Influence of the extrinsic curvature on two-dimensional nematic films

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Nematic films are thin fluid structures, ideally two dimensional, endowed with an in-plane degenerate nematic order. In this paper we examine a generalization of the classical Plateau problem to an axisymmetric nematic film bounded by two coaxial parallel rings. At equilibrium, the shape of the nematic film results from the competition between surface tension, which favors the minimization of the area, and the nematic elasticity, which instead promotes the alignment of the molecules along a common direction. We find two classes of equilibrium solutions in which the molecules are uniformly aligned along the meridians or parallels. Depending on two dimensionless parameters, one related to the geometry of the film and the other to the constitutive moduli, the Gaussian curvature of the equilibrium shape may be everywhere negative, vanishing, or positive. The stability of these equilibrium configurations is investigated.

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I. INTRODUCTION

Fluid films, such as soap films or lipid membranes, often give rise to shapes, as beautiful as they are complex, that have fascinated scientists of all times and of several areas of science. Since, as known, the energy of an idealized two-dimensional fluid film is proportional to the area it occupies, with the surface tension being the constant of proportionality, the minimizers of the energy and area functionals are the same. For this reason the problem of determining minimal surfaces with given boundaries (raised first by Euler) has relevance not only in geometry but also in physics and engineering. As a classical example, a soap film attached to two twin coaxial parallel rings takes the shape of a catenoid, the only nonplanar minimal surface of revolution. On the other hand, the study of ultrathin structures subjected to the simultaneous action of various forces gives rise to new Plateau-like problems whose solutions, besides being of interest from the mathematical-physic point of view, may be used to engineer new devices controlling the geometric properties of soft shells.

An insightful approach to study the interplay between orientational order and geometry is given by nematic films. These are fluid films endowed with an in-plane nematic order provided by elongated molecules, which may freely glide and/or rotate while keeping their axes lying on the local tangent plane. The recent review by Zhang *et al.* [1] reports how liquid crystalline vesicles exhibit a large variety of shapes due to the interplay between in-plane liquid crystalline order and bending elasticity. Chen and Kamien [2] found axisymmetric equilibrium shapes of nematic films by minimizing a combination of surface tension and nematic elastic energies. They showed that the nematic order is able to support a rich class of shapes in addition to the classical constant mean curvature surfaces. In the same energetic framework, Giomi [3] searched for axisymmetric interfaces whose boundaries are two given coaxial rims and argued that only two branches of solution are allowed: the catenoidal shape when the surface tension is the dominant effect, and the pseudospherical hyperboloidal shape when the nematic elasticity plays a predominant role. In Ref. [3] it has been shown that the competition between nematic elasticity and surface tension induces a first-order phase transition between the two branches. More recently, the same problem has been reexamined in terms of forces by Barrientos *et al.* [4].

It ought to be said that in all the studies quoted above only the contribution due to the intrinsic curvatures of the flux lines of the director field has been accounted for in the elastic free energy of the nematics. Such a contribution is related to the spatial variations of the director field on the curved substrate. More recently, it has been demonstrated that also the extrinsic curvature terms, i.e., curvatures related to the geometry of the substrate itself, are relevant in the energetic balance [5–8]. The potential applications of these new theories in soft matter and their elegant mathematical formalism have produced a vivid research activity in the communities of both theoretical physicists [9–13] and applied mathematicians [14–18].

In this paper, we revise the variational problem studied in Ref. [3] in the light of the correction to the two-dimensional nematic free energy proposed in a previous work of ours [6]. This correction includes terms accounting for the extrinsic curvature of the nematic film, which are instead missing in Ref. [3]. As a result of the competition between the nematic elasticity and the surface tension, equilibrium shapes with positive, vanishing, or negative Gaussian curvature can be obtained depending on the magnitudes of the constitutive parameters, the radius of the bounding rings and the distance

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between them. The inclusion of the extrinsic curvature terms in the energy functional, on the one hand, makes the solutions obtained by Giomi [3] no longer admissible, on the other hand, it opens to new scenarios in which the boundary anchoring is crucial in the determination of the equilibrium shapes.

The paper is organized as follows. In Sec. II we introduce the model for the energy functional of a two-dimensional (2D) nematic film, and write down the equilibrium equations and appropriate sets of boundary conditions. The specialization of the equilibrium problem to axisymmetric shapes and homogeneous alignments of the molecules of the nematics is considered in Sec. III, where, depending on the (uniform) alignment of the director field, the equilibrium equations are solved numerically or analytically. In Sec. IV, we study the local stability of the solutions considering both in-plane strong anchoring and natural boundary conditions on the nematic director. Section V contains instead some concluding remarks. The paper is closed by two Appendices in which we illustrate the notation adopted throughout the paper, report lengthy calculations and derive rigorously the equilibrium equations.

II. MODEL

We assume that the nematic film is schematized by a regular surface S with unit normal field $\boldsymbol{\nu}$. We denote $\mathbf{n}$ the nematic director and assume it to be a smooth unit vector field tangent to S. The interplay between the geometry of the film and the director field will be studied by minimizing the following energy functional

$$W = \int_S \left(\gamma + \frac{k}{2}|\nabla_s \mathbf{n}|^2\right) dA, \tag{1}$$

where γ is the surface tension, k the elastic constant of the nematics and ∇_s indicates the surface gradient. We anticipate from the beginning that the choice of the differential operator ∇_s strongly affects the shape of the equilibrium configuration. In most of the existing literature on nematic shells or films, the energy formula is usually expressed in terms of the covariant derivative (commonly denoted D) instead of the surface gradient. What should be the most appropriate form of the energy is still on debate. In favor of our constitutive model for the free energy it must be said that the energy formula (1) can be derived from the classical three-dimensional Frank's model by means of a perturbation analysis. Specifically, regarding the nematic film as a thin fluid layer whose thickness is much smaller than the minimum radius of curvature of S, to leading order the Frank free-energy density approximates to

$$2w_F = k_1(\mathrm{div}_s \mathbf{n})^2 + k_2(\mathbf{n}\cdot \mathrm{curl}_s \mathbf{n})^2 + k_3|\mathbf{n}\times \mathrm{curl}_s \mathbf{n}|^2, \tag{2}$$

where div_s and curl_s are, respectively, the surface divergence and surface curl [5]. Next, note that, under the one-constant approximation ($k_1 = k_2 = k_3 = k$), (2) reduces to $(k/2)|\nabla_s \mathbf{n}|^2$. To appreciate the differences between our model and that studied by Giomi [3], observe that the surface gradient and the covariant derivative of the director field $\mathbf{n}$ are related through the simple relation $\nabla_s \mathbf{n} = D\mathbf{n} + \boldsymbol{\nu}\otimes \mathbf{L}\mathbf{n}$, where $\mathbf{L}$ is the extrinsic curvature tensor of S. Consequently, we have $|\nabla_s \mathbf{n}|^2 = |D\mathbf{n}|^2 + |\mathbf{L}\mathbf{n}|^2$. It is then evident that our free-energy density exhibits an extra term reflecting the coupling of the extrinsic curvature of the film with the nematic order.

