Dependence of extreme events on spatial location

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To model the dependence of extreme events on locations, we consider extreme events of Brownian particles in a potential. We find that barring the exception of very large potentials and/or very small regions, in general, the probability of extreme events increases with the potential. Our approach is general and can be useful for studying several complex systems.

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I. INTRODUCTION

Extreme events are ubiquitous in nature [1]. Most of the mathematical analyses of extreme events focus on independent random variables [2]. However, it is well known that there is a location dependence to several extreme events, with some locations being more prone to extreme events than the others. For example, there are some regions on the earth's surface which are more prone to earthquakes than others [3]. Similar dependence on locations is found for many other extreme events such as tsunamis, volcanos, cyclones [3], floods, storms, draughts [4], etc. Obviously, there are physical reasons for this location dependence.

A question one may like to address is whether one can develop a general formalism to understand such a correlation between the region under consideration and the probability of extreme events. It is crucial that one brings in the relevant physics of the problem while addressing this task. The present work is a step in this direction.

Brownian motion is a very important physics problem which has applications in several disciplines and can act as a base model for several problems [5–7]. Hence, we use it for our analysis of extreme events. Our model is the motion of a Brownian particle in a potential. The potential allows us to introduce a location-dependent physical parameter in the dynamics. Our analysis follows that of Refs. [8,9] for random walks on networks. An extreme event can be defined as that which exceeds a prescribed quantity above a threshold. Here, it must be noted that the threshold may not be uniform everywhere and will in general depend on the location. For example, the definition of extreme cold weather in the Sahara Desert can be very different from that in the North Pole. Thus, the threshold needs to be defined while keeping the local conditions in mind. In our model, the inherent fluctuations in the model lead to the crossing of this threshold, and it is not obtained by any external driving force. Thus, they are inherent to the system.

II. BROWNIAN PARTICLES IN A POTENTIAL

For a Brownian particle in a potential $V(x)$, we consider Smoluchowski equation [5,10] given by

$$\frac{\partial Q(x,t)}{\partial t} = -\frac{\partial}{\partial x}S(x,t),$$
$$S(x,t) = \left[\frac{1}{m\gamma}F(x) - D\frac{\partial}{\partial x}\right]Q(x,t), \quad (1)$$

so that

$$\frac{\partial Q(x,t)}{\partial t} = \frac{1}{m\gamma}\left[-\frac{\partial}{\partial x}F(x) + kT\frac{\partial^2}{\partial x^2}\right]Q(x,t), \quad (2)$$

where $Q(x,t)$ is the probability density of Brownian particle, $S(x,t)$ is the probability current, $F(x) = -\frac{dV}{dx}$ is the force due to potential V, γ is the friction coefficient, $D = \frac{kT}{m\gamma}$ is the diffusion constant, k is the Boltzmann constant, m the particle mass, and T is the temperature of the heat bath.

The stationary solution when the probability current is zero is [5]

$$Q_{st}(x) = Ae^{-\Phi(x)}, \quad (3)$$

where $\Phi(x) = V(x)/kT$. The constant A can be determined by the normalization condition

$$\int_a^b Q_{st}(x)dx = 1,$$

where a and b are the limits of normalization.

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The probability of finding a Brownian particle in a region $R = \{x \in (c,d)\}$ is given by

$$p(R) = \int_c^d Q_{st}(x)dx. \tag{4}$$

Consider N such independent and noninteracting Brownian particles [8]. We consider the probability $p(n,R)$ that there are n particles in the region R and the remaining $N-n$ particles are outside this region. Since the particles are independent and noninteracting, the probability $p(n,R)$ is given by

$$p(n,R) = \binom{N}{n} p^n (1-p)^{N-n},$$

where we write $p = p(R)$ for brevity. The mean number of particles and its variance for a given region are given by

$$\bar{n} = Np, \quad \sigma^2 = Np(1-p). \tag{5}$$

Extreme events are rare and normally occur in the tail of the probability distribution. Using this intuitive notion of extreme events, we define extreme events for region R as those where the number of particles in the region R is greater than some threshold value q, i.e., $n > q$. Since this threshold should give extreme events in the tail of probability distribution, we define the threshold as [8]

