Magnetic energy transient growth in the subcritical Kazantsev model

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We study average magnetic field growth in a mirror-symmetrical Kazantsev turbulent flow near the dissipative scales. Our main attention is directed to a subcritical regime, when an exponential decrease of magnetic energy is usually expected. We show that instead of damping, transient energy growth can be obtained, for example, in stochastic processes supported by the large-scale magnetic fields. We calculate the longitudinal correlation functions and demonstrate that they can tend to nonzero stationary solutions, whose localization width is inversely proportional to the square of the magnetic Reynolds numbers and with amplitude depending on the closeness of these numbers to the critical value. We present the local generation effect without any external support, predicted by Zeldovich in 1956. Numerically solving the initial-boundary Kazantsev problem on the nonuniform grids, we simulate this process by implicit schemes and discuss the possible consequences of subcritical growth for dynamo theory.

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I. INTRODUCTION

The magnetospheres of planets and stars are believed to be formed by large-scale hydrodynamic processes, which transform the kinetic energy of turbulent flows into the energy of magnetic fields [1]. Such transformations are usually considered as threshold phenomena, in which the process is possible if the dynamo intensity is larger than some critical value [2–5], whereas in subcritical regimes this process decays similarly to nonuniform temperature fields. However, in 1956 Zeldovich made an important remark that there should be a fundamental difference between subcritical magnetic field processes and temperature perturbations [6]. He noticed that despite the maximum principle for parabolic equations that affirms the monotonic decreasing of maximum temperature [7], the analogous principle for the magnetic energy should be invalid, because of tangling of mixed magnetic lines. In other words, an increase of magnetic energy can be expected even in subcritical cases.

Such subcritical regimes have not been studied in detail by the astrophysical dynamo community, mentioned only in several books as a probable phenomenon (see, e.g., Refs. [1,8,9]) and investigated for some stationary flows (see Refs. [10,11]). A possible reason can be connected with the substantial supercriticality of dynamo regimes for most celestial bodies [12]. Nevertheless, this phenomenon seems deserving of attention, even if it is temporal, because nowadays this research is conducted in association with contemporary progress in dynamo experiments (see, e.g., Refs. [13–15]). Indeed, in the frame of laboratory physics the critical value of the generation regime is hard to reach: too large of velocities and scales are required, so mostly only the subcritical generation can be realized. Thus for practical applications in liquid metal mechanisms or magnetic storage devices any results about subcritical behavior of magnetic fields are surely essential and important.

The aim of this paper is to consider subcritical cases in the frame of the simplest small-scale dynamo model [16]. Assuming an isotropic reflectively invariant and instantaneously correlated velocity field of turbulent electrically conducting flow, suggested by Kazantsev in 1967 (see, e.g., Ref. [17]), we solve numerically the evolutional equation for the longitudinal correlation function $M(r,t)$ of a random magnetic field $\mathbf{B}(r,t)$:

$$\frac{\partial M}{\partial t} = \frac{2}{r^4}\frac{\partial}{\partial r}\left(r^4\eta\frac{\partial M}{\partial r}\right) + \frac{2M}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial \eta}{\partial r}\right). \quad (1)$$

The mirror symmetry and isotropy of this model allows us to write the magnetic correlation tensor in the form

$$\langle B^i(r)\, B^j(0)\rangle = \left(M + \frac{r}{2}\frac{\partial M}{\partial r}\right)\delta^{ij} - \frac{\partial M}{\partial r}\frac{r^i r^j}{2r}. \quad (2)$$

The particular solution of Eq. (1) is determined by the boundary conditions and by the magnetic Reynolds number Rm, included

