# Simplest breather

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The simplest Klein-Gordon (KG) breather is a compacton on a string subject to the force of gravity in a frictionless V-shaped trough. Its dynamics, spectrum, and energy are discussed and it is compared to sine-Gordon breathers. A generalization of this problem consists of a charged string subject to the electrostatic force of two semi-infinite coplanar charged planes separated by a gap of constant width. For motion in the midplane between these planes, the string's displacement $u(x,t)$ satisfies the nonlinear KG equation $(\partial_t^2 - \partial_x^2)u = -\tan^{-1} u$ in dimensionless form. Simulations of this equation reveal long-lived, breatherlike states, or "pseudobreathers," which preserve shape and speed to high accuracy when Lorentz-transformed to simulate collisions.

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## I. INTRODUCTION

The theory of nonlinear waves rests largely upon the inverse scattering transform (IST). ISTs allow exact solutions of the three most naturally arising equations, the Korteweg-deVries (KdV) [1], cubic Schroedinger [2], and sine-Gordon (SG) [3] equations. ISTs are not simple, but they show why solitons and breathers exist and have the remarkable particlelike property of retaining their initial shape and speed after collisions. Among all categories of nonlinear wave equations, Klein-Gordon (KG) equations are uniquely interesting because of their Lorentz invariance. Moreover, SG breathers and their variations arise in widely diverse continuous and discrete physical systems [4], including DNA [5] and Bose-Einstein condensates [6]. More than 50 years after the IST was introduced, KG breathers remain the subject of intensive research. Most fields of theoretical physics have a relatively simple set of examples, prototypes that illustrate or exemplify their essential nature. Is there a simplest KG breather?

This article presents the simplest KG breather, by which is meant an exact solution of the simplest nonlinear KG equation that possesses a self-trapped state. It is a compacton, meaning it vanishes outside of some region.

Kruskal has shown that the only integrable Klein-Gordon equations are those whose nonlinear term is sin or sinh [7]. The simplest breather therefore is not a true breather. It would not preserve shape and speed exactly after a collision. It is also noted that traveling waves for the equation governing the simplest breather, Eq. (1) below, are necessarily oscillatory. That is, Eq. (1) admits no localized, solitary traveling waves.

Scalar nonlinear KG equations in one spatial dimension govern an idealized mechanical string subject to an external nonlinear restoring force. This could be the force of gravity on a string sliding without friction on the walls of a trough. The simplest breather may be thought of as a localized, self-trapped, and nearly sinusoidal mode of a mechanical string subject to the force of gravity in a frictionless V-shaped trough. It is explained with elementary mechanics.

This suggests an intuitively natural extension of the simplest breather to the electrostatic system discussed in Sec. III. In this system, a charged string obeys the nonlinear KG Eq. (21) below, reading $(\partial_t^2 - \partial_x^2)u = -\tan^{-1}u$. This appears to have escaped attention in the literature. Simulations of approximate breather-like solutions of this equation reveal long-lived states. When "boosted" by Lorentz transforms, these behave very much like true breathers in collisions, meaning those with an associated IST, by preserving their initial shape and speed to high accuracy. These states are called "pseudobreathers" in the following.

A breather may be a bound soliton-antisoliton state corresponding to mirror-image complex eigenvalues in an associated IST. This is the case for the KdV [1,8] and SG [3] equations. Alternatively, a breather may be interpreted as a kink-antikink resonance, as with the SG equation[7] and the $\phi^4$ or Landau-Ginsburg equation [9–11], although it may be noted that the latter does not admit a true breather [12].

For nonintegrable KG equations, nonlinear dispersion can explain the self-trapping of a breather intuitively. If the potential energy function giving rise to the external restoring force on the string has negative curvature asymptotically, analogous to "softening on-site nonlinearity" in discrete systems, then the cutoff or resonant frequency (at which the wave number vanishes) of large-amplitude traveling waves is smaller than that of linear, small-amplitude traveling waves.

In this case, the frequency of a large-amplitude breather is smaller than the cutoff frequency for small-amplitude traveling waves, and therefore the small-amplitude wings of a breather act as mirrors. They are driven below the linear resonant frequency. This view has been invoked for a hydrodynamic breather [13]. It is also cited for an approximate breather on a positively charged dielectric string whose equilibrium

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position is midway between identical, fixed, parallel, negatively charged lines and which moves perpendicularly to their plane [14]. A natural question is whether the self-trapping of the simplest breather is explained similarly.

At first sight, the simplest breather should be trivial. It involves only elementary mechanics and is only quadratic in distance and time. Nevertheless, its shape, algebraic expression, mechanics, and spectrum are not self-evident. Although not the only compacton arising in a continuous rather than a discrete system [15], it is the large-amplitude limit of any breather for a KG equation whose external restoring force asymptotically approaches a constant, or "saturates."

Saturable media are widespread. The density of ions saturates in plasma cavitons [16–18] and many optical media have saturable Kerr nonlinearities [19]. The liquid $CS_2$ is an example of the latter, having prolate spheroidal molecules whose "easy" axis of polarization aligns increasingly with the electric field of linearly polarized light at greater intensity. Other examples include alkali metal vapors [20] and ferromagnetic materials [21].

The governing Eq. (1) below is at the boundary between two classes of nonlinear KG equations with conservative external restoring forces, namely those for which the curvature of the external potential energy function is asymptotically positive and those for which it is asymptotically negative. The potential in the latter case has the shape of seagull wings.

The charged string mentioned above [14] and Eq. (1) below both have a saturable nonlinear restoring force and do not possess solitary, shape-invariant traveling-wave solutions. A stationary breather may be "boosted" to any speed less than the speed of waves on a free string by a Lorentz transform in which this speed replaces the speed of light in vacuum, allowing the study of collisions.

Section II treats the algebraic form, mechanics, Fourier representation, and energy of the simplest breather. Dispersion relations for traveling and standing waves are derived, the latter yielding the breather as a special case. Section III comments on standing waves, compares the simplest KG breather with the SG breather, notes the impossibility of simulating the simplest breather, and introduces the electrostatic generalization of the simplest breather. Simulations of collisions between these pseudobreathers are shown, mainly to demonstrate their existence.