A. Equilibrium equations

The Euler-Lagrange equations associated with the energy functional (1) can be readily derived by following consolidated variational schemes [19]. Specifically, denoting $\boldsymbol{\sigma}$, $\mathbf{T}$, and $\mathbf{G}$ the stress, couple-stress, and microtorque tensors, respectively, and $\mathbf{g}$ and the microcouple density acting on the nematic molecules, the balance equations of forces, and macro- and microtorques read

$$\mathrm{div}_s \boldsymbol{\sigma} = \mathbf{0}, \tag{3a}$$

$$\mathrm{div}_s \mathbf{T} - \boldsymbol{\varepsilon}\boldsymbol{\sigma} = \mathbf{0}, \tag{3b}$$

$$\mathbf{t}\cdot(\mathrm{div}_s \mathbf{G} - \mathbf{g}) = 0, \tag{3c}$$

where $\mathbf{t} \equiv \boldsymbol{\nu}\times\mathbf{n}$ represents the conormal vector, and $\boldsymbol{\varepsilon}$ is the Ricci alternator. Specializing the analytical scheme introduced in Ref. [19] to the energy functional (1) gives

$$\boldsymbol{\sigma} = \left(\gamma + \frac{k}{2}|\nabla_s \mathbf{n}|^2\right)\mathbf{P} - k[(\nabla_s \mathbf{n})^T \nabla_s \mathbf{n} + (\boldsymbol{\nu}\cdot\Delta_s \mathbf{n})\boldsymbol{\nu}\otimes\mathbf{n}], \tag{4a}$$

$$\mathbf{T} = k[\boldsymbol{\nu}\otimes(\nabla_s \mathbf{n})^T \mathbf{t} - \mathbf{t}\otimes\mathbf{L}\mathbf{n}], \tag{4b}$$

$$\mathbf{G} = k\nabla_s \mathbf{n}, \quad \mathbf{g} = \mathbf{0}, \tag{4c}$$

where $\mathbf{P} \equiv \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$ denotes the projection onto the tangent plane, $\otimes$ the tensor product, and $\Delta_s \equiv \mathrm{div}_s \nabla_s$ is the Laplace-Beltrami differential operator on S.

Observe first that in view of (4) and (3c) the balance equation of macromoments (3b) is identically satisfied (see Appendix B for details). Next, inserting (4c) into (3c) yields the equilibrium equation for the in-plane orientation of $\mathbf{n}$ in the simple form

$$\mathbf{t}\cdot\Delta_s \mathbf{n} = 0. \tag{5}$$

Adopting the most common terminology in the literature, we shall refer to (5) as the director equation. We now denote $\mathbf{e}_1$ and $\mathbf{e}_2$ the principal directions on S, and parametrize the director through the convex angle α contained between $\mathbf{e}_1$ and $\mathbf{n}$ as

$$\mathbf{n} = \cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2. \tag{6}$$

In this way, the director equation (5) can be rewritten as

$$\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega} + 2H\tau_{\mathbf{n}} = 0, \tag{7}$$

where $\boldsymbol{\omega}$ is the vector parametrizing the spin connection on S, H is the mean curvature, and $\tau_{\mathbf{n}}$ is the geodesic torsion [20] of the flux lines of the director field.

By substituting (4a) into (3a) and projecting along $\boldsymbol{\nu}$, we arrive at the so-called shape equation

$$2H\left(\gamma + \frac{k}{2}|\nabla_s \mathbf{n}|^2\right) - k\{(\nabla_s \mathbf{n})^T(\nabla_s \mathbf{n})\cdot\mathbf{L} + \mathrm{div}_s[(\boldsymbol{\nu}\cdot\Delta_s \mathbf{n})\mathbf{n}]\} = 0. \tag{8}$$

On the other hand, the projection of (4a) onto the tangent plane yields (see Appendix B)

$$(\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega} + 2H\tau_{\mathbf{n}})(\nabla_s\alpha - \boldsymbol{\omega}) = \mathbf{0}, \tag{9}$$

which, as an immediate consequence of the director equation (7), is identically satisfied.

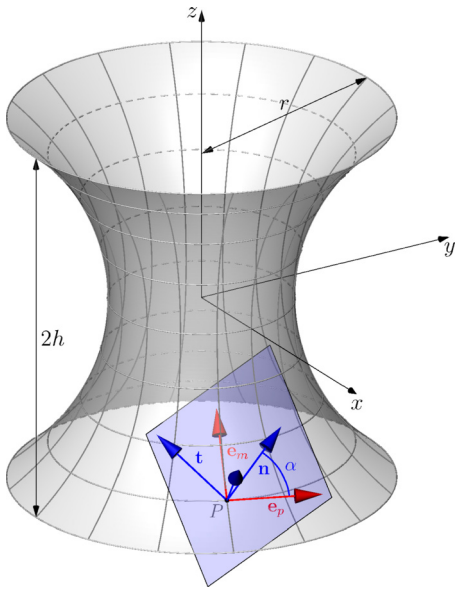

FIG. 1. Schematic representation of an axisymmetric 2D nematic film. At any point P we define both the Darboux frame $\{\mathbf{n},\mathbf{t},\boldsymbol{\nu}\}$ and the orthonormal basis $\{\mathbf{e}_p,\mathbf{e}_m,\boldsymbol{\nu}\}$, with $\mathbf{e}_p$ and $\mathbf{e}_m$ being the principal directions.

B. Boundary conditions

For 2D nematic films with boundary, the shape and director equations must be supplemented by appropriate boundary conditions. Here, we shall assume that the nematic film is simply supported, that is the boundary is fixed, while the surface can freely rotate about the tangent to the boundary. This obviously entails that the component of the macrotorque along the unit tangent vector field $\mathbf{l}$ must vanish, viz, orienting $\mathbf{l}$ such that $\boldsymbol{\nu}\times\mathbf{l}$ coincides with the in-plane outward normal $\mathbf{k}$ to ∂S,

$$\mathbf{l}\cdot\mathbf{T}\mathbf{k}=0 \quad \text{on } \partial S. \tag{10}$$

As the boundary conditions on the director are concerned, we will take into consideration the following two cases:

(i) natural boundary conditions, which are valid whenever the molecules of the nematics may freely rotate about the normal $\boldsymbol{\nu}$ at the boundary, and impose the following restriction on the microtorque $\mathbf{G}$ [21]:

$$\mathbf{t}\cdot\mathbf{G}\mathbf{k}=0 \quad \text{on } \partial S. \tag{11}$$

(ii) in-plane strong anchoring boundary conditions, which are valid whenever the in-plane direction of $\mathbf{n}$ is fixed at the boundary. In other words, whenever the angle α is prescribed at the boundary.

C. Axisymmetric shapes

Hereinafter, we shall limit our analysis to 2D nematic films schematised by axisymmetric surfaces of genus 1, bounded by two fixed coaxial circular rings of radius r placed at distance $2h$ one each other as displayed in Fig. 1. For this class of surfaces the parallels and meridians (with tangent directions $\mathbf{e}_p$ and $\mathbf{e}_m$, respectively) are lines of curvature, namely $\mathbf{e}_p$ and $\mathbf{e}_m$ are principal directions, and the vector field $\boldsymbol{\omega}$ is divergence free and tangent to the boundary [22]. Observe now that, according to the convention on the orientation of the unit tangent vector field $\mathbf{l}$ agreed in the previous subsection, at the upper (lower) boundary $\mathbf{l}\equiv\mathbf{e}_p$ ($\mathbf{l}\equiv-\mathbf{e}_p$) and $\mathbf{k}\equiv\mathbf{e}_m$ ($\mathbf{k}\equiv-\mathbf{e}_m$). Thus, for 2D axisymmetric films, in view of (4b), the boundary condition (10) reads

$$k(\mathbf{L}\mathbf{n}\cdot\mathbf{e}_m)(\mathbf{t}\cdot\mathbf{e}_p)=0 \quad \text{on } \partial S. \tag{12}$$

For the sake of simplicity, we limit further our analysis to uniform equilibrium alignments, i.e., homogeneous solutions to the director equation. Within this ansatz, Eq. (7) reduces to

$$H\tau_{\mathbf{n}}=0, \tag{13}$$

that is satisfied on the catenoid (the only surface of revolution bounded by the two given coaxial rings with vanishing mean curvature) irrespective of the (uniform) alignment of the molecules, or when the alignment of the director field is such that the geodesic torsion $\tau_{\mathbf{n}}$ vanishes identically. However, the catenoid satisfies the shape equation (8) if and only if $k=0$, that is when the functional (1) reduces to the energy of a soap film. The classical result on the equilibrium shape of a soap film attached to two coaxial rings with the same radius is then recovered. More interestingly, the equation $\tau_{\mathbf{n}}=0$ implies that the director field is aligned along a principal direction on S. This means that, on the axisymmetric surface at hand, at equilibrium the only two uniform alignments are those with $\mathbf{n}$ oriented along the parallels ($\alpha\equiv\alpha_p=0$) or the meridians ($\alpha\equiv\alpha_m=\pi/2$).