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in the model through a function of magnetic viscosity

$$\eta(r) = \frac{1}{\text{Rm}} + \frac{F(0)}{3} - \frac{F(r)}{3}, \qquad (3)$$

where the longitudinal correlation function of the random velocity field $F(r)$ is assumed to be known. The problem is dimensionless, since the function $F(r)$ is localized on the scale $r \in [0,1]$ (see, e.g., Refs. [4,18–21]). For the delta-correlated in time velocity tensor $\sim\delta(t)$ both analytical and numerical investigations prove the existence of such a critical value $\text{Rm}_{cr} \sim 58$, from which the small-scale generation of the magnetic field begins. In other words, for supercritical regimes $\text{Rm} > \text{Rm}_{cr}$ the correlation function $M(r,t)$ exponentially increases, and for subcritical values $\text{Rm} < \text{Rm}_{cr}$ it exponentially decays. Note that this subcritical interval $\text{Rm} < 58$ includes the typical laboratory Reynolds numbers $\text{Rm} \sim 20$, so the subcritical conditions of the possible magnetic field growth do not seem unrealistic.

Previously for supercritical cases $\text{Rm} > \text{Rm}_{cr}$ it was natural to reduce the differential partial equation (1) to the eigenvalue problem [22], so the results obtained earlier usually describe only the exponential behavior and do not take into account the initial data. Moreover, the region was traditionally assumed unbounded with zero boundary conditions at $r \to \infty$, in other words without large-scale correlations (the main analytical and numerical results, estimates of small-scale dynamo rates, and analysis of symmetrical and asymmetrical cases for supercritical regimes can be found in, e.g., Refs. [4,9,19]). Here by numerical methods we solve the evolutional Kazantsev equation and study the dynamics of the function $M(r,t)$, developed from the nonzero initial distribution $M_0(r)$ or from the nonzero boundary conditions at large distances $r \gg 1$. For numerical realizations we use the ideas of the special substitution, suggested in Ref. [19], which transform Eq. (1) to a Schrödinger-like problem and then solve it on nonuniform grids by implicit schemes (see, e.g., Ref. [23]). We show that for the subcritical values $\text{Rm} < \text{Rm}_{cr}$ the initial large-scale correlations cannot lead to exponential decay, as was found without large-scale support, but to saturation up to the nonzero stationary solution. It will be better to connect this process with a transient growth, not with a subcritical dynamo effect, assuming that in subcritical regimes the growth reaches a saturated state due to some nonlinear effects. We show that the amplitude of such an ultimate solution strongly depends on both the activity of the external magnetic field and parameters of the velocity random field. We estimate the rate of approach to the stationary level, compare it with analytical results on dissipative $r \ll 1$ and energy $r \gg 1$ scales, and show that the situation when the subcritical longitudinal correlation function $M(r,t)$ temporally grows (even without permanent support at the large scales) is also possible.

Finally, note that we clearly recognize the delicate and controversial features of the Kazantsev model. In particular, the condition of short correlation times contradicts our understanding of different correlation intervals on different scales, so it can be used only as the first approximation (see, e.g., Ref. [24], and concerning the problem for flows with finite correlation times see Ref. [25]). Moreover, this model does not include the decay of magnetic field due to the boundary effects and assumes implicitly that the size of the computational region is much larger than the typical correlation length, because it is well known that a realistic dynamo operates in regions of finite size where dynamo action is impossible for 2D flow [9], whereas the isotropic and unbounded flows can lead to the generation (see, e.g., Ref. [26]). Fortunately, here the magnetic field growth is concentrated only in a limited range of spatial and timescales, so it seems reasonable to use this model for our purposes.

II. METHOD AND REGIMES

We further relate the average magnetic energy density with the longitudinal correlation function, rewriting the correlation tensor (2) as the scalar product

$$\langle \mathbf{B}(r,t)\,\mathbf{B}(0,t)\rangle = \frac{1}{r^2}\frac{\partial(r^3 M(r,t))}{\partial r}.$$

To find $M(r,t)$ we use the replacement proposed for the first time in Ref. [19]:

$$M(r,t) = \frac{\phi(r,t)}{r^2\eta^{1/2}}. \qquad (4)$$

This substitution (4) allows us to use the zero boundary condition $\phi(0,t) = 0$ at the point $r = 0$, assuming that the correlation function $M(0,t)$ is bounded.

