Beating effects of vector solitons in Bose-Einstein condensates

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We study the beating effects of solitons in multicomponent coupled Bose-Einstein condensate systems. Our analysis indicates that the period of beating behavior is determined by the energy eigenvalue difference in the effective quantum well induced by solitons, and the beating pattern is determined by the eigenstates of a quantum well, which are involved in the beating behavior. We show that the beating solitons correspond to linear superpositions of eigenstates in some quantum wells, and the correspondence relations are identical for solitons in both an attractive interaction and a repulsive interaction condensate. This provides a possible way to understand the beating effects of solitons for attractive and repulsive interaction cases in a unified way, based on the knowledge of quantum eigenstates. Moreover, our results demonstrate many different beating patterns for solitons in multicomponent coupled condensates, in sharp contrast to the beating dark soliton reported before. The beating behavior can be used to test the eigenvalue differences in certain quantum wells, and more abundant beating patterns are expected to exist in more component-coupled systems.

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I. INTRODUCTION

The multicomponent coupled Bose-Einstein condensate (BEC) provides a good platform to study the dynamics of vector solitons [1]. Many different vector solitons have been obtained in the two-component coupled BEC systems, such as the bright-bright soliton [2], the bright-dark soliton [3], the dark-bright soliton [4–7], and the dark-dark soliton [8]. Bright-bright and bright-dark solitons usually exist in the coupled BEC with attractive interactions [9–12], while dark-bright and dark-dark solitons usually exist in the coupled BEC with repulsive interactions [7,13,14]. Recently, it was shown that it is possible to find a dark-antidark soliton in the BEC with unequal inter- and intraspecies interaction strengths [15–17]. All those previously reported solitons were stable and had no beating effects, but some of them can be used to generate beating solitons. For example, beating dark solitons were shown to exist in the two-component coupled BEC with equal inter- and intraspecies repulsive interactions [18–20], which were generated from the dark-bright soliton with the SU(2) symmetry property [21]. Based on abundant vector solitons for more component cases, it is naturally expected that there should be more exotic beating patterns for more component coupled BEC systems.

On the other hand, there are also beating antidark solitons in the two-component BEC with attractive interactions [see Fig. 1(a)], which can be seen from the results for a rogue wave and breathers in coupled systems with attractive interactions [22,23]. Then, we note that the beating patterns of a dark soliton and an antidark soliton exhibit many similar properties, even though they exist in different interaction cases. Can we find a mechanism to understand the beating effects of them in a unified way? As far as we know, the beating effects have not been discussed systemically to uncover the underlying mechanisms. It is essential to discuss the beating patterns in detail and to find some fundamental mechanisms for these different beating behaviors.

In this paper, we study the beating effects of vector solitons in detail. The analysis suggests that the beating effects of solitons is determined by the energy eigenvalue difference and corresponding eigenstates in the effective quantum wells. The quantum wells have identical forms for both attractive and repulsive interactions cases. In this way, we show that beating antidark solitons and beating dark solitons correspond to the same eigenproblems in a quantum well. Their beating period and pattern can be understood in a unified way, based on the well-known knowledge of the linear superposition of quantum eigenstates. Furthermore, we demonstrate that there are some new beating patterns in three-component or four-component coupled BEC systems with the aid of SU(2) and SU(3) symmetry, such as beating bright solitons with a double-hump, beating dark solitons with a double-valley, and beating bright solitons with a triple-hump, in sharp contrast to the well-known beating dark solitons. These results can be further extended to discuss more component involved cases, and the beating behaviors can be understood well based on the knowledge of quantum eigenstates in quantum mechanics. We further discuss beating solitons in an arbitrary N-component coupled BEC. The beating period can also be used to test the energy eigenvalue differences in some certain quantum wells.

The paper is organized as follows. In Sec. II, we discuss the relations between beating solitons in a two-component coupled BEC with attractive interactions and repulsive interactions. A unified way to understand the beating behaviors is argued. The beating period is determined by the eigenvalue difference of energy in an effective quantum well. In Sec. III, the discussions are extended to three-component cases. More different beating patterns are demonstrated. We further discuss the beating effects of a soliton in an arbitrary N-component coupled

