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Rapid Communications

Product-sum universality and Rushbrooke inequality in explosive percolation

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We study explosive percolation (EP) on an Erdös-Rényi network for product rule (PR) and sum rule (SR). Initially, it was claimed that EP describes discontinuous phase transition; now it is well accepted as a probabilistic model for thermal continuous phase transition (CPT). However, no model for CPT is complete unless we know how to relate its observable quantities with those of thermal CPT. To this end, we define entropy and specific heat, redefine susceptibility, and show that they behave exactly like their thermal counterparts. We obtain the critical exponents ν, α, β, and γ numerically and find that both PR and SR belong to the same universality class and obey Rushbrooke inequality.

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The notion of percolation is omnipresent in many seemingly disparate natural and man-made systems [1]. Examples include spread of forest fire, flow of fluid through porous media, spread of biological and computer viruses, etc. [2–4]. Besides such direct applications, percolation is best known as a model for phase transition. One of the simplest models for percolation is the classical random percolation (RP) on an Erdös-Rényi (ER) network in which one starts with N labeled nodes that are initially all isolated [5]. Then at each step a link, say e_{ij}, is picked at random from all the possible pairs of links and occupies it to connect nodes i and j provided they are not already connected. As the number of occupied links $n = tN$ increases from zero we find that clusters, i.e., contiguous nodes connected by occupied links, are formed and on the average grown. In the process, the largest cluster $s_{\max}$ undergoes a transition across $t_c = 0.5$ from minuscule size ($s_{\max} \sim \log N$) to giant size ($s_{\max} \sim N$). The emergence of such threshold value t_c is found to be accompanied by a sudden change in the order parameter P, the ratio of the largest cluster to the network size, such that $P = 0$ at $t \leqslant t_c$ and $P > 0$ at $t > t_c$ in the limit $N \to \infty$. This is reminiscent of the second-order or continuous phase transition (CPT).

In 2009, Achlioptas *et al.* proposed a class of percolation model in which two links instead of one are picked randomly at each step [6]. However, ultimately only one of the links, which results in the smaller clustering, is occupied and the other is discarded for future picking. One of the key features of this rule, which is now known as the Achlioptas process, is that it discourages the growth of the larger clusters and encourages the smaller ones which inevitably delays the transition. Eventually, when it reaches near the critical point it is so unstable that occupation of one or two links triggers an explosion of growth. It leads to the emergence of a giant cluster with a bang and hence it is called "explosive percolation" (EP). Indeed, the corresponding P, in contrast to its classical counterpart, undergoes such an abrupt transition that it was at first mistaken as a discontinuity and suggested to exhibit the first-order or discontinuous transition. Their results jolted the scientific community through a series of claims, unclaims, and counterclaims [7–16]. It is now well settled that the explosive percolation transition is actually continuous but with first-order-like finite-size effects [15–19].

In general, scientists use a theoretical model, just like architects use a geometric model before building a large, expensive structure, because it provides useful insights into the real-world systems. However, modeling is only useful if we know how to relate its various observable quantities to those of the real-world systems. To this end, Kasteleyn and Fortuin mapped the percolation problem onto the q-state Potts model and suggested one-to-one correspondence between some of its observables to the thermal quantities of the Potts model [20]. Owing to that mapping we know that P is the order parameter, and mean cluster size $\langle s\rangle$ or the second moment of the distribution function n_s (the number of cluster of size s per site) is the susceptibility. However, this mapping could not help to obtain an exact equivalent counterpart of entropy and specific heat. In thermal CPT, the entropy S and the order parameter (OP) complement each other as S, which measures the degree of disorder, is maximum where OP is zero and OP, which measures the extent of order, is maximum where S is zero. A similar behavior is also expected in percolation in order to elucidate whether it is also an order-disorder transition or not. Universality is another aspect that we find common in the thermal CPT and in the random percolation. In the case of EP, we still do not know even one example of two processes which belong to the same universality class. However, some interesting variants of the EP model have been discussed in Ref. [21] which are expected to belong to different universality classes. Another interesting aspect of thermal CPT is that its critical exponents α, β, and γ obey the Rushbrooke inequality $\alpha + 2\beta + \gamma \geqslant 2$ which reduces to equality under the static scaling hypothesis [22]. Whether it holds in explosive percolation or not, is also an interesting issue.

In this Rapid Communication, we investigate EP on the ER networks for product rule (PR) and sum rule (SR) and find their critical exponents numerically. First, we define susceptibility χ as the ratio of the successive jump ΔP of P and the magnitude of successive intervals Δt. Then, we obtain the critical exponents ν of the correlation length, γ of χ, and β of P. Note that $\langle s\rangle$ exhibits the expected divergence only if the largest cluster size is excluded from it and even then it gives too large a value of γ. Realizing these drawbacks, many researchers are already considering alternative definitions [9,23,24]. Second, we define entropy H for EP and find that it is continuous across the whole spectrum of the control parameter t which clearly reveals that EP transition is indeed continuous in nature. We then define specific heat as $C = q\frac{dH}{dq}$ where $q = (1-t)$ and find that it diverges with positive critical exponent α. The most intriguing and unexpected findings of this work is that PR and SR belong to the same universality class. Besides, we find that the elusive Rushbrooke inequality holds in EP. Recently, using the the same definitions for entropy, specific heat, and susceptibility we have shown that the Rushbrooke inequality also holds in the random percolation [25].