$$q = \bar{n} + m\sigma, \tag{6}$$

where m is some real positive number. The value of m decides the extent of rarity of extreme events. As the value of m increases, the event becomes more rare. We can now write the probability of extreme events in region R as

$$\mathcal{F}(p) = \sum_{k=\lfloor q\rfloor+1}^{N} \binom{N}{k} p^k (1-p)^{N-k}, \tag{7}$$

$$= I_p(\lfloor q\rfloor + 1, N - \lfloor q\rfloor), \tag{8}$$

where $I_p(..,..)$ is the regularized incomplete β function and $\lfloor q\rfloor$ is the largest integer less than q [11]. The extreme value probability depends on (p,N,q) and not on how the region R is defined. Note that only two of (p,N,q) are independent due to the relation (6) between them.

Figure 1 shows a plot of extreme event probability $\mathcal{F}(p)$, as a function of p for $m = 4$ and $N = 100,1000,10\,000$. The extreme event probability shows an interesting oscillatory behavior with peaks corresponding to positive integer values of the threshold q. Let p_k, $k = 1,2\ldots$ be the values of p corresponding to the threshold values $q = k$. Using Eqs. (5) and (6), we obtain

$$k - Np_k = m\sqrt{Np_k(1-p_k)}. \tag{9}$$

This leads to a quadratic equation for p_k,

$$p_k^2 N(N+m^2) - p_k N(2k+m^2) + k^2 = 0. \tag{10}$$

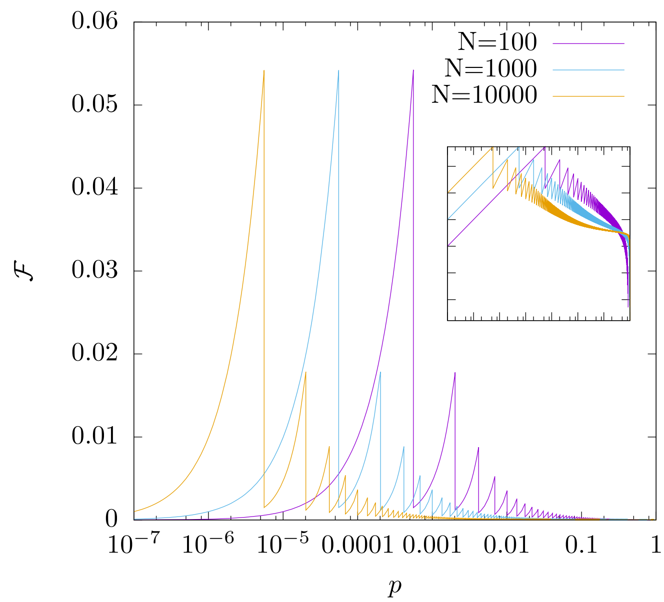

FIG. 1. A plot of extreme event probability $\mathcal{F}(p)$ as a function of p for $m = 4$ and different values of $N = 100,1000,10\,000$. The inset shows the same figure in log-log scale. The sharp changes occur at $p_k, k = 1,2,3,\ldots$ such that the corresponding thresholds q are integers, $q = k$.

Hence the values of p_k corresponding to the peak values are

$$p_k = \frac{(2k+m^2) \pm m\sqrt{m^2+4k-4k^2/N}}{2N(1+m^2/N)}. \tag{11}$$

For large N ($N >> m^2,k^2$), we get

$$p_k \simeq \frac{(2k+m^2) \pm m\sqrt{m^2+4k}}{2N}. \tag{12}$$

Between $0 < p < p_1$, i.e., $q < 1$, the probability of extreme events increases as p increases and then abruptly falls. Again for $p_1 < p < p_2$ $(1 < q < 2)$, the probability of extreme events increases as p increases. This behavior is observed for all intervals $p_k < p < p_{k+1}$ $(k < q < k+1)$. In addition to this oscillatory behavior, there is an overall decrease in the probability of extreme events for $p > p_1$. In Ref. [8], this overall decrease led to the conclusion that the extreme event probability for nodes with smaller degree is in general larger than that of nodes with larger degree.

