

**Stresses in curved nematic membranes**

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Ordering configurations of a director field on a curved membrane induces stress. In this work, we present a theoretical framework to calculate the stress tensor and the torque as a consequence of the nematic ordering; we use the variational principle and invariance of the energy under Euclidean motions. Euler-Lagrange equations of the membrane as well as the corresponding boundary conditions also appear as natural results. The stress tensor found includes attraction-repulsion forces between defects; likewise, defects are attracted to patches with the same sign in Gaussian curvature. These forces are mediated by the Green function of the Laplace-Beltrami operator of the surface. In addition, we find nonisotropic forces that involve derivatives of the Green function and the Gaussian curvature, even in the normal direction to the membrane. We examine the case of axial membranes to analyze the spherical one. For spherical vesicles we find the modified Young-Laplace law as a consequence of the nematic texture. In the case of spherical cap with defect at the north pole, we find that the force is repulsive with respect to the north pole, indicating that it is an unstable equilibrium point.

DOI: [10.1103/PhysRevE.97.052706](https://doi.org/10.1103/PhysRevE.97.052706)**I. INTRODUCTION**

When extrinsic couplings of Frank's energy describing liquid crystals on curved membranes are neglected, one finds that defects interact with each other through the Green function of the Laplace-Beltrami operator of the surface [1,2]; they also have interactions with the membrane itself and a bulk term appears describing the interaction of the Gaussian curvature of the membrane mediated by the Green function. Clearly, these interactions induce stresses along the membrane, which in turn responds by modifying its shape: the interest in determining the shape of biological membranes because it is related to specific functions of the cell [3,4]. The distribution of stress along the membrane plays a relevant role, whether its shape can change or remain fixed. If the shape of the membrane is frozen, the amount of topological charge is determined precisely by the topology of the membrane through the Hopf-Poincaré and Gauss-Bonnet theorems [5,6]. The nematic texture with defects determines how the stress is distributed along the membrane.

The stress tensor has been calculated in several different ways: in Ref. [7] and using a variational principle the authors find it in the case of fluid membranes; in Ref. [8] and using an elegant and general geometric formalism, the authors find this tensor for very general schemes that can be applied to the relevant case of elastic membranes coated with nematic textures; in Ref. [9] the author finds the stress tensor of the bending energy, examining deformations with respect to a flat membrane. Remarkably, in Ref. [10] the author finds this tensor by using auxiliary variables, avoiding the tedious calculations of deforming the geometric objects involved.

The first main result of this article is the covariant stress tensor of Frank's energy in the so-called limit of one constant, denoted  $\kappa_A$ . Although extrinsic effects is a subject of great

interest [11–14], in the model we examine, the extrinsic couplings are not taken into account, but instead interactions between topological defects and the Gaussian curvature of the surface are explicitly introduced. This model can be seen as the dominant approximation of an effective energy that includes extrinsic corrections in curvature.

The stress tensor found exhibits the forces in the nematic membrane: Two like (unlike) charge defects repel (attract) each other. Defects are attracted to patches with the same sign in Gaussian curvature, the interaction being through the Green function ( $-1/\nabla^2$ ) of the surface. We also find nonisotropic forces that involve derivatives of the Green function and the Gaussian curvature, a result that exhibits more complex nonisotropic forces than those described above.

Using this theoretical framework, we also find the covariant Euler-Lagrange equation for the nematic energy. This equation describes the shape of the membrane that is coupled with the configuration of the director field. It is the covariant form of the von Kármán equation [15], to which it is reduced when we use the Monge approach. In the calculation of deformations of the nematic energy, we have found that the tangential deformations do not imply only a boundary term, that is because this energy is not invariant under reparametrizations: the presence of the nematic texture implies elastic stresses tangent to the membrane. Moreover, when the variational principle is implemented, the boundary conditions for a free edge appear naturally. We write these conditions in terms of geometrical information of the edge curve.

As a relevant example, we obtain the stress tensor in the case of axially symmetric membranes. If the membrane is closed, we find the corresponding Young-Laplace law, Eq. (54), which gives us the relationship with the pressure difference  $P$  between inside and outside. Although this is also a relevant result of this work, this expression still depends on the nematic texture on the vesicle.

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Therefore, we analyze the spherical case that has been studied not only from a theoretical point of view but also experimentally [16–19]. Placing +1 defect at each pole of the spherical vesicle, we find the relationship of  $P$  with the radius  $R$  of the membrane, the surface tension  $\sigma$  and the nematic constant  $\kappa_A$  to be

$$P = \frac{2\sigma}{R} \left( 1 - \frac{\alpha\kappa_A}{2\sigma R^2} \right), \quad (1)$$

where  $\alpha \in (0, 1)$  is constant that depends of the nematic texture. Notice the negative sign of the nematic correction, unlike the positive sign by bending rigidity [20]: while the elastic force of the membrane adds to the surface tension, the nematic force subtracts it.

Taking into account that elastic membranes have  $\sigma \sim 10^{-2}$  J/m<sup>2</sup> and for typical liquid crystals  $\kappa_A \sim 4.1 \times 10^{-21}$  J, to have a nematic correction of at least 10%, the sphere must be  $R \sim 0.7$  nm. Nevertheless, some liquid crystals have  $\kappa_A \sim 10^{-11}$  N and  $\sigma \sim 10^{-5}$  J/m<sup>2</sup>, [21,22]; for these liquid crystals we have a nematic correction of 50% if  $R \sim 1$   $\mu$ m, a reasonable size in which the nematic correction can be observed

If the membrane is not closed, the stress tensor is conserved on its surface. We must also take into account that in addition to the integrated Gaussian curvature over the area, the edge curve determines the topological charge through the Gauss-Bonnet theorem. We analyze two examples within the spherical cap, one of them with charge +1 and the other one with charge +1/2 at the north pole. The result we get is that the force on any horizontal loop is repulsive with respect to the point defect at the pole.

The rest of the paper is organized as follows: in Sec. II we give a brief review to describe the Frank energy on a curved surface, in the limit of one constant. In Sec. III we obtain the response of the energy to small deformations of the embedding function. To avoid confusion in the reading we have separated the calculation of the normal and tangential deformation. In Sec. IV the boundary conditions are obtained. The key point here is to project the edge deformations along the Darboux basis. By using the invariance of the energy under translations and rotations, in Sec. V we find the stress tensor and the torque. In Sec. VI, the case of membranes with axial symmetry is examined, and then some results for the spherical case are obtained. We finished the article with a brief summary in Sec. VII. Most of the long calculations have been written in several appendices at the end of the paper.

## II. NEMATIC ENERGY

Let us consider a surface in  $\mathbb{R}^3$  of coordinates  $\mathbf{x} = (x^1, x^2, x^3)$ . The surface is parametrized by  $\xi^a$ , through the embedding functions  $\mathbf{x} = \mathbf{X}(\xi^a)$ . The induced metric on the surface is given by  $g_{ab} = \mathbf{e}_a \cdot \mathbf{e}_b$ , the euclidean inner product in  $\mathbb{R}^3$  of the tangent vectors  $\mathbf{e}_a = \partial_a \mathbf{X}$  to the surface. The unit normal vector to the surface is defined as  $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2 / \sqrt{g}$ , where  $g = \det g_{ab}$ . The covariant derivative compatible with the induced metric will be denoted  $\nabla_a$ .

Frank's energy describes the ordering of a unit director field  $\boldsymbol{\eta}$ . This energy includes the effect of splay, twist, and bend the

field along the surface. In the limit of one coupling constant the Frank energy can be written as [23]

$$F = \frac{\kappa_A}{2} \int_{\mathcal{M}} dA (\nabla_a \boldsymbol{\eta}^b)^2. \quad (2)$$

The integral involves the infinitesimal area element on the patch  $\mathcal{M}$  given by  $dA = \sqrt{g} d^2\xi$ , and the coupling with the extrinsic curvature has been neglected; nevertheless, by using theoretical and numerical simulations methods, some recent works have taken into account extrinsic effects [11,12,24].

A convenient alternative route to describe this field theory is in terms of the spin connection  $\Omega = \mathbf{e}^a \Omega_a$ , a vector valued function defined in the tangent space of the surface [25], whose fundamental property is its relationship with the Gaussian curvature

$$\nabla \times \Omega = \mathcal{R}_G \mathbf{n}. \quad (3)$$

We define an orthonormal basis  $\boldsymbol{\epsilon}_\alpha$ ,  $\alpha = \{1, 2\}$ , such that the field  $\boldsymbol{\eta}$  can be written in terms of the angle  $\Theta$  with  $\boldsymbol{\epsilon}_1$ :

$$\begin{aligned} \boldsymbol{\eta} &= \eta^\alpha \boldsymbol{\epsilon}_\alpha \\ &= \cos \Theta \boldsymbol{\epsilon}_1 + \sin \Theta \boldsymbol{\epsilon}_2. \end{aligned} \quad (4)$$

The spin connection is defined by  $\boldsymbol{\epsilon}_1 \cdot \nabla_a \boldsymbol{\epsilon}_2 = \Omega_a$ , and with that we have an alternative way of writing the nematic energy Eq. (2) as [25]

$$F = \frac{\kappa_A}{2} \int_{\mathcal{M}} dA g^{ab} (\partial_a \Theta - \Omega_a) (\partial_b \Theta - \Omega_b). \quad (5)$$

