Stability in a fiber bundle model: Existence of strong links and the effect of disorder

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The present paper deals with a fiber bundle model which consists of a fraction α of infinitely strong fibers. The inclusion of such an unbreakable fraction has been proven to affect the failure process in early studies, especially around a critical value α_c. The present work has a twofold purpose: (i) a study of failure abruptness, mainly the brittle to quasibrittle transition point with varying α and (ii) variation of α_c as we change the strength of disorder introduced in the model. The brittle to quasibrittle transition is confirmed from the failure abruptness. On the other hand, the α_c is obtained from the knowledge of failure abruptness as well as the statistics of avalanches. It is observed that the brittle to quasibrittle transition point scales to lower values, suggesting more quasi-brittle-like continuous failure when α is increased. At the same time, the bundle becomes stronger as there are larger numbers of strong links to support the external stress. High α in a highly disordered bundle leads to an ideal situation where the bundle strength, as well as the predictability in failure process is very high. Also, the critical fraction α_c, required to make the model deviate from the conventional results, increases with decreasing strength of disorder. The analytical expression for α_c shows good agreement with the numerical results. Finally, the findings in the paper are compared with previous results and real-life applications of composite materials.

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I. INTRODUCTION

The study of strength and predictability in failure processes has been the center of discussion in engineering and material science for decades [1,2]. A material with higher strength and less abruptness in failure process is usually considered to be an ideal material for many purposes. Mainly there are two factors that determine the mode of failure: (i) strength of disorder and (ii) range of interaction for stress release within a material. When a material is subjected to an external stress, the final breaking is characterized by the appearance of defects such as dislocations or microcracks. The microcracks can produce elastic waves, detectable by a piezoelectric microphone [3]. Experimentally it is believed that heterogeneities (like microcracks) present in materials leads to precursor activities that often show scale-free behavior in energy release during the failure process [4–6]. There are a number of external parameters that affect the heterogeneity within a material and hence affect the failure mode. External parameters such as temperature [7] and pressure [7], and internal parameters such as crystal defects (mainly dislocation) [8,9], porosity [10], and the driving process (strain rate) [11] play crucial roles in determining both strength and failure abruptness. Apart from the above factors, there is another special technique that affects the mode of failure: the mixing of materials known as composites [1,2,12–14]. In material science, composites are prepared by mixing two materials of different properties in a certain proportion to create a third one that usually comes with greater strength [15] and toughness [16] than the component materials. Fiber reinforced composites [17–23] are a good example in such case where fibers with high strength are embedded with a carrier matrix. There are some recent works [24–26] on the fiber bundle model in comparison with composite materials. The present work shows how the strength and failure abruptness of the composite fiber bundle model (FBM) [27–30] (prepared by mixing a fraction of infinitely strong fibers with conventional breakable fibers) is affected in the presence of a variable strength of disorder. The FBM is basically a disordered system, guided through threshold activated dynamics. We will discuss the model in detail in Sec. II.

The existence of a strong link plays a crucial role while studying the stability during the failure process in the fiber bundle model. Recent studies show that the existence of an unbreakable fraction affects the burst statistics in the global load sharing [31] as well as local load sharing schemes [32]. The average size of maximum burst shows an abrupt change around a critical fraction α_c of strong links. Also, recently it was observed that the introduction of disorder greatly affects the failure process in the model [33]. In that study, a critical strength of disorder was observed that separates brittlelike abrupt failure (where the total bundle breaks in a single avalanche without any necessity of stress increment) from nonabrupt quasi-brittle-like failure (where the bundle shows precursor activities prior to failure with increasing external stress). Such critical disorder strength is a function of system size [34] in the local load sharing scheme. Although the disorder plays a significant role in the local load sharing (LLS) model, the uniqueness of such critical disorder is lost here due to the system size effect [34–36]. In light of this knowledge, I have concentrated only on the mean-field limit, for both analytical as well as numerical studies.

The present work has two main purposes that revolve around the idea that a disordered system with a variable strength of disorder contains a fraction of infinitely strong links. The first

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purpose of the paper will be to study how the existence of strong links affects the failure process, especially the failure abruptness. The study mainly includes the behavior of the brittle to quasibrittle transition point [33,37] as we treat the fraction α of strong links as the controlling parameter. Two extreme limits of this variable can be understood: (i) $\alpha = 0$, which is the conventional limit and (ii) $\alpha = 1$, where each and every fiber has infinite strength (hence can support infinite load) and the model does not evolve at all. The present study discusses the nature of failure in all possible α values. The other part of the paper is dedicated to understanding the response of α_c as the disorder is varied. A previous study [31] in the mean-field limit claims that for uniform threshold distribution [0,1] (introducing a constant disorder strength), half of the bundle should have infinite strength to change the conventional avalanche behavior: scale-free decay with a universal exponent $-5/2$ [38–40]. Here I have studied how α_c changes when the strength of disorder is continuously varied.

In the next section a description of the model is provided, followed by the analytical results in the mean-field limit (Sec. III). Section IV is dedicated to the numerical results performed with 10^5 fibers and a large set ($\approx 10^4$) of configurations. Finally, in Sec. V, a brief discussion on the work is provided.