Moreover, the replacement (4) transforms Eq. (1) to the form of a well-known Schrödinger-type problem for a particle in a potential well:

$$\frac{\partial\phi}{2\partial t} = \eta\frac{\partial^2\phi}{\partial r^2} + U(r)\phi,$$

where the potential

$$U(r) = \frac{1}{2}\frac{\partial^2\eta}{\partial r^2} + \frac{2}{r}\frac{\partial\eta}{\partial r} - \frac{2\eta}{r^2} + \frac{1}{4\eta}\left(\frac{\partial\eta}{\partial r}\right)^2. \qquad (5)$$

To define the magnetic viscosity $\eta(r)$ by the longitudinal correlation function (3) we use the special Gaussian form for the velocity correlations $F(r) = \exp(-3r^2/5)$. The details and explanations of this reasonable and traditional choice for the random velocity field can be found in, e.g., Refs. [4,19,20].

So we suggest that the correct boundary condition $M(\infty,t) = M_\infty$ does not depend on time. Usually the Kazantsev problem is combined with the zero condition $M_\infty = 0$, which assumes an absence of large-scale correlations in the small-scale model. This is a quite rational assumption, because the function $M(r,t)$ in supercritical regimes $\text{Rm} > \text{Rm}_{cr}$ is usually localized in a very narrow region $0 < r < 1/\text{Rm}^{1/2}$. However, nobody can deny the possibility of nonzero large-scale correlations $M_\infty \neq 0$, for example, maintained by an external permanent fields or originating from the initial nonzero distribution. One more instance of a nonzero large-scale correlation can arise in mirror-asymmetrical cases, where a large-scale field can be generated for any Rm, thus M_∞ can be supported by the mean field processes.

In this work we consider only two modes of the numerical experiment:

First mode: with nonzero right-hand condition $M_\infty \neq 0$ and with uniform initial distribution $M(r,0) = M_\infty$

Second mode: with zero right condition $M_\infty = 0$ and with large initial Gauss-type perturbation $M(r,0)$, localized at an

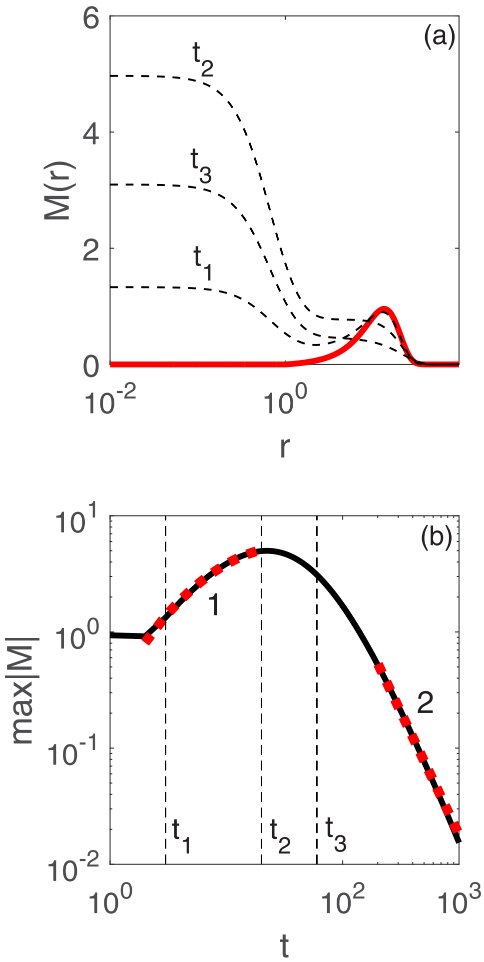

FIG. 1. *Second mode:* (a) the evolution of the Kazantsev solution from nonzero initial perturbation (red line). Different snapshots are marked by t_1, t_2, and t_3. (b) The maximum of the correlation function as a function of time. Line 1 approximates the energy growth [$\max|M| = 5.6 - 6.1\exp(-0.13t)$]; line 2 approximates the damping ($\max|M| \sim t^{-2.1}$).

energy scale $r \gg 1$. Of course, such an initial perturbation is unrealistic, because the maximum of real correlation function should be at the point $r = 0$. However, calculations show similar evolution for various initial distributions: if $M(r,0)$ is a set of small-scale perturbations, then the solution $M(r,t)$ immediately begins to fade, and if $M(r,0)$ is localized at large scales, then the solution rapidly grows to a stationary level, and only after that does it begin to damp (see the example in Fig. 1).