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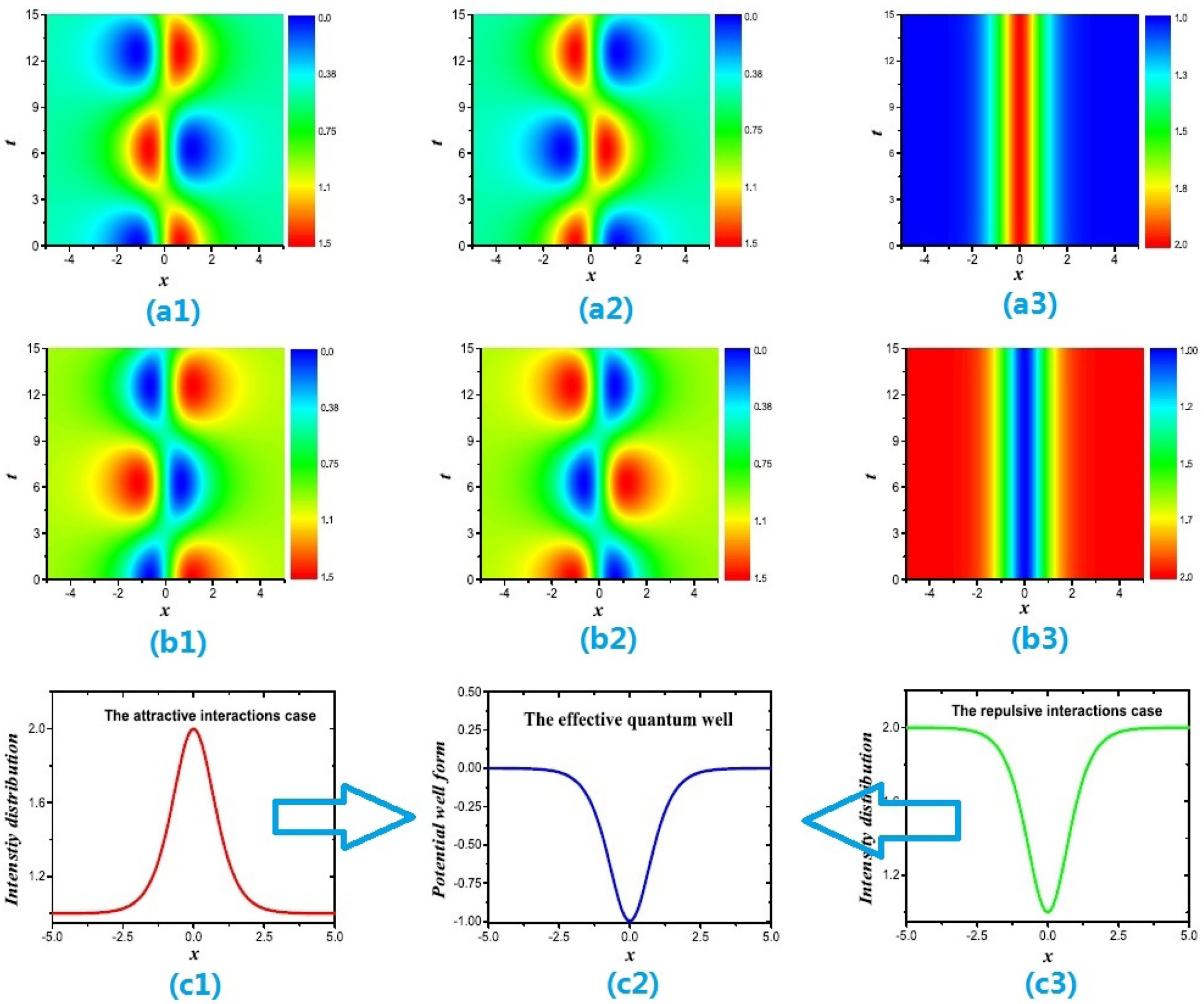

FIG. 1. (a1)–(a3) Beating antidark soliton for two-component coupled BEC with attractive interactions. The three figures show the density evolution of component ψ_1,ψ_2 and the superposition of them, respectively. (b1)–(b3) Beating dark soliton for two-component coupled BEC with repulsive interactions. The three figures show the density evolution of component ψ_1,ψ_2 and the superposition of them, respectively. Parts (c1)–(c3) show that the beating solitons in repulsive and attractive cases correspond to the superposition of identical eigenstates in a quantum well $-f\ \mathrm{sech}^2[\sqrt{f}x]$. Our analysis indicates that the beating effects are induced by the coherence superposition of eigenstates in the quantum well. The beating period is determined by the energy eigenvalue difference in the quantum well. These characters hold well for the beating solitons in both attractive and repulsive interaction cases. The parameters in soliton solutions are $a=1$ and $f=1$.

BEC. In Sec. IV, we summarize our results and present some discussions.

II. THE BEATING EFFECTS OF TWO-COMPONENT SOLITONS

We first study the beating effects of vector solitons in a two-component coupled BEC. The dynamical equation can be written as the following dimensionless coupled model:

$$i\frac{\partial\psi_1}{\partial t}+\frac{1}{2}\frac{\partial^2\psi_1}{\partial x^2}+\sigma(|\psi_1|^2+|\psi_2|^2)\psi_1=0,$$
$$i\frac{\partial\psi_2}{\partial t}+\frac{1}{2}\frac{\partial^2\psi_2}{\partial x^2}+\sigma(|\psi_1|^2+|\psi_2|^2)\psi_2=0, \qquad (1)$$