## II. THEORY

The subsections below treat, in order, the dimensionless form of the nonlinear KG equation for the simplest breather, its algebraic form, dynamics, spectrum, energy, and the dispersion relations for traveling and standing waves, the simplest breather being a special case of the latter.

Let a gravitational field point down along the axis of symmetry of a trough whose cross-section has the shape of the letter "V." A string of infinite length lies along the bottom of the trough when at rest. Then the magnitude of the external gravitational restoring force per unit length is independent of the amplitude of the string's displacement from the bottom of the trough, save when it vanishes. The restoring force itself changes sign when the string crosses the bottom of the trough. The bottom of the trough may be assumed to have a very small smooth region, making the restoring force a continuous function, but without significant effect on the string's motion for amplitudes of interest. This region is ignored below.

### A. Dimensionless form of the equation

Let $u(x,t)$ be the string's displacement, measured along the wall itself, from the bottom of the trough described above. Let the string have linear mass density $\rho$ and tension $\tau$. Then $u(x,t)$ obeys the nonlinear KG equation

$$\rho \partial_t^2 u - \tau \partial_x^2 u = -\rho g \,\mathrm{sgn} u, \tag{1}$$

where $g$ is the acceleration of gravity projected onto the wall by the factor $\sin\alpha$, where $\alpha$ is the angle of the wall with respect to horizontal, and the "sign-function" is defined by $\mathrm{sgn} u = -1,0,1$, respectively, for $u < 0$, $u = 0$, $u > 0$.

Equation (1) has a natural scale for distance, namely the tension divided by the force per unit length owing to gravity, which we denote by $d \equiv \tau/\rho g$. Equation (1) has dimensionless form with dimensionless displacement $u/d$, dimensionless distance $x/d$, and dimensionless time $\sqrt{g/d}\,t$. On replacing $u$, $x$, and $t$ by their dimensionless counterparts, Eq. (1) becomes

$$\left(\partial_t^2 - \partial_x^2\right) u = -\mathrm{sgn} u. \tag{2}$$

This equation has no free parameters, and hence its breathers are scaled replicas of the unit-amplitude breather. The following is limited primarily to the consideration of Eq. (2), but Eq. (1) is revisited when it is helpful for discussing certain aspects of the breather, such as its energy.

### B. Description and algebraic form

The breather is first described and its algebraic form presented. A derivation is given in Sec. IIC. In dimensionless variables, let $T$ denote its period, $A$ its maximum amplitude, and $2L$ its two-sided width. These obey

$$T = 4\sqrt{A} \tag{3}$$

and

$$L = T/2 = 2\sqrt{A}. \tag{4}$$

There is no small-amplitude, linear approximation. From Eq. (2), the resonant frequency increases without bound as $A \to 0$. If we let $k = \pi/L$ and $\omega = 2\pi/T$, then Eqs. (3) and (4) give $k = \omega$, the linear dispersion relation for a free string.

Figure 1 shows the right-hand half of the breather, itself an even function of $x$, for the case $A = 1$, corresponding to $T = 4$ and $L = 2$, at the times $t = 0$ (solid) and $T/8$ (dashed). The breather is initialized at maximum displacement with speed $\partial_t u = 0$.

For $0 < t < T/4$, the first quarter-period, the breather has three regions labeled 1, 2, and 3 in Fig. 1 with boundaries at $x_1(t) = L/2 - t = \sqrt{A} - t$ and $x_2(t) = L/2 + t = \sqrt{A} + t$. Region 1 has constant curvature $-1$ and acceleration $-2$. Region 2 is linear, and hence its acceleration is $-1$. Its slope $\partial_x u$ varies linearly with $t$. Region 3 is motionless and has constant curvature 1. Region 2 grows in length from 0 to $L$ during the first quarter-cycle.

For $x \geqslant 0$, in the first quarter-cycle, the breather may be written

$$u(x,t) = \begin{cases} A - t^2 - x^2/2, & 0 \leqslant x \leqslant x_1, \\ A/2 - t^2/2 + (t - \sqrt{A})(x - \sqrt{A}), & x_1 \leqslant x \leqslant x_2, \\ (x - 2\sqrt{A})^2/2, & x_2 \leqslant x \leqslant 2\sqrt{A}, \\ 0, & x \geqslant 2\sqrt{A}. \end{cases} \tag{5}$$

The extension of the form of $u(x,t)$ in Eq. (5) to other points and times is straightforward and need not be shown. Equation (5) agrees with the description of the breather. It may be shown that Eq. (5) and its extensions satisfy Eq. (2) and that $u$, $\partial_x u$, and $\partial_t u$ are continuous at $x = \pm L = \pm 2\sqrt{A}$, $x_1(t)$, and $x_2(t)$ for all $t$.

### C. Dynamics

The breather is the sum of two motions. One is a free-falling overall oscillation of a straight string with slope 0, amplitude $A/2$, acceleration $\pm 1$, and period $T = 4\sqrt{A}$. This is the position of the center of mass (CM) of the breather and identical to the motion of a point mass. It is a particular solution of Eq. (2). In the first quarter-cycle, the CM displacement, say $u_{\rm CM}(t)$, is given by

$$u_{\rm CM}(t) = A/2 - t^2/2. \tag{6}$$

At $t = T/4 = \sqrt{A}$, $u_{\rm CM} = 0$.

The other motion is a homogenous standing-wave solution of Eq. (2), in phase with the CM motion at $x = 0$, equal to the sum of identical counterpropagating parabolic traveling waves, each with amplitude $A/4$, period $T = 4\sqrt{A}$, and wavelength $2L = T = 4\sqrt{A}$. The standing wave and the CM displacement both vanish at $t = T/4$. When they do not vanish, their sum at any given instant is either positive or negative, that is, the breather is entirely on one side of the trough or the other at any instant.