Note that the first model allows us to simulate the contribution in the subcritical regime from an external large-scale magnetic field given by, e.g., the geomagnetic field present in any laboratory experiment. The second model excludes the large-scale external field as a source and considers the small-scale seed field only. Our analysis of the external source is not exhaustive because such a seed can destroy isotropy of the magnetic amplified field; however we ignore this option in the context of this paper.

Taking into account the boundary condition at the right hand end [for nonzero M_∞: $\phi(r,t) \sim r^2$] and simultaneously the strong localization of solution at the interval $[0, 1/\mathrm{Rm}^{1/2}]$, we use a quasi-uniform grid:

$$\hat{r}_i = b\, r_i \frac{c-1}{c-r_i}, \tag{6}$$

where $r_i \in [0,1]$ is uniform.

Using various grid parameters shows us that the solutions are sensitive to having a sufficient number of nodes near zero and at large r, but do not strongly depend on the value b (if it is large enough, e.g., $b \gg 10$). Therefore, for all results presented we take $b = 100$ and, thus, set the right boundary condition near the point $r = 100$. Larger values of b lead to larger values of $\phi \sim r^2$ at the right boundary, which enhances calculation errors. We restrict ourselves to a grid with 2000 nodes and $c = 1.007$. A hyperbolic initial-boundary value problem is solved with a purely implicit scheme and with shifted derivatives (various technical details of implicit counting on quasiuniform grids, and questions of the accuracy and convergence of difference methods can be found, e.g., in Refs. [27,28]). According to the computed auxiliary function $\phi(r,t)$, the correlation function $M(r,t)$ is calculated at each time step (over $\Delta t = 1$). Also solving the stationary Kazantsev problem for $M_\infty \neq 0$, we study the typical dependencies of a stationary solution $M_{st}(r)$ on values M_∞ and Rm. The main results of these calculations for various parameters are discussed in the next section.

III. RESULTS AND ESTIMATES

A. First mode

For subcritical Rm the longitudinal correlation function $M(r,t)$ grows near the dissipative scale $r \ll 1$ with the permanent support of a correlated magnetic field at the large scale $M_\infty \neq 0$. Figure 2(a) demonstrates the function $M(r,t)$, transforming from the initial uniform data to a bell-shaped form and then tending to the stationary solution $M_{st}(r)$ (bold black line). Figure 1(b) shows the time dependence of the difference between the stationary $M_{st}(0)$ and dynamical amplitudes $M(0,t)$ for various values of $\mathrm{Rm} = 30$, 40, and 50. The curves consist of two parts: rapidly and slowly decreasing. They are plotted with a log scale, so we can say that these dependences appear exponential for the initial parts, decreasing with growth of Rm, and with the rates of the second parts not dependent on Rm. The fast growth interval contains all three time snapshots t_1, t_2, and t_3, marked in the top panel. Note that this situation differs from the supercritical dependences, where the exponential rate logarithmically grows and tends to the constant (3/4) for $\mathrm{Rm} \to \infty$ (see, e.g., [4,19,29]).

The second part with slow saturation can be also confirmed by the analytical estimates. Indeed, expanding the magnetic viscosity in a Taylor series for small r as $\eta(r) = 1/\mathrm{Rm} + r^2/5 + o(r^2)$, and substituting this expansion in Eq. (5), we obtain a parabolic equation for the difference between stationary $\phi_{st}(r)$ and dynamical functions $\phi(r,t)$:

$$\frac{1}{2}\frac{\partial(\phi-\phi_{st})}{\partial t} = \frac{1}{\mathrm{Rm}}\frac{\partial^2(\phi-\phi_{st})}{\partial r^2} - \frac{2}{\mathrm{Rm}}\frac{(\phi-\phi_{st})}{r^2} \tag{7}$$