Euler-Lagrange equation of the field  $\nabla^a (\nabla_a \Theta - \Omega_a) = 0$  implies that a scalar field  $\chi$  exists such that  $-\varepsilon^{ab} \nabla_b \chi = \nabla^a \Theta - \Omega^a$ , where  $\varepsilon^{ab} = \epsilon^{ab} / \sqrt{g}$ . The presence of topological defects screening by the Gaussian curvature of the membrane is the source of this field:

$$-\nabla^2 \chi = \rho_D(\xi) - \mathcal{R}_G, \quad (6)$$

where  $\rho_D(\xi) = \sum_i q_i \delta(\xi - \xi_i)$  is the charge density. A formal solution of Eq. (6) can be written as

$$\chi = \sum_i q_i G(\xi, \xi^i) - \mathcal{U}, \quad (7)$$

where  $G(\xi, \zeta)$  denotes the Green function associated with the Laplace-Beltrami operator on the surface such that

$$-\nabla^2 G(\xi, \zeta) = \frac{\delta(\xi - \zeta)}{\sqrt{g}} \quad (8)$$

and

$$\mathcal{U}(\xi) = \int_{\mathcal{M}} dA_\zeta G(\xi, \zeta) \mathcal{R}_G(\zeta) \quad (9)$$

defines the geometric potential. The energy can thus be written as

$$\begin{aligned} F &= \int_{\mathcal{M}} dA (\nabla_a \chi)^2 \\ &= \int_{\mathcal{M}} dA \nabla_a (\chi \nabla^a \chi) + \int_{\mathcal{M}} dA \chi (-\nabla^2) \chi. \end{aligned} \quad (10)$$

The first integral in Eq. (10) is a boundary term and the second one is the bulk term that can be developed as

$$\begin{aligned} \int_{\mathcal{M}} dA \chi (-\nabla^2) \chi &= \int_{\mathcal{M}} dA \chi(\xi) [q_i \delta(\xi - \xi^i) - \mathcal{R}_G] \\ &= q_i q_j G(\xi^i, \xi^j) + q_i \mathcal{U}(\xi^i) \\ &\quad + \int_{\mathcal{M}} dA \mathcal{U}(\xi) \mathcal{R}_G(\xi). \end{aligned} \quad (11)$$

From this we see that defects interact with each other through the Green function, we also see that the geometric potential plays the role of an external electric field. The last term is the interaction energy between the Gaussian curvature mediated by the Green function.

In the next section, the shape equation and boundary conditions of the functional energy

$$\mathcal{H} = \sigma \int_{\mathcal{M}} dA + \kappa_A \int_{\mathcal{M}} dA \chi (-\nabla^2) \chi + \sigma_b \oint_C ds \quad (12)$$

will be obtained;  $\sigma$  is the surface tension of the membrane patch  $\mathcal{M}$ , and  $\sigma_b$  is the linear tension of its boundary  $C$ .

### III. SHAPE EQUATIONS AND NOETHER CHARGES

To find the shape equation, we obtain the response of the energy Eq. (12), to small deformations of the embedding functions,  $\mathbf{X} \rightarrow \mathbf{X} + \delta\mathbf{X}$ . We project the deformation into its tangential and normal to the surface:

$$\begin{aligned} \delta\mathbf{X} &= \delta_{\parallel}\mathbf{X} + \delta_{\perp}\mathbf{X} \\ &= \Phi^a \mathbf{e}_a + \Phi \mathbf{n}. \end{aligned} \quad (13)$$

As a first step, we get from Eq. (6):  $-\delta\nabla^2\chi = \delta\rho_D - \delta\mathcal{R}_G$ .

Now, when the area of the surface is modified, the total defects can also be modified. Nevertheless, if the total area remains fixed, local deformations of the surface implies deformations of the charge density without further changes in the total defects. Thus, since the total charge  $Q = \int_{\mathcal{M}} dA \rho_D$  is preserved, we have that  $\delta Q = \int_{\mathcal{M}} (\delta dA) \rho_D + \int_{\mathcal{M}} dA \delta\rho_D = 0$ , in such a way that locally,

$$\delta\rho_D = -\rho_D(\nabla_a \Phi^a + K\Phi), \quad (14)$$

where we used the area deformation,  $\delta dA = dA(\nabla_a \Phi^a + K\Phi)$ .

Let us first get the normal variation of the nematic energy. This deformation can be obtained by using the commutator  $[\delta_{\perp}, \nabla^2]\chi = J_{\perp}$  where  $J_{\perp} = -2K^{ab}\Phi\nabla_a\nabla_b\chi + \nabla_b[(Kg^{ab} - 2K^{ab})\Phi]\nabla_a\chi$ , see Ref. [26], so that we can write  $-\nabla^2\delta_{\perp}\chi = \delta_{\perp}\rho_D - \delta_{\perp}\mathcal{R}_G + J_{\perp}$  and deformation of the energy gets

$$\begin{aligned} \delta_{\perp}F &= - \int_{\mathcal{M}} dA K(\rho_D + \mathcal{R}_G)\chi\Phi \\ &\quad + \int_{\mathcal{M}} dA [J_{\perp} - 2\delta_{\perp}\mathcal{R}_G]\chi, \end{aligned} \quad (15)$$

where we used the normal deformation of the charge density, according to Eq. (14):  $\delta_{\perp}\rho_D = -\Phi K\rho_D$ . Deformation of the Gaussian curvature has also been calculated as [26]

$$\delta_{\perp}\mathcal{R}_G = -\mathcal{R}_G K\Phi + (K^{ab} - g^{ab}K)\nabla_a\nabla_b\Phi. \quad (16)$$

After some algebra and several integrations by parts, we have

$$\delta_{\perp}F = \int_{\mathcal{M}} dA \mathcal{E}_{\perp} \Phi + \int_{\mathcal{M}} dA \nabla_a Q_{\perp}^a, \quad (17)$$

where the Euler-Lagrange derivative of the nematic energy and the Noether charge  $Q_{\perp}^a$  are given by

$$\begin{aligned} \mathcal{E}_{\perp} &= 2(Kg^{ab} - K^{ab})\nabla_a\nabla_b\chi + (2K^{ab} - Kg^{ab})\nabla_a\chi\nabla_b\chi, \\ Q_{\perp}^a &= -2(K^{ab} - Kg^{ab})\chi\nabla_b\Phi + [(Kg^{ab} - 2K^{ab})\chi\nabla_b\chi \\ &\quad + 2(K^{ab} - g^{ab}K)\nabla_b\chi]\Phi. \end{aligned} \quad (18)$$

This expression for the Noether charge has not been completed; tangential deformation is needed, and as we shall see, it is not just a boundary term.

Let us now get the tangential deformation. For the scalar curvature we have (see Appendix)

$$\delta_{\parallel}\mathcal{R} = \Phi^a \nabla_a \mathcal{R}. \quad (19)$$

Notice that the tangential deformation  $\delta_{\parallel}F$  is not only a boundary term, this happens because the nematic energy is not reparameterization invariant. The presence of the director field breaks out this property of the bending energy. To prove this, we see that the commutator with the Laplacian is given by  $[\delta_{\parallel}, \nabla^2]\chi = J_{\parallel}$ , where now

$$J_{\parallel} = (-\nabla^2\Phi^a + \mathcal{R}_G\Phi^a)\nabla_a\chi - 2(\nabla^a\Phi^b)\nabla_a\nabla_b\chi. \quad (20)$$

By using this commutator we have that  $-\nabla^2\delta_{\parallel}\chi = J_{\parallel} + \delta_{\parallel}\rho_D - \delta_{\parallel}\mathcal{R}_G$ , and thus the tangential deformation does depend on the Green function. By using that  $\delta_{\parallel}\rho_D = -\rho_D\nabla_a\Phi^a$  and proceeding as in the case of the normal deformation, we have

$$\delta_{\parallel}F = \int_{\mathcal{M}} dA (\mathcal{E}_a\Phi^a + \nabla_a Q_{\parallel}^a), \quad (21)$$

where we have identified

$$\begin{aligned} \mathcal{E}_a &= 2(\rho_D + \mathcal{R}_G\chi)\nabla_a\chi, \\ Q_{\parallel}^a &= \Phi^b[\nabla^a(\chi\nabla_b\chi) - 2\chi\nabla^a\nabla_b\chi - \delta_b^a(\rho_D + \mathcal{R}_G)\chi] \\ &\quad - \chi\nabla_b\chi\nabla^a\Phi^b. \end{aligned} \quad (22)$$

To obtain the Euler-Lagrange equation of the energy Eq. (12), we write its bulk deformation

$$\delta\mathcal{H} = \int_{\mathcal{M}} dA \mathcal{E} \cdot \delta\mathbf{X} + \int_{\mathcal{M}} dA \nabla_a Q^a, \quad (23)$$

where the Euler-Lagrange derivative

$$\mathcal{E} = (\kappa_A \mathcal{E}_{\perp} + \sigma K)\mathbf{n} + \mathcal{E}_a \mathbf{e}^a, \quad (24)$$

and the Noether charges in  $Q^a = \kappa_A Q_{\perp}^a + (\kappa_A Q_{\parallel}^a + \sigma\Phi^a)$  are given by Eqs. (18) and (22). In equilibrium we have  $\mathcal{E} = 0$ , and therefore its components must vanish:  $\mathcal{E}_{\perp} + \sigma K = 0 = \mathcal{E}_a$ .

An interesting fact occurs if there are no defects on the membrane; in such a case we have that  $\chi = -\mathcal{U}$  and  $\mathcal{E}_a = 0$  implies that  $\nabla_a\mathcal{U} = 0$ , so that the Euler-Lagrange equation simplifies to

$$K(\sigma + 2\kappa_A \mathcal{R}_G) = 0, \quad (25)$$

and therefore, minimal surfaces or hyperboliclike surfaces are solutions to the Euler-Lagrange equation [13,27,28]. Notice that this result has been obtained by deforming the energy

functional  $\mathcal{H}$ , Eq. (12), which contains the function  $\chi$ . If instead of doing that one deforms Eq. (5), which involves  $\Omega_a$ , we get an apparently different result [29]. We will tackle this interesting point in a future work.