where ψ_1 and ψ_2 denote the two-component fields in the coupled BEC systems [1]. In this case, the inter- and intraspecies interactions have equal strength, and the model can be solved exactly by the Darboux transformation, the Hirota bilinear method, etc. Bright-bright and bright-dark solitons usually exist in the coupled BEC with attractive interactions $\sigma=1$ [9–12], while dark-bright and dark-dark solitons usually exist in the coupled BEC with repulsive interactions $\sigma=-1$ [7,13,14]. The bright-bright soliton or the dark-dark soliton cannot be used to generate beating behavior, since the bright solitons or dark solitons in two components have identical distribution profiles and chemistry potential values. It has been shown that dark-bright and bright-dark solitons can be used to generate beating solitons. A dark-bright soliton can generate a beating soliton, as shown in Fig. 1(b). It is seen that the beating behavior just emerges in each component, but there are no beating effects for the whole density of the two components, and the superposition density profile is a stable dark soliton. Therefore, it has been called a beating dark soliton [18–20]. The beating antidark soliton can also be generated from the well-known bright-dark solitons. Correspondingly, the superposition density profile is an antidark soliton [shown in Fig. 1(a)], therefore it is called a beating antidark soliton. To find a unified way to understand them, we represent them separately for the attractive and repulsive interaction cases. The beating antidark soliton can be given as follows with $\sigma = 1$:

$$\psi_1 = -(\sqrt{f+a^2}\,\mathrm{sech}[\sqrt{f}x]\,e^{if/2t} + a\tanh[\sqrt{f}x]) \times \frac{1}{\sqrt{2}}e^{ia^2t}, \tag{2}$$

$$\psi_2 = -(\sqrt{f+a^2}\,\mathrm{sech}[\sqrt{f}x]\,e^{if/2t} - a\tanh[\sqrt{f}x]) \times \frac{1}{\sqrt{2}}e^{ia^2t}, \tag{3}$$

where a denotes the amplitude of the plane-wave background for a dark-soliton component. The beating dark soliton can be written as follows with $\sigma = -1$:

$$\psi_1(x,t) = -(a\ \mathrm{sech}[\sqrt{f}x]\,e^{if/2t} + \sqrt{f+a^2}\ \tanh[\sqrt{f}x]) \times \frac{1}{\sqrt{2}}e^{-i(a^2+f)t}, \tag{4}$$

$$\psi_2(x,t) = -(a\ \mathrm{sech}[\sqrt{f}x]\,e^{if/2t} - \sqrt{f+a^2}\ \tanh[\sqrt{f}x]) \times \frac{1}{\sqrt{2}}e^{-i(a^2+f)t}, \tag{5}$$

where $\sqrt{f+a^2}$ denotes the amplitude of the plane-wave background for a dark-soliton component. The beating period is obviously determined by the chemical potential difference for solitons [20]. The beating behaviors are distinctive for attractive and repulsive interaction cases. We would like to find a unified way to understand them, since the beating behavior for a dark soliton and an antidark soliton always has an identical oscillation period if the parameter f of solitons is chosen identically. Why does this character hold for the two different cases, one for the attractive case and one for the repulsive case? There may be many different ways to understand this point. We would like to present one possible way to understand this character, based on the relations between solitons and eigenstates in a quantum well. We will show that the classical linear superposition of eigenstates in quantum mechanics can be used to explain perfectly the beating behavior of solitons in both the attractive and repulsive interaction cases.

Calculating that $|\psi_1|^2 + |\psi_2|^2 = a^2 + f\ \mathrm{sech}^2[\sqrt{f}x]$ in a two-component coupled NLS with the attractive interaction case, and then substituting it into Eqs. (1) and (2) with $\sigma = 1$, one can find that the beating soliton solution is related with the eigenproblem in a quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. The corresponding eigenequation is

$$-\frac{1}{2}\frac{\partial^2\psi_j}{\partial x^2} - f\,\mathrm{sech}^2[\sqrt{f}x]\psi_j = \mu_j\psi_j. \tag{6}$$

TABLE I. The correspondence between soliton states in a two-component BEC and eigenstates in a quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. It is seen that the solitons in the attractive and repulsive cases correspond to identical eigenvalues in the quantum well. The beating period is determined by the eigenvalue difference, which can be understood well using knowledge of quantum mechanics. "AI," "QW," and "RI" denote attractive interaction BEC, quantum well, and repulsive interaction BEC respectively.

Solitons in an AI	Eigenvalues in a QW	Solitons in an RI
dark soliton	0	dark soliton
bright soliton	$-f/2$	bright soliton

We can prove directly that the bright soliton and the dark soliton, which are used to generate a beating antidark soliton, are both eigenstates in the quantum well $-f\,\mathrm{sech}^2[\sqrt{f}x]$. It is seen that the bright soliton corresponds to the eigenvalue $-f/2$ in the quantum well, and the dark soliton corresponds to eigenvalue zero in the quantum well. This agrees well with the results in quantum wells [24,25]. For a beating dark soliton in the repulsive case, $|\psi_1|^2 + |\psi_2|^2 = a^2 + f\tanh^2[\sqrt{f}x]$ can be used to find related eigenstates in a quantum well for the repulsive cases. With the help of $\tanh^2(x) = 1 - \mathrm{sech}^2(x)$, we can rewrite the potential form as $a^2 + f\tanh^2[\sqrt{f}x] = a^2 + f - f\,\mathrm{sech}^2[\sqrt{f}x]$. After transferring the constant terms to chemistry potential terms, the effective quantum well also becomes $-f\,\mathrm{sech}^2[\sqrt{f}x]$. Interestingly, *the eigenproblem for the repulsive case is identical to that in the attractive case, i.e., they have the identical eigenequation*. This is demonstrated in Fig. 1(c). It is seen that the eigenvalues of a bright soliton and a dark soliton, which are used to generate a beating dark soliton, are also $-f/2$ and 0 in the quantum well. These characters are summarized in Table I. In this way, we can establish the correspondence between solitons and eigenstates in a quantum well. This provides possibilities to explain the beating effects of solitons based on the knowledge of eigenstates in quantum mechanics.