One of the possible solutions,

$$(\phi-\phi_{st}) \sim r\sin\left(r\mu\mathrm{Rm}^{1/2}\right)\exp(-2\mu^2 t),$$

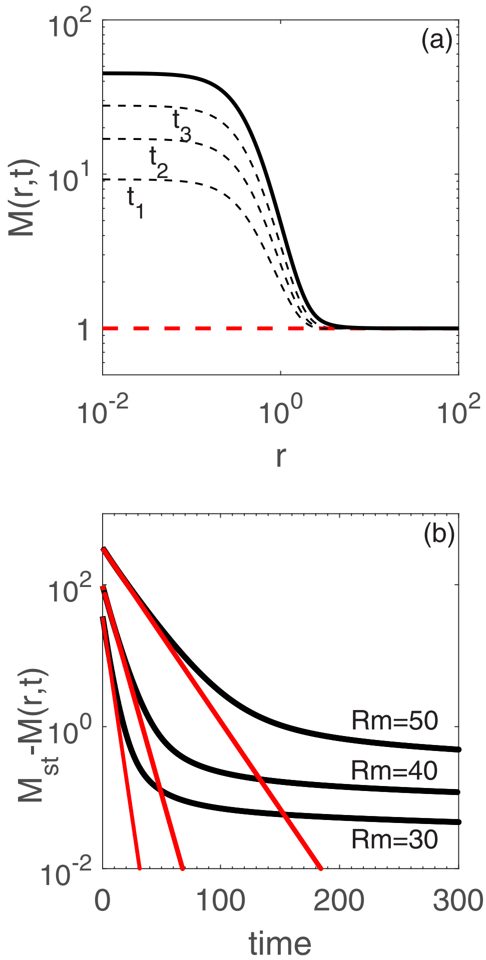

FIG. 2. First mode: (a) The stationary solution (bold line) and the dynamical solutions for different time snapshots $t_1 = 1$, $t_2 = 2$ and $t_3 = 4$. (b) The dependences of the maximal deviation between stationary and dynamical solutions for $\mathrm{Rm} = 5$, 30, and 50. Red lines approximate initial intervals of rapid magnetic energy growth.

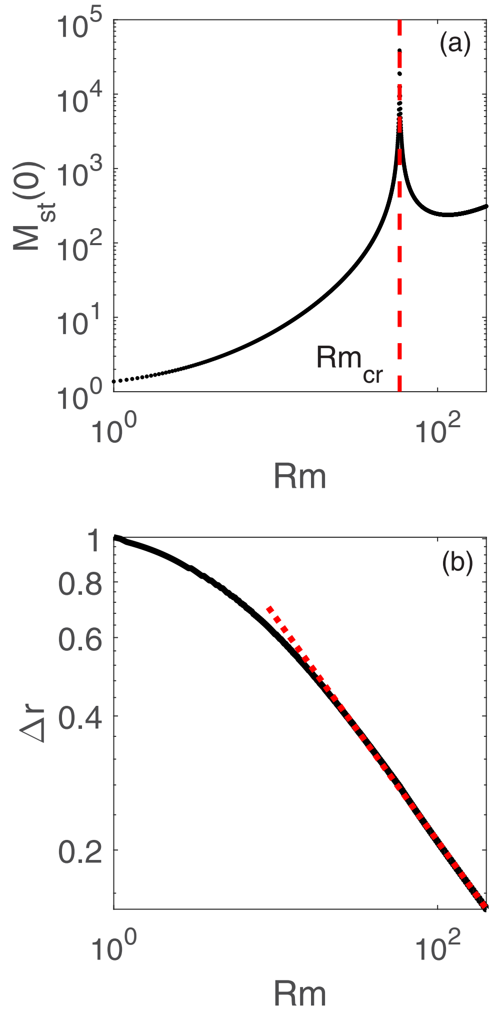

FIG. 3. (a) The dependence of the amplitude of the stationary solution on Rm (red line denotes the critical value Rm_{cr}). (b) The dependence of the width of the stationary solution on Rm and its approximation $\Delta r = -0.01 + 1.5\sqrt{2/\mathrm{Rm}}$, marked by the red line.

shows that this difference decreases with rate independent of Rm, if the localization area $\sim 1/\mathrm{Rm}^{1/2}$ does not depend on time.