III. EQUILIBRIUM SHAPES

To determine the equilibrium shapes of the nematic film when the molecules are oriented along the parallels or meridians, we express the position vector $\mathbf{r}$ using cylindrical coordinates, $\mathbf{r}=(\rho(z)\cos\varphi,\rho(z)\sin\varphi,z)$, with $\rho(z)>0$ for all $z\in[-h,h]$, $\varphi\in[0,2\pi]$, and introduce the dimensionless quantities $\varrho=\rho/r$ and $\zeta=z/h$. Thanks to such parametrization and nondimensionalization the energy functional (1) may be rewritten as

$$W=2\pi\gamma r^2\int_{-1}^{1}w(\varrho,\varrho',\varrho'',\alpha,\alpha')d\zeta, \tag{14}$$

where the prime denotes differentiation with respect to ζ, and

$$w=\left\{1+c\left[\frac{\varrho^2\alpha'^2+\varrho'^2+\xi^2\cos^2\alpha}{\varrho^2(\xi^2+\varrho'^2)}+\frac{\xi^2\varrho''^2\sin^2\alpha}{(\xi^2+\varrho'^2)^3}\right]\right\}\times\varrho\sqrt{\xi^2+\varrho'^2}. \tag{15}$$

The dimensionless parameters $\xi\equiv h/r$ and $c\equiv k/(2\gamma r^2)$ in (15) give, respectively, a measure of the slenderness and the ratio between the magnitudes of the surface tension and elastic stiffness of the nematic film.

The shape equations (8) corresponding to the two homogeneous equilibrium alignments, $\alpha\equiv\alpha_i$ ($i=p,m$), can be written as

$$\left(\frac{d}{d\zeta^2}\frac{\partial w}{\partial\varrho''}-\frac{d}{d\zeta}\frac{\partial w}{\partial\varrho'}+\frac{\partial w}{\partial\varrho}\right)\Bigg|_{\alpha\equiv\alpha_i}=0. \tag{16}$$

Equation (16) with $i = m$ is a fourth-order ordinary differential equation (ODE), whereas for $i = p$ (16) is a second-order ODE. Since the boundary is assumed fixed, (16) must be solved subject to the boundary conditions

$$\varrho(-1) = \varrho(1) = 1. \tag{17}$$

These two boundary conditions are sufficient to determine the equilibrium shapes when $\alpha \equiv \alpha_p$. Two more boundary conditions are instead necessary when the molecules are oriented along the meridians. Since within the parametrization and nondimensionalization adopted here the boundary condition (12) reduces to

$$\varrho'' \sin^2\alpha = 0 \quad \text{at } \zeta = \pm 1, \tag{18}$$

the two additional boundary conditions to add to (16) with $i = m$ are

$$\varrho''(-1) = \varrho''(1) = 0. \tag{19}$$

It is worth noting that when the molecules are oriented along the parallels the boundary conditions (18) are identically satisfied.

For the sequent stability analysis of the equilibria it is convenient to specify also the boundary conditions on the angle α. The natural anchoring boundary conditions result in the Neumann conditions

$$\alpha'(\pm 1) = 0, \tag{20}$$

while assuming that the molecules of the nematics are forced to align tangentially to the delimiting rims leads to the Dirichlet boundary conditions

$$\alpha(\pm 1) = 0. \tag{21}$$

Obviously, both the uniform alignments $\alpha \equiv \alpha_i$ $(i = p,m)$ meet the boundary conditions (20), whereas $\alpha \equiv \alpha_p$ is the only uniform equilibrium alignment, which satisfies the boundary conditions (21).

A. Director field aligned along the parallels

Let us now examine the equilibrium configurations in details and start with the case $\alpha \equiv \alpha_p$. In this case, the shape equation (16) reads

$$(\varrho^2 + c)\varrho\varrho'' - (\varrho^2 - c)(\varrho'^2 + \xi^2) = 0, \tag{22}$$

and, as discussed above, has to be solved subject to the Dirichlet boundary conditions (17). The resulting boundary value problem (BVP) can be solved exactly to yield

$$\varrho_\pm(\zeta) = \frac{\sqrt{2 - a^2c \pm 2\sqrt{1 - a^2c}\cosh(\xi a\zeta)}}{a}, \tag{23}$$

where the solution with the subscript $+$ $(-)$ refers to the case $c < 1$ $(c > 1)$. When $c = 1$ the solution of the BVP is the cylindrical shape $\varrho \equiv 1$. The positive constant a in (22) is a root of the equation

$$\xi = \frac{1}{a}\operatorname{arccosh}\left[\pm\frac{a^2(c+1) - 2}{2\sqrt{1 - a^2c}}\right]. \tag{24}$$

For any fixed values of $c \geqslant c_\star \approx 0.0257$ and $\xi > 0$ Eqs. (24) can be solved uniquely for $a > 0$. On the contrary, if $0 < c <$

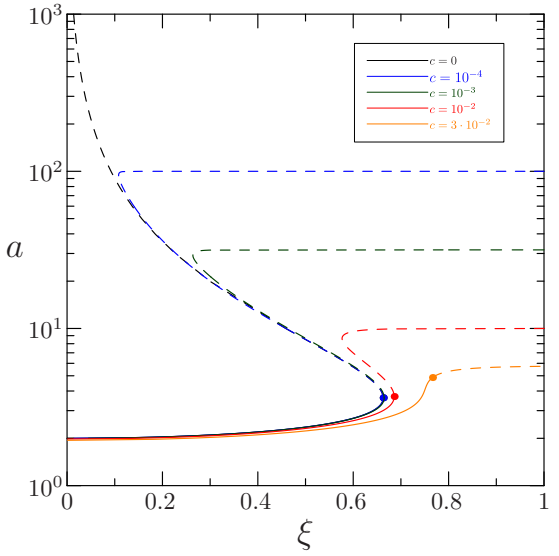

FIG. 2. Solutions to (24) for different values of c. Solid lines correspond to stable equilibrium configurations in the case of in-plane strong anchoring (see Sec. IV B). The dashed lines correspond instead to unstable equilibria.

$c_\star$, depending on the value of ξ, Eq. (24) may admit one, two or three roots (see Fig. 2). In the limiting case $c = 0$, (24) admits two roots for any $\xi < \xi_\star \equiv \max_{s>0} \frac{\operatorname{arccosh}(s^2-1)}{\sqrt{2}s} \approx 0.663$, exactly one root if $\xi = \xi_\star$ and no root if $\xi > \xi_\star$.

Figure 3 displays equilibrium shapes at different c. The cylindrical configuration $(c = 1)$ separates the equilibrium shape with inward concavity $(0 \leqslant c < 1)$ from those with an outward concavity $(c > 1)$. Consequently, at equilibrium the Gaussian curvature of the nematic film is negative if $0 \leqslant c < 1$, vanishing if $c = 1$ and positive if $c > 1$. In the particular case $c = 0$, i.e., in the absence of the nematic order, the equilibrium shape is a catenoid. On the contrary, in the limit as c tending to infinity, that is when the effects due to the nematic elasticity are dominant, the equilibrium shape tends to a portion of a sphere. This result has an

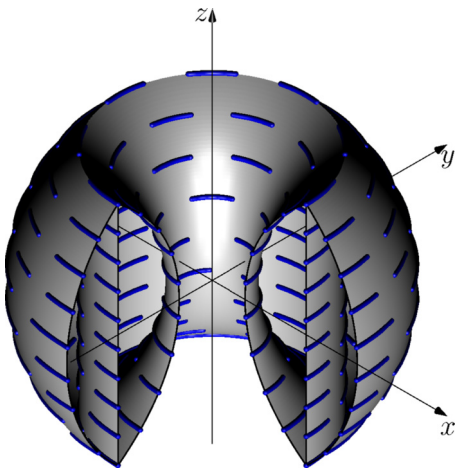

FIG. 3. Equilibrium configurations with the molecules of the nematics oriented along parallels for different values of c.

intuitive explanation. At equilibrium, the molecules of the nematics are aligned along circles whose radii is as large as possible to diminish the bending energy. This effect is in competition with the boundary conditions, which fix the radius of the boundary circles, and the surface tension, which pushes the nematic film to minimize its area and then towards the catenoidal configuration. Thus, whenever the nematic elasticity represents the dominant contribution to the energy of the nematic film the circles far from the boundaries have larger radii, which lends the equilibrium configuration a bulgy shape.