From Eq. (12), we find that $p_k N$ does not depend on N for large N. In Fig. 2, we plot the extreme event probability $\mathcal{F}(p)$, as a function of pN/N_0 where $N_0 = 10\,000$, the largest value of N that we have used. We see that for smaller values of p, the plots for different values of N overlap, though there are deviations for large values of p. Figure 3 shows a plot of $\mathcal{F}(p)$ as a function of p, for $N = 1000$ and $m = 1,2,3,4,6$. As expected, the probability of extreme events decreases as m increases. The peaks p_k shift to lower values of probability as m increases. However, the overall behavior remains the same. For smaller values of m, e.g., $m = 1$, the extreme event probability distribution becomes flatter for $p > p_1$. This is expected, since for $m = 0$, $p_k = 0.5$, $k = 1,2,\ldots$, and the corresponding probability of extreme events is $\mathcal{F}_k = \mathcal{F}(p_k) = 0.5$. Clearly, the extreme events are no longer rare. Thus, we must choose

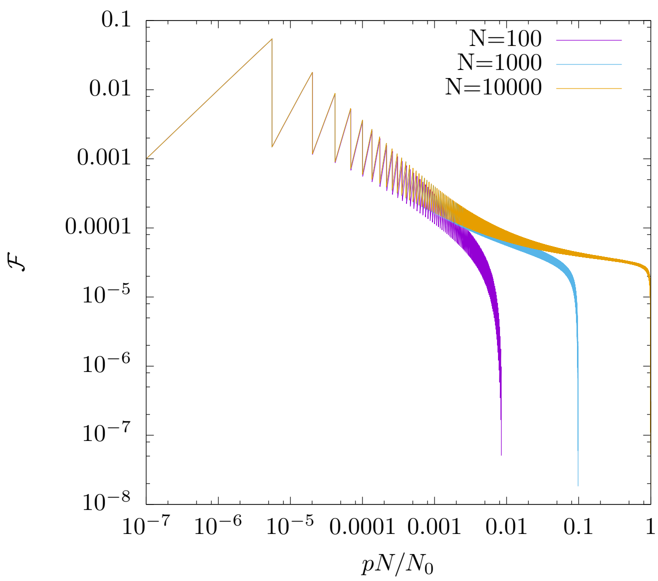

FIG. 2. A plot of extreme event probability $\mathcal{F}(p)$ as a function of pN/N_0, $N_0 = 10\,000$ for $m = 4$ and different values of $N = 100,1000,10000$.

larger values of m to get rare extreme events. As m increases, there is an overall decrease in $\mathcal{F}(p)$ as p increases for $p > p_1$.

III. EXTREME EVENT PROBABILITY FOR SOME POTENTIALS

We now consider some simple potentials [12] to illustrate the dependence of extreme event probability on the potential. As seen from Figs. 1 and 3, for $p < p_1$ the extreme event probability increases, while for $p > p_1$ it shows an overall decrease with oscillations. This complex behavior makes it difficult to analyze the results for different potentials.