As we will see below, from the Noether charge  $Q^a$  we can find both the stress tensor and the torque; these can be found when writing explicitly a translation and rotation of the embedding function. Before that, let us find the boundary conditions that appear naturally in the variational principle.

#### IV. BOUNDARY CONDITIONS

According to the previous section, in equilibrium shapes, deformation of energy Eq. (12), including the boundary terms, is given by

$$\delta\mathcal{H} = \kappa_A \oint_{\mathcal{C}} ds l_a Q^a + \sigma \oint_{\mathcal{C}} ds l_a \Phi^a + \sigma_b \delta \oint_{\mathcal{C}} ds, \quad (26)$$

and thereby the boundary conditions will be obtained by doing  $\delta\mathcal{H} = 0$ .

The calculation involves the Darboux basis adapted to the boundary  $\mathcal{C}$  parametrized by arc length  $s$  [30]. Deformation of the boundary can be projected as

$$\begin{aligned} \delta\mathbf{X} &= \Phi^a \mathbf{e}_a + \Phi \mathbf{n} \\ &= \phi \mathbf{T} + \psi \mathbf{l} + \Phi \mathbf{n}, \end{aligned} \quad (27)$$

where we have defined the scalar functions  $\Phi^a T_a = \phi$  and  $\Phi^a l_a = \psi$ . Therefore, deformation of the unit tangent can be written as

$$\begin{aligned} \delta\mathbf{T} &= \dot{\phi} \mathbf{T} + \dot{\psi} \mathbf{l} + \dot{\Phi} \mathbf{n} + \phi \dot{\mathbf{T}} + \psi \dot{\mathbf{l}} + \Phi \dot{\mathbf{n}} \\ &= (\dot{\phi} - \kappa_g \psi - \kappa_n \Phi) \mathbf{T} + (\dot{\psi} + \kappa_g \phi + \tau_g \Phi) \mathbf{l} \\ &\quad + (\dot{\Phi} + \kappa_n \phi - \tau_g \psi) \mathbf{n}, \end{aligned} \quad (28)$$

where  $\kappa_g$  is the geodesic curvature,  $\kappa_n$  the normal curvature, and  $\tau_g$  the geodesic torsion of the boundary, see Appendix E. The point means derivative with respect to arc length. Then we obtain [31]

$$\begin{aligned} \delta \oint_{\mathcal{C}} ds &= \oint_{\mathcal{C}} ds \mathbf{T} \cdot \delta\mathbf{T} \\ &= \oint_{\mathcal{C}} ds (\dot{\phi} - \kappa_g \psi - \kappa_n \Phi) \\ &= \Delta\phi - \oint_{\mathcal{C}} ds (\kappa_g \psi + \kappa_n \Phi), \end{aligned} \quad (29)$$

where  $\Delta\phi = 0$  for a closed curve. Thus,  $\delta L$  does not include deformation along the unit tangential vector. According to Eqs. (18) and (22) we have  $l_a Q^a = l_a (Q_{\perp}^a + Q_{\parallel}^a)$ . If we write

$$\begin{aligned} Q_{\perp}^a &= M^{ab} \nabla_b \Phi + M^a \Phi, \\ Q_{\parallel}^a &= N^a_b \Phi^b + N_b \nabla^a \Phi^b, \end{aligned} \quad (30)$$

where

$$\begin{aligned} M^{ab} &= 2(Kg^{ab} - K^{ab})\chi, \\ M^a &= [(Kg^{ab} - K^{ab})(\chi - 2) - K^{ab}\chi] \nabla_b \chi, \\ N^{ab} &= \nabla^a \chi \nabla^b \chi - \chi \nabla^a \nabla^b \chi - g^{ab}(\rho_D + \mathcal{R}_G)\chi, \\ N^a &= -\chi \nabla^a \chi, \end{aligned} \quad (31)$$

we have the boundary conditions, see Appendix E:

$$\begin{aligned} -\kappa_A \frac{d}{ds} (l_a M^{ab} T_b) + \kappa_A l_a M^a - \sigma_b \kappa_n &= 0, \\ \kappa_A (l_a N^{ab} l_b + N^b \nabla_l l_b) + \sigma - \sigma_b \kappa_g &= 0, \\ l_a M^{ab} l_b = 0, \quad N^b l_b = 0, \quad N^b T_b = 0, \end{aligned} \quad (32)$$

where we have used that on the boundary  $\nabla_a \Phi = l_a \nabla_l \Phi + T_a \dot{\Phi}$  and  $\nabla_b \Phi^a = T_b \dot{\Phi}^a + l_b \nabla_l \Phi^a$ , and the fact that on the boundary the independent deformations are given by the scalar functions  $\psi, \phi, \Phi$ .

#### V. STRESS AND TORQUE

How the stress is distributed along a membrane is the information that is encoded in the stress tensor [7,32]. To find it, we write the deformation of the energy as

$$\delta\mathcal{H} = \int_{\mathcal{M}} dA \mathcal{E} \cdot \delta\mathbf{X} + \int_{\mathcal{M}} dA \nabla_a Q^a, \quad (33)$$

where the Euler-Lagrange derivative  $\mathcal{E} = (\mathcal{E}_{\perp} + \sigma K) \mathbf{n} + \mathcal{E}_a \mathbf{e}^a$  and the Noether charges  $Q^a = Q_{\perp}^a + Q_{\parallel}^a$  are given by Eqs. (18) and (22). In equilibrium we have that  $\mathcal{E} = 0$ , which implies  $\mathcal{E}_{\perp} = 0 = \mathcal{E}_a$ .

If the energy is invariant under reparametrizations, then its tangential deformation is a boundary term and  $\mathcal{E}_a$  vanish identically; however, if the energy does not have this invariance, as in the case of the nematic energy, these terms are not trivial as we see in Eq. (21).

However, invariance of energy under translations implies that  $\delta H = 0$ , so that locally we have

$$\mathcal{E} = \nabla_a \mathbf{f}^a, \quad (34)$$

where  $\mathbf{f}^a$  is the stress tensor. In equilibrium, the conservation law of the stress  $\nabla_a \mathbf{f}^a = 0$  is fulfilled and thus  $\mathbf{F} = \oint_{\mathcal{C}} ds \mathbf{f}^a l_a$ , is a conserved vector field along the surface; it is identified as the force acting on the curve  $\mathcal{C}$  parametrized by arc length  $s$  with normal  $l_a$ . The tangential derivatives  $\mathcal{E}_b$  will be relevant when coupled with crystalline order through the strain deformation [15,33].

In the case of a membrane that encloses a certain volume  $V$ , we must add the term  $PV$  to the energy, where  $P$  is the pressure difference between the interior and the exterior. In that case the stress tensor is not conserved but  $\nabla_a \mathbf{f}^a = P \mathbf{n}$ , in such a way that

$$\oint_{\mathcal{C}} ds \mathbf{f}^a l_a = \int_{\mathcal{M}} dA P \mathbf{n}. \quad (35)$$

#### A. Stress

Under an infinitesimal translation  $\delta\mathbf{X} = \mathbf{a}$ , we have that  $\Phi = \mathbf{a} \cdot \mathbf{n}$  and  $\Phi^a = \mathbf{a} \cdot \mathbf{e}^a$ ; we also see that  $\nabla_b \Phi = \mathbf{a} \cdot K_b^c \mathbf{e}_c$ . Substituting in Eqs. (18) and (22), we find the stress tensor as

$$\mathbf{f}^a = (f_{\perp}^{ab} + f_{\parallel}^{ab}) \mathbf{e}_b + (f_{\perp}^a + f_{\parallel}^a) \mathbf{n}, \quad (36)$$

where the coefficients are given by

$$\begin{aligned} f_{\perp}^{ab} &= -g^{ab}(\sigma + 2\chi \mathcal{R}_G), \\ f_{\parallel}^{ab} &= \chi \nabla^a \nabla^b \chi - \nabla^a \chi \nabla^b \chi + g^{ab}(\rho_D + \mathcal{R}_G)\chi, \end{aligned}$$

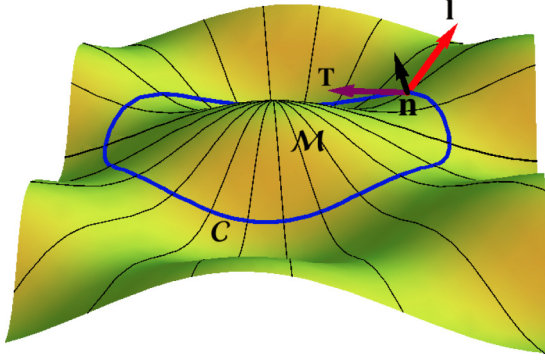


FIG. 1. A +1 defect on the top of a mountain. At any point of the curve  $C$ , the force per unit length along  $\mathbf{l}$  is given by  $-(\sigma + \kappa_A \chi \mathcal{R}_G)$ . In addition, there is the anisotropic force  $\kappa_A l^a l^b (\chi \nabla_a \nabla_b \chi - \nabla_a \chi \nabla_b \chi)$  along  $\mathbf{l}$ . Darboux frame adapted to the curve is shown:  $\mathbf{T}$  the unit tangent, the unit normal to the surface  $\mathbf{n}$ , and  $\mathbf{l} = \mathbf{T} \times \mathbf{n}$ .

$$\begin{aligned} f_{\perp}^b &= -(K g^{ab} - 2K^{ab}) \chi \nabla_a \chi - 2(K^{ab} - g^{ab} K) \nabla_a \chi, \\ f_{\parallel}^b &= -K^{ab} \chi \nabla_a \chi. \end{aligned} \quad (37)$$

We have verified that the relationship Eq. (34) with the Euler-Lagrange derivatives is fulfilled, this guarantees that both the expression for the stress tensor and the shape equation are self-consistent.