Therefore, the beating solitons are fundamentally linear superposition forms of eigenstates in a quantum well. In quantum mechanics, arbitrary linear superpositions of eigenstates are always the solution of the linear Schrödinger equation. But there is a slight difference in that the linear superposition coefficients cannot be arbitrary for Eqs. (1) and (2), since they are nonlinear partial equations. The linear superposition of two eigenstates will have a beating behavior, and the beating period is determined by the eigenvalue difference [24]. Namely, the beating period is

$$T = \frac{2\pi}{\Delta}, \tag{7}$$

where Δ denotes the eigenvalue difference of energy between the beating behavior of the two eigenstates. For an example, the beating period T in Fig. 1 is about 12.56 in scaling units, and the energy difference is $\Delta = \frac{1}{2}$ in scaling units, which agrees well with the quantitative relation. Therefore, *there are no beating effects for superpositions of degenerated solitons, for which the solitons have identical eigenvalues.* This provides a good way to understand the beating effects of solitons in a BEC.

Most of the previous experiments in a BEC demonstrated that two-component solitons could be produced well using different density and phase-modulation techniques [6,7,13,14]. Very recently, three-component soliton states were further observed in a spinor BEC system [26]. Motivated by these developments, we would like to discuss the beating effects of solitons in a three-component coupled BEC.

III. THE BEATING EFFECTS OF MULTICOMPONENT SOLITONS

The three-component coupled BEC can be described by the following dynamical equations:

$$i\frac{\partial \psi_j}{\partial t}+\frac{1}{2}\frac{\partial^2\psi_j}{\partial x^2}+\sigma(|\psi_1|^2+|\psi_2|^2+|\psi_3|^2)\psi_j=0, \quad (8)$$

where ψ_j's ($j=1,2,3$) denote the three-component fields in the coupled BEC systems [1]. Most of the previously reported three-component solitons are degenerate solitons or partially degenerated solitons [9–11,26], such as a bright-bright-bright soliton, a bright-dark-dark soliton, a dark-bright-bright soliton, a dark-dark-dark soliton, etc. For example, for a dark-bright-bright soliton, the dark soliton component has one node and the two bright soliton components have identical eigenstates (they are degenerate). The superpositions of degenerate solitons cannot produce any beating effects, but the nondegenerate solitons can produce beating effects that are similar to those in the two-component cases [9–11,26]. This is because of the dark soliton and the bright soliton in partly degenerate solitons for the three-component case, which correspond to identical eigenstates in the quantum well to those for the two-component case. Therefore, we do not show them in detail, and these characters can be seen in previously reported soliton solutions.

In fact, the vector solitons in three-component coupled systems can have nondegenerate solitons [27,28]. The dark soliton in one component can have a double-valley structure, and one bright soliton component can also have a node that makes the bright soliton have a double-hump. These are quite different from the ones observed in [26]. Similarly, we can generate a beating soliton from the eigenstates in a quantum well $-3f\,\mathrm{sech}^2[\sqrt{f}x]$. It is found that the solitons in the three components correspond to identical eigenstates in the quantum well for the attractive and repulsive interaction cases. Namely, the eigenstates of the linear Schrödinger equation,

$$-\frac{1}{2}\frac{\partial^2\psi_j}{\partial x^2}-3f\,\mathrm{sech}^2[\sqrt{f}x]\psi_j=\mu_j\psi_j, \quad (9)$$

can be used to generate three-component nondegenerate solitons for both the attractive and repulsive interaction cases. This makes the beating patterns for the attractive interaction case similar to those in the repulsive case. Therefore, we mainly discuss the beating solitons in a three-component coupled BEC with repulsive interactions. Similar behaviors are expected in the attractive cases.