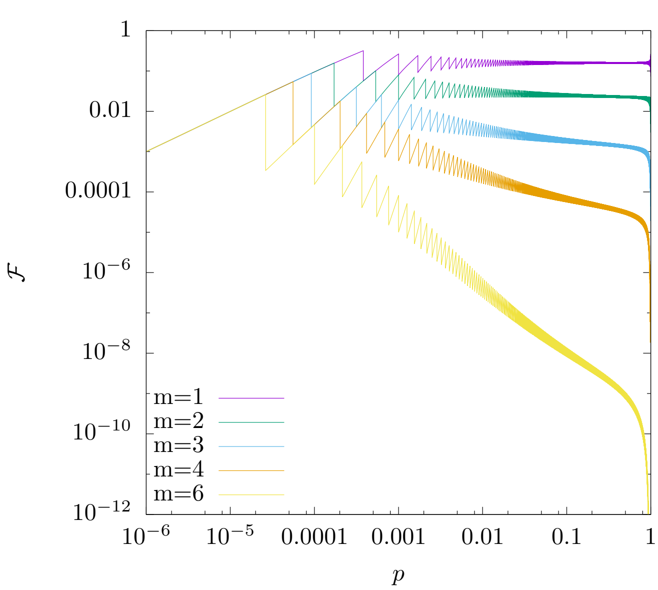

FIG. 3. A plot of extreme event probability $\mathcal{F}(p)$ as a function of p for $N = 1000$ and different values of $m = 1,2,3,4,6$.

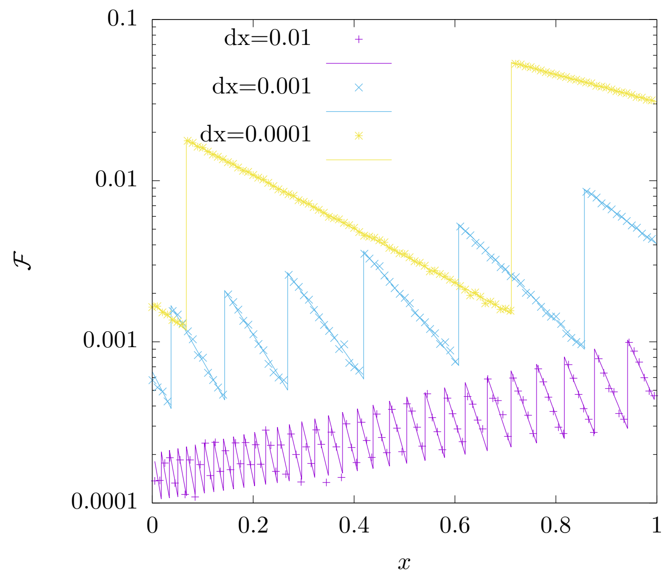

FIG. 4. A plot of extreme event probability $\mathcal{F}(p)$ as a function of x for linear potential and different values of $dx = 0.01,0.001,0.0001$. Here, $N = 1000, m = 4, c = 2$. The continuous curves are the theoretical curves obtained using Eqs. (8), (6), and (16). The symbols are values obtained from stochastic simulation using the Langevin equation, Eq. (A1).

Consider a linear potential

$$V(x) \propto x, \quad 0 \leqslant x \leqslant 1 \tag{13}$$

$$= \infty, \quad x < 0, \text{ and } x > 1, \tag{14}$$

$$\Phi(x) = V(x)/kT = cx, \tag{15}$$

where c is some constant depending on the temperature. The probability of a particle being in a small interval $R = (x - dx/2, x + dx/2)$ is [Eqs. (3) and (4)]

$$p(R) = \frac{2e^c}{e^c - 1} \sinh(cdx/2) e^{-cx}. \tag{16}$$

Figure 4 shows the probability of extreme events as a function of x, for $dx = 0.01,0.001,0.0001$ (continuous curves). First, we note that for a given dx, on the average, the probability increases as the potential increases. Second, the probability increases as dx decreases, i.e., the width of the region decreases. We also do a stochastic simulation of Brownian motion (see the Appendix) and the values obtained are shown in Fig. 4 by symbols. We see a good agreement between the theoretical curves and the results of stochastic simulation.