II. DESCRIPTION OF THE MODEL

After being introduced by Pierce in 1926 [27], the fiber bundle model has been proven to be important yet arguably the simplest model to study failure process. A conventional fiber bundle model consists of fibers or Hookean springs, attached between two parallel plates. The plates are pulled apart by a force F, creating a stress $\sigma = F/L$ on L fibers. Once the stress crosses the breaking threshold of a particular fiber, chosen from a random distribution, that fiber breaks irreversibly. The stress of a broken fiber is then redistributed either globally among all surviving fibers (global load sharing or GLS scheme) or among the surviving nearest neighbors only (local load sharing or LLS scheme). For the GLS scheme [27,28] no stress concentration occurs anywhere around the failed fibers as the stress of the failed fibers is shared among all surviving fibers democratically. On the other hand, in the LLS scheme [41–45], stress concentration is observed near a broken patch (series of broken fibers) and increases with the length of such patches. After such redistribution, the load per fiber increases, initiating failure of more fibers, and starting an avalanche. At the end of an avalanche, either all fibers are broken (suggesting global failure) or the bundle comes to a stable state with few broken fibers where an increment of external stress is required to make the model evolve further. The last applied stress just before global failure is considered to be the nominal stress or strength of the bundle.

In this work the conventional model is modified by considering a fraction of total fibers to be infinitely strong and therefore can support any amount of stress without breaking. This kind of work is already studied in the fiber bundle model with both GLS [31] and LLS [32] schemes. I have carried out the study with varying disorder in the mean field or GLS limit and observed how the above findings are affected when the strength of disorder is varied.

If there are initially L fibers in the model, then among them let us assume the α fraction (αL number of fibers) is unbreakable and does not contribute to the evolution of the model. A certain amount of applied stress breaks some fibers among the remaining $(1-\alpha)$ fraction and increases the stress per fiber that leads to avalanches. The infinitely strong fibers carry the extra stress (due to redistribution) without breaking and do not contribute to the avalanche process. The behavior of the model is mainly determined by $(1-\alpha)L$ conventional fibers that have random but finite breaking thresholds. So, whenever we talk about failure abruptness, we indicate abrupt failure of the $(1-\alpha)$ fraction. Also the nominal or critical stress is determined by the minimum applied stress at which this $(1-\alpha)$ fraction breaks with α fraction of intact fiber. A further increase in applied stress does not change anything in the bundle and hence is not worth observing.

The next section contains some analytical results for the model, dealing with the variation of α_c with disorder δ as well as the behavior of the brittle to quasibrittle transition point with varying α values.

III. ANALYTICAL APPROACH

For analytical calculation let us assume that a fraction of fibers α in the model is too strong to break. A stress σ_0 is applied externally creating a stress per fiber σ. Here I have shown the analytical calculations for a uniform distribution of threshold stress. Also, a different case with power-law threshold distribution is adopted in order to approach the high-disorder limit. This will be discussed later in this paper. Since the α fraction of the fibers are infinitely strong, it will not take part in describing the dynamics of the system. So, later in this paper, whenever we discuss the evolution of the model (analytically or numerically), it is actually the evolution of the $(1-\alpha)$ breakable fraction.

A. Critical fraction for strong links

In the case of a particular distribution $P(\sigma)$ of threshold strength values, we can relate the externally applied stress (σ_0) with the local stress per fiber (σ) as

$$\sigma_0 = (1-\alpha)[1-P(\sigma)]\sigma + \alpha\sigma. \tag{1}$$

The second part of Eq. (1) shows the stress carried by the infinitely strong fibers while the first part gives the stress carried by the conventional fibers after a certain redistribution (depending on σ_0).

For a uniform distribution of width 2δ and mean at 0.5 we get $P(\sigma) = \frac{\sigma-a}{2\delta}$, δ being the strength of disorder and $a(= 0.5-\delta)$ is the minimum of the threshold distribution. For that particular case, Eq. (1) can be written as

$$\sigma_0 = (1-\alpha)\left[1-\frac{(\sigma-a)}{2\delta}\right]\sigma + \alpha\sigma. \tag{2}$$

This will give a parabolic curve at $\alpha = 0$. For other α values there will be a curve with a maximum at the unstable point of the model, i.e., at the critical point of stress per fiber (σ_c). This point is given by

$$\left.\frac{d\sigma_0}{d\sigma}\right|_{\sigma_0=\sigma_c} = 0. \tag{3}$$

Inserting value σ_0 from Eq. (2) and using $a = (0.5 - \delta)$, we get

$$1 - \frac{\sigma_c}{\delta}(1-\alpha) - \frac{0.5-\delta}{2\delta}(1-\alpha) = 0$$

$$\text{or,} \quad \sigma_c(\alpha) = \frac{\delta}{1-\alpha} - \left(\frac{1}{4} - \frac{\delta}{2}\right), \tag{4}$$

Now the maximum value σ can attain is $(0.5 + \delta)$, which is the maximum of the threshold distribution, and after this point we cannot get any maximum of the curve. This is the point where α reaches its critical value. So at $\alpha = \alpha_c$, we get $\sigma_c = (0.5 + \delta)$. Implying this condition we get this critical fraction of strong links in terms of disorder:

$$\alpha_c = \frac{3 - 2\delta}{3 + 2\delta}. \tag{5}$$

So as we go for higher and higher δ values, α_c decreases and we need a lesser fraction of unbreakable fibers to satisfy this critical condition.