We use two parameters to characterize a stationary distribution $M_{st}(r)$: the amplitude of the stationary solution $M_{st}(0)$ and its width Δr on the half-amplitude level $M_{st}(0)/2$. The amplitude $M_{st}(0)$ depends both on the right hand boundary condition M_∞ and on the magnetic Reynolds number Rm; the localization Δr depends only on Rm. Due to the linearity of the problem the dependence on M_∞ is direct, so we choose $M_\infty = 1$. The dependences on Rm for $M_{st}(0)$ and Δr are shown in the top and bottom panels of Fig. 3. One can see that if the dependence $M_{st}(0)$ is complex, then the dependence Δr is inversely proportional to $\mathrm{Rm}^{1/2}$ (for $\mathrm{Rm} > 1$). An analogous dependence for the localization scale was earlier obtained for growing solutions of the Kazantsev model in supercritical regimes $\mathrm{Rm} > \mathrm{Rm}_{cr}$ [19,29,30]. For very small Rm the stationary solution occupies the entire region of velocity field correlations [0,1].

When Rm approaches the critical value Rm_{cr}, the amplitude of the stationary solution $M_{st}(0)$ rapidly increases as $(\mathrm{Rm}_{cr} - \mathrm{Rm})^{-1}$ (see the top panel of Fig. 3, where the vertical line shows $\mathrm{Rm}_{cr} = 58.3$; the approximation is calculated but not shown here). In the supercritical regime it changes sign: $M_{st}(0)$ becomes negative. However, for $\mathrm{Rm} > \mathrm{Rm}_{cr}$ the Kazantsev problem becomes incorrect due to the nonuniqueness of exponentially growing solutions, so we remove such cases from our consideration. The rapid growth of $M(0,t)$ near $_{cr}$ means that in a near-critical regime, the influence of the external magnetic field on the small-scale processes should be quite significant, and correspondingly the inverse effect will be inessential. For small $\mathrm{Rm} < 1$, the amplitude $M_{st}(0)$ tends to M_∞, so possible influences between the large and small scales become comparable.

B. Second mode

In this mode the function $M(r,t)$ evolves from the initial shifted Gauss-type bell form [see the example in Fig. 2(a)]. Previous work shows that in the absence of external magnetic fields (large-scale correlations) any nonzero Kazantsev solutions eventually damp if $\mathrm{Rm} < \mathrm{Rm}_{cr}$. The rate of this damping can be estimated from Eq. (1) under the assumption $r \gg 1$. In this case, the value of the magnetic viscosity can be estimated as $\eta(r) \sim 1/\mathrm{Rm} + 1/3$, and the Kazantsev equation can be rewritten in the form

$$\frac{\partial M}{\partial t} = \frac{2\eta}{r^4}\frac{\partial}{\partial r}\left(r^4 \frac{\partial M}{\partial r}\right)$$

with the solution

$$M(r,t) \sim \frac{1}{|8\eta t - 1|^{5/2}} \exp\left(\frac{-r^2}{|8\eta t - 1|}\right). \qquad (8)$$

We see that for large t the amplitude of function $M(r,t)$ decreases as $t^{-5/2}$ and spreads like $t^{1/2}$. In other words, the equation at large scales behaves similarly to the heat equation, and $M(r,t)$ decreases as a power law. For small-scale processes, this decreasing distribution defines the nonzero M_∞ serving as an external support. This external support leads to rapid small-scale growth (compare with the first mode). Thus, even in the absence of a permanent support $M_{inf} \neq 0$, the longitudinal correlation function can grow for some time. Therefore, an increase in the small-scale magnetic energy is possible for subcritical regimes without any additional support but for limited times, depending on the initial large-scale distribution.