B. Director field aligned along the meridians

When the molecules of the nematics are oriented along the meridians, the free-energy density depends on ϱ'' and, as mentioned above, the associated shape equation is a fourth-order ODE. The related BVP [(16)–(19)] can be solved only numerically. In addition, only the natural anchoring boundary conditions $\alpha'(\pm 1)$ are compatible with this homogeneous alignment.

As in the previous case, when c vanishes the equilibrium shape is catenoidal. For greater c the nematic elasticity is more significant and the equilibrium shape departs from the catenoid maintaining an inward concavity (and hence the negativeness of the Gaussian curvature), though. Also in the case $\alpha \equiv \alpha_m$ our results are in good agreement with the physical intuition. In fact, since the flux lines of the director field are open curves (the meridians), the bending energy attains the absolute minimum when the flux lines are straight. On the other hand, the effect of the surface tension encourages the meridians to be catenaries with inward concavity. The equilibrium configurations in Fig. 4 result then from the competition of these two effects. In the limiting case as $c \to +\infty$ the dominant nematic elasticity lends the equilibrium configuration the cylindrical shape.

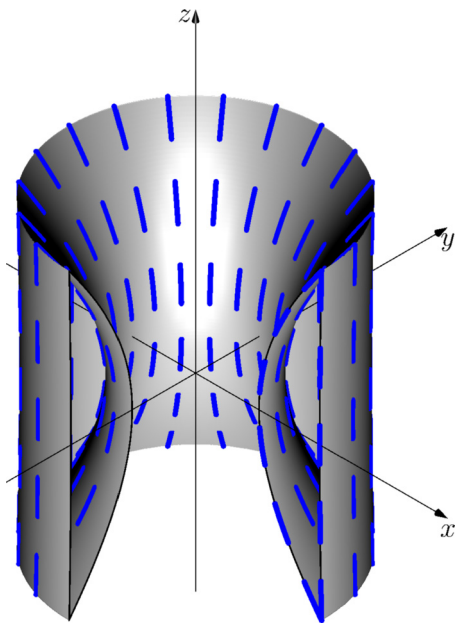

FIG. 4. Equilibrium configurations with the molecules of the nematics oriented along the meridians for different values of c.

IV. STABILITY

Let us denote S_p and S_m the shapes corresponding to the homogeneous alignments $\alpha \equiv \alpha_p$ and $\alpha = \alpha_m$, respectively, and let (S_p,α_p) and (S_m,α_m) denote the two resulting equilibrium configurations. We now discuss the stability of the two equilibrium configurations under natural and in-plane strong anchoring boundary conditions.

A. Natural anchoring boundary conditions

Both the classes of equilibrium shape analyzed in Sec. III are compatible with the natural anchoring boundary conditions. The direct calculation of the energy of the two equilibria shows that (S_m,α_m) requires less energy than (S_p,α_p) for any values of the dimensionless parameters c and ξ. On the other hand, the study of the positive definiteness of the second variations at the two equilibria reveals that (S_m,α_m) is stable, that is the configuration (S_m,α_m) is a local minimizer of the energy functional (14)–(15), whereas (S_p,α_p) is unstable.

For the sake of brevity and simplicity of presentation we omit the details regarding the positive definiteness of the second variation of the energy functional at (S_m,α_m). We instead focus on the equilibrium configuration (S_p,α_p). After some manipulations, the second variation of the energy functional (14)–(15) at (S_p,α_p) can be written as

$$\delta^2 W[u,\vartheta] = \underbrace{\frac{\xi^2}{2}\int_{-1}^{1}\left[u'^2+\frac{2\xi^2(3c-\varrho_\pm^2)}{(\varrho_\pm^2+c)^2}u^2\right]d\zeta}_{\equiv\delta^2_{sh}W[u]} + \underbrace{\int_{-1}^{1}\frac{c\varrho_\pm}{\sqrt{\varrho_\pm'^2+\xi^2}}\left[\vartheta'^2-\frac{4c\xi^2}{(\varrho_\pm^2+c)^2}\vartheta^2\right]d\zeta}_{\equiv\delta^2_{na}W[\vartheta]}, \quad (25)$$

where $u \in \mathcal{H}^1_0([-1,1]) \equiv \{f \in \mathcal{H}^1([-1,1]) : f(\pm 1) = 0\}$, with $\mathcal{H}^1([-1,1])$ being the Hilbert space of functions defined in $[-1,1]$ whose square is integrable together with the square of its weak first derivative, $\vartheta \in \mathcal{X} \equiv \{f \in \mathcal{H}^1([-1,1]) : f'(\pm 1) = 0\}$, and $\varrho_\pm$ is given by (23). As an immediate consequence of (25) we deduce that (S_p,α_p) is stable if and only if the quadratic functionals $\delta^2_{sh}W[u]$ and $\delta^2_{na}W[\vartheta]$ are both positive definite. However, obviously, $\delta^2_{na}W[\vartheta_c] < 0$ for any nonzero constant ϑ_c. This implies that $\delta^2_{na}W[\vartheta]$ is not positive definite and thus (S_p,α_p) is an unstable equilibrium configuration.

B. In-plane strong anchoring boundary conditions

We now assume that the molecules of the nematics are constrained to align themselves tangentially to the boundaries. We then consider the Dirichlet boundary conditions (21). As mentioned before, the only homogeneous alignment compatible with these boundary conditions is that with the molecules oriented along the parallels.

In contrast to the case of natural boundary conditions, in the case at issue the equilibrium configuration (S_p,α_p) may be stable for some values of c and ξ. To validate such a claim, we

see that, setting

$$\theta = \vartheta \sqrt{\frac{\varrho_\pm}{\sqrt{\varrho_\pm'^2 + \xi^2}}}, \tag{26}$$

the second variation at (S_p,α_p) can be rewritten as $\delta^2 W[u,\theta] = \delta^2_{sh}[u] + \delta^2_{sa}[\theta]$, with $\delta^2_{sh}[u]$ as in (25),

$$\delta^2_{sa} W[\theta] = c \int_{-1}^{1} \left\{ \theta'^2 - \frac{c[(2\varrho_\pm^2 + c)(\varrho_\pm'^2 + \xi^2) + \xi^2 \varrho_\pm^2]}{\varrho_\pm^2(\varrho_\pm^2 + c)^2} \theta^2 \right\} d\zeta, \tag{27}$$

and $u,\theta \in \mathcal{H}^1_0([-1,1])$. As in the case of natural anchoring boundary conditions, (S_p,α_p) is stable if and only if $\delta^2_{sh} W[u]$ and $\delta^2_{sa} W[\theta]$ are positive-definite quadratic functionals.

Following standard arguments in calculus of variations, a necessary and sufficient condition for $\delta^2_{sh} W[u]$ to be positive-definite is that the interval $[-1,1]$ contains no interior points conjugate to -1 (see, for instance, Ref. [23], page 111). For each $c > 0$ we then determine the least positive value of ξ, say $\xi^{(sh)}_{\rm cr}(c)$, such that both the boundary value problem

$$u'' - \frac{2\xi^2(3c^2 - \varrho_\pm^2)}{(\varrho_\pm^2 + c)^2} u = 0, \qquad u(-1) = 0, \quad u(1) = 0, \tag{28}$$

and the normalization condition $u'(-1) = 1$ (see Ref. [23], page 106) are satisfied. Clearly, for $\xi < \xi^{(sh)}_{\rm cr}(c)$, the boundary value problem (28) admits only the trivial solution. Thus, the interval $[-1,1]$ contains no interior points conjugate to -1 and, consequently, $\delta^2_{sh} W[u]$ is positive definite.

