 \pm 0.06$, and $\tau_r = 2.43 \pm 0.05$. These scaling behaviors for the geometrical quantities of the system, which is surely not scale invariant, are very interesting. For the critical systems, it is well known that $\gamma_{xy} = \frac{\tau_y - 1}{\tau_x - 1}$. The fact that this hyperscaling relation is not true for our system ($\gamma_{lr}^{\text{hyperscaling}} \equiv \frac{\tau_r - 1}{\tau_l - 1} = 1.12$ and $\gamma_{Sr}^{\text{hyperscaling}} \equiv \frac{\tau_r - 1}{\tau_s - 1} = 2.6$) is due to the fact that our system is not self-affine. It is notable that the obtained value for τ_l is compatible with the ungated graphene $(\mu = 0).$

V. CONCLUSION

In this paper we have considered the graphene out of (and in the vicinity of) the Dirac point, i.e., for finite chemical potentials. To this end we have employed the Thomas-Fermi-Dirac (TFD) theory and have solved it numerically for finite μ s. We have mapped the problem to a rough surface system and have calculated the relevant functions which are of the scaling form for the rough surfaces. Importantly, we have calculated multipoint charge correlation functions (D_n), the roughness function (W_n) (n = 2, 3, and 4), and also the Fourier power spectrum function (S_a).

The electron density is the solution of the nonlinear Eq. (1). We see that in the limit $V_{xc} = 0$ and $\mu = 0$ this equation leads to scale-invariant charge density, i.e., when $r \rightarrow \lambda r$, then $n(r) \rightarrow n(\lambda r) = \lambda^{-2}n(r)$. This scale invariance is marginally preserved also in the nonzero exchange-correlation potential V_{xc} , i.e., the following equation is obtained after implying the scale transformation ($\mu = 0$):

$$\xi(\lambda)\operatorname{sgn}(n)\sqrt{|\pi n|} + \frac{r_s}{2}\int d^2\mathbf{r}' \frac{n(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + r_s V_{xc}[n] + r_s V_D(\mathbf{r}) = 0, \qquad (11)$$

in which $\xi(\lambda) \equiv 1 - \beta r_s \ln \lambda$ [see Eq. (5) of Ref. [16]]. Therefore, in the limit $\mu = 0$, one expects scale invariance, as shown in that reference. The non-Gaussianity of n is attributed to the special nonlinear form of Eq. (1) in which the second root of *n* is proportional to the disorder potential V_D (as the noise). As shown in Ref. [16], the distribution of the solution of this equation is not Gaussian at all, and is of the form of Eq. (7). We observe that the non-Gaussianity has its roots in the special form of the master equation for the density of electrons in graphene. For the graphene system out of the Dirac point (when μ becomes nonzero), the impurity potential in not bare Coulomb potential, but instead is of the form of Eq. (3). In this limit the mentioned scale invariance breaks down by two effects: first due to the form of unbare Coulomb potential and second the direct effect of μ in Eq. (1). Therefore, no power-law behavior is expected in this limit.

Our main finding is that the functions are decomposable to two parts and the function of μ is factored out. D_n and W_n show double-logarithmic behaviors with r and L, whereas $\ln S_q$ has a linear behavior of the third power of $\ln q$. The μ functions behave in most cases with the second power of μ . Based on our observations, the nonaffinity is not a finite-size effect. For example, $\langle C_2^2 \rangle / \langle C_4 \rangle$ does not approach the value 1/3 (that is for a Gaussian random field) in any μ and any system size. Also $\langle C_b^3 \rangle$ does not approach zero in any limit or by enlarging the system size.

We have also analyzed the geometrical functions. To this end, we have cut the samples from some equalspacing planes from which nonintersecting stochastic curves and loops result. Each loop has its own length (l), gyration radius (r), cluster mass (S), and area inside (a). Our observations support that, despite the fact that the system is not self-affine, these quantities show critical (power-law) behaviors and the resulting critical exponents are μ independent. These exponents are compatible with the corresponding exponents for ungated graphene ($\mu = 0$), i.e., geometrical properties of the system are independent of μ and also the spatial scale $1/q_{\rm TF}$. Therefore, although μ changes the mean density \bar{n} , the geometrical properties of the system which are determined with respect to the new reference $\bar{n}(\mu)$ are robust against μ . It can be understood noting the fact that all functions of r (spatial distance) and μ are decomposable to two pure functions, leading to the fact that the spatial structure of the system is not affected by μ .

At the end we emphasize that the criticality in this system for $\mu = 0$ has its roots in the underlying randomness and also the (marginal) scale invariance of the master equation, i.e., Eq. (1). The scaling properties of the underlying external noise guarantee this scale invariance and the corresponding critical behaviors, i.e., for the same system with the other type of noise, this scale invariance may disappear. All disorders with the same scaling properties have the same effects and same critical properties. Therefore, the role of disorder is summarized in its scaling properties.

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