B. Study of failure abruptness

To understand the failure process in the model we have to construct the recursion relation for the fraction of unbroken bonds. Let us assume that the total N_u number of unbroken fibers is the combination of N_u^s numbers of infinitely strong fibers and N_u^w numbers of conventional (or weak) fibers. This makes $N_u^s/N_u = \alpha$ and $N_u^w/N_u = 1 - \alpha$. In a recursion relation N_u^s does not have any role. Then, for a uniform distribution with mean at 0.5 and width 2δ the equation of the fraction of unbroken fibers can be given by

$$(1-\alpha) - n_u^w = \int_a^{\sigma_0/(n_u^w+\alpha)} p(\sigma)d\sigma$$

$$= \frac{1}{2\delta}\int_a^{\sigma_0/(n_u^w+\alpha)} d\sigma. \tag{6}$$

Here $p(\sigma) = dP(\sigma)/d\sigma$. Also n_u^w is the fraction of unbroken bonds corresponding to applied stress σ_0 and a $(= 0.5 - \delta)$ is the minimum of the distribution. Equation (6) will give a quadratic equation of n_u^w,

$$\left(n_u^w\right)^2 - n_u^w\left(1 + \frac{a}{2\delta} - 2\alpha\right) + \left(\alpha^2 + \frac{\sigma_0}{2\delta} - \alpha - \frac{a\alpha}{2\delta}\right) = 0. \tag{7}$$

The solution to the above equation will be

$$n_u^w = \frac{1}{2}\left[\left(1 + \frac{a}{2\delta} - 2\alpha\right) \pm\sqrt{\left(1 + \frac{a}{2\delta} - 2\alpha\right)^2 - 4\left(\alpha^2 + \frac{\sigma_0}{2\delta} - \alpha - \frac{a\alpha}{2\delta}\right)}\right]. \tag{8}$$

Since at critical point two solutions of Eq. (8) cannot exist, it suggests at critical point the rooted part of the above equation will vanish. In that case, the critical fraction unbroken will be given by

$$n_c = \left(n_u^w\right)_c = \frac{1}{2}\left(1 + \frac{a}{2\delta} - 2\alpha\right). \tag{9}$$

For the case $\alpha = 0$ above, the results reduce to the conventional results in the model [33], where all fibers can break. An abrupt brittlelike failure is seen in the model when $n_c = 1$. In such a case, the whole $(1 - \alpha)$ fraction breaks in a single avalanche without any increment of external stress. For $n_c < 1$, the model breaks through a number of stable states. Inserting $n_c = 1$ and $a = 0.5 - \delta_c$ we get the value of δ_c from the above equation:

$$\delta_c = \frac{1}{2}\frac{1}{(4\alpha + 3)}. \tag{10}$$

For $\alpha = 0$ we get $\delta_c = 1/6$, which is the exact result we obtained for the conventional fiber bundle model in the mean-field limit (using uniform threshold distribution) [33].

IV. NUMERICAL RESULTS

Numerically, the model has been studied with system size 10^5 and a large set ($\sim 10^4$) of configurations. Most of the results are generated in the mean-field limit with uniform distribution, although the last part of the numerical results are discussed with power-law threshold distribution to confirm the universality as well as to approach the high-disorder limit. Previous numerical studies suggest that there exists a critical fraction α_c of strong links above which the avalanche statistics deviates from the mean-field results. The value of α_c is quite high (=0.5) [31] with the GLS scheme and drops to a very low value ($\sim$0.05) [32] in the presence of local stress concentration. In this paper, I have studied this α_c in detail with varying strength of disorder in the mean-field limit. Also, the stability of the model during the failure process is discussed with varying α values.

Below, the numerical results are discussed where the threshold values are chosen from a uniform distribution of half-width δ and mean at 0.5. δ expresses the strength of disorder.

A. Stability during failure process

A fraction of unbroken bonds just before the global failure (n_c) is proven to be a good measure of failure abruptness in recent studies [33,34,46]. $n_c = 1$ corresponds to a brittlelike abrupt failure as the total model is intact just before the global failure. After the application of a stress, large enough to break the weakest fiber, the bundle becomes unstable and breaks in a single avalanche without any prior warning. $n_c < 1$ suggests that the bundle goes through a number of stable states before global failure (similar to quasibrittle failure). At each stable state an increment in applied stress is required. In this section we have discussed the behavior of n_c with varying disorder, for different fraction α. As expected, at low disorder, n_c remains at 1.0 and the failure process is brittlelike abrupt. In this region all the conventional fibers ($1 - \alpha$ fraction) break in at a constant external stress in a single avalanche. On the other hand, at high disorder $n_c < 1$ and the model goes through a number of stable states prior to failure point.