Following similar arguments one can determine a necessary and sufficient condition for the positive definiteness of $\delta^2_{sa} W[\theta]$. Specifically, denoting $\xi^{(sa)}_{\rm cr}(c)$ the least positive value of ξ for which both the boundary value problem

$$\theta'' + \frac{c[(2\varrho_\pm^2 + c)(\varrho_\pm'^2 + \xi^2) + \xi^2 \varrho_\pm^2]}{\varrho_\pm^2(\varrho_\pm^2 + c)^2} \theta = 0, \qquad \theta(-1) = 0, \quad \theta(1) = 0, \tag{29}$$

and the normalization condition $\theta'(-1) = 1$ are satisfied, $\delta^2_{sa} W[\theta]$ is positive definite if and only if $\xi < \xi^{(sa)}_{\rm cr}(c)$.

We now observe that $\delta^2_{sh} W[u]$ and $\delta^2_{sa} W[\theta]$ are both positive definite, and hence $\delta^2 W[u,\vartheta]$ is positive definite, if and only if $\xi < \xi_{\rm cr}(c) \equiv \min\{\xi^{(sh)}_{\rm cr}(c), \xi^{(sa)}_{\rm cr}(c)\}$.

The critical curve $\xi = \xi_{\rm cr}(c)$ displayed in Fig. 5 has been determined numerically by using MATLAB BVP4C solver. When the molecules of the liquid crystal are anchored tangentially at the boundaries, the equilibrium configuration (S_p,α_p) is locally stable if and only if $\xi < \xi_{\rm cr}(c)$. Beyond this critical threshold, (S_p,α_p) is no longer a local minimizer of the energy functional (14)–(15) and the equilibrium solutions bifurcate to configurations with nonhomogeneous alignments as depicted in Fig. 5.

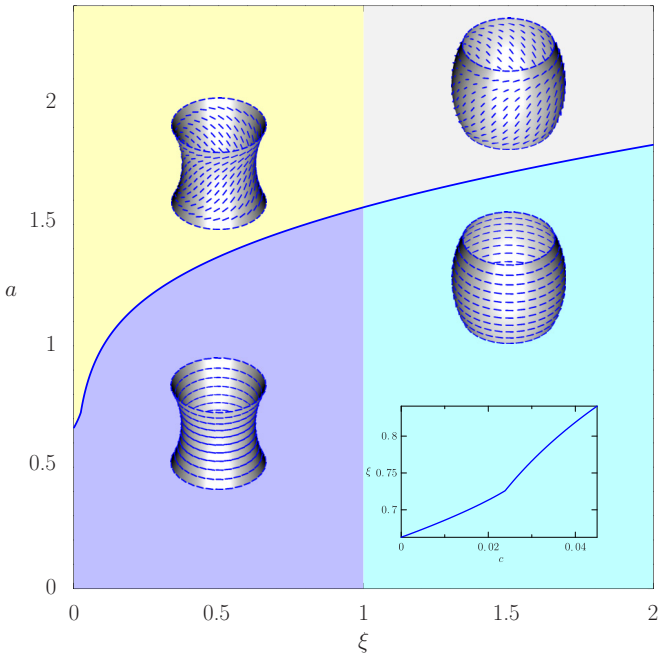

FIG. 5. Critical threshold for the stability of (S_p,α_p) as a function of c. The equilibrium configuration (S_p,α_p) is stable if and only if $\xi \leqslant \xi_{\rm cr}(c)$. Beyond this critical value, at a stable equilibrium configuration, the alignment of the molecules must be inhomogeneous.

V. CONCLUSIONS

In summary, we have investigated the equilibrium problem of fluid films endowed with nematic order. We have showed that, as a result of the competing effects due to surface tension and orientational order, the equilibrium shape of the nematic film may have positive, vanishing or negative Gaussian curvature. We have presented the case of a surface bounded by two coaxial parallel rims and studied the existence, uniqueness, and stability of the equilibrium configurations with homogeneous alignments of the molecules of the nematics. Specifically, we have considered two different sets of boundary conditions on the director field: natural and in-plane strong anchoring. In both cases we have determined locally stable equilibria, i.e., local minimizers of the energy functional.

Our analysis, though not exhaustive, shows that the inclusion of terms accounting for the extrinsic curvature in the energy functional renders the equilibrium problem of nematic films complex and intriguing at the same time. Existence and uniqueness of solutions to the equilibrium equations corresponding to nonuniform alignments of the molecules of the nematics and the search for global minimizers of the energy functional represent challenges for future analytical and numerical investigations. Another problem worth of investigation is the generalization of this problem in the framework of the two-dimensional Frank's formula (2), relaxing then the one constant approximation. Motivated by the recent results by Sonnet and Virga [24], we think that such a generalization leads to a more intricate scenario in the energy landscape.

Finally, our study lays the foundations for the design of devices capable to control the shape of nematic films. To this aim, note the analogy of the nematic films studied here with the soft elastic sheets where surfaces with both positive, vanishing, or negative Gaussian curvature can be produced by tuning the amount of local growth or swelling [25]. In the case of nematic

films, an external electric or magnetic field may control the curvatures of the equilibrium shapes.

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APPENDIX A: SURFACE DIFFERENTIAL OPERATORS

1. Notation

We first introduce the terminology and notation adopted throughout the paper. Let $\mathcal{E}$ be a three-dimensional Euclidean point space and $\mathcal{V}$ be the Euclidean vector space associated to $\mathcal{E}$. The elements of $\mathcal{V}$ are three-dimensional vectors, which are here denoted by lowercase boldface letters. The scalar, vector, and tensor products of two vectors $\mathbf{u}$ and $\mathbf{v}$ are denoted $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ and $\mathbf{u}\otimes\mathbf{v}$, respectively. In components, adopting the Einstein summation convention, we have $\mathbf{u}\cdot\mathbf{v} = u_i v_i$, $(\mathbf{u}\times\mathbf{v})_i = \varepsilon_{ijk}u_j v_k$, $(\mathbf{u}\otimes\mathbf{v})_{ij} = u_i v_j$, with ε_{ijk} being the alternating symbol.

Second-order tensors are linear maps from $\mathcal{V}$ to $\mathcal{V}$ itself and are denoted by capital boldface letters. The set of all second-order tensors is denoted $\mathrm{Lin}(\mathcal{V})$. The composition of two second-order tensors $\mathbf{A}$, $\mathbf{B}$ is the tensor $\mathbf{AB}$ with components $(\mathbf{AB})_{ij} = A_{ih}B_{hj}$. Once again, sum over repeated indices is understood. The trace is the linear operator $\mathrm{tr} : \mathrm{Lin}(\mathcal{V}) \to \mathbb{R}$, which assigns to a second-order tensor $\mathbf{A}$ the scalar obtained by saturation of the two indices of $\mathbf{A}$, *viz* $\mathrm{tr}\mathbf{A} \equiv A_{ii}$. The superscript suffix T to a second-order tensor indicates transposition: the transpose of $\mathbf{A} \in \mathrm{Lin}(\mathcal{V})$ is the second-order tensor $\mathbf{A}^T$ with components $(\mathbf{A}^T)_{ij} = A_{ji}$. Thanks to the definitions of trace and transposition the bilinear map, which assigns to two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ the quantity $\mathbf{A}\cdot\mathbf{B} \equiv \mathrm{tr}(\mathbf{A}^T\mathbf{B}) = A_{ij}B_{ij}$ is a scalar product in $\mathrm{Lin}(\mathcal{V})$.

Tensors of order $n > 2$ are multilinear maps from $\mathcal{V}^n$ to $\mathbb{R}$. However, in this paper we consider only one tensor of order greater than 2: the Ricci alternator $\boldsymbol{\varepsilon}$, which is a third-order tensor with components ε_{ijk}. Finally, in composing tensors of different orders we agree to write the lower-order tensor on the right and saturate all its indices. As examples of this convention, regarding vectors as tensors of order one, $\mathbf{Av}$ is a vector with components $(\mathbf{Av})_i = A_{ij}v_j$, $\boldsymbol{\varepsilon}\mathbf{A}$ is a vector with components $(\boldsymbol{\varepsilon}\mathbf{A})_i = \varepsilon_{ijk}A_{jk}$, and $\boldsymbol{\varepsilon}\mathbf{v}$ is a second-order tensor with components $(\boldsymbol{\varepsilon}\mathbf{v})_{ij} = \varepsilon_{ijk}v_k$.