Percolation is all about clusters as every observable quantity of it is related, this way or another, to clusters by virtue of definition. Initially, all the labeled nodes are considered isolated so that every node is a cluster of its own size. The process starts by picking two distinct links, say e_{ij} and e_{kl}, randomly at each step. To apply the PR, we then calculate the products, $\Pi_{ij} = s_i \times s_j$ and $\Pi_{kl} = s_k \times s_l$, of the size of the clusters contained in the two nodes on either side of each link. The link with the smaller value of the products Π_{ij} and Π_{kl} is occupied. On the other hand, if we find $\Pi_{ij} = \Pi_{kl}$, then we occupy one of the two links at random with equal probability. In the case of SR, we take the sum $\Sigma_{ij} = s_i + s_j$ and $\Sigma_{kl} = s_k + s_l$ instead of the product and do the rest exactly in the same way as we did for PR. Each time we occupy a link, either the size of an existing cluster grows due to occupation of an intercluster link or the cluster size remains the same due to the addition of an intracluster link. In either case, the growth of large clusters is always disfavored, which is in sharp contrast to its RP counterpart. Thus, the emergence of a giant cluster is considerably slowed down but eventually when it happens, it happens abruptly but without discontinuity.

We first investigate not the order parameter P itself but its successive jump ΔP within successive intervals $\Delta t = 1/N$. The idea of the largest jump size $\Delta P_{\max}$ was first introduced by Nagler *et al.* and jump size ΔP in general by Manna [26,27]. We use it to define susceptibility as

$$\chi(t) = \frac{\Delta P}{\Delta t}, \tag{1}$$

which essentially becomes the derivative of P since in the limit $N \to \infty$ we have $\Delta t \to 0$. Note that Kasteleyn and Fortuin first suggested that the mean cluster size without the largest cluster is susceptibility [20]. Lately, the variance of the order parameter $\sqrt{\langle P^2\rangle - \langle P\rangle^2}$ has also been defined as the susceptibility [13,16,23]. Both the definitions give almost the same γ value which is far too high to satisfy the Rushbrooke equality. For instance, in the case of random percolation, the γ value itself is greater than 2. It justifies why a new definition for susceptibility, that would give small enough γ value so that the Rushbrooke equality is obeyed, is necessary.

In general, an observable X is said to obey finite-size scaling (FSS) if it satisfies

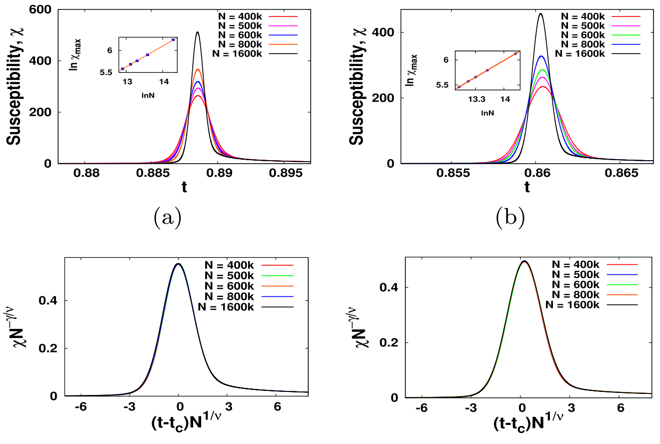
$$X(t,N) \sim N^{a/\nu}\phi[(t-t_c)N^{1/\nu}], \tag{2}$$

where $a = \alpha$ if X is specific heat, $a = -\beta$ if X is the order parameter, and $a = \gamma$ if X is the susceptibility. We shall now verify whether χ obeys the FSS or not. To that end, we first plot χ versus t for both the PR and the SR model in Figs. 1(a) and 1(b). According to Eq. (2) $\chi_{\max}$ at $t = t_c$ increases following a power law $\chi_{\max} \sim N^{\gamma/\nu}$. We then plot $\log(\chi_{\max})$ vs $\log(N)$ [see insets of Figs. 1(a) and 1(b)], and find straight lines with slopes $\gamma/\nu = 0.476(2)$ for PR and $\gamma/\nu = 0.478(2)$ for SR. Following the procedures in Ref. [28], we also get a rough estimate of the exponent $1/\nu = 0.535(5)$ for PR and $1/\nu = 0.534(1)$ for SR. The FSS theory further suggests that if we now plot $\chi N^{-\gamma/\nu}$ vs $(t-t_