At low α values, where a very small fraction of the fibers are strong, n_c decreases to 0.5 when the disorder reaches $\delta = 0.5$. Such behavior remains unchanged in the region $\alpha < 0.5$. As we go to high values for α, n_c starts saturating after a certain disorder value. The saturation occurs as the remaining fibers are strong enough to bear any stress without breaking. In this section we will mainly concentrate on the point where n_c

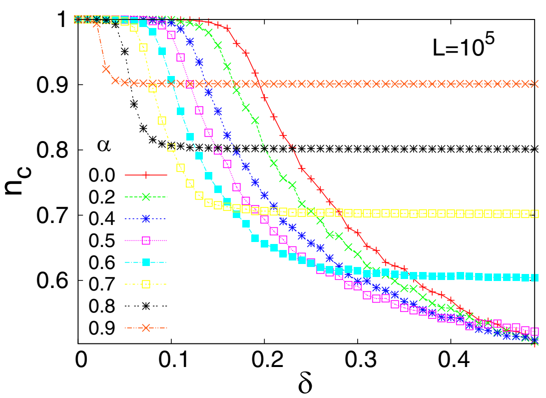

FIG. 1. Variation of critical fraction unbroken n_c with strength of disorder δ for different α values. For low α, n_c remains at 1 at low disorder strength and gradually decreases to 0.5 with increasing δ. For high α, due to the existence of unbreakable fibers, n_c saturates to a certain value (>0.5) beyond a certain strength of disorder.

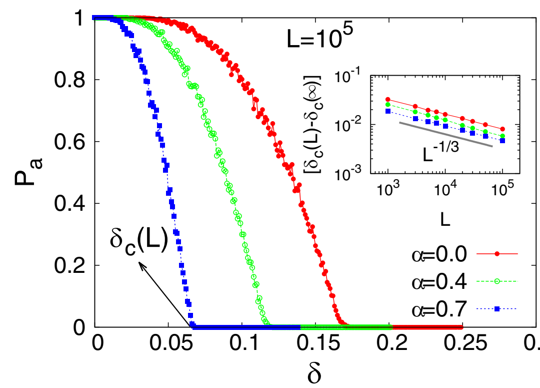

FIG. 2. P_a vs δ for $\alpha = 0.0$, 0.4, and 0.7. For $\delta < \delta_c$, $P_a > 0$ and hence there is a nonzero probability of abrupt failure. The inset shows that $\delta_c(L)$ approaches its thermodynamic limit $\delta_c(\infty)$ as $\delta_c(L) = \delta_c(\infty) + L^{-1/3}$. The scaling remains invariant with respect to α.

deviates from 1 and hence from abrupt failure. The disorder at which such deviation takes place is denoted as δ_c and can be denoted as the brittle to quasibrittle transition point. Two extreme limits of the model correspond to $\alpha = 0$ and $\alpha = 1$. The former corresponds to the conventional mean-field limit where δ_c is expected to be around 1/6 [33] with uniform threshold distribution (mean 0.5 and half-width δ). On the other hand, the later one corresponds to a situation where each and every fiber is unbreakable and the model does not evolve at all. Figure 1 clearly shows that δ_c approaches lower values as we increase α. This in turn, reduces the window of disorder strength within which an abrupt failure is expected. As a result, at high α we will start getting stable states during the failure process, even at low strength of disorder.

B. Strong link and disorder dependence in probability of abrupt failure

To understand the predictability and stability during the failure process, I have studied the probability of abrupt failure with varying disorder and α. Since a fraction α is unbreakable here, the study only shows the probability of abrupt failure for the remaining $(1-\alpha)$ fraction. A recent study [33] already shows how the predictability is affected by disorder in the mean-field limit. Here such behavior is studied with a varying fraction of strong links.

The probability of abrupt failure, P_a, is basically defined as the ratio of how many times the model goes through abrupt failure (breaks in a single avalanche) to the total number of observations. Figure 2 shows P_a as a function of disorder strength δ for different fractions α. At low disorder P_a remains at 1 and the failure is abrupt for each and every observation. With increasing δ, P_a gradually decreases to zero. The region $P_a > 0$ is denoted as brittle as there exists a nonzero probability of abrupt failure. $\delta_c(L)$ is defined as the critical disorder for a particular system size L beyond which $P_a = 0$. Figure 2 clearly shows a decreasing $\delta_c(L)$ when α is increased. Also, the fall of P_b becomes more and more sharp. This in turn supports our previous claim of decreasing abrupt failure with α values. Also, $\delta_c(L)$ scales down with increasing system size as $\delta_c(L) = \delta_c(\infty) + L^{-\zeta}$, where $\zeta = 0.33 \pm 0.005$. This exponent has been observed earlier in the fiber bundle model. For example, in the mean-field limit the relaxation time diverges close to critical stress with the same exponent when the system size is increased [47]. $\delta_c(\infty)$ is the brittle to quasibrittle transition point at the thermodynamic limit. The inset shows that the above scaling of $\delta_c(L)$ remains unchanged even when α is varied.

In Fig. 3, the behavior of P_a is discussed while both the parameters α and δ are varied simultaneously. The color scheme for P_a can be understood as follows:

(1) Yellow stands for the condition $P_a = 1$. The failure process is abrupt here in each and every observation.

(2) Black corresponds to $P_a = 0$ and the failure process is always quasi-brittle-like nonabrupt.

(3) The region $0 < P_a < 1$ is shown in the other color gradients. The probability of abrupt failure is variable in this region and decreases with both α and δ.