A simple way to understand the increase in the extreme event probability with dx is to consider the limiting cases. First, consider the case when dx becomes the entire range of x of the system. Clearly, the probability that a particle is in the entire range is one, giving zero fluctuations, and the extreme event probability is zero. Next, consider dx to be small, so that the average number of particles in dx is of the order of one. Then the likely occupancies of 0, 1, 2, 3, ... correspond to a large fluctuation as compared to the average value and hence a large extreme event probability [13]. Our definition of extreme

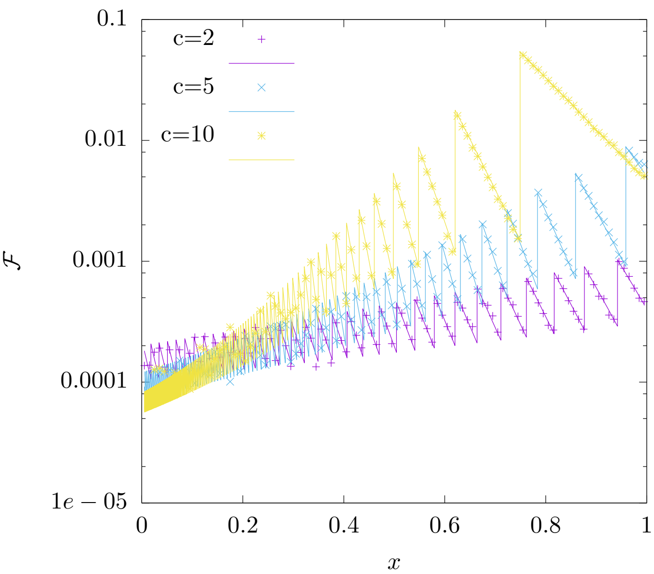

FIG. 5. A plot of extreme event probability $\mathcal{F}(p)$ as a function of x for linear potential and different values of $c = 2,5,10$. Here, $N = 1000, m = 4, dx = 0.01$. The continuous curves are the theoretical curves obtained using Eqs. (8), (6), and (16). The symbols are values obtained from stochastic simulation using the Langevin equation, Eq. (A1).

events using the threshold $q(p)$ [Eq. (6)] is able to catch the above intuitive notion of extreme events in the two limiting cases.

Figure 5 plots the probability of extreme events as a function of x, for $c = 2,5,10$ using continuous curves. We also show the results of stochastic simulation of Brownian particles for $a = 4$ in Fig. 6(a). The stochastic simulation results for other values of a are not shown to avoid clutter of points. For x approximately less that 0.2, the probability is larger for smaller slope of the potential and there is a crossover, and for x approximately larger than 0.2, the probability is larger for larger slope.

We now consider the sinusoidal potential

$$V(x) = B\sin(2\pi x), \tag{17}$$

$$\Phi(x) = V(x)/kT = a\sin(2\pi x), \tag{18}$$

where $a = B/kT$ is a constant depending on the temperature. We then consider the probability of a particle being in a small interval $R = (x - dx/2, x + dx/2)$ of width dx. Figure 6 shows the probability of extreme events as a function of x, for $a = 1,2,3,4$ in Fig. 6(a) and for $a = 4,5,6,7$ in Fig. 6(b). Figure 6(a) corresponds to the region $p > p_1$, where p_1 corresponds to $q = 1$, i.e., the first and the largest peak of F in Fig. 1. We see that on the average the extreme event probability near a maximum of potential is larger than that near a minimum. On the other hand, in Fig. 6(b) we see a slow reversal of behavior near $x = 0.25$ as the amplitude c increases. This corresponds to the region $p < p_1$ in Fig. 1. In this region, the average number of particles is much less than one. The actual number of particles is $0,1,2,\ldots$. When the average number of particles becomes less than one, the probability of having one or two particles decreases as p decreases and hence the probability of extreme events decreases. This is the reason for the reversal of behavior observed in Fig. 6(b) near the maxima.