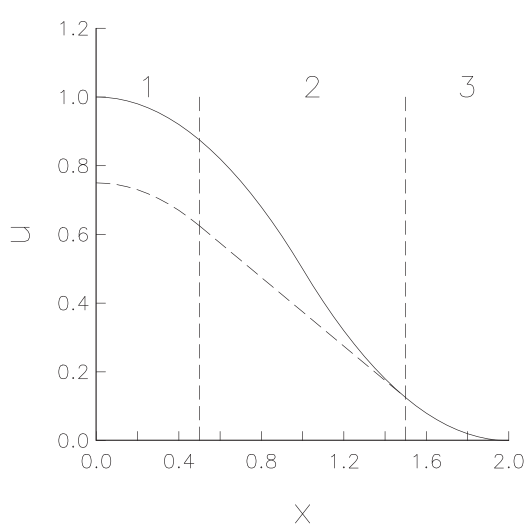


FIG. 1. The breather is shown at $t = 0$ (solid) and $t = T/8 = 0.5$ (dashed) for amplitude $A = 1$, for which the period is $T = 4$ and the half-width is $L = 2$. Dashed vertical lines are at boundaries $x_1(t)$ and $x_2(t)$ between regions 1, 2, and 3 at $t = 0.5$, namely, $x_1(t) = L/2 - t = 0.5$ and $x_2(t) = L/2 + t = 1.5$.

When added to the CM motion, the minima of the standing wave at $x = \pm L$ vanish, touching the bottom of the trough. The breather's initial shape is such that, at the minima, and only at the minima, the curvature of the standing wave alone (more precisely, the upward force from the bottom of the trough) gives an acceleration 1, canceling the free-fall CM motion and leaving the minima undisturbed at the bottom of the trough throughout a cycle. More precisely, the string at $x = \pm L$ is indistinguishable from a free-hanging, static string at the point where its displacement first vanishes. The force from the bottom of the trough at this point replaces that from the string's positive curvature to balance the force of gravity.

To show that the breather has the decomposition claimed above, consider the algebraic forms of the traveling waves. Let $X_+ \equiv x - t$ and $X_- \equiv x + t$, and let the forward and backward parabolic traveling waves be denoted by $u_+(X_+)$ and $u_-(X_-)$, respectively. Figure 2 shows either of these at $t = 0$ near $x = 0$. The parabolic wave in Fig. 2, denoted by $f(x)$, is given near the origin by

$$\begin{aligned} f_1(x) &= (A/4)(1 - 4x^2/L^2), \quad -L/2 \leqslant x \leqslant L/2, \\ f_2(x) &= (A/4)[-1 + 4(x - L)^2/L^2], \quad L/2 \leqslant x \leqslant 3L/2. \end{aligned} \tag{7}$$

Figure 3 shows regions 1, 2, and 3 and the values on the characteristics near the origin in the plane $(x, t)$. In region 1, within triangle ABC, $X_+$ and $X_-$ are both on $(-L/2, L/2)$. Therefore, $u_+$ and $u_-$ both have the form of $f_1$ in Eq. (7), that is, $u_+ = f_1(X_+)$ and $u_- = f_1(X_-)$. Adding the standing wave

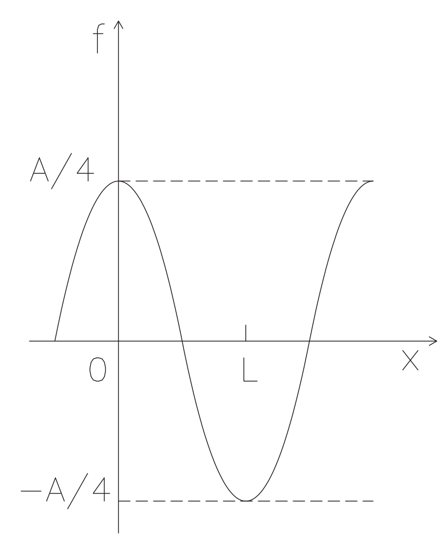


FIG. 2. Each of the two counterpropagating parabolic traveling waves $u_+$ and $u_-$ defined in the text is shown near $x = 0$ at $t = 0$.

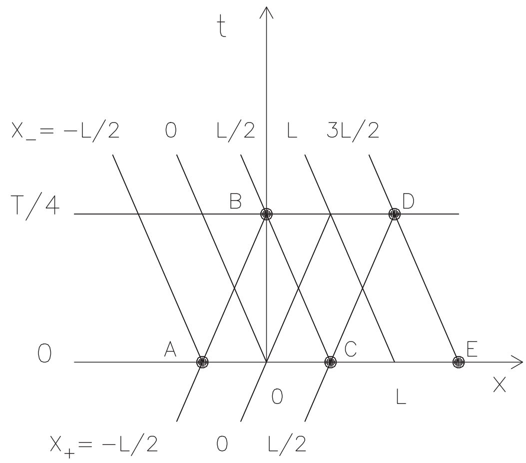


FIG. 3. Region 1 in Fig. 1 is bounded by A,B,C; 2 by B,C,D; and 3 by C,D,E. Values on characteristic lines $X_+ = x - t$ and $X_- = x + t$ are shown. The breather extends from $x = -L$ to $x = L$, so only the left half of CDE is used.

and the CM displacement in Eq. (6) gives the expression in Eq. (5) for region 1.

In region 2, within triangle BCD, $-L/2 \leqslant X_+ \leqslant L/2$, but $L/2 \leqslant X_- \leqslant 3L/2$. Therefore, $u_+ = f_1(X_+)$ and $u_- = f_2(X_-)$. These have opposite curvatures that cancel to leave a linear function of $x$. Adding the standing wave and the CM motion yields the expression in Eq. (5) in region 2.

In region 3, triangle CDE, both $X_+$ and $X_-$ are on $(L/2,3L/2)$, so the form of $f_2$ is used for both $u_+$ and $u_-$. Summing the standing wave and the CM motion yields the expression in Eq. (5) in region 3.