The eigenstates $\mathrm{sech}^2[\sqrt{f}x]$, $\mathrm{sech}[\sqrt{f}x]\tanh[\sqrt{f}x]$, and $(1-3\tanh^2[\sqrt{f}x])$ correspond to the eigenvalues $-2f$, $-f/2$, and 0, respectively, in the quantum well $-3f\,\mathrm{sech}^2[\sqrt{f}x]$. From the nodes of the eigenstates, we know that these eigenstates are the ground state, the first-excited state, and the second-excited state, which correspond to eigenvalues $-2f$, $-f/2$, and 0, respectively. This can be used to construct nondegenerate vector solitons for a three-component coupled nonlinear Schrödinger equation with attractive or repulsive interactions. Since the nonlinear equations have some additional constraints on the coefficients of three wave functions, we introduce some new coefficients for them, namely $\phi_1(x)=a_3\,\mathrm{sech}^2[\sqrt{f}x]$, $\phi_2(x)=b_3\,\mathrm{sech}[\sqrt{f}x]\tanh[\sqrt{f}x]$, and $\phi_3(x)=c_3(1-3\tanh^2[\sqrt{f}x])$. We can identify the values of a_3, b_3, and c_3 with the constraint condition $|\psi_1|^2+|\psi_2|^2+|\phi_3(x)|^2=a^2+3f\tanh^2[\sqrt{f}x]$. Then the static vector nondegenerate soliton of the three-component BEC with repulsive interactions $\sigma=-1$ can be given as $\psi_{1s}=\frac{1}{2}\sqrt{3(a^2-f)}\,\mathrm{sech}^2[\sqrt{f}x]e^{-it(a^2+f)}$, $\psi_{2s}=\sqrt{3(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}[\sqrt{f}x]e^{-it(a^2+\frac{5f}{2})}$, and $\psi_{3s}=\frac{1}{2}\sqrt{(a^2+3f)}(1-3\tanh^2[\sqrt{f}x])e^{-it(a^2+3f)}$. From the nodes of these eigenstates, we know that ψ_{1s}, ψ_{2s}, and ψ_{3s} are the ground state, the first-excited state, and the second-excited state in the quantum well, respectively. Their linear superpositions can generate many different beating solitons, with the aid of the SU(2) or SU(3) symmetry admitted by the three-component coupled nonlinear equations.

For the SU(2) symmetry case, the transformation matrix has many different forms. Their beating patterns are similar, with identical beating periods. As an example, we choose $S_{3\times3}=\begin{pmatrix}-\sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} & 0\\ -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0\\ 0 & 0 & 1\end{pmatrix}$. The linear transformation $S_{3\times3}(\psi_{1s},\psi_{2s},\psi_{3s})^T$ (T denotes the transpose of a matrix) can be used to construct many different beating solitons. First, $S_{3\times3}(\psi_{1s},\psi_{2s},\psi_{3s})^T$ describes a superposition of the ground state and the first-excited state in the quantum well. The dynamical processes of the beating soliton are shown in Fig. 2(a). It is seen that the beating behaviors are demonstrated on zero background, in contrast with the beating dark soliton and the beating antidark soliton shown above. Their superposition is a bright soliton with a double-hump, therefore this beating soliton is called a beating bright soliton with a double-hump. This is similar to the case for which the beating dark soliton is named. The beating period is about $T=4.18$, which agrees well with $\frac{4\pi}{3}$. Second, $S_{3\times3}(\psi_{1s},\psi_{3s},\psi_{2s})^T$ describes a superposition of the ground state and the second-excited state in the quantum well. The dynamical processes for the beating soliton are shown in Fig. 2(b). It is seen that the beating behaviors emerge on a plane-wave background, which is similar to the beating dark soliton discussed before. But there is a sharp difference, namely the superposition of the beating components is a dark soliton with a double-valley. Therefore, this beating soliton is called a beating dark soliton with a double-valley, in contrast to the beating dark soliton. Third, $S_{3\times3}(\psi_{2s},\psi_{3s},\psi_{1s})^T$ describes a superposition of the first-excited state and the second-excited state in the quantum well. The dynamical processes for the beating soliton are shown in Fig. 2(c). It is seen that beating patterns also emerge on a plane-wave background, and their superposition is a dark soliton. The dark soliton also has one valley, therefore this beating soliton is a beating dark soliton. It should be noted that the beating pattern is different from the beating dark soliton in the two-component cases, since the eigenstates are different from those in the two-component cases.