Let  $\mathbf{x}(s) = \mathbf{X}[\xi^a(s)]$  be a curve  $C$  parametrized by arc length on the surface, see Fig. 1; as before, we identify the Darboux basis adapted to it:  $\mathbf{T} = T^a \mathbf{e}_a$ , its tangent vector, and  $\mathbf{l} = l^a \mathbf{e}_a$ , the outward pointing unit vector, such that  $\mathbf{l} = \mathbf{T} \times \mathbf{n}$ . The force per unit of length can be written as

$$\mathbf{l}^a l_a = F_T \mathbf{T} + F_l \mathbf{l} + F_n \mathbf{n}, \quad (38)$$

where  $F_T = l_a T_b f^{ab}$ ,  $F_l = l_a l_b f^{ab}$ , and  $F_n = l_a f^a$ . We get

$$\begin{aligned} F_l &= -f + l_a l_b (\chi \nabla^a \nabla^b \chi - \nabla^a \chi \nabla^b \chi), \\ F_T &= l_a T_b (\chi \nabla^a \nabla^b \chi - \nabla^a \chi \nabla^b \chi), \\ F_n &= l_b (K^{ab} - g^{ab} K) (\chi - 2) \nabla_a \chi. \end{aligned} \quad (39)$$

Note that  $F_l$  includes  $f = \sigma - (\rho_D - \mathcal{R}_G) \chi$ . This force can be written explicitly:

$$\begin{aligned} -f &= -\sigma + \sum_{i \neq j} q_i q_j \delta(\mathbf{x} - \mathbf{x}^i) G(\mathbf{x}, \mathbf{x}^j) - \sum_i q_i G(\mathbf{x}, \mathbf{x}^i) \mathcal{R}_G \\ &\quad - \sum_i q_i \delta(\mathbf{x} - \mathbf{x}^i) \mathcal{U} + \mathcal{U} \mathcal{R}_G. \end{aligned} \quad (40)$$

The second term is the force on the charge  $q_i$  due to  $q_j$ ; it is given by  $q_i q_j G(\mathbf{x}^i, \mathbf{x}^j)$ , and this force is repulsive (attractive) between defects with like (unlike) charge. Similarly, the third term is the force on the point  $\mathbf{x}$  (of Gaussian curvature  $\mathcal{R}_G$ ), caused by the presence of  $q_i$  at the point  $\mathbf{x}^i$ : defects are attracted to points with the same sign of Gaussian curvature. These interactions are mediated by the Green function. The fourth term is a self-force at the point  $\mathbf{x}^i$  with the Gaussian curvature at the same point.

The total force along  $\mathbf{l}$  includes the anisotropic stress  $\kappa_A l_a l_b (\chi \nabla^a \nabla^b \chi - \nabla^a \chi \nabla^b \chi)$ , along  $\mathbf{l}$  and  $\mathbf{T}$ . Finally, there is also a force  $F_n$  along the unit normal to the surface as given in Eq. (39). None of these forces has been reported so far.

## B. Torque

Taking now an infinitesimal rotation  $\delta \mathbf{X} = \mathbf{b} \times \mathbf{X}$ , we have that  $\Phi = \mathbf{b} \cdot \mathbf{X} \times \mathbf{n}$  and  $\Phi_a = \mathbf{b} \cdot \mathbf{X} \times \mathbf{e}_a$ . Therefore, we can write

$$\begin{aligned} \nabla^b \Phi &= \mathbf{b} \cdot (\mathbf{e}^b \times \mathbf{n} + K^{ab} \mathbf{X} \times \mathbf{e}_a) \\ &= \mathbf{b} \cdot (\varepsilon^{ab} \mathbf{e}_a + K^{ab} \mathbf{X} \times \mathbf{e}_a). \end{aligned} \quad (41)$$

Similarly, we have

$$\nabla_b \Phi_a = \mathbf{b} \cdot (\varepsilon_{ba} \mathbf{n} - K_{ab} \mathbf{X} \times \mathbf{n}), \quad (42)$$

where now  $\varepsilon_{ab} = \sqrt{g} \epsilon_{ab}$ . Deformation of the energy under a rotation is then given by [7]

$$\delta \mathcal{H} = \int_{\mathcal{M}} dA \mathcal{E} \cdot (\mathbf{b} \times \mathbf{X}) + \int_{\mathcal{M}} dA \nabla_a \mathbf{m}^a, \quad (43)$$

where

$$\mathbf{m}^b = \mathbf{X} \times \mathbf{f}^b + \mathbf{s}^b, \quad (44)$$

$\mathbf{f}^b$  being the stress tensor Eq. (36), and

$$\mathbf{s}^b = 2(K_a^b - \delta_a^b K) \chi \varepsilon^{ac} \mathbf{e}_c + \varepsilon^{ab} \chi \nabla_a \chi \mathbf{n}. \quad (45)$$

In equilibrium we have  $\mathcal{E} = 0$  so that  $\mathbf{m}^a$  is conserved as a consequence of invariance under rotations. The first term in Eq. (44) is the orbital torque, while  $\mathbf{s}^b$  can be seen as an intrinsic torque. If we use the fact that  $\varepsilon^{ac} \mathbf{e}_c = l^a \mathbf{T} - T^a \mathbf{l}$ , then we obtain the intrinsic torque in the Darboux basis along a curve on the membrane.

## VI. AXIAL NEMATIC MEMBRANES

Let us see the case of axial surfaces parametrized as

$$\begin{aligned} \mathbf{X}(l, \phi) &= (\rho(l) \cos \phi, \rho(l) \sin \phi, h(l)) \\ &= \rho \boldsymbol{\rho} + h \mathbf{k} \end{aligned} \quad (46)$$

where  $\boldsymbol{\rho} = (\cos \phi, \sin \phi, 0)$  is a unit radial vector field, and  $\mathbf{k} = (0, 0, 1)$ . The tangent vectors to the surface can be found to be

$$\begin{aligned} \mathbf{e}_l &= (\rho' \cos \phi, \rho' \sin \phi, h') \\ &= \rho' \boldsymbol{\rho} + h' \mathbf{k}, \\ \mathbf{e}_\phi &= (-\rho \sin \phi, \rho \cos \phi, 0) \\ &= \rho \boldsymbol{\phi}, \end{aligned} \quad (47)$$

where  $\boldsymbol{\phi} = (-\sin \phi, \cos \phi, 0)$  is the unit azimuthal vector and  $'$  denotes derivative with respect to  $l$ . The induced metric on the surface can be written as

$$g_{ab} d\xi^a d\xi^b = dl^2 + \rho^2 d\phi^2, \quad (48)$$

where we have taken the parameter  $l$  along the meridians to be the arc length such that  $h'^2 + \rho^2 = 1$ . The unit normal to the surface  $\mathbf{n} = \boldsymbol{\phi} \times \mathbf{e}_l$ , is given by

$$\begin{aligned} \mathbf{n} &= (h' \cos \phi, h' \sin \phi, -\rho') \\ &= h' \boldsymbol{\rho} - \rho' \mathbf{k}. \end{aligned} \quad (49)$$

The second fundamental form can be written as

$$K_{ab} d\xi^a d\xi^b = -\frac{\rho''}{h'} dl^2 + \rho h' d\phi^2, \quad (50)$$

whereas the mean curvature  $K = h'/\rho - \rho''/h'$ , and the Gaussian curvature  $\mathcal{R}_G = -\rho''/\rho$ . Let  $\epsilon_1 = \boldsymbol{\phi}$  and  $\epsilon_2 = \rho'\boldsymbol{\rho} + h'\mathbf{k}$  be the unit basis so that the components of the spin connection are given by  $\Omega_l = 0$  and  $\Omega_\phi = \rho'$ . Along a horizontal curve we have  $l_l = \mathbf{1} \cdot \mathbf{e}_l = 1$  and  $T_l = 0$ , so that in these coordinates the coefficients Eq. (38) of the force  $l_l \mathbf{f}'$  per unit length on a horizontal loop can also be written as

$$l_l \mathbf{f}' = (F_l \rho' + F_n h') \boldsymbol{\rho} + (F_l h' - F_n \rho') \mathbf{k}, \quad (51)$$

where we have

$$F_l = -\sigma + \left( \rho_D + \frac{\rho''}{\rho} \right) \chi + [\chi \chi'' - (\chi')^2],$$

$$F_n = \left( -\frac{h'}{\rho} \right) (\chi - 2) \chi', \quad F_T = 0. \quad (52)$$

We note that although  $\rho = \rho(l)$  by the axial symmetry, in a general setting, the presence of the nematic texture implies that the coefficients depend on both variables  $(l, \phi)$  on the surface, through the function  $\chi$ . This force has radial and vertical components. The total vertical force on the loop is then

$$\mathbf{F}(l) = \mathbf{k} \int_0^{2\pi} d\phi \rho (F_l h' - F_n \rho')$$

$$= \mathbf{k} [h' \langle F_l \rangle - \rho' \langle F_n \rangle], \quad (53)$$

where we have denoted  $\langle F \rangle = \int_0^{2\pi} d\phi \rho F$ . If the membrane is a closed surface we must take into account the pressure difference  $P$  between the inside and outside to the nematic membrane. Equation (35) is then

$$2\rho(h' F_l - \rho' F_n) = -P\rho^2, \quad (54)$$

where we have taken  $\rho(0) = 0$ . This equation must be satisfied for each value of  $l$  in the domain considered; it is the corresponding Young-Laplace law.