### D. Fourier representation

We use the decomposition of the breather into a CM motion and counterpropagating traveling waves to obtain its Fourier series. It is convenient first to let $A = 4$, so that the amplitude of the parabolic wave in Fig. 2 is unity. The coefficients $f_n$ in its Fourier (cosine) series,

$$f(x) = \sum_{n=1}^{\infty} f_n \cos(k_n x), \tag{8}$$

where $k_n = \pi n/L$, vanish if $n$ is even and for odd $n$ are given by

$$f_n = \frac{32}{(\pi n)^3} \sin\left(\frac{n\pi}{2}\right). \tag{9}$$

It follows that the contribution to $u(x,t)$ from the $n$th harmonic (for odd $n$) of the standing wave, say $f_n(x,t)$, is given by

$$\begin{aligned} f_n(x,t) &= (-1)^{(n-1)/2} \frac{32}{\pi^3 n^3}[\cos(k_n x - \omega_n t) + \cos(k_n x + \omega_n t)] \\ &= (-1)^{(n-1)/2} \frac{64}{\pi^3 n^3} \cos(k_n x)\cos(w_n t), \end{aligned} \tag{10}$$

where $\omega_n = 2\pi n/T$ is the angular frequency of the $n$th harmonic. The symbol $f$ has been redefined here after being introduced in Eq. (7), but the distinction between its different meanings should be evident from the subscript of Fourier coefficients and the explicit arguments for traveling waves.

The CM motion has the same shape as the curve in Fig. 2. Hence, if it had unit amplitude, rather than amplitude $A/2$, then the CM motion would have a Fourier series in which the coefficient of $\cos(\omega_n t)$ were $f_n$ in Eq. (9). On multiplying the sum of the traveling waves for unit amplitude in Eq. (10) by $A/4$, multiplying the CM motion in Eq. (6) for unit amplitude by $A/2$, and adding the products, one has

$$\begin{aligned} u(x,t) &= \frac{16A}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3}(-1)^{(n-1)/2}[1 + \cos(k_n x)]\cos(\omega_n t), \\ &|x| \leqslant L, \end{aligned} \tag{11}$$

with the index of summation odd here and below. This vanishes at $x = \pm L$ because $k_n L = n\pi$ and $n$ is odd.

The parabolic wave in Fig. 2 is nearly sinusoidal. As a measure of this and to verify that Eqs. (7) and (9) are consistent, consider Parseval's theorem,

$$\int_{-L}^{L} f^2(x)dx = L \sum_n f_n^2.$$

Evaluating the left-hand side of this equation using Eq. (7) gives $16L/15$. Substituting the coefficients $f_n$ in Eq. (9) into the right-hand side gives

$$\frac{16L}{15} = L \sum_{n=1}^{\infty} \left(\frac{32}{\pi^3 n^3}\right)^2.$$

This reduces to the identity [22]

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{960} \simeq 1.0014,$$

showing that Eqs. (7) and (9) are consistent. It is interesting that only about 0.14% of the "energy" in $f(x)$ is in the harmonics, and almost all of this is in the third harmonic.

### E. Energy

The energy of the breather is $8\rho g^2 A^{3/2}/3c^2$, where $c^2 = \tau/\rho$. By Eq. (3), the energy varies in proportion to $1/\omega^3$. This formula may be verified by calculating the energy at $t = T/4$, when the energy is purely kinetic. It is then the integral of $\rho(\partial_t u)^2/2$ with respect to $x$ on $(-L,L)$. From Eq. (5), restored to dimensional form, one has $\partial_t u = g(x - L)/c$ on $(0,L)$ at $t = T/4$. This says that the speed varies linearly with $x$ when the string crosses the bottom of the trough. On $(0,L)$ the density of kinetic energy at $t = T/4$ is $\rho g^2(x - L)^2/2c^2$. The density of kinetic energy at $t = T/4$ increases quadratically with $x$ moving inwards, away from $\pm L$ and toward $x = 0$.

The potential energy per unit length owing to the slope and resultant stretching of the string is $\tau(\partial_x u)^2/2$. From the dimensional form of Eq. (5), it follows that at $t = 0$, when the energy is entirely potential, this is equal to $\rho g^2 x^2/2c^2$ on $(0,L/2)$. It suffices to consider only one-quarter of the breather's length. Integrating this with respect to $x$ on $(0,L/2)$ and multiplying by 4, one finds this potential energy is equal to one-fourth of the total energy. By integrating the gravitational potential energy per unit length, namely $\rho g u$, by Eq. (5) again, with respect to $x$ at $t = 0$, one may show independently that the gravitational potential energy is three-fourths of the total energy at maximum displacement.

### F. Dispersion relationships

For traveling waves, let $u = u(\theta)$, where $\theta = kx - \omega t$ is a dimensionless phase, for some constants $k$ and $\omega$, with $x$ and $t$ being the dimensionless distance and time. Inserting this into Eq. (2) gives

$$(\omega^2 - k^2)u'' = -\mathrm{sgn}u, \tag{12}$$

where a prime ($'$) denotes differentiation with respect to the argument.

For $\omega^2 > k^2$, the solution of Eq. (12) of interest is a periodic series of linked parabolic half-cycles. If $\omega^2 < k^2$, then Eq. (12) has no physical solution. For a traveling wave with amplitude $A$ and peak at $x = 0$ at $t = 0$, the solution of Eq. (12) over the first quarter-cycle is given by

$$u(\theta) = A - \frac{\theta^2}{2(\omega^2 - k^2)}. \tag{13}$$

On letting $u(\pi/2) = 0$, which is equivalent to choosing a spatial period of $2\pi/k$ and temporal period of $2\pi/\omega$, Eq. (13) gives the dispersion relationship

$$k^2 = \omega^2 - \frac{\pi^2}{8A}. \tag{14}$$

Equation (14) implies that the cutoff or resonant frequency $\omega_c$ for traveling waves of amplitude $A$ is

$$\omega_c = \pi\sqrt{\frac{1}{8A}}. \tag{15}$$

This is the same as the frequency-amplitude relationship for the CM motion in Eq. (6), as is to be expected for $k = 0$, when $A$ is replaced by the amplitude $A/2$ of the CM motion.