A number of numerical experiments with different initial distributions confirm that such dynamo growth actually takes place [see the example presented in Fig. 2(a)]. In the bottom panel, which depicts the amplitude $\max[M(r,t)]$ as a function of time, we at first see rapid growth up to the stationary solution as $\sim[1 - \exp(-0.13t)]$ (compare with the first mode), and then a slow power law decrease with rate tending to $t^{-2.1}$ (all approximations were realized by the NLLS method). Thus we can state that for nonzero initial distributions $M(r,0)$ and for subcritical magnetic Reynolds numbers, a small-scale transient generation (Zeldovich effect) can be expected in both the presence and absence of external support.

IV. DISCUSSION AND CONCLUSIONS

The traditional small-scale dynamo models predict stable magnetic energy growth only for large enough magnetic Reynolds numbers $\mathrm{Rm} > \mathrm{Rm}_{cr}$; however, the subcritical regimes $\mathrm{Rm} < \mathrm{Rm}_{cr}$ are not as trivial as at first glance. This situation connects with general study of magnetic field time dependency in a special exponential form $\sim\exp(\lambda t)$, which can lead only to two cases: exponential increase or exponential decay. In such "traditional" approaches one does not take into account the regimes with nonexponential, local in time growth, when the small-scale magnetic fields are formed under the large-scale magnetic field support.

We consider here the classical Kazantsev model for mirror-symmetrical delta-correlated flows with Gaussian correlation function and show that even for subcritical Reynolds numbers the local growth of magnetic fields is possible. Therefore, from the physical point of view it means that the use of conductive mediums in external magnetic fields, or in fields generated by large-scale processes, can be accompanied by magnetic energy growth near the dissipative scales and by mutual interaction between small and large scales.

The region of subcritical energy growth near the dissipative scales follows from the existence of a stationary nonzero solution of the Kazantsev equation in the presence of a large-scale correlated magnetic field. The stationary longitudinal correlation function $M_{st}(r)$ has a bell-shaped form with width changing as $\mathrm{Rm}^{-1/2}$. The ratio of amplitudes of correlation functions on small scales $M(0,t)$ and large scale M_∞ can be quite significant: numerical results show that it can indefinitely increase for Reynolds numbers approaching the critical value Rm_{cr} (even for arbitrary small M_∞). It means that near the critical conditions the external magnetic fields strongly influence the small-scale process, and conversely the small-scale fields have almost no effect on large scales. On the other hand, far from the critical situation $\mathrm{Rm} \ll \mathrm{Rm}_{cr}$ the relationship between different scales increases, because the ratio $M(0,t)/M_\infty$ tends to unity.

The dependence for subcritical behavior $M(r,t)$ on time is also different at large and small scales. For $r \ll 1$ the approach to the stationary solution is exponential, while for $r \gg 1$ the amplitude of the correlation function power law decreases by analogy with solutions of the heat equation. Thus there is the possible situation where the correlation function $M(r,t)$ increases rapidly at small scales with the support of a large-scale slow decreasing of initial distribution. Of course, after saturation we should observe a slow power law-like decrease due to decay of external (large-scale) support. It is clear that the duration of magnetic energy growth and amplitude will be defined by the initial data, external magnetic fields, and magnetic Reynolds number, but for the near-critical values $\mathrm{Rm} \sim \mathrm{Rm}_{cr}$ this local-in-time field generation can be quite continuous and significant. Summing up the results we can confirm that magnetic energy growth can be expected in both the supercritical and subcritical regimes, but in the subcritical case it is bounded by an external support, and in the supercritical case it can grow without limit, in the frame of the general Kazantsev model.

The subcritical results obtained have many disputable points: the equality of correlation times at different scales, the simultaneous existence of space isotropy and correlated external field, the possibility and uniqueness of expression for the correlation tensor (2), the influence of mirror asymmetry and magnetic helicity, which should exist in any real random flow, among others. We do not have the opportunity to consider all these questions here but hope that the behavior of the longitudinal function $M(r,t)$ described here will be confirmed in laboratory experiments and by direct numerical simulations of turbulent conductive fluids in future studies. We believe that the isolation of subcritical magnetic field growth can be a reasonable milestone on the way to laboratory realization of small-scale dynamos.

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