### A. Spherical particles

Without nematic texture in the membrane such that  $F_l = -\sigma$  and  $F_n = 0$ , Eq. (54) reduces to  $2\rho h' \sigma = P\rho^2$ . By using that  $h' = \sqrt{1 - \rho'^2}$  and taking the simplest case such that  $P$  is a constant, we obtain

$$\rho(l) = \frac{2\sigma}{P} \sin\left(\frac{Pl}{2\sigma}\right), \quad (55)$$

which is the representation of a sphere with radius  $R = 2\sigma/P$ ; this is the corresponding Young-Laplace equation, which relates the surface tension  $\sigma$ , the pressure  $P$ , and the radius of the sphere  $R$ . Let us find the corresponding law in the presence of the nematic texture. From Eq. (54), we see that it is necessary to calculate the function  $\chi$  that involves the Green function on the sphere. To this, write the metric in isothermal coordinates

$$ds^2 = \omega(dr^2 + r^2 d\phi^2), \quad (56)$$

where  $r > 0$ ,  $\phi \in [0, 2\pi]$ , and  $\omega$  is the conformal factor [34]. Comparison with the induced metric in axial coordinates Eq. (48) gives

$$dl^2 = \omega dr^2, \quad \omega r^2 = \rho^2. \quad (57)$$

That is,  $\log r = \int dl/\rho + C$ . Let  $\xi = (l, \phi)$  and  $\zeta = (\ell, \varphi)$  and write the Green function that satisfies the equation

$$-\left[ \frac{1}{\rho} \partial_l (\rho \partial_l) + \frac{1}{\rho^2} \partial_\phi^2 \right] G(\xi, \zeta) = \frac{1}{\rho} \delta(l - \ell) \delta(\phi - \varphi), \quad (58)$$

replacing with isothermal coordinates  $u = (r, \phi)$ , gets into

$$-\nabla^2 G = -\frac{1}{\omega} \nabla_u^2 G(u, u') = \frac{1}{\omega r} \delta(r - r') \delta(\phi - \phi'). \quad (59)$$

The last equality in Eq. (59) implies the Green function in isothermal coordinates,

$$G(u, u') = -\frac{1}{4\pi} \log[r(l)^2 + r(\ell)^2 - 2r(l)r(\ell) \cos(\phi - \varphi)]. \quad (60)$$

If the surface is closed, the singularities that appear into the Green function can be eliminated if we subtract both  $\bar{G}(\xi) = (1/A) \int dA_\zeta G(\xi, \zeta)$  and  $\bar{G}(\zeta)$ . Let us look explicitly at the example of the sphere; parametrize it as

$$\rho(l) = R \sin(l/R), \quad h(l) = -R \cos(l/R), \quad (61)$$

where  $l \in [0, \pi R]$ . If we choose  $r(\pi R/2) = R$ , then we have

$$r(l) = R \tan\left(\frac{l}{2R}\right), \quad (62)$$

and we can obtain

$$\bar{G}(\xi) = \frac{1}{A} \int_0^{\pi R} d\ell \rho(\ell) \int_0^{2\pi} d\phi G(\xi, \zeta)$$

$$= -\frac{1}{8\pi R^2} \int_0^{\pi R} d\ell \rho(\ell) \log r_{>}^2$$

$$= \frac{1}{4\pi} \log \cos^2\left(\frac{l}{2R}\right), \quad (63)$$

where  $r_{>}$  refers to the larger value between  $r(l)$  and  $r(\ell)$ . The Green function can then be written as

$$G(\xi, \zeta) = -\frac{1}{4\pi} \log \left[ \sin^2(l/2R) \cos^2(\ell/2R) \right.$$

$$+ \left. \sin^2(\ell/2R) \cos^2(l/2R) \right.$$

$$\left. - \frac{1}{2} \sin(l/R) \sin(\ell/R) \cos(\phi - \varphi) \right]. \quad (64)$$

Therefore, as shown in Appendix F, the geometric potential is simply given by  $\mathcal{U} = 1$ . Thus, with a charge  $+1$  at each pole, the function  $\chi$  can be written as

$$\chi(l) = -\frac{1}{4\pi} \log \left[ \sin^2\left(\frac{l}{2R}\right) \cos^2\left(\frac{l}{2R}\right) \right] - 1. \quad (65)$$

Notice that as a consequence of topological defects at the poles, singularities in Eq. (65) appear, see Fig. 2.

Now, since Eq. (54) is fulfilled for  $l \in (0 + \epsilon, \pi R - \epsilon)$ , where  $\epsilon$  is related with the core of defects, it can be rewritten as

$$P = \frac{2\sigma}{R} \left( 1 - \frac{\alpha \kappa_A}{R^2 2\sigma} \right), \quad (66)$$

where  $\alpha$  is a fixed number  $\in (0, 1)$ , which is obtained from

$$\alpha = \dot{\chi}(\chi - 2) - \chi + (\chi \ddot{\chi} - \dot{\chi}^2), \quad (67)$$

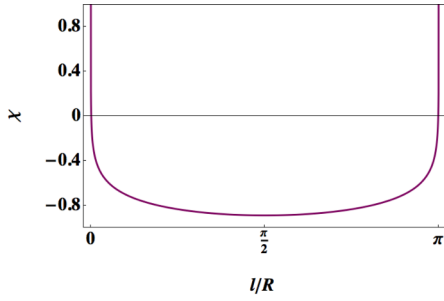


FIG. 2. The function  $\chi$  in Eq. (65) with  $+1$  defect at each pole where singularities appear.

where the dot means derivative with respect to  $x = l/R$ . We see that the surface tension has been modified by the presence of the nematic texture with  $+1$  defects at the poles, see Fig. 3. As mentioned in the Introduction, for spherical membranes with  $R \sim 1 \mu\text{m}$ , coated with some liquid crystals, the nematic correction is about 50%.

In the case of a spherical cap  $\mathcal{M}$ , Gauss-Bonnet implies that

$$\int_{\mathcal{M}} dA \mathcal{R}_G + \oint_{\mathcal{C}} \kappa_g ds = 2\pi Q, \quad (68)$$

where  $\kappa_g$  is the Gaussian curvature of the boundary curve  $\mathcal{C}$  parametrized by arc length  $s$ . The sum of these integrals is equivalent to the charge  $Q$  of defects into the surface. Integration of the Gaussian curvature gives

$$\begin{aligned} \int_{\mathcal{M}} dA \mathcal{R}_G &= \frac{2\pi}{R^2} \int_0^{l_0} dl R \sin(l/R) \\ &= 4\pi \sin^2(l_0/2R). \end{aligned} \quad (69)$$

If the boundary is the parallel  $l = l_0$ , then we find

$$\oint_{\mathcal{C}} \kappa_g ds = 2\pi \cos(l_0/R), \quad (70)$$

and therefore, the total charge on the spherical cap is given by

$$Q = 4 \sin^2(l_0/2R) - 1. \quad (71)$$

For a half sphere  $l_0 = \pi R/2$ , we have  $Q = 1$ , in such a case, the boundary is a geodesic curve with  $\kappa_g = 0$ ; a cap with  $l_0 = \pi R/3$  as boundary point has a nematic texture with  $Q = 0$ . If  $l_0 = 2\pi R/3$ , then we have  $Q = 2$ . Notice that  $Q = 1/2$  if  $l_0 = 2R \arcsin(\sqrt{3}/2) \sim 5\pi R/12$  and there is not  $l_0$  such

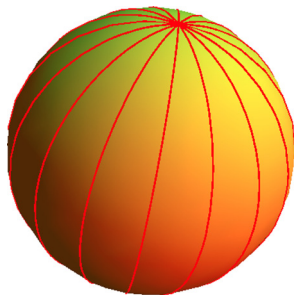


FIG. 3. A spherical *particle* with radius  $R$  and  $+1$  defects on opposite sides; with this nematic texture, the relationship between the parameters is given by the Young-Laplace law, Eq. (66).

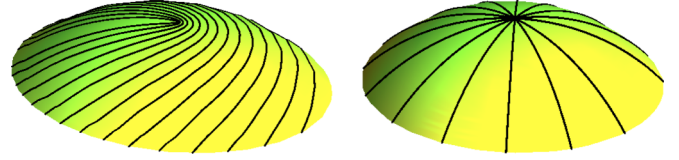


FIG. 4. Nematic texture on spherical sheets with boundary at  $l_0 = 2R \arcsin(\sqrt{3}/2)$  and  $l_0 = \pi R/2$  respectively. Gauss-Bonnet theorem implies defects with  $q = +1/2$  and  $q = +1$  on them.

that  $Q = -1$ . Two of these caps with their nematic texture are shown in Fig. 4. For each of these spherical shells the Green function is given by Eq. (60), while  $r(l)$  is given by Eq. (62), but now we must impose boundary conditions on the Green function at  $l = l_0$ . Under Dirichlet boundary conditions it reads

$$G(\xi, \zeta) = -\frac{1}{4\pi} \log \left[ \frac{r^2(l) + r(\ell)^2 - 2r(l)r(\ell) \cos(\phi - \varphi)}{\frac{r(l)^2 r(\ell)^2}{r_0^2} + r_0^2 - 2r_0 r(\ell) \cos(\phi - \varphi)} \right], \quad (72)$$

where  $r_0 = r(l_0)$ . After making some integrations we can find the geometric potential  $\mathcal{U}$  as

$$\begin{aligned} \mathcal{U} &= -\log \left[ \frac{\cos^2(l/2R) \sin^2(l/2R)}{\sin^2(l_0/2R)} \right] \\ &+ \cos(l/R) \log \left[ \frac{\tan^2(l/2R)}{\tan^2(l_0/2R)} \right]. \end{aligned} \quad (73)$$

For a half spherical cap,  $r_0 = R$  and  $\sin(l_0/2R) = 1/\sqrt{2}$ , and thus we get

$$\begin{aligned} \mathcal{U} &= -\log[2 \cos^2(l/2R) \sin^2(l/2R)] \\ &+ \cos(l/R) \log[\tan^2(l/2R)]. \end{aligned} \quad (74)$$