By Eq. (3), the frequency of a breather of amplitude $A$, say $\omega_b$, is

$$\omega_b = \frac{\pi}{2\sqrt{A}}. \tag{16}$$

This exceeds the cutoff frequency $\omega_c$ for traveling waves with amplitude $A$ in Eq. (15). For the breather to decay, however, the small-amplitude wings must emit traveling waves at the frequency $\omega_b$. Note that the breather's frequency for amplitude $A$ is equal to the cutoff frequency for traveling waves with amplitude $A/2$. This is the amplitude of the CM motion. This provides an intuitively satisfying explanation of self-trapping for the simplest breather. It is just barely or precisely self-trapped in that its center of mass oscillates at resonance.

For standing waves, let $u(x,t) = u(x)\cos\omega t$, ignoring harmonics for the moment. Substituting this into Eq. (2) and taking the Fourier (cosine) transform gives

$$-\omega^2 u(x) - u''(x) = -\frac{2}{T}\int_0^T \mathrm{sgn}[u(x)\cos(\omega t)]\cos(\omega t)dt, \tag{17}$$

where $T = 2\pi/\omega$. Over one period $T$, the integral on the right-hand side of Eq. (17) is four times the integral over a quarter-cycle, which by inspection is $\mathrm{sgn}[u(x)]/\omega$. This leads to

$$u''(x) = -\omega^2 u(x) + \frac{4}{\pi}\mathrm{sgn}[u(x)]. \tag{18}$$

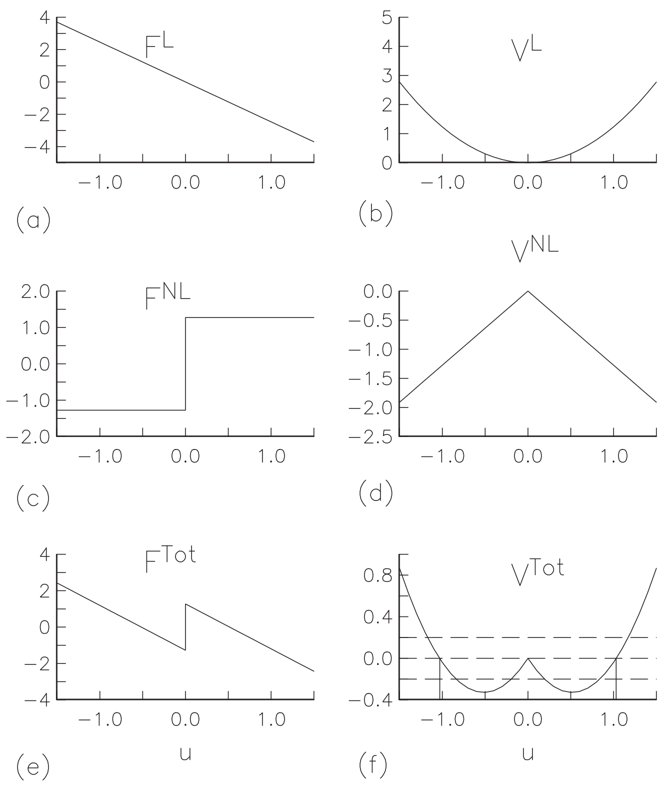


FIG. 4. (a) and (b) are, respectively, the linear restoring force $F^L$ and its corresponding potential $V^L$, as defined in the text and Eqs. (18) and (19), for the classical mechanical particle analogous to a standing wave; (c) and (d), respectively, the nonlinear force $F^{NL}$ and nonlinear potential $V^{NL}$; and (e) and (f), respectively, the total force $F^{\mathrm{Tot}}$ and total potential $V^{\mathrm{Tot}}$. Dashed lines in (f) at 0.2, 0, and $-0.2$ correspond respectively to unbiased orbits, the breather, and biased orbits. Tick marks at $\pm u_c$ are the turning points for the breather given by Eq. (20).

Equation (18) is Newton's equation of motion for a classical mechanical particle at position $u(x)$, with $x$ analogous to time, subject to two conservative forces, a linear restoring force $F^L(u) = -\omega^2 u$ and a nonlinear repulsive force $F^{NL}(u) = (4/\pi)\mathrm{sgn}(u)$. The corresponding potentials are respectively $V^L = \omega^2 u^2/2$ and $V^{NL}(u) = -4|u|/\pi$. Their sum is the total potential

$$V^{\mathrm{Tot}}(u) = \frac{1}{2}\omega^2 u^2 - \frac{4}{\pi}|u|. \tag{19}$$

These forces and potentials are plotted in Fig. 4 for the choice $\omega = \pi/2$, corresponding to $A = 1$ in Eq. (3).

The total potential in Eq. (19) has two wells for any nonzero value of $\omega$. Hence, there are three classes of orbits for the analogous classical mechanical particle for any value of $\omega$. Figure 5 shows these in the space $(u,u')$ for energies of $-0.2, 0, 0.2,$ and $0.4$.