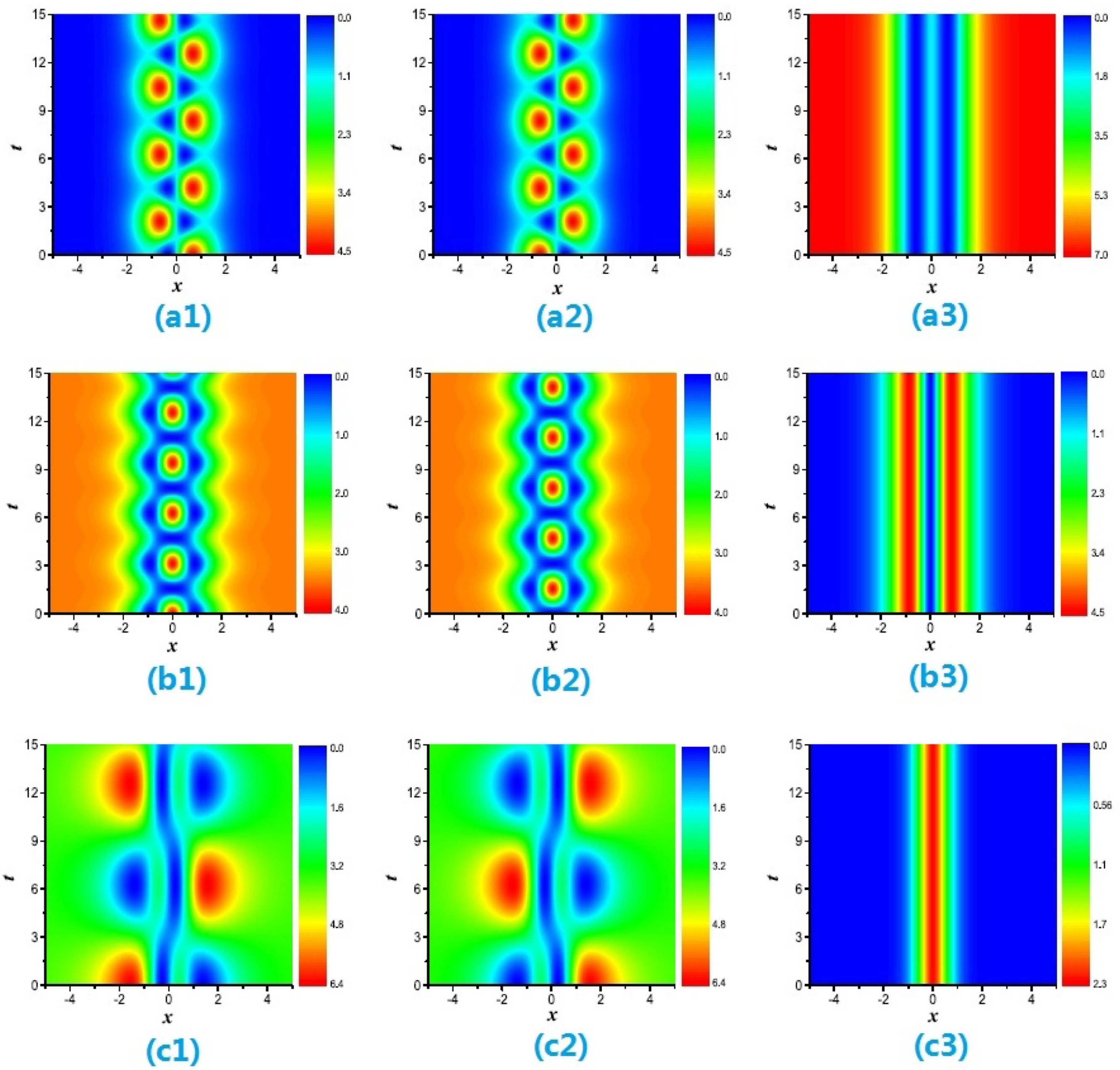

FIG. 2. The beating solitons in a three-component coupled BEC with repulsive interactions, which are generated from SU(2) symmetry. Parts (a1)–(a3) show the evolution of a beating bright soliton with a double-hump in the three components, respectively, which are superpositions of the ground state and the first-excited states in the quantum well $-3f\,\mathrm{sech}^2[\sqrt{f}x]$. Parts (b1)–(b3) show the evolution of a beating dark soliton with a double-valley in the three components, respectively, which are superpositions of the ground state and the second-excited states in the quantum well. Parts (c1)–(c3) show the evolution of a beating dark soliton in the three components, respectively, which are superpositions of the first-excited states and the second-excited states in the quantum well. The parameters in the soliton solutions are $a = 2$ and $f = 1$.

For SU(3) symmetry, the transformation matrix can be chosen as $S_{3\times3} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}}\exp(\frac{i\pi}{3}) & \sqrt{\frac{1}{3}}\exp(\frac{i2\pi}{3}) \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}}\exp(\frac{i2\pi}{3}) & -\sqrt{\frac{1}{3}}\exp(\frac{i\pi}{3}) \end{pmatrix}$ as an example without losing generality. The linear transformation $S_{3\times3}(\psi_{1s},\psi_{2s},\psi_{3s})^T$ describes the superposition of the ground state, the first-excited state, and the second-excited state. The dynamical processes for the beating soliton are shown in Fig. 3. The beating pattern becomes more complicated, since the beating period involves more periodic functions in this case. Their superposition is a dark soliton that has one valley. Therefore, it is also a beating dark soliton, but its beating pattern is different from all previously reported ones. This suggests that more beating patterns can be found in cases with more components, since a coupled BEC with more components corresponds to deeper quantum wells, which have more eigenstates with many different nodes.

Explicitly, an N-component coupled BEC with equal nonlinear interaction strengths has SU(N) symmetry. The eigenstates in the corresponding quantum wells and the related