2. Differential operators on S: The extrinsic curvature tensor

The nematic film is represented by a regular oriented surface S of $\mathcal{E}$. Scalar, vector, and tensor fields are functions defined on S, which assigns to each point $p \in S$ an element of $\mathbb{R}$, $\mathcal{V}$ or $\mathrm{Lin}(\mathcal{V})$, respectively.

At each point p, S is endowed with a two-dimensional linear space $\mathcal{T}_p$ called the tangent space of S at p. The normal $\boldsymbol{\nu}(p)$ at $p \in S$ is one of the two unit vectors spanning the orthogonal complement of the tangent space. Since S is orientable, at each point p we can choose an orientation of the normal so that the resulting unit vector field $\boldsymbol{\nu} : S \to \mathcal{V}$ is differentiable. The perpendicular projection onto the tangent plane, $\mathbf{P} \equiv \mathbf{I} - \boldsymbol{\nu}\otimes\boldsymbol{\nu}$, with $\mathbf{I}$ being the identity tensor, is then a differentiable tensor field.

A vector field $\mathbf{v}$ on S is tangential if $\mathbf{v}(p) \in \mathcal{T}_p$ for all $p \in S$. A tensor field $\mathbf{A}$ on S is tangential if, at each point $p \in S$, $\mathbf{A}(p)\mathbf{w} \in \mathcal{T}_p$ for all $\mathbf{w} \in \mathcal{V}$, and $\mathbf{A}(p)\boldsymbol{\nu}(p) = \mathbf{0}$.

Let ϕ be a differentiable scalar field on S. The surface gradient of ϕ is the tangential vector field $\nabla_s\phi \equiv \mathbf{P}\nabla\phi$. Similarly, the surface gradient of a differentiable vector field $\mathbf{v}$ is the tensor field $\nabla_s\mathbf{v} \equiv (\nabla\mathbf{v})\mathbf{P}$. The trace of $\nabla_s\mathbf{v}$ gives the surface divergence of $\mathbf{v}$, i.e.,

$$\mathrm{div}_s\mathbf{v} \equiv \mathrm{tr}(\nabla_s\mathbf{v}) = \nabla\mathbf{v}\cdot\mathbf{P}, \tag{A1}$$

while twice the axial vector corresponding to the skew-symmetric part of $\nabla_s\mathbf{v}$ gives the surface curl of $\mathbf{v}$, i.e.,

$$\mathrm{curl}_s\mathbf{v} \equiv -\boldsymbol{\varepsilon}\nabla_s\mathbf{v}. \tag{A2}$$

The tensor field $\mathbf{L} \equiv -\nabla_s\boldsymbol{\nu}$ is symmetric and tangential. At each point $p \in S$, we may then regard $\mathbf{L}(p)$ as a linear map from $\mathcal{T}_p$ to the tangent plane at p itself whose eigenvalues c_1 and c_2 and corresponding unit eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are the principal curvatures and directions at p, respectively. The first two principal scalar invariants of $\mathbf{L}$,

$$2H \equiv \mathrm{tr}(\mathbf{L}) = -\mathrm{div}_s\boldsymbol{\nu} = c_1 + c_2, \tag{A3}$$

and

$$K \equiv \tfrac{1}{2}[(\mathrm{tr}\mathbf{L})^2 - \mathrm{tr}\mathbf{L}^2] = c_1c_2, \tag{A4}$$

are the mean and Gaussian curvatures of S, respectively. Since $\mathbf{L}$ is a tangential tensor field, the Cayley-Hamilton theorem implies that

$$\mathbf{L}^2 - 2H\mathbf{L} + K\mathbf{P} = \mathbf{0}. \tag{A5}$$

Let $\mathbf{n}$ be a tangent unit vector field. The normal curvature and the geodesic torsion along $\mathbf{n}$ are defined, respectively, as

$$c_\mathbf{n} \equiv \mathbf{n}\cdot\mathbf{Ln} \quad \text{and} \quad \tau_\mathbf{n} \equiv -\mathbf{t}\cdot\mathbf{Ln}, \tag{A6}$$

where $\mathbf{t} = \boldsymbol{\nu}\times\mathbf{n}$. Similarly, $c_\mathbf{t} \equiv \mathbf{Lt}\cdot\mathbf{t}$ is the normal curvature along $\mathbf{t}$. From this definition and (A6) the extrinsic curvature tensor $\mathbf{L}$ can be written as

$$\mathbf{L} = c_\mathbf{n}\mathbf{n}\otimes\mathbf{n} - \tau_\mathbf{n}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n}) + c_\mathbf{t}\mathbf{t}\otimes\mathbf{t}, \tag{A7}$$

by which we readily deduce that

$$c_\mathbf{n} + c_\mathbf{t} = 2H \quad \text{and} \quad c_\mathbf{n}c_\mathbf{t} - \tau_\mathbf{n}^2 = K. \tag{A8}$$

We conclude this section by reporting some identities that will be useful in deriving the equilibrium equations. Let f, $\mathbf{u}$, $\mathbf{w}$, and $\mathbf{S}$ be differentiable fields on S, with f being scalar valued, $\mathbf{u}$ and $\mathbf{w}$ vector valued, and $\mathbf{S}$ tensor valued. The following identities hold

$$\nabla_s(f\mathbf{u}) = \mathbf{u}\otimes\nabla_s f + f\nabla_s\mathbf{u}, \tag{A9a}$$

$$\mathrm{div}_s(f\mathbf{S}) = \mathbf{S}\nabla_s f + f\mathrm{div}_s\mathbf{S}, \tag{A9b}$$

$$\mathrm{div}_s(\mathbf{S}^T\mathbf{u}) = (\mathrm{div}_s\mathbf{S})\cdot\mathbf{u} + \mathbf{S}\cdot\nabla_s\mathbf{u}, \tag{A9c}$$

$$\mathrm{div}_s(\mathbf{u}\otimes\mathbf{w}) = (\nabla_s\mathbf{u})\mathbf{w} + (\mathrm{div}_s\mathbf{w})\mathbf{u}, \tag{A9d}$$

$$(\nabla_s\mathbf{u})\mathbf{u} = \mathrm{curl}_s\mathbf{u}\times\mathbf{u} + \tfrac{1}{2}\nabla_s(|\mathbf{u}|^2), \tag{A9e}$$

$$\mathrm{curl}_s(f\mathbf{u}) = \nabla_s f \times \mathbf{u} + f\,\mathrm{curl}_s\mathbf{u}, \tag{A9f}$$

$$\mathrm{curl}_s\nabla_s f = -\boldsymbol{\nu}\times\mathbf{L}\nabla_s f. \tag{A9g}$$

APPENDIX B: DERIVATION OF THE EQUILIBRIUM EQUATIONS

In Ref. [5] we proved that the surface gradients of the principal directions are given as follows

$$\nabla_s\mathbf{e}_1 = k_1\mathbf{e}_2\otimes\mathbf{e}_1 + k_2\mathbf{e}_2\otimes\mathbf{e}_2 + c_1\boldsymbol{\nu}\otimes\mathbf{e}_1, \tag{B1a}$$

$$\nabla_s\mathbf{e}_2 = -k_1\mathbf{e}_1\otimes\mathbf{e}_1 - k_2\mathbf{e}_1\otimes\mathbf{e}_2 + c_2\boldsymbol{\nu}\otimes\mathbf{e}_2, \tag{B1b}$$

where k_1 and k_2 are the geodesic curvatures of the curvature lines of S, i.e., the integral curves of the principal directions on S. Then combining (6), (A9a), and (B1) the surface gradients of the director and conormal fields are, respectively,

$$\nabla_s\mathbf{n} = \mathbf{t}\otimes(\nabla_s\alpha - \boldsymbol{\omega}) + \boldsymbol{\nu}\otimes\mathbf{L}\mathbf{n}, \tag{B2a}$$

$$\nabla_s\mathbf{t} = -\mathbf{n}\otimes(\nabla_s\alpha - \boldsymbol{\omega}) + \boldsymbol{\nu}\otimes\mathbf{L}\mathbf{t}, \tag{B2b}$$

where $\boldsymbol{\omega} = -k_1\mathbf{e}_1 - k_2\mathbf{e}_2$ is the vector parametrizing the spin connection on S. From (B2), we readily deduce that

$$\mathrm{curl}_s\mathbf{n} = (\nabla_s\alpha - \boldsymbol{\omega})\times\mathbf{t} - \boldsymbol{\nu}\times\mathbf{L}\mathbf{n}, \tag{B3a}$$

$$\mathrm{curl}_s\mathbf{t} = -(\nabla_s\alpha - \boldsymbol{\omega})\times\mathbf{n} - \boldsymbol{\nu}\times\mathbf{L}\mathbf{t}, \tag{B3b}$$

and

$$|\nabla_s\mathbf{n}|^2 = |\nabla_s\alpha - \boldsymbol{\omega}|^2 + |\mathbf{L}\mathbf{n}|^2. \tag{B4}$$