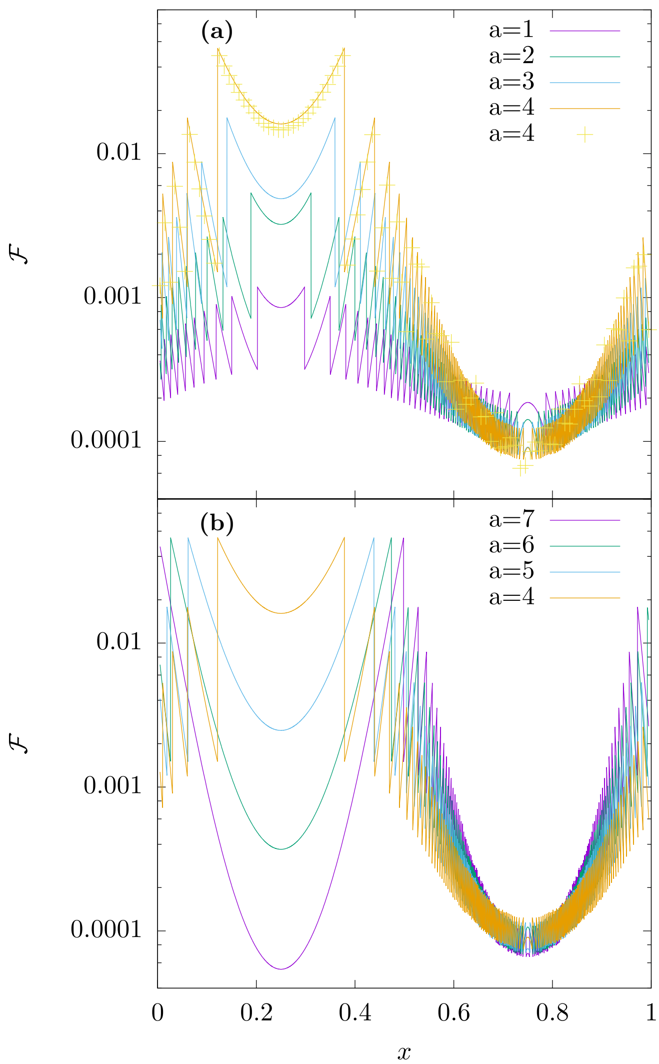

FIG. 6. A plot of extreme event probability $\mathcal{F}(p)$ as a function of x for sinusoidal potential and different values of the magnitude of potential: (a) $a = 1,2,3,4$ and (b) $a = 4,5,6,7$. The curve for $a = 4$ is drawn in both panels (a) and (b) for easy comparison. Here, $N = 1000, m = 4, dx = 0.01$. The continuous curves are the theoretical curves obtained using Eqs. (8) and (6). The symbols are values obtained from stochastic simulation using the Langevin equation, Eq. (A1). We show the stochastic simulation values only for $a = 4$ in Fig. 6(a) to avoid clutter of points.

The two examples above show that except for the exception above which occurs for large potentials and/or very small intervals, the probability of extreme events increases with potential; i.e., on the average the probability of extreme events is larger for larger potentials and smaller for smaller potentials. We have considered some more potentials such as quadratic potential and periodic step function, but they do not appear to provide any additional insight.

The above result may not appear to be obvious. But, it is important to note that the extreme events depend on the fluctuations and not the average value.

IV. DISCUSSION AND CONCLUSION

We have presented a model for analyzing extreme events for a Brownian particle in a potential. We study the probability of observing extreme events in a region and this probability shows oscillations and exponential decay. However, if we see the general average behavior then we find that in general, the larger potential has a larger probability of extreme events while smaller potential has a smaller probability of extreme events. Exception is observed when the average number of particles in an interval is less than one, which may be obtained for very large potentials and/or very small intervals.

In our model, the inherent fluctuations in the model are responsible for giving an extreme event and the extreme event is not obtained by any external driving force. Thus, the extreme event is an integral part of the system and it will always occur.

In our model, we consider N independent noninteracting Brownian particles. There are two ways of looking at it. (a) One way is to treat N independent noninteracting particles as members of an ensemble as in statistical mechanics. This is the meaning that we have used in this paper. When one wants probabilities for the outcome of an experiment, one conducts the experiment several times (independent noninteracting experiments) and then uses the resulting probabilities to make a probabilistic prediction for the result of a single experiment. (b) Another way is to treat N particles as one system. In this case, the correlations and interactions between particles can play an important role. Here, we have not addressed this problem but it can be an extension of the present work.