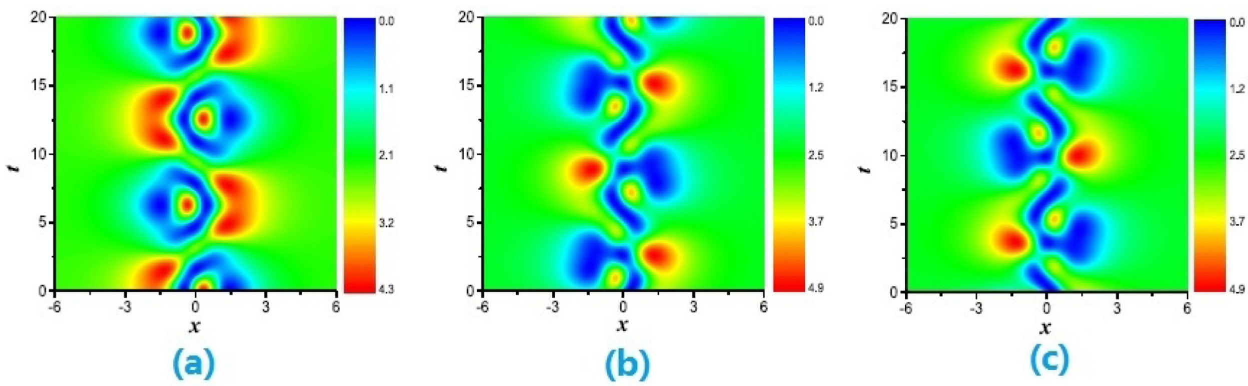

FIG. 3. The beating dark soliton in a three-component coupled BEC with repulsive interactions, which are generated from SU(3) symmetry. Parts (a)–(c) show the evolution of beating dark solitons in the three components, respectively. The beating effects come from the superposition of the ground state, the first-excited state, and the second-excited state in the quantum well $-3f\ \mathrm{sech}^2[\sqrt{f}x]$. The parameters in the soliton solutions are $a=2$ and $f=1$.

unitary matrix can be used to construct beating solitons. In particular, the forms of the unitary matrix can be chosen to have SU(M) ($M \leqslant N$) symmetry, which can be used to obtain very abundant different beating patterns. As examples, we further demonstrate that many different beating patterns can be generated from a nondegenerate soliton for a four-component coupled BEC with repulsive interactions. Meanwhile, it should be noted that a partially degenerate soliton for multicomponent cases can also be used to generate many different beating patterns. We do not discuss the partially degenerate soliton cases in detail anymore. The dynamical equations for the four-component coupled BEC with repulsive interactions are $i\psi'_t + \frac{1}{2}\psi'_{xx} - \psi'^{\dagger}\psi'\psi' = 0$, where $\psi = (\psi_1,\psi_2,\psi_3,\psi_4)^T$. In a similar way, we obtain nondegenerate vector solitons from the quantum well $6f\tanh^2[\sqrt{f}x]$. Namely, the fundamental soliton solution can be derived as $\psi_1(x,t) = \frac{1}{2}\sqrt{\frac{5}{2}(a^2-3f)}\,\mathrm{sech}^3[\sqrt{f}x]e^{-i(a^2+3f/2)t}$, $\psi_2(x,t) = \frac{1}{2}\sqrt{15(a^2+2f)}\tanh[\sqrt{f}x]\,\mathrm{sech}^2[\sqrt{f}x]e^{-i(a^2+4f)t}$, $\psi_3(x,t) = \frac{1}{2}\sqrt{\frac{3}{2}(a^2+5f)}(5\tanh^2[\sqrt{f}x]-1)\,\mathrm{sech}[\sqrt{f}x]e^{-i(a^2+11f/2)t}$, and $\psi_4(x,t) = \frac{1}{2}\sqrt{a^2+6f}\tanh[\sqrt{f}x](5\tanh^2[\sqrt{f}x]-3)e^{-i(a^2+6f)t}$. We introduce a matrix $S_{4\times4}$ to investigate the beating effects of these solitons based on the symmetry properties of dynamical equations. The forms of the unitary matrix can be chosen to have SU(M) ($M \leqslant 4$) symmetry, which enables us to obtain very abundant different beating patterns. As an example, we show one case to demonstrate abundant beating patterns with the aid of SU(3). If $S_{4\times4} = \begin{pmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{3}}\exp(\frac{i\pi}{3}) & \sqrt{\frac{1}{3}}\exp(\frac{i2\pi}{3}) & 0 \\ \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}}\exp(\frac{i2\pi}{3}) & -\sqrt{\frac{1}{3}}\exp(\frac{i\pi}{3}) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, the linear transformation $S_{4\times4}\psi$ can also be used to construct many different localized waves. Different from the above cases, for which beating solitons just emerge in two components, the beating behaviors are demonstrated in three components in this case. We will show one case of them as an example to demonstrate the striking dynamics, in contrast to the vector soliton reported before. The solution $S_{4\times4}(\psi_1,\psi_2,\psi_3,\psi_4)^T$ can be obtained directly from the nondegenerate solitons. The dynamics of solitons in the four components are shown in Fig. 4(a). It is seen that beating solitons emerge in three components in this case, in contrast to those in Figs. 2 and 3. The beating pattern can be seen as a beating bright soliton with a triple-hump. Similarly, we can also exchange the components to generate different beating solitons. The dynamics of the soliton solution $S_{4\times4}(\psi_4,\psi_2,\psi_3,\psi_1)^T$ are shown in Fig. 4(b). It is shown that the beating patterns become more complicated than those in Fig. 2. This is because the superposition forms involve more eigenvalues of the same quantum well, and more eigenvalues will produce more oscillation behaviors. If the elements of the matrix $S_{4\times4}$ are all nonzero, then we can obtain more complicated beating patterns, since the beating effects will be induced by the interference between more eigenstates.