Figure 3 shows that at higher α, while going from the yellow to the black region, we cross the color gradient at a lower disorder strength.

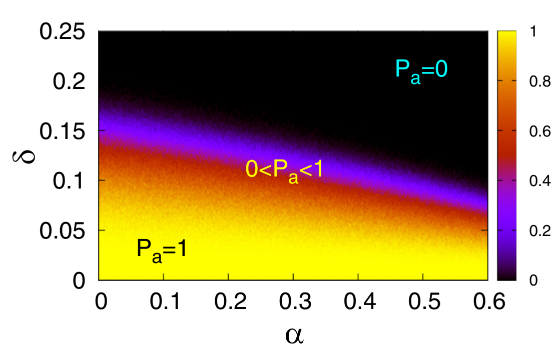

FIG. 3. P_a as a function of both disorder δ and fraction of strong link α. The yellow and black colors correspond to pure abrupt and pure nonabrupt failure, respectively. Within the color gradient, the abruptness in the failure process is a function of both δ and α.

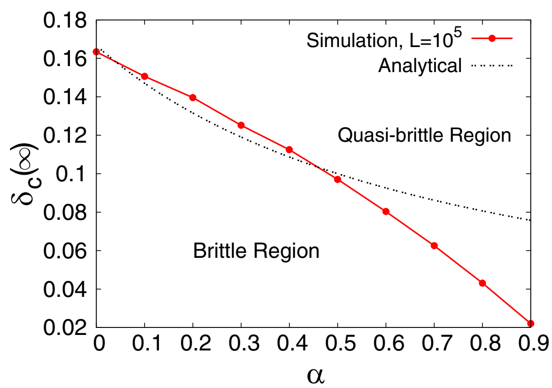

FIG. 4. Comparison between theoretical and numerical findings of $\delta_c(\infty)$ for different α values. $\delta_c(\infty)$ decreases with increasing α, making the quasibrittle response more and more prominent.

C. Brittle to quasibrittle transition point

We have now reached the point where we can discuss the brittle to quasibrittle transition point $\delta_c(\infty)$ with continuously varying α values. In Fig. 4 we have shown this numerical variation along with the analytical finding given by Eq. (10). Although the analytical and numerical values of $\delta_c(\infty)$ do not show good agreement, both agree to a decreasing behavior of $\delta_c(L)$ with increasing α. Why the analytical and numerical results do not match is not quite clear at this moment. The above disagreement is quite prominent for higher α values. As per Fig. 4, $\delta_c(L)$ starts from 1/6 at $\alpha = 0$ (the conventional limit) and decreases to 0.02 for $\alpha = 0.9$. In this high α limit, the failure process is predictable almost for all strength of disorder.

D. Strength of the bundle

A different way to understand the advantage of introducing a fraction of unbreakable fibers is to monitor how the strength of the bundle is affected by it. Figure 5 shows that the strength of the bundle increases as we include more and more unbreakable fibers.

We already know that $\alpha = 0$ leads to the conventional limit of the model, where with decreasing strength of disorder δ, the failure process becomes more brittle but at the same time the strength of the bundle increases (see Ref. [48]). We cannot approach an ideal limit where both strength and predictability in the failure process is high. So, in that conventional limit we have to compromise, either in terms of strength or predictability in failure abruptness. Figure 5 suggests that, as we go to a relatively high α value the response of σ_c against δ reverses and instead of decreasing σ_c starts increasing with δ. Combining this finding along with the results of failure abruptness we can conclude that at high α the model operates in an ideal situation where the strength of the bundle is high and the failure process is highly predictable due to quasi-brittle-like continuous failure.

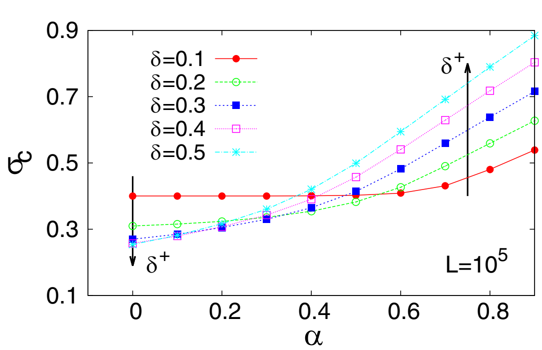

FIG. 5. Variation of critical strength σ_c with α for different strength of disorder δ. σ_c increases monotonically with α for any δ value. The response of σ_c against disorder is also modified as α is increased.

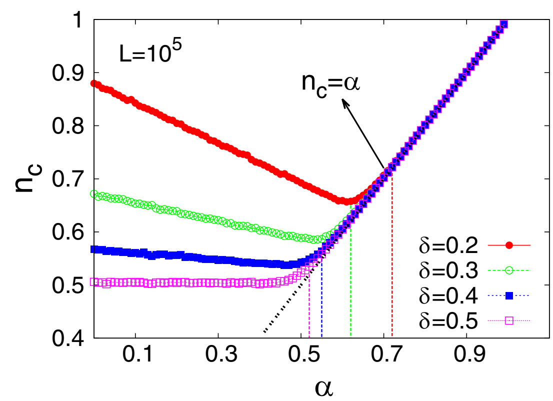

FIG. 6. Variation of n_c with continuously increasing α parameter for different strength of disorder: $\delta = 0.2$, 0.3, 0.4, and 0.5. α_c is the point where n_c deviates from the straight line $n_c = \alpha$. α_c is observed to decrease with increasing δ values.