The energy of the particle is $(u')^2/2 + V^{\mathrm{Tot}}(u)$. For positive energy, such as the upper dashed line at 0.2 in Fig. 4(f), there are unbiased oscillations with turning points symmetrically on either side of $u = 0$ whose amplitudes exceed the critical value $u_c$ given by

$$u_c = \frac{8}{\pi\omega^2}, \tag{20}$$

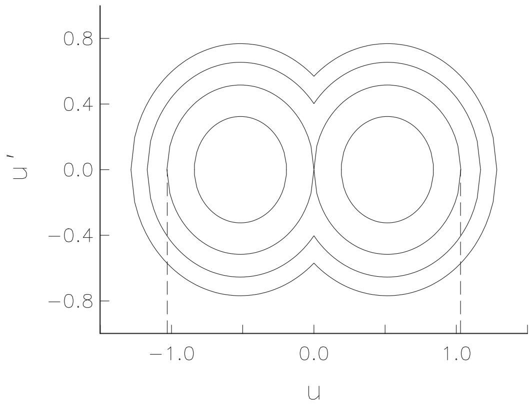


FIG. 5. Three types of orbits for the double-well potential in Fig. 4(f) are shown in the $(u,u')$ plane, or phase-space in classical mechanics, within a constant of proportionality. Positive energies of 0.4 and 0.2 give the two unbiased orbits. A negative energy of $-0.2$ gives the biased orbit. An energy of 0 gives the "separatrix" orbit passing through (0,0) corresponding to the breather. Dashed lines are at $\pm u_c$ in Eq. (20). Points at $(u = \pm u_c/2 \simeq \pm 0.5, u' = 0)$ (not shown) would correspond to a stationary particle at the bottom of a well.

at which $V^{\rm Tot}(u) = 0$ by Eq. (19). For the parameters in Fig. 1, we have $u_c = 8/\pi(\pi/2)^2 = 32/\pi^3 \simeq 1.03$.

For negative energy, such as the lower dashed line at $-0.2$ in Fig. 4(f), there are biased oscillations with turning points symmetric about $u_c/2$ and amplitude between $u_c/2$ and $u_c$. The analogous particle executes simple harmonic motion because the potential is a quadratic. The spatial period of these biased oscillations therefore is independent of their amplitude. As could have been noted earlier, these oscillations are linear waves on a massive but weightless string in the eyes of the free-falling CM frame.

The third class of orbit is for an energy 0, the middle dashed line in Fig. 4(f), and corresponds to a breather. The turning points of the analogous particle are at $u = 0$ and $u = u_c$. The approximate standing-wave theory says that the amplitude of the breather is $u_c$ in Eq. (20). This differs from the true amplitude, $A = 1$. For the choice $\omega = \pi/2$, the ratio of amplitudes is $A/u_c = 32/\pi^3 \simeq 1.03$.

Similarly, Eq. (20) says that the amplitude of a biased standing wave with constant displacement, when the string is straight, is equal to $u_c/2 = 4/\pi\omega^2$, corresponding to the minimum allowed energy in Fig. 4(f). The true amplitude at the cutoff frequency for traveling waves from Eq. (15) for the case of free-fall is $A = \pi^2/8\omega^2$. These amplitudes have the same ratio as the exact and approximate amplitudes of the breather.

An error of 3% for the amplitude in the standing-wave approximation is reasonable. The potential energy density owing to gravity is proportional to $u(x,t)$. Equation (11) says that the contribution to the gravitational potential energy from the $n$th harmonic is proportional to $1/n^3$. An identity [22] reads

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \cdots = \frac{7}{8}\zeta(3) \simeq 1.0518,$$

where $\zeta$ is the Riemann $\zeta$ function, implying that the neglected harmonics contain about 5% of the gravitational potential energy, roughly the error in the predicted amplitude.

A similar calculation of the potential energy owing to slope using an identity for the sum of the reciprocals of the fourth powers of the consecutive odd integers [22] shows that the harmonics contain about 1.5% of the total energy. Section IIE noted that this potential energy is only one-fourth of the total energy.

## III. DISCUSSION

The approximate analytic technique for treating standing waves above is the same as that used to derive exact solutions of the Akhiezer-Polovin equations, that is, Maxwell's equations for a relativistic cold collisionless plasma, for circularly polarized standing waves [23]. As mentioned, the same method was used in an approximate theory for a charged string [14]. The present case differs from those cases by the absence of a small-amplitude, linear cutoff frequency.

In the other cases, the total potential for the analogous classical mechanical particle has a single well for sufficiently large frequency. For sufficiently large frequency, the standing-wave solutions in these cases are therefore unbiased. In contrast, the total potential for the classical mechanical particle analogous to the simplest breather, as mentioned, always has two wells, and hence allows biased standing waves. The discontinuities in Fig. 4(c) and 4(e) would be smoothed in a physical system.

The breathers in the other cases extend to $x = \pm\infty$ because the analogous classical mechanical particle with energy 0 ascends a hill with a rounded summit. This causes the period of oscillation to be infinite. More accurately, the half-period is infinite when the analogous particle is released at its maximum value $u_c$ at "time" $x = 0$. In contrast, the period of oscillation of the classical mechanical particle corresponding to the simplest breather is finite, making it a compacton.

The shapes of the simplest breather and the SG breather behave differently when "boosted." The Theory of Special Relativity says that if frame $S'$ moves to the right at speed $v$ relative to frame $S$, or the "laboratory frame," then in the eyes of frame $S$ there is a difference of $\gamma\beta x/c$, where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$, and $c$ is the speed of light in vacuum, between clocks fixed in $S'$ separated by a distance $x$ in the eyes of frame $S$ along the direction of motion, with the right-hand clock lagging the left-hand one.

Let $2L'$ denote the full-width of a breather, as seen by $S'$, that is stationary in $S'$. Its full-width in $S$ is then $2L'/\gamma$. Letting $x = 2L'/\gamma$ in the time-delay equation shows that the delay between the two clocks at the edges of the breather is $2L'\beta/c$, where $c$ now denotes the speed of waves on a free string and in dimensionless variables is unity. As $\beta \to 1$, this becomes $2L'$. This is equal to the period of the breather in $S'$ according to Eq. (4). In this limit, the moving breather executes one complete cycle along the $x$ axis in the eyes of $S$ at any instant. It would appear to swim, with the trailing edge shaking the blob ahead to move it along, at a frequency and length tending to 0 as $\beta \to 1$.