IV. CONCLUSION AND DISCUSSION

In summary, we show that the beating patterns of solitons are determined by the eigenvalue difference and corresponding eigenstates in the effective quantum wells. In particular, the effective quantum wells have identical forms for both attractive and repulsive interactions. In this way, we show that a beating antidark soliton and a beating dark soliton correspond to the same eigenproblems in a quantum well for a two-component coupled BEC. Their beating period and pattern can be understood in a unified way, based on the well-known knowledge of the linear superposition of quantum eigenstates. These characters hold well for cases with more components. A brief discussion is also given for beating solitons in an arbitrary N-component coupled BEC. As examples, we demonstrate that there are some new beating patterns in three-component and four-component coupled BEC systems, such as beating bright solitons with a double-hump, and beating dark solitons with more humps or valleys, in sharp contrast to the beating dark solitons reported before. We note that internal vibrations of vector solitons were investigated numerically in [29,30]. The vibration behavior could be related with the beating effects discussed above, since the states obtained by a variational method may involve several eigenstates admitted by the systems.

From the results of beating effects, we know that the beating period is determined by the corresponding energy eigenvalue

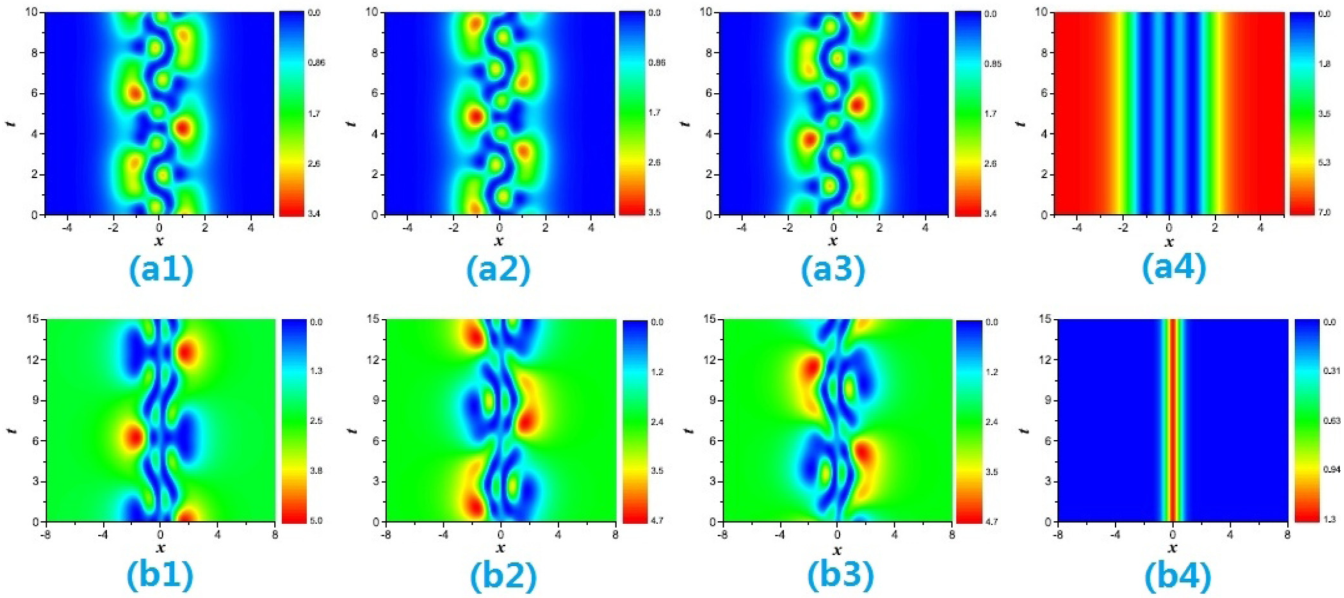

FIG. 4. The evolution of beating solitons generated from nondegenerate solitons in a four-component coupled BEC with repulsive interactions. Parts (a1)–(a4) show the evolution of the beating bright soliton case in the four components, respectively. It is shown that *the beating effects of solitons emerge in three components*, and there is a stable dark soliton with a triple-valley in the other component. Parts (b1)–(b4) show the evolution of the beating dark soliton case in the four components, respectively. It is shown that the beating effects of dark solitons emerge in three components, and there is a stable bright soliton with a single hump in the other component. The parameters in the soliton solutions are $a = 1$ and $f = 1$.

difference in the effective quantum wells. On the other hand, one can produce the initial density and phase distribution for beating solitons in BEC systems through the well-developed density and phase-modulation techniques [6,7,13,14,26]. The beating period would be measured directly in real experiments [31]. Therefore, it is possible to measure the energy eigenvalue difference in many different quantum wells in multicomponent BEC systems, with the aid of the beating period $T = \frac{2\pi}{\Delta}$, where Δ denotes the energy eigenvalue difference in the quantum wells.

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