From (A9f) and (B1) the surface curl of the vector parametrizing the spin connection is found to be

$$\begin{aligned}\mathrm{curl}_s\boldsymbol{\omega} &= -\mathrm{curl}_s(k_1\mathbf{e}_1 + k_2\mathbf{e}_2)\\ &= -(\nabla_s k_1\times\mathbf{e}_1 + \nabla_s k_2\times\mathbf{e}_2)\\ &\quad - |\boldsymbol{\omega}|^2\boldsymbol{\nu} + k_1c_1\mathbf{e}_2 - k_2c_2\mathbf{e}_1\\ &= -(\nabla_s k_2\cdot\mathbf{e}_1 - \nabla_s k_1\cdot\mathbf{e}_2 + |\boldsymbol{\omega}|^2)\boldsymbol{\nu}\\ &\quad + (\mathbf{e}_2\otimes\mathbf{e}_1 - \mathbf{e}_1\otimes\mathbf{e}_2)\mathbf{L}\boldsymbol{\omega}\\ &= -\boldsymbol{\nu}\times\mathbf{L}\boldsymbol{\omega} + K\boldsymbol{\nu},\end{aligned} \tag{B5}$$

where the identity

$$-(\nabla_s k_2\cdot\mathbf{e}_1 - \nabla_s k_1\cdot\mathbf{e}_2 + |\boldsymbol{\omega}|^2) = \boldsymbol{\nu}\cdot\mathrm{curl}_s\boldsymbol{\omega} = K \tag{B6}$$

has been used. We refer the reader to Appendix C in Ref. [5] for the proof of (B6). Finally, with the aid of (A9g) and (B5) we conclude that

$$\mathrm{curl}_s(\nabla_s\alpha - \boldsymbol{\omega}) = -\boldsymbol{\nu}\times\mathbf{L}(\nabla_s\alpha - \boldsymbol{\omega}) - K\boldsymbol{\nu}. \tag{B7}$$

Next, on using (A9c), (A5), (B2), and the definition of the extrinsic curvature tensor we deduce that

$$\begin{aligned}\mathbf{n}\cdot\Delta_s\mathbf{n} &= \mathbf{n}\cdot\mathrm{div}_s(\nabla_s\mathbf{n})\\ &= \mathrm{div}_s[(\nabla_s\mathbf{n})^T\mathbf{n}] - \nabla_s\mathbf{n}\cdot\nabla_s\mathbf{n} = -|\nabla_s\mathbf{n}|^2,\end{aligned} \tag{B8}$$

$$\begin{aligned}\mathbf{t}\cdot\Delta_s\mathbf{n} &= \mathbf{t}\cdot\mathrm{div}_s(\nabla_s\mathbf{n})\\ &= \mathrm{div}_s[(\nabla_s\mathbf{n})^T\mathbf{t}] - \nabla_s\mathbf{n}\cdot\nabla_s\mathbf{t}\\ &= \mathrm{div}_s(\nabla_s\alpha - \boldsymbol{\omega}) - \mathbf{L}\mathbf{n}\cdot\mathbf{L}\mathbf{t}\\ &= \Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega} + 2H\tau_{\mathbf{n}},\end{aligned} \tag{B9}$$

thanks to which the director equation (5) can be rewritten as (7), and

$$\begin{aligned}\boldsymbol{\nu}\cdot\Delta_s\mathbf{n} &= \boldsymbol{\nu}\cdot\mathrm{div}_s(\nabla_s\mathbf{n})\\ &= \mathrm{div}_s[(\nabla_s\mathbf{n})^T\boldsymbol{\nu}] + \nabla_s\mathbf{n}\cdot\mathbf{L}\\ &= \mathrm{div}_s(\mathbf{L}\mathbf{n}) + \mathbf{L}\mathbf{t}\cdot(\nabla_s\alpha - \boldsymbol{\omega}).\end{aligned} \tag{B10}$$

We now report the identity

$$\mathrm{div}_s\mathbf{L} = 2[\nabla_s H + (2H^2 - K)\boldsymbol{\nu}], \tag{B11}$$

the proof of which is contained in Appendix A of Ref. [19]. As a consequence of (A9c), (B2b), (B11) and the symmetry of the extrinsic curvature tensor $\mathbf{L}$, we have

$$\begin{aligned}\mathrm{div}_s(\mathbf{L}\mathbf{t}) &= \mathrm{div}_s\mathbf{L}\cdot\mathbf{t} + \mathbf{L}\cdot\nabla_s\mathbf{t}\\ &= 2\nabla_s H\cdot\mathbf{t} - \mathbf{L}\mathbf{n}\cdot(\nabla_s\alpha - \boldsymbol{\omega}).\end{aligned} \tag{B12}$$

On the other hand, from (A7), (A8), (A9b), and (B2) we have

$$\begin{aligned}\mathrm{div}_s(\mathbf{L}\mathbf{t}) &= \mathrm{div}_s(-\tau_{\mathbf{n}}\mathbf{n} + c_{\mathbf{t}}\mathbf{t})\\ &= -\nabla_s\tau_{\mathbf{n}}\cdot\mathbf{n} + \nabla_s c_{\mathbf{t}}\cdot\mathbf{t} - (c_{\mathbf{t}}\mathbf{n} + \tau_{\mathbf{n}}\mathbf{t})\cdot(\nabla_s\alpha - \boldsymbol{\omega})\\ &= 2\nabla_s H\cdot\mathbf{t} - \nabla_s\tau_{\mathbf{n}}\cdot\mathbf{n} - \nabla_s c_{\mathbf{n}}\cdot\mathbf{t}\\ &\quad + (\boldsymbol{\nu}\times\mathbf{L}\mathbf{t})\cdot(\nabla_s\alpha - \boldsymbol{\omega}).\end{aligned} \tag{B13}$$

Then, combining (B12) and (B13) yields

$$\nabla_s\tau_{\mathbf{n}}\cdot\mathbf{n} + \nabla_s c_{\mathbf{n}}\cdot\mathbf{t} = (\mathbf{L}\mathbf{n} + \boldsymbol{\nu}\times\mathbf{L}\mathbf{t})\cdot(\nabla_s\alpha - \boldsymbol{\omega}). \tag{B14}$$

From (A5), (A7), (A9f), (B3), and (B14) we obtain

$$\begin{aligned}\mathrm{curl}_s(\mathbf{L}\mathbf{n}) &= \nabla_s c_{\mathbf{n}}\times\mathbf{n} + c_{\mathbf{n}}\mathrm{curl}_s\mathbf{n} - \nabla_s\tau_{\mathbf{n}}\times\mathbf{t} - \tau_{\mathbf{n}}\mathrm{curl}_s\mathbf{t}\\ &= -(\nabla_s c_{\mathbf{n}}\cdot\mathbf{t} + \nabla_s\tau_{\mathbf{n}}\cdot\mathbf{n})\boldsymbol{\nu}\\ &\quad + (\nabla_s\alpha - \boldsymbol{\omega})\times(\boldsymbol{\nu}\times\mathbf{L}\mathbf{n}) - \boldsymbol{\nu}\times\mathbf{L}^2\mathbf{n}\\ &= -[(\boldsymbol{\nu}\times\mathbf{L}\mathbf{t})\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\boldsymbol{\nu} - 2H\boldsymbol{\nu}\times\mathbf{L}\mathbf{n} + K\mathbf{t}.\end{aligned} \tag{B15}$$