Consider the SG breather, which in dimensionless form may be written [3]

$$u(x,t) = 4\tan^{-1}\left[\frac{k\cos(\omega t)}{\omega\cosh(kx)}\right],$$

with $0 < \omega < 1$ and $k = \sqrt{1-\omega^2}$. As a measure of the one-sided width $W$ of the SG breather in its own rest-frame, we may choose the value of $x$ for which $\cosh(kx) = 4$, corresponding to $kW \simeq 2$. The half-width is then $W \simeq 2/k$ in the breather's rest-frame. The full width in the laboratory frame is then about $4/\gamma k$. The clocks at its edges differ by the amount $4\beta/k$, which gives a phase difference of $4\beta\omega/k$. As $\beta \to 1$, this becomes $4\omega/k = 4\omega/\sqrt{1-\omega^2}$, which becomes infinite as $\omega \to 1$.

The simplest breather and the SG breather also differ in regard to the relationship between their amplitude and width. By Eq. (4), the width $L$ of the simplest breather increases monotonically with amplitude $A$, whereas the width of the SG breather decreases monotonically with amplitude. The details of this are omitted since it follows straightforwardly from the expression for the sine-Gordon breather above.

As noted, the simplest breather is not physically realizable, but moreover the governing Eq. (1), or its dimensionless form Eq. (2), cannot be solved numerically. The most straightforward finite-difference scheme [24] for solving a nonlinear KG equation, the method of characteristics, uses finite-difference equations for forward and backward waves defined as $uf \equiv (\partial_t - \partial_x)u$ and $ub \equiv (\partial_t + \partial_x)u$. One writes the wave equation as coupled first-order partial differential equations in hyperbolic or "normal" form, which for the simplest breather read

$$(\partial_t + \partial_x)uf = -\mathrm{sgn}u,$$
$$(\partial_t - \partial_x)ub = -\mathrm{sgn}u,$$

together with $\partial_t u = (uf + ub)/2$. For the time-centered implicit finite-difference system based on these equations to be stable, the "source term" on the right-hand side of these equations must be differentiable. The function $\mathrm{sgn}\,u$ is not differentiable at $u = 0$.

The simplest breather has a natural extension that comes from noting that Eq. (2) also governs a uniformly charged string oscillating perpendicularly to a uniformly charged plane through an infinitesimal slit in the plane, with the plane's charge opposite in sign to that on the string. This system is likewise unphysical, but the slit may be replaced by a gap of uniform width to obtain the realizable system shown in Fig. 6. Two semi-infinite, coplanar, uniformly charged planes with surface charge density $-\sigma < 0$ are separated by a gap of width $2a$. In practice, the planes would of course be plates of finite thickness, but similarly to the small smooth region at the bottom of the V-shaped trough, the finite thickness of the plates and the associated edge-effects are assumed to have no significant effect on the string's motion at amplitudes of interest.

The electric field in the $xz$ plane in Fig. 6 lies along the $z$ axis by symmetry. Ignoring edge effects owing to the finite thickness of the plane, one can verify that this electric field is equal to $-4\sigma\tan^{-1}(z/a)$ in cgs-esu. Letting $\lambda$ denote the linear charge density of the string, it follows that, in cgs-esu, the electrostatic restoring force per unit length on the string is given by $-4\sigma\lambda\tan^{-1}(z/a)$. In the limit of small amplitude, the resonant angular frequency (squared) of this system is $4\sigma\lambda/\rho a$, where $\rho$ is the linear mass density of the string. In dimensionless form, the displacement of the string obeys a nonlinear KG equation that appears not to have been previously

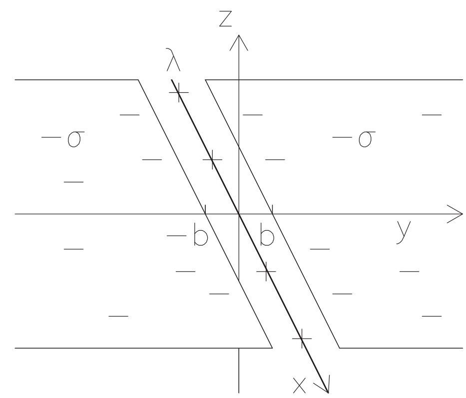


FIG. 6. In this cut-away view, a positively charged string with linear charge density $\lambda > 0$ lies along the $x$ axis when at rest. It is assumed to oscillate in the $xz$ plane, midway between two semi-infinite uniformly negatively charged coplanar planes with surface charge density $-\sigma < 0$ separated by a gap of width $2a$.

addressed,

$$(\partial_t^2 - \partial_x^2)u = -\tan^{-1}u. \tag{21}$$

The prior analytic technique for treating standing waves [14,23] and used here was used to approximate the shape of breatherlike, self-trapped states, or pseudobreathers for Eq. (21) [25]. That is, a standing-wave solution was approximated by $u(x,t) = u(x)\cos(\omega t)$, ignoring harmonics. This was substituted into Eq. (21) and the Fourier (cosine) transform was taken. This gives Newton's equation of motion for a classical mechanical particle in a conservative double-well potential if $\omega < 1$, where $x$ plays the role of time.

For a fixed value of $\omega$, given the particle's energy, its trajectory or orbit can be found by numerical integration. For a specific $\omega < 1$, the trajectory of the particle when its energy vanishes give the shape $u(x)$ of the pseudobreather to a first approximation. It may be mentioned that the width of the pseudobreather as a function of amplitude has a single local

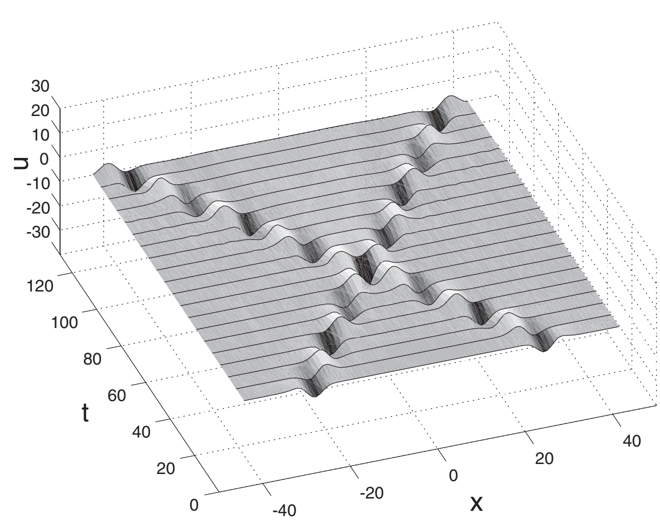


FIG. 7. A simulated collision is shown between counterpropagating approximate breathers with amplitude 5 and speeds of $\pm0.5$.