If the boundary is at the point  $l_0 = 2R \arcsin(\sqrt{3}/2)$ , we obtain the geometric potential as

$$\begin{aligned} \mathcal{U} &= -\log \left[ \frac{8}{3} \cos^2(l/2R) \sin^2(l/2R) \right] \\ &+ \cos(l/R) \log \left[ \frac{5}{3} \tan^2(l/2R) \right]. \end{aligned} \quad (75)$$

Figure 5 shows these geometric potentials: To minimize the energy, defects must be at  $l = 0$ ; nevertheless, as we shall see, it is an unstable equilibrium point. For a half sphere such that  $l_0 = \pi R/2$  and  $\ell = 0$  (defect at the pole  $\zeta = \zeta_N$ ), we have

$$G(\xi, \zeta_N) = -\frac{1}{4\pi} \log[\tan^2(l/2R)]. \quad (76)$$

If the boundary is at  $l_0/R = 2 \arcsin(\sqrt{3}/2)$  and the defect at the north pole, we obtain the Green function as

$$G(\xi, \xi_0) = -\frac{1}{4\pi} \log \left[ \frac{5}{3} \tan^2(l/2R) \right]. \quad (77)$$

Since the membrane is not closed, then  $\mathbf{F}$  in Eq. (51) is a conserved quantity; in particular, we evaluate it at the equator of the half sphere. In this case,  $F_l$  can be written as

$$F_l = -\sigma - \frac{1}{R^2} (\chi - \chi \ddot{\chi} + \dot{\chi}^2), \quad (78)$$

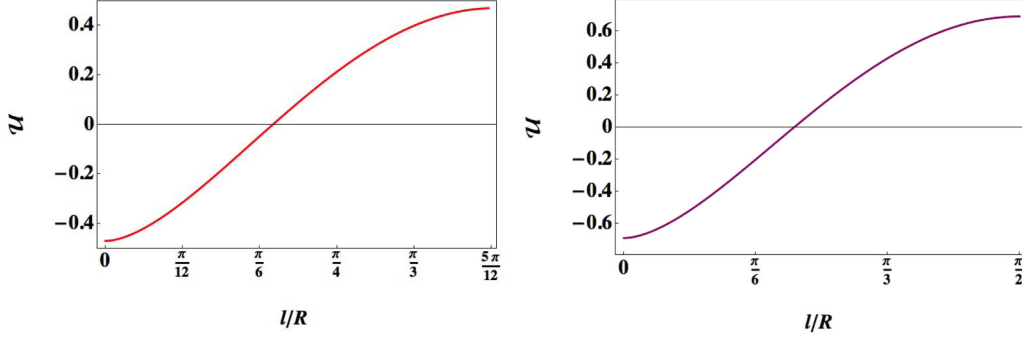


FIG. 5. The geometric potential  $\mathcal{U}$  for the spherical caps in Fig. 4. The point  $l = 0$  being the north pole, where the defect is placed. The force  $\mathcal{R}_G \mathcal{U}$  before than the root  $x_0 = l_0/R$  is attractive to the defect point and repulsive after this point.

where the dot means derivative with respect to  $x = l/R$ . The force on a horizontal loop is thereby given by

$$\mathbf{F} = \langle F_l \rangle \mathbf{k} = -2\pi R\sigma \left( 1 + \frac{\kappa_A}{\sigma R^2} \frac{C}{2} \right) \mathbf{k}, \quad (79)$$

where  $C \sim 0.72$  for half sphere with  $q = 1$  at the north pole, and  $C \sim 0.49$  for spherical cap with defect  $q = +1/2$  at the pole. This force acts to elongate the shape of the membrane towards cylindrical forms [35–37].

## VII. SUMMARY

In this work we have introduced a framework to calculate both the stress tensor and the torque induced by nematic ordering on curved membranes. Using the variational principle and differential geometry of surfaces, we obtain the Euler-Lagrange equations and boundary conditions. Taking advantage of invariance under translations and rotations, we find the corresponding Noether charges; from these we obtain the stress tensor and the torque, respectively. We find repulsive (attractive) forces between defects with like (unlike) charge; defects are attracted to points with the same sign of Gaussian curvature. These forces are mediated by the Green function of the Laplace-Beltrami operator of the surface. Furthermore, we find anisotropic forces that involve derivatives of both the Green function and the Gaussian curvature. Extrinsic geometry only plays a role in the forces along the normal direction to the surface. We present these results in a coordinate independent way. We next applied this framework to the case of membranes with axial symmetry to analyze the spherical case. For a spherical vesicle with defects at the poles we find the modified Young-Laplace law. We find that for certain liquid crystals, the nematic corrections to the Young-Laplace law will be at least 50%, if the radius of the vesicle  $R \sim 1-10 \mu\text{m}$ , a reasonable size in micropipette experiments [38]. For spherical layers with a defect at the north pole we find that the force at any point is repulsive with respect to the pole, which implies that it is an unstable equilibrium point.

It is possible that this *nematic* force be relevant in the description of nanoparticles embedded onto spherical nematic vesicles [39]. As we will show in a future report, it is possible to extend this theoretical framework to take into account the effect of extrinsic couplings, a fact that may be relevant for

both the texture of the nematic and the membrane shape itself [40].

## ACKNOWLEDGMENTS

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## APPENDIX A: TANGENTIAL DEFORMATION OF SCALAR CURVATURE

We need the deformation of the scalar curvature,

$$\delta \mathcal{R} = g^{ab} \delta \mathcal{R}_{ab} + \delta g^{ab} \mathcal{R}_{ab}. \quad (\text{A1})$$

The first term in Eq. (A1) can be calculated in terms of deformations of the Christoffel symbols

$$g^{ab} \delta \mathcal{R}_{ab} = g^{ab} \nabla_c (\delta \Gamma_{ab}^c) - g^{ab} \nabla_b (\delta \Gamma_{ca}^c), \quad (\text{A2})$$

where we can write

$$\delta \Gamma_{ab}^c = \frac{1}{2} g^{cd} (\nabla_b \delta g_{ad} + \nabla_a \delta g_{bd} - \nabla_d \delta g_{ab}). \quad (\text{A3})$$

Using the fact that the induced metric transforms as  $\delta_{\parallel} g_{ab} = \nabla_a \Phi_b + \nabla_b \Phi_a$ , so that the tangential deformation of the Christoffel symbols are given by

$$g^{ab} \delta_{\parallel} \Gamma_{ab}^c = \frac{1}{2} g^{cd} (2\nabla^2 \Phi_d + [\nabla_a, \nabla_d] \Phi^a + [\nabla_b, \nabla_d] \Phi^b) = \nabla^2 \Phi^c + \mathcal{R}_a{}^c \Phi^a \quad (\text{A4})$$

$$g^{ab} \delta_{\parallel} \Gamma_{ca}^c = \frac{1}{2} g^{cd} g^{ab} ([\nabla_c, \nabla_d] \Phi_a + [\nabla_a, \nabla_d] \Phi_c + [\nabla_a, \nabla_c] \Phi_d + 2\nabla_c \nabla_a \Phi_d) = g^{cd} g^{ab} \nabla_c \nabla_a \Phi_d - \mathcal{R}_c{}^b \Phi^c, \quad (\text{A5})$$

where the commutator  $[\nabla_a, \nabla_b] \Phi^c = \mathcal{R}^c{}_{dab} \Phi^d$  has been used. By taking the corresponding gradients, and using the fact that  $\nabla_b \nabla_c \nabla^b \Phi^c = \nabla_c \nabla^2 \Phi^c$ , we can write Eq. (A2) as

$$g^{ab} \delta_{\parallel} \mathcal{R}_{ab} = 2\nabla_a (\mathcal{R}_c{}^a \Phi^c). \quad (\text{A6})$$

Adding this result into Eq. (A1) we obtain Eq. (19).

## APPENDIX B: THE COMMUTATOR $[\delta_{\parallel}, \nabla^a] f$

Deformation of a second derivative can be written as

$$[\delta_{\parallel}, \nabla_a \nabla_b] f = -(\nabla_c f) \delta_{\parallel} \Gamma_{ab}^c, \quad (\text{B1})$$



so that

$$[\delta_{\parallel}, \nabla^2]f = -g^{ab}(\nabla_c f)\delta_{\parallel}\Gamma_{ab}^c + (\delta_{\parallel}g^{ab})\nabla_a\nabla_b f, \quad (\text{B2})$$

by using  $g^{ab}\delta_{\parallel}\Gamma_{bc}^a$  from Appendix A, we find Eq. (20).

### APPENDIX C: MONGE GAUGE

In the representation a la Monge, where the embedding function is  $\mathbf{X}(x, y) = (x, y, f(x, y))$ , the induced metric can be written as  $g_{ab} = \delta_{ab} + \nabla_a f \nabla_b f$  and its inverse

$$g^{ab} = \delta^{ab} - \frac{\nabla_a f \nabla_b f}{1 + (\nabla f)^2}. \quad (\text{C1})$$

The normal vector to the surface is given by  $\mathbf{n} = \frac{(-\nabla_a f, 1)}{\sqrt{1 + (\nabla f)^2}}$ .

The extrinsic curvature is then

$$K_{ab} = -\frac{\nabla_{ab}^2 f}{\sqrt{1 + (\nabla f)^2}}, \quad (\text{C2})$$

and the mean curvature is

$$K = -\frac{\nabla^2 f}{\sqrt{1 + (\nabla f)^2}} + \frac{\nabla_a f \nabla_b f \nabla_{ab}^2 f}{(1 + (\nabla f)^2)^{3/2}}. \quad (\text{C3})$$

To lower order and without defects we can write the shape equation as

$$\begin{aligned} &(\partial_y^2 f)\partial_x^2 \mathcal{U} + (\partial_x^2 f)\partial_y^2 \mathcal{U} - 2(\partial_{xy}^2 f)\partial_{xy}^2 \mathcal{U} \\ &+ \frac{1}{2}[\partial_x^2 f - \partial_y^2 f][(\partial_y \mathcal{U})^2 - (\partial_x \mathcal{U})^2] \\ &- 2(\partial_{xy}^2 f)(\partial_x \mathcal{U})(\partial_y \mathcal{U}) = 0. \end{aligned} \quad (\text{C4})$$

When the corresponding term of the bending energy is added, the von Kármán equation is obtained.