We are now in position to derive the equilibrium equations (8) and (9). We first project (3a), with $\boldsymbol{\sigma}$ as in (4a), along the normal $\boldsymbol{\nu}$ and obtain

$$\begin{aligned}0 = \boldsymbol{\nu}\cdot\mathrm{div}_s\boldsymbol{\sigma} &= \mathrm{div}_s(\boldsymbol{\sigma}^T\boldsymbol{\nu}) + \boldsymbol{\sigma}\cdot\mathbf{L}\\ &= -k\mathrm{div}_s[(\boldsymbol{\nu}\cdot\Delta_s\mathbf{n})\mathbf{n}] + 2H\left(\gamma + \frac{k}{2}|\nabla_s\mathbf{n}|^2\right)\\ &\quad - k(\nabla_s\mathbf{n})^T(\nabla_s\mathbf{n})\cdot\mathbf{L},\end{aligned} \tag{B16}$$

that is equation (8). Next, since from (A3) and (A9d) one deduces that

$$\begin{aligned}\mathrm{div}_s\left[\left(\gamma + \frac{k}{2}|\nabla_s\mathbf{n}|^2\right)\mathbf{P}\right] &= 2H\left(\gamma + \frac{k}{2}|\nabla_s\mathbf{n}|^2\right)\boldsymbol{\nu}\\ &\quad + \frac{k}{2}\nabla_s(|\nabla_s\mathbf{n}|^2),\end{aligned} \tag{B17}$$

on using (A9d), (A9e), (B4), (B7), (B10), and (B15), the projection of (3a), with $\boldsymbol{\sigma}$ as in (4a), onto the tangent plane yields

$$\begin{aligned}\mathbf{0} &= \mathbf{P}\mathrm{div}_s[(\nabla_s\mathbf{n})^T(\nabla_s\mathbf{n}) + (\boldsymbol{\nu}\cdot\Delta_s\mathbf{n})\boldsymbol{\nu}\otimes\mathbf{n}] - \tfrac{1}{2}\nabla_s(|\nabla_s\mathbf{n}|^2)\\ &= \mathbf{P}\mathrm{div}_s[(\nabla_s\alpha - \boldsymbol{\omega})\otimes(\nabla_s\alpha - \boldsymbol{\omega}) + \mathbf{L}\mathbf{n}\otimes\mathbf{L}\mathbf{n}]\\ &\quad - (\boldsymbol{\nu}\cdot\Delta_s\mathbf{n})\mathbf{L}\mathbf{n} - \tfrac{1}{2}\nabla_s(|\nabla_s\mathbf{n}|^2)\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{P}[\mathrm{curl}_s(\nabla_s\alpha - \boldsymbol{\omega}) \times (\nabla_s\alpha - \boldsymbol{\omega}) + \mathrm{curl}_s(\mathbf{Ln}) \times \mathbf{Ln}] \\
&\quad + \tfrac{1}{2}\nabla_s(|\nabla_s\alpha - \boldsymbol{\omega}|^2 + |\mathbf{Ln}|^2 - |\nabla_s\mathbf{n}|^2) \\
&\quad + (\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega})(\nabla_s\alpha - \boldsymbol{\omega}) + [\mathrm{div}_s(\mathbf{Ln}) - \boldsymbol{\nu}\cdot\Delta_s\mathbf{n}]\mathbf{Ln} \\
&= -K\boldsymbol{\nu}\times(\nabla_s\alpha - \boldsymbol{\omega}) + (\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega})(\nabla_s\alpha - \boldsymbol{\omega}) \\
&\quad - [(\boldsymbol{\nu}\times\mathbf{Lt})\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\boldsymbol{\nu}\times\mathbf{Ln} - [\mathbf{Lt}\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\mathbf{Ln} \\
&= (\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega} + 2H\tau_{\mathbf{n}})(\nabla_s\alpha - \boldsymbol{\omega}),
\end{aligned}
\tag{B18}
$$

where the last equality is a consequence of the fact that, in the light of (A7) and (A8),

$$
\begin{aligned}
&[(\boldsymbol{\nu}\times\mathbf{Lt})\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\boldsymbol{\nu}\times\mathbf{Ln} + [\mathbf{Lt}\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\mathbf{Ln} \\
&\quad = -[\mathbf{n}\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\big[(c_{\mathbf{n}} + c_{\mathbf{t}})\tau_{\mathbf{n}}\mathbf{n} + \big(c_{\mathbf{n}}c_{\mathbf{t}} - \tau_{\mathbf{n}}^2\big)\mathbf{t}\big] \\
&\qquad + [\mathbf{t}\cdot(\nabla_s\alpha - \boldsymbol{\omega})]\big[\big(c_{\mathbf{n}}c_{\mathbf{t}} - \tau_{\mathbf{n}}^2\big)\mathbf{n} - (c_{\mathbf{n}} + c_{\mathbf{t}})\tau_{\mathbf{n}}\mathbf{t}\big] \\
&\quad = -2H\tau_{\mathbf{n}}(\nabla_s\alpha - \boldsymbol{\omega}) - K\boldsymbol{\nu}\times(\nabla_s\alpha - \boldsymbol{\omega}).
\end{aligned}
\tag{B19}
$$

The derivation of (9) is then complete.

Observe now that the stress tensor (4a) is not symmetric and, in view of (B10) and the symmetry of $\mathbf{L}$, twice the axial vector corresponding to the skew-symmetric part of $\boldsymbol{\sigma}$ is

$$
\begin{aligned}
\boldsymbol{\varepsilon}\boldsymbol{\sigma} &= -k(\boldsymbol{\nu}\cdot\Delta_s\mathbf{n})\mathbf{t} \\
&= -k[\mathrm{div}_s(\mathbf{Ln}) + \mathbf{L}(\nabla_s\alpha - \boldsymbol{\omega})\cdot\mathbf{t}]\mathbf{t}.
\end{aligned}
\tag{B20}
$$

On the other hand, from (A9d), (B2), (A5), (A7), and, again, the symmetry of the extrinsic curvature tensor $\mathbf{L}$, the surface divergence of the macrotorque tensor (4b) reads

$$
\begin{aligned}
\mathrm{div}_s\mathbf{T} &= -k\mathbf{L}(\nabla_s\mathbf{n})^T\mathbf{t} + k\mathrm{div}_s[(\nabla_s\mathbf{n})^T\mathbf{t}]\boldsymbol{\nu} \\
&\quad - k(\nabla_s\mathbf{t})\mathbf{Ln} - k\mathrm{div}_s(\mathbf{Ln})\mathbf{t} \\
&= -k\mathbf{L}(\nabla_s\alpha - \boldsymbol{\omega}) + k(\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega})\boldsymbol{\nu} \\
&\quad + k[\mathbf{L}(\nabla_s\alpha - \boldsymbol{\omega})\cdot\mathbf{n}]\mathbf{n} - k(\mathbf{L}^2\mathbf{n}\cdot\mathbf{t})\boldsymbol{\nu} - k\mathrm{div}_s(\mathbf{Ln})\mathbf{t} \\
&= k(\Delta_s\alpha - \mathrm{div}_s\boldsymbol{\omega} + 2H\tau_{\mathbf{n}})\boldsymbol{\nu} \\
&\quad - k[\mathrm{div}_s(\mathbf{Ln}) + \mathbf{L}(\nabla_s\alpha - \boldsymbol{\omega})\cdot\mathbf{t}]\mathbf{t}.
\end{aligned}
\tag{B21}
$$

Thus, in the light of (7), (B20) and (B21), the equation of balance of macrotorques is identically satisfied.

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