E. Estimation of α_c from failure abruptness

To estimate the critical fraction of strong link we have studied the failure abruptness with varying fraction α of infinitely strong fibers (see Fig. 6). We know that for a certain α, n_c remains at 1 below δ_c (where δ_c is a decreasing function of α itself) and gradually decreases as δ is increased beyond δ_c. In this section we have studied how n_c responds to a continuous change in α values when the disorder strength δ is kept constant. For reference we have shown the locus of $n_c = \alpha$ in Fig. 6. This line corresponds to the strong link dominated region. $n_c = \alpha$ suggests that the bundle breaks continuously until there remains α infinitely strong fibers only. The failure process is extremely stable in this limit. At high α values, the fraction unbroken shows this above behavior. As we decrease α, n_c deviates from the straight line (given by $n_c = \alpha$) below a critical fraction α_c. When we decrease the strength δ, such deviation takes place at lower α values and hence α_c decreases.

The system size effect of α_c is also studied. α_c is observed to show almost no change while the system size is increased. The studies are carried out over the range $10^3 \leqslant L \leqslant 10^5$. Over such range of system size, α_c changes roughly by an amount 0.01. Due to such L-independent behavior, it is safe to treat the above α_c's as $\alpha_c(L \to \infty)$, their values in the thermodynamic limit.

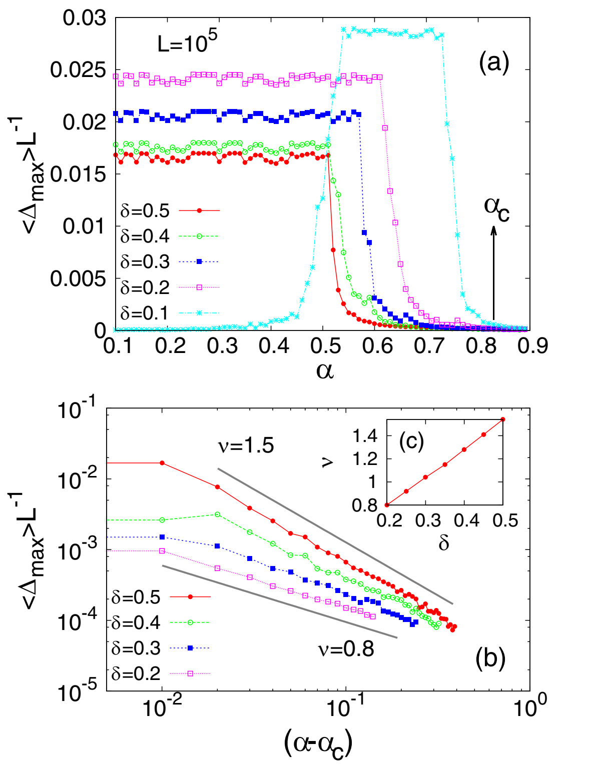

FIG. 7. (a) Variation of average maximum burst size $\langle\Delta_{\max}\rangle$ with α. For $\delta > \delta_c$, $\langle\Delta_{\max}\rangle$ falls sharply around α_c. For $\delta < \delta_c$, $\langle\Delta_{\max}\rangle$ shows a nonmonotonic behavior but still shows an abrupt change as we go beyond α_c. (b) $\langle\Delta_{\max}\rangle$ diverges around the critical fraction α_c as $\langle\Delta_{\max}\rangle/L \sim (\alpha - \alpha_c)^{-\nu}$. (c) The inset shows a linearly increasing ν with disorder strength δ.

F. Behavior of maximum burst

To understand the existence of α_c and its variation with disorder δ, I have also studied how the maximum burst behaves at different conditions. A burst size is defined here as the number of fibers broken in between two consecutive stress increments. The final burst during the failure process has been neglected in the above study. The maximum of the burst is chosen among the rest of the avalanches. $\langle\Delta_{\max}\rangle$ is defined as the average over 10^4 such maximum burst values. Figure 7 shows how $\langle\Delta_{\max}\rangle$ varies with α at different disorder strengths. For $\delta > \delta_c(\infty)$, $\langle\Delta_{\max}\rangle$ saturates at a nonzero value for small α [see Fig. 7(a) for $\delta = 0.5$, 0.4, 0.3, and 0.2]. As the model crosses α_c, $\langle\Delta_{\max}\rangle$ shows a sudden decrease and reaches zero gradually. As we decrease δ, this sudden jump in $\langle\Delta_{\max}\rangle$ starts taking place at higher α, suggesting a shift of α_c toward high values. For $\delta < \delta_c$, $\langle\Delta_{\max}\rangle$ shows a nonmonotonic behavior. $\langle\Delta_{\max}\rangle$ remains at a very low value for small α, reaches a maximum, and falls back again close to α_c. Figure 7(b) offers a closer look at the divergence of $\langle\Delta_{\max}\rangle$ around $\alpha = \alpha_c$. The following scaling is observed:

$$\frac{\langle\Delta_{\max}\rangle}{L} \sim (\alpha - \alpha_c)^{-\nu}, \tag{11}$$

where the exponent ν has a disorder dependence. As we decrease disorder from $\delta = 0.5$, ν starts to decrease from 1.5 (matches with the earlier claim by Hidalgo *et al.* [31]) and reaches 0.8 at $\delta = 0.2$. We have restricted our study to $\delta > \delta_c(\infty)$ as below this the disorder α_c value is quite high and there will be very few points before the model stops evolving. The variation of ν with δ is shown in Fig. 7(c). The increment of ν is almost linear with strength of disorder δ.