### APPENDIX D: DEFORMATION OF THE NEMATIC ENERGY

Write the nematic energy of the membrane  $\chi$  as

$$F = -\int_{\mathcal{M}} dA \chi \nabla^2 \chi, \quad (\text{D1})$$

where the field  $\chi$  satisfies the equation

$$-\nabla^2 \chi = \rho_D - \mathcal{R}_G, \quad (\text{D2})$$

and  $\rho_D$  is the charge density. Deformation of Eq. (D1) can be written as

$$\delta F = -\int_{\mathcal{M}} [(\delta dA)\chi \nabla^2 \chi \underbrace{- dA(\delta\chi)\nabla^2 \chi}_{II} \underbrace{- dA\chi(\delta\nabla^2 \chi)}_{III}]. \quad (\text{D3})$$

In the second term, deformation of the field  $\delta\chi$  can be calculated as

$$\delta\chi = \int_{\mathcal{M}'} dA' G(\xi, \xi')(J' + \delta\rho_D' - \delta\mathcal{R}_G'), \quad (\text{D4})$$

which is because  $-\delta\nabla^2 \chi = \delta\rho_D - \delta\mathcal{R}_G$ , so that if the commutator  $[\delta, \nabla^2]f = J$ , we have  $-\nabla^2 \delta\chi = J + \delta\rho_D - \delta\mathcal{R}_G$ ,

and thus Eq. (D4) follows. The integrals in Eq. (D3) can then be written as

$$\begin{aligned} II &= \int_{\mathcal{M}} dA \chi (J + \delta\rho_D - \delta\mathcal{R}_G), \\ III &= \int_{\mathcal{M}} dA \chi (\delta\rho_D - \delta\mathcal{R}_G). \end{aligned} \quad (\text{D5})$$

We have then  $II + III = \int_{\mathcal{M}} dA \chi (J + 2\delta\rho_D - 2\delta\mathcal{R}_G)$ , and therefore we can write

$$\delta F = -\int_{\mathcal{M}} (\delta dA)\chi \nabla^2 \chi + \int_{\mathcal{M}} dA \chi (J + 2\delta\rho_D - 2\delta\mathcal{R}_G). \quad (\text{D6})$$

Once again, let us calculate separately. For the normal deformation, the first integral in Eq. (D6) becomes

$$-\int_{\mathcal{M}} (\delta_{\perp} dA)\chi \nabla^2 \chi = -\int_{\mathcal{M}} dA [K\chi \nabla^2 \chi] \Phi.$$

In the second integral, we substitute  $J_{\perp}$  and several integrations by parts to obtain

$$\begin{aligned} \int_{\mathcal{M}} dA J_{\perp} \chi &= -\int_{\mathcal{M}} dA \chi \{ [2K^{ab}\nabla_a\nabla_b \chi + (\nabla_a K)(\nabla^a \chi)] \Phi \\ &\quad + (2K^{ab} - Kg^{ab})\nabla_a \chi \nabla_b \Phi \}, \end{aligned}$$

which can be written as

$$\begin{aligned} &= -\int_{\mathcal{M}} dA \chi [2K^{ab}\nabla_a\nabla_b \chi + (\nabla_a K)(\nabla^a \chi)] \Phi \\ &\quad + \int_{\mathcal{M}} dA \nabla_b [(2K^{ab} - Kg^{ab})\chi \nabla_a \chi] \Phi \\ &\quad - \int_{\mathcal{M}} dA \nabla_b [(2K^{ab} - Kg^{ab})\chi (\nabla_a \chi) \Phi]. \end{aligned} \quad (\text{D7})$$

We also have that

$$2\int_{\mathcal{M}} dA \chi \delta_{\perp} \rho_D = -2\int_{\mathcal{M}} dA \chi \rho_D K \Phi. \quad (\text{D8})$$

The last integral in Eq. (D6) can be calculated as

$$\begin{aligned} &-2\int_{\mathcal{M}} dA_{\mathcal{M}} \chi \delta_{\perp} \mathcal{R}_G \\ &= 2\int_{\mathcal{M}} dA \chi \mathcal{R}_G K \Phi - 2\int_{\mathcal{M}} dA \chi (K^{ab} - g^{ab}K)\nabla_a \nabla_b \Phi, \end{aligned}$$

and after some integrations by parts we get

$$\begin{aligned} &= 2\int_{\mathcal{M}} dA [\mathcal{R}_G K \chi - (K^{ab} - g^{ab}K)\nabla_a \nabla_b \chi] \Phi \\ &\quad - \int_{\mathcal{M}} dA 2\nabla_a [(K^{ab} - g^{ab}K)(\chi \nabla_b \Phi - \Phi \nabla_b \chi)]. \end{aligned} \quad (\text{D9})$$

The normal deformation is therefore

$$\delta_{\perp} F = \int_{\mathcal{M}} dA \mathcal{E}_{\perp} \Phi + \int_{\mathcal{M}} dA \nabla_a \mathcal{Q}_{\perp}^a, \quad (\text{D10})$$

where the normal Euler-Lagrange derivative and the Noether charge are given, respectively, by

$$\begin{aligned}\mathcal{E}_\perp &= -K\chi\nabla^2\chi - \chi[2K^{ab}\nabla_a\nabla_b\chi + (\nabla_a K)(\nabla^a\chi)] \\ &\quad + \nabla_b[(2K^{ab} - Kg^{ab})\chi\nabla_a\chi] - 2\chi\rho_D K \\ &\quad + 2[\mathcal{R}_G K\chi - (K^{ab} - g^{ab}K)\nabla_a\nabla_b\chi] \\ &= 2(Kg^{ab} - K^{ab})\nabla_a\nabla_b\chi + (2K^{ab} - Kg^{ab})\nabla_a\chi\nabla_b\chi, \\ \mathcal{Q}_\perp^a &= -(2K^{ab} - Kg^{ab})\chi(\nabla_a\chi)\Phi \\ &\quad - 2(K^{ab} - g^{ab}K)(\chi\nabla_b\Phi - \Phi\nabla_b\chi).\end{aligned}\quad (\text{D11})$$

The tangential deformation can be calculated in a similar way. By using the tangential deformation of the area we have

$$\begin{aligned}-\int_{\mathcal{M}}(\delta_\parallel dA)\chi\nabla^2\chi &= -\int_{\mathcal{M}}dA\chi\nabla^2\chi(\nabla_a\Phi^a) \\ &= -\int_{\mathcal{M}}dA\nabla_a(\chi\nabla^2\chi\Phi^a) \\ &\quad + \int_{\mathcal{M}}dA\nabla_a(\chi\nabla^2\chi)\Phi^a.\end{aligned}\quad (\text{D12})$$

We also obtain that the integral

$$\begin{aligned}\int_{\mathcal{M}}dA J_\parallel\chi &= \int_{\mathcal{M}}dA\chi[(-\nabla^2\Phi^a + \mathcal{R}_G\Phi^a)\nabla_a\chi \\ &\quad - 2(\nabla^a\Phi^b)\nabla_a\nabla_b\chi]\end{aligned}$$

can be rewritten after integrations by parts

$$\begin{aligned}&= -\int_{\mathcal{M}}dA_{\mathcal{M}}\nabla^a[\nabla_a\Phi^b\chi\nabla_b\chi] + \int_{\mathcal{M}}dA\nabla_a[\Phi^b\nabla^a(\chi\nabla_b\chi)] \\ &\quad - \int_{\mathcal{M}}dA\Phi^a\nabla^2(\chi\nabla_a\chi) + \int_{\mathcal{M}}dA\Phi^a\mathcal{R}_G\chi\nabla_a\chi \\ &\quad - 2\int_{\mathcal{M}}dA\nabla^a(\Phi^b\chi\nabla_a\nabla_b\chi) + 2\int_{\mathcal{M}}dA\Phi^b\nabla^a(\chi\nabla_a\nabla_b\chi).\end{aligned}\quad (\text{D13})$$

The next integration can be written as

$$\begin{aligned}2\int_{\mathcal{M}}dA\chi\delta_\parallel\rho_D &= -2\int_{\mathcal{M}}dA\chi\rho_D\nabla_a\Phi^a \\ &= -2\int_{\mathcal{M}}dA\nabla_a(\chi\rho_D\Phi^a) + 2\int_{\mathcal{M}}dA\nabla_a(\chi\rho_D)\Phi^a,\end{aligned}\quad (\text{D14})$$

and finally we get

$$-2\int_{\mathcal{M}}dA\chi\delta_\parallel\mathcal{R}_G = -2\int_{\mathcal{M}}dA\chi\Phi^a\nabla_a\mathcal{R}_G.\quad (\text{D15})$$

So that we obtain the tangential derivative and the Noether charge as

$$\begin{aligned}\mathcal{E}_a &= \nabla_a(\chi\nabla^2\chi) - \nabla^2(\chi\nabla_a\chi) + \mathcal{R}_G\chi\nabla_a\chi \\ &\quad + 2\nabla^b(\chi\nabla_b\nabla_a\chi) + 2\nabla_a(\chi\rho_D) - 2\chi\nabla_a\mathcal{R}_G \\ &= 2(\rho_D + \mathcal{R}_G\chi)\nabla_a\chi, \\ \mathcal{Q}_\parallel^a &= -(\chi\nabla^2\chi\Phi^a) - [\nabla^a\Phi^b\chi\nabla_b\chi] + [\Phi^b\nabla^a(\chi\nabla_b\chi)] \\ &\quad - 2(\Phi^b\chi\nabla_a\nabla_b\chi) - 2(\chi\rho_D\Phi^a). \\ &= \Phi^b[\nabla^a(\chi\nabla_b\chi) - 2\chi\nabla^a\nabla_b\chi - \delta_b^a(\rho_D + \mathcal{R}_G)\chi] \\ &\quad - \chi\nabla_b\chi\nabla^a\Phi^b.\end{aligned}\quad (\text{D16})$$