Since we have analyzed the motion of a Brownian particle, we expect our model to be useful for the problem of transport in a potential. Besides transport in a potential, there are other problems which use potential formulation, e.g., potential energy surface or potential landscape for problems such as chemical reactions, spin glasses, etc. [14,15]. Also, in some problems, one may be able to define optimizing functions similar to potential. Our analysis may be applicable to such problems with suitable modifications. We have mentioned several problems such as tsunami, volcanoes, floods, and storms in the introduction. It will be interesting to see whether any of them can be obtained from a potential function.

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APPENDIX: STOCHASTIC SIMULATION

We have carried out the stochastic simulation of Brownian particles and here we give a brief outline of the procedure we followed. While we use the Smoluchowski equation (a version of the Fokker-Plank equation) to get the probability distribution, we need to use the Langevin equation to simulate motion of Brownian particles.

A general Langevin equation has the form

$$\dot{x}(t) = h(x,t) + g(x,t)\Gamma(t), \tag{A1}$$

where the Langevin force is given by

$$\langle \Gamma(t) \rangle = 0; \quad \langle \Gamma(t)\Gamma(t') \rangle = 2\delta(t-t'). \tag{A2}$$

The noise strength is absorbed in the function $g(x,t)$. The corresponding probability distribution obeys the Fokker-Planck equation.

$$\frac{\partial Q(x,t)}{\partial t} = -\frac{\partial}{\partial x} S(x,t), \tag{A3}$$
$$S(x,t) = \left[D^{(1)} - \frac{\partial}{\partial x} D^{(2)} \right] Q(x,t),$$
$$\frac{\partial Q(x,t)}{\partial t} = \left[-\frac{\partial}{\partial x} D^{(1)} + \frac{\partial^2}{\partial x^2} D^{(2)} \right] Q(x,t)$$

where $S(x,t)$ is the probability current and

$$D^{(1)} = h(x,t) + \frac{\partial g(x,t)}{\partial x} g(x,t), \tag{A4}$$
$$D^{(2)} = g^2(x,t). \tag{A5}$$

For a Brownian particle in a potential we consider the Smoluchowski equation. In the Smoluchowski equation for a particle in a potential $V(x)$ and high friction, we take

$$g^2(x,t) = \frac{kT}{m\gamma}, \tag{A6}$$
$$h(x,t) = \frac{1}{m\gamma} F(x) = -\frac{1}{m\gamma} V'(x). \tag{A7}$$

Thus,

$$D^{(1)} = \frac{1}{m\gamma} F(x), \tag{A8}$$
$$D^{(2)} = D = \frac{kT}{m\gamma}. \tag{A9}$$

Thus, we get the Smoluchowski equation

$$\frac{\partial Q(x,t)}{\partial t} = \frac{1}{m\gamma} \left[-\frac{\partial}{\partial x} F(x) + kT \frac{\partial^2}{\partial x^2} \right] Q(x,t). \tag{A10}$$

This is Eq. (2) of the text. The stationary solution when the probability current is zero is given by Eq. (3). We have used the stationary solution in our theoretical calculations.

We note that when using the Langevin equation we need additional parameters. While the stationary solution involves only the potential normalized by the temperature, we also need the friction coefficient γ and the noise strength $q_n = 2\gamma kT/m$ [5]. These additional parameters have to be chosen so as to get the correct asymptotic stationary solution for the probability distribution, Eq. (3).

For the numerical simulations, we evolve N random walkers starting from randomly chosen initial positions, using the Langevin equation. The evolution is done using numerical integration of stochastic differential equations as described in Ref. [16]. The potential is included using Eq. (A6). The parameters were varied to obtain the correct asymptotic stationary solution, Eq. (3). The final parameters used are $kT/m\gamma = 0.1$ and $q_n = 4.9$. To obtain the extreme event probability, we bin the space in regions of width dx. If the number of random walkers in a bin at a given time exceeds the threshold of

Eq. (6), we treat it as an extreme event. Time average gives the probability of extreme events.

The simulation results for the extreme event probabilities are shown in Figs. 4, 5, and 6.

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