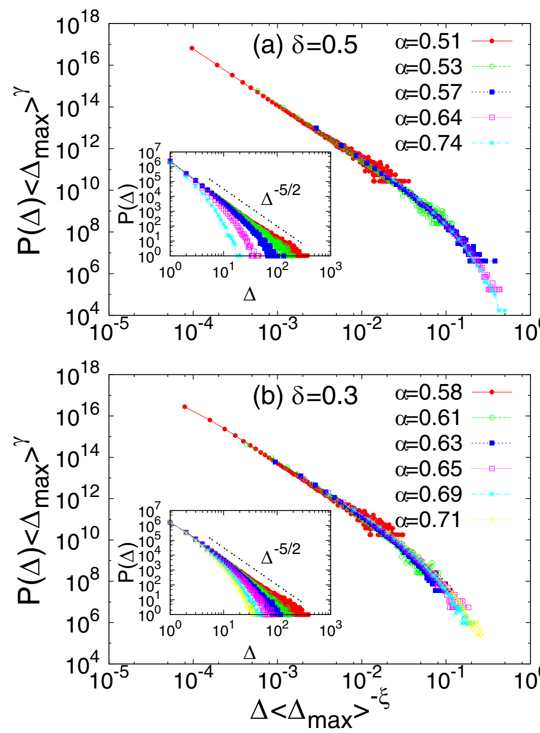

FIG. 8. Burst size distribution for (a) $\delta = 0.5$ and (b) 0.3. The inset shows the unscaled distribution. For system size scaling the following scaling behavior is adopted: $P(\Delta) = \langle\Delta_{\max}\rangle^{-\gamma}\Phi[\frac{\Delta}{\langle\Delta_{\max}\rangle^{\xi}}]$.

G. Burst size distribution

Finally, we have studied the distribution of burst size Δ and how it scales at different disorder values. The size of a burst holds the same definition as previously: the number of fibers broken in between consecutive stress increments. Figure 8 shows the burst size distribution $P(\Delta)$ for $\delta = 0.5$ and 0.3. The figures in the inset show the unscaled burst distribution. For $\alpha < \alpha_c$, the model barely deviates from the conventional behavior and hence produces a scale-free avalanche: $P(\Delta) = \Delta^{-5/2}$ [38–40]. As we go beyond α_c, an exponential cutoff is added with the distribution, although the power of the scale-free behavior remains unaltered. Such exponential cutoff takes place at lower and lower Δ values as α is increased beyond α_c. The main purpose of the scaled figure is to construct a master curve for the region $\alpha > \alpha_c$. Since $\langle\Delta_{\max}\rangle$ is described by the exponential cutoff (if the cutoff is at lower Δ, $\langle\Delta_{\max}\rangle$ also decreases), we have treated $\langle\Delta_{\max}\rangle$ as a scaling parameter to obtain the following scaling:

$$P(\Delta) = \langle\Delta_{\max}\rangle^{-\gamma}\,\Phi\left(\frac{\Delta}{\langle\Delta_{\max}\rangle^{\xi}}\right), \tag{12}$$

TABLE I. Scaling exponents γ and ξ for different δ are tabulated.

δ	γ	ξ	γ/ξ ($\approx\kappa$)
0.5	3.25	1.25	2.60
0.4	3.40	1.35	2.52
0.3	3.45	1.38	2.50

where the function Φ is given by

$$\Phi\left(\frac{\Delta}{\langle\Delta_{\max}\rangle^{\xi}}\right) = \Delta^{-\kappa} e^{-(\Delta/\langle\Delta_{\max}\rangle^{\xi})}, \quad (13)$$

where $\kappa = 5/2$. The exponents γ and ξ vary with disorder strength δ, obeying the following relation: $\gamma = \kappa\xi$.

Table I shows such variation of γ and ξ for $\delta = 0.5$, 0.4, and 0.3. As δ increases, both γ and ξ increase, keeping a universal value for κ ($\approx 5/2$) independent of disorder strength.

H. Variation of α_c with disorder

The above behavior of n_c as well as the study of $\langle\Delta_{\max}\rangle$ leads to the same α_c values beyond which the strong links play a crucial role in the evolution of the model.

Figure 9 explicitly shows the variation of α_c with δ. The value of α_c was observed at 0.5 earlier for $\delta = 0.5$. The results of the present paper show an increment in α_c as the disorder strength is decreased. This suggests that as the disorder is decreased we have to reach higher α values to make the model deviate from its conventional limit.