## APPENDIX E: DARBOUX FRAME

For the second integral we recall the Darboux basis adapted to the boundary  $\mathcal{C}$  parametrized by arc length. Defining  $\mathbf{T}$  as its tangent vector, such that  $\mathbf{T} = T^a\mathbf{e}_a$ , we also define  $\mathbf{l} = \mathbf{T} \times \mathbf{n}$  as the normal unit to the boundary, tangent to the surface. We have that

$$\begin{aligned}\dot{\mathbf{T}} &= \kappa_n\mathbf{n} + \kappa_g\mathbf{l}, \\ \dot{\mathbf{l}} &= -\kappa_g\mathbf{T} - \tau_g\mathbf{n}, \\ \dot{\mathbf{n}} &= -\kappa_n\mathbf{T} + \tau_g\mathbf{l}.\end{aligned}\quad (\text{E1})$$

In these equations, we have defined the normal curvature

$$\begin{aligned}\kappa_n &= \dot{\mathbf{T}} \cdot \mathbf{n}, \\ &= (\dot{T}^a\mathbf{e}_a - K_{ab}T^aT^b\mathbf{n}) \cdot \mathbf{n}, \\ &= -K_{ab}T^aT^b,\end{aligned}\quad (\text{E2})$$

and its geodesic curvature

$$\begin{aligned}\kappa_g &= \dot{\mathbf{T}} \cdot \mathbf{l}, \\ &= \kappa_g^a\mathbf{e}_a \cdot \mathbf{l}, \\ &= (\dot{T}^a + \Gamma_{bc}^aT^bT^c)l_a.\end{aligned}\quad (\text{E3})$$

The second equation in Eqs. (E1) defines the geodesic torsion

$$\tau_g = \dot{\mathbf{n}} \cdot \mathbf{l}, = K_{ab}T^a l^b.\quad (\text{E4})$$

Let us calculate the deformations in the Darboux frame. Deformation of the boundary is given by

$$\begin{aligned}\delta\mathbf{X} &= \phi\mathbf{T} + \psi\mathbf{l} + \Phi\mathbf{n}, \\ &= \Phi^a\mathbf{e}_a + \Phi\mathbf{n},\end{aligned}\quad (\text{E5})$$

that is  $\Phi^a T_a = \phi$  and  $\Phi^a l_a = \psi$ . Therefore, deformation of the unit tangent can be written as

$$\begin{aligned}\delta\mathbf{T} &= \dot{\phi}\mathbf{T} + \dot{\psi}\mathbf{l} + \dot{\Phi}\mathbf{n} + \phi\dot{\mathbf{T}} + \psi\dot{\mathbf{l}} + \Phi\dot{\mathbf{n}}, \\ &= (\dot{\phi} - \kappa_g\psi - \kappa_n\Phi)\mathbf{T} + (\dot{\psi} + \kappa_g\phi + \tau_g\Phi)\mathbf{l} \\ &\quad + (\dot{\Phi} + \kappa_n\phi - \tau_g\psi)\mathbf{n}.\end{aligned}\quad (\text{E6})$$

Then we obtain

$$\begin{aligned}\delta\oint_{\mathcal{C}}ds &= \oint_{\mathcal{C}}ds\mathbf{T} \cdot \delta\mathbf{T} \\ &= \oint_{\mathcal{C}}ds(\dot{\phi} - \kappa_g\psi - \kappa_n\Phi), \\ \delta L &= \Delta\phi - \oint_{\mathcal{C}}ds(\kappa_g\psi + \kappa_n\Phi),\end{aligned}\quad (\text{E7})$$

where  $\Delta\phi = 0$  for a closed curve. Thus,  $\delta L$  does not include deformation along the unit tangential vector. Write

$$\begin{aligned}\mathcal{Q}_\perp^a &= M^{ab}\nabla_b\Phi + M^a\Phi, \\ \mathcal{Q}_\parallel^a &= N^a{}_b\Phi^b + N_b\nabla^a\Phi^b,\end{aligned}\quad (\text{E8})$$

where

$$\begin{aligned}M^{ab} &= 2(Kg^{ab} - K^{ab})\chi, \\ M^a &= [(Kg^{ab} - K^{ab})(\chi - 2) - K^{ab}\chi]\nabla_b\chi, \\ N^{ab} &= \nabla^a\chi\nabla^b\chi - \chi\nabla^a\nabla^b\chi - g^{ab}(\rho_D + \mathcal{R}_G)\chi, \\ N^a &= -\chi\nabla^a\chi.\end{aligned}\quad (\text{E9})$$

The we can obtain

$$\begin{aligned}
l_a Q^a &= l_a(Q_\perp^a + Q_\parallel^a) \\
&= l_a(M^{ab}\nabla_b\Phi + M^a\Phi) + l_a(N^{ab}\Phi_b + N^b\nabla^a\Phi_b), \\
&= l_a M^{ab}l_b\nabla_l\Phi + l_a M^{ab}T_b\dot{\Phi} + l_a M^a\Phi \\
&\quad + (l_a N^{ab}T_b + N^b\nabla_l T_b)\phi + (l_a N^{ab}l_b + N^b\nabla_l l_b)\psi \\
&\quad + N^b l_b\nabla_l\psi + N^b T_b\nabla_l\phi, \tag{E10}
\end{aligned}$$

where we have used that on the boundary

$$\begin{aligned}
\nabla_a\Phi &= \mathbf{e}_a \cdot \nabla\Phi \\
&= (l_a\mathbf{I} + T_a\mathbf{T}) \cdot \nabla\Phi \\
&= l_a\nabla_l\Phi + T_a\dot{\Phi}, \tag{E11}
\end{aligned}$$

that is  $\nabla_l\Phi = l^a\nabla_a\Phi$  and  $\dot{\Phi} = T^a\nabla_a\Phi$ . We also have that

$$\nabla_b\Phi^a = T_b\dot{\Phi}^a + l_b\nabla_l\Phi^a. \tag{E12}$$

Note that on the boundary the independent deformations are given by the scalars functions  $\psi, \phi, \Phi$ . Then we have that

$$\delta\mathcal{H} = \oint ds[\kappa_A l_a Q^a + (\sigma - \sigma_b \kappa_g)\psi - \sigma_b \kappa_n \Phi]. \tag{E13}$$

#### APPENDIX F: GREEN FUNCTION AND GEOMETRIC POTENTIAL ON THE SPHERE

To find the Green function on the sphere, we need

$$I = \int_0^{\pi R} d\ell \rho(\ell) \log r_>. \tag{F1}$$

We split the integral as

$$\begin{aligned}
I &= \log r(l) \int_0^l d\ell \rho(\ell) + \int_l^{\pi R} d\ell \rho(\ell) \log r(\ell) \\
&= \log[R \tan(l/2R)] \int_0^l d\ell R \sin(\ell/R) \\
&\quad + \int_l^{\pi R} d\ell R \sin(\ell/R) \log[R \tan(\ell/2R)] \\
&= -R^2 \log[R \tan(l/2R)][\cos(l/R) - 1] \\
&\quad + R^2 \int_{l/R}^{\pi} dx \sin x \log(R \tan x/2). \tag{F2}
\end{aligned}$$

Here the integral can be obtained as

$$\begin{aligned}
&\int_{l/R}^{\pi} dx \sin x \log(R \tan x/2) \\
&= \log R - \log[\sin(l/2R) \cos(l/2R)] \\
&\quad + \cos(l/R) \log[R \tan(l/2R)]. \tag{F3}
\end{aligned}$$

When substituting we obtain Eq. (63). The Green function is then given by

$$\begin{aligned}
G(\xi, \zeta) &= -\frac{1}{4\pi} \log[r(l)^2 + r(\ell)^2 - 2r(l)r(\ell) \cos(\phi - \varphi)] \\
&\quad - \frac{1}{4\pi} \log[\cos^2(l/2R) \cos^2(\ell/2R)], \tag{F4}
\end{aligned}$$

which no longer contains singularities. By using the Green function Eq. (F4), we can evaluate the geometric potential as

$$\mathcal{U}(\xi) = \int dA_\zeta G(\xi, \zeta) \mathcal{R}_G(\ell). \tag{F5}$$

The Gaussian curvature of the sphere is given by  $\mathcal{R}_G = 1/R^2$ , such that

$$\mathcal{U} = \frac{1}{R^2} \int_0^{\pi R} d\ell \rho(\ell) \int_0^{2\pi} d\varphi G(\xi, \zeta). \tag{F6}$$

As an intermediate step we obtain

$$\mathcal{U} = \log \cos^2(l/2R) - I_1, \tag{F7}$$

where  $I_1$  is written as

$$I_1 = \frac{1}{4\pi R^2} \int dA_\zeta \log[\cos^2(l/2R) \cos^2(\ell/2R)]. \tag{F8}$$

We split this integral as

$$\begin{aligned}
4\pi R^2 I_1 &= 2\pi \log \cos^2(l/2R) \int_0^{\pi R} d\ell \rho(\ell) \\
&\quad + 2\pi \int_0^{\pi R} d\ell \rho(\ell) \log \cos^2(\ell/2R) \\
&= 4\pi R^2 \log \cos^2(l/2R) - 4\pi R^2, \tag{F9}
\end{aligned}$$

in such a way that when substituting into Eq. (F7) we get  $\mathcal{U} = 1$ .

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