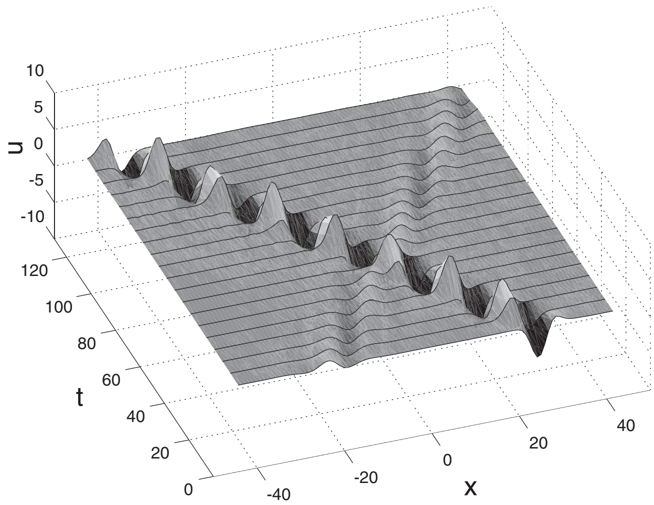


FIG. 8. A simulated collision between counterpropagating approximate breathers with amplitudes 1 and 5 and speeds ±0.5 is shown.

minimum, and approaches $\infty$ as its amplitude approaches 0 and $\infty$. The term "pseudobreather" is used since the nonlinear KG Eq. (21), as noted above for Eqs. (1) and (2), does not possess an IST [7].

Simulations of these approximate pseudobreathers (not shown) reveal that they are long-lived. They were "boosted" by Lorentz transforms to investigate collsions. The simulation in Fig. 7 shows a symmetric collision between approximate pseudobreathers with amplitude 5 and speeds ±0.5. The amplitude of a breather is a Lorentz invariant since the string oscillates perpendicularly to the velocity. Figure 8 shows another collision, this between breathers with amplitudes of 1 and 5 and speeds of ±0.5. Both Figs. 7 and 8 show 20 snapshots of the shape of the string equally spaced in time. These results show that the breathers' initial shape and speed are very nearly preserved. Few if any other nonintegrable KG equations allow such effectively true breathers. In Fig. 8, it seems as if the frequency of the smaller-amplitude right-going breather is smaller than that of the larger-amplitude left-going one. This results from "strobing." That is, the period of a breather with amplitude 1 is roughly equal to the time between snapshots.

Simulations could study the creation and control of these pseudobreathers by varying the gap-width $2a$ along the $x$ axis. Such simulations could be similar to simulations of breatherlike states, akin to cavitons, found in overdense regions of a cold, collisionless, inhomogeneous electron plasma reflecting a linearly polarized narrowband planewave pulse at normal incidence, taking into account relativistic electron mass but ignoring the Lorentz force $e(\mathbf{v}\,\mathbf{x}\,\mathbf{B})/c$, where $\mathbf{B}$ denotes the magnetic field [26]. Recall that the plasma or Langmuir frequency (squared) is given by $4\pi ne^2/m$, where $n$ is the number density of free electrons, $e$ the electron's charge, and $m$ the electron's mass. An increase in $m$ is equivalent to a decrease in $n$.

Some work has addressed longitudinal inhomogeneities on graphene nanotubes [27], but research on the creation or control of breathers on nanotubes [28] seems not to have considered the system in Fig. 6, nor does it seem to arise elsewhere.

## IV. CONCLUSION

The above is far from a complete treatment of the simplest breather. There is no discussion, for example, of the power flow during one period, or the relative energy in the two forms of potential energy for the simplest breather and the SG breather, nor is there a comparison between the dispersion relations for traveling waves for Eq. (2) and (21). These waves are asymptotically identical as the amplitude increases.

Because it involves elementary mechanics and is thus intuitively accessible, the simplest breather might suggest new insights or techniques for analyzing self-attracting or self-trapped entities or be a starting point for analyzing other breathers. The treatment of the pseudobreather was only approximate. It would seem that the most interesting question is whether it has an exact analytic expression. A natural question is whether there is a next simplest breather.

The theory of the electrostatic pseudobreather is deferred [25] and the results above are intended mainly to show simply that they exist. A nonintegrable nonlinear KG equation, in particular one with no localized solitary traveling-wave solutions, may possess breatherlike solutions with the particlelike properties of solitons and breathers for integrable equations to high accuracy.

It may also be noted in passing that the electrostatic system in Fig. 6 has an exact gravitational analog in which the charged planes are simply planes of uniform mass density exerting some force per unit length on a massive string.

These pseudobreathers pose a number of questions: Their treatment above is only approximate. Can they be expressed exactly in terms of standard analytic functions? Do they emerge from arbitrary initial data in a fashion similar to solitons in integrable systems? What theory might describe them? Do they obey any simple rules in collisions? Are there any other closely related pseudobreathers? Could they be realized on graphene nanotubes for measurement, memory, or computation? Can they be generalized to higher spatial dimensions? The shift of the trajectory of the smaller breather in Fig. 8 might suggest how a shift-register could be implemented, for example.

Another question is whether breather-like states arise if the inverse tangent in Eq. (21) is replaced by the hyperbolic tangent, a function whose shape is a smoothed step-function similar to that of the inverse tangent. It might be interesting to study the nonlinear KG equations that govern the string when the surface charge density $-\sigma$ on the planes varies with $y$. The present theory could also be extended to take into account emission or absorption of radiation by the pseudobreather.

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