I. High-disorder limit

We have already discussed the effect of low disorder strength on α_c. In this section we want to focus on the question: what happens if the strength of disorder is very high? Understandably, at high disorder some links are already so strong that no additional strong link might be required at all to make the model deviate from the conventional limit. Such high-disorder limit for the model is achieved by choosing a power-law distribution (say, with power -1) for the threshold values instead of the uniform one. The threshold strength values are chosen between $10^{-\eta}$ and 10^{η}, η being the amount of disorder here.

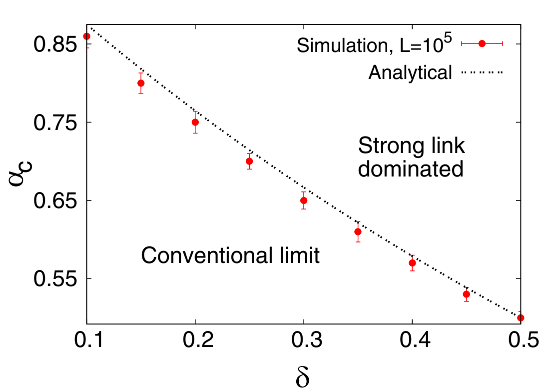

FIG. 9. Variation of α_c with strength of disorder in the case of uniform threshold distribution. For $\alpha < \alpha_c$, the model operates in the conventional limit. Beyond α_c the strong links play a crucial role.

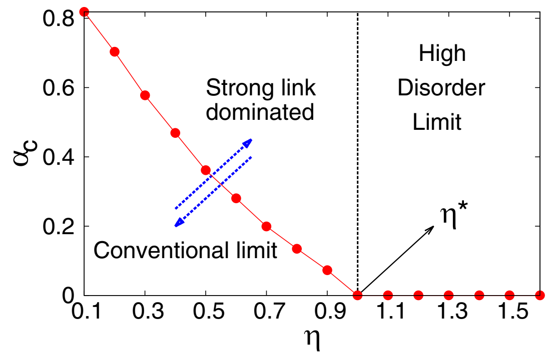

FIG. 10. α_c vs η (strength of disorder) in the case of power-law threshold distribution. Beyond η^*, some fibers are themselves so strong that no strong link is required to stabilize the model. Below η^* we get two regions: conventional and strong link dominated, depending on whether $\alpha > \alpha_c$ or $\alpha < \alpha_c$.

For the power-law threshold distribution α_c shows a decreasing behavior with disorder η. Figure 10 shows that as we go beyond η^* we obtain $\alpha_c = 0$. The region $\eta > \eta^*$ acts as the high-disorder limit for the model, where the conventional burst statistics is not observed even in the absence of any kind of additional strong links. At very high η, the threshold values are so distinct from each other, we hardly observe any big avalanches. In this region almost each and every fiber breaks with external stress increment only. On the other hand, for $\eta < \eta^*$, the model either operates in the conventional limit, producing a scale- free avalanche size distribution (with exponent $-5/2$) or is dominated by the strong links, depending on whether α is less or greater than α_c.

V. DISCUSSIONS

In the present work I have studied the effect of disorder in the failure process of a heterogeneous system in light of the framework of the fiber bundle model. A model is adopted where a certain fraction of the fibers are infinitely strong. The model has its similarity with a composite material as it is a mixture of two types of fibers: α fraction unbreakable and $(1-\alpha)$ fraction with a certain breaking threshold. It is observed that with increasing α not only failure abruptness decreases but also it increases the strength of the bundle. The brittle to quasibrittle transition point takes place at lower-disorder strength values when α is increased and that in turn reduces the brittlelike abrupt failure in the model. The critical fraction of strong links (α_c), beyond which the behavior of the model deviates from the conventional results, is observed to be a function of disorder strength δ (or η). Beyond α_c, the strength of the bundle is an increasing function of disorder. In the limit $\alpha > \alpha_c$, we approach an ideal situation for the model with increasing disorder where the strength is very high and failure abruptness is very low. The existence of α_c is captured through a sudden decrease in the average maximum avalanche during the failure process. Besides, the scaling of avalanche size distribution produces a master curve in the region $\alpha > \alpha_c$. The universality of the results is verified with three different threshold distributions: uniform, power law, and Weibull.

The material properties, mostly the strength of a material, have been studied extensively for different composites under different conditions. For example, the strength of a composite is evaluated with fiber length or fiber content [49], orientation of the fibers [50], fiber diameter [51], fiber matrix stress transfer [52], etc. In summary, the present study, consistent with a previous work [31], shows that there exists a critical density of infinitely strong links in a disordered system like the fiber bundle model above in which the model deviates from its conventional limit, causing higher strength and more stable failure. Additionally, I have shown that the existence of such critical density is observed irrespective of the disorder strength. The study can also be repeated with a more realistic stress release algorithm that includes the effect of stress localization near a broken link. Specifically, the present study can be studied in linear elastic fracture mechanics in terms of pinning to dipping transition, where the strong links can be considered to be pinned permanently, irrespective of an external driving parameter. Also, the study of such infinitely strong elements in other statistical models of fracture such as spring bundle model [53,54], random resistor network [55–57], etc., can be helpful to compare the numerical results with the results of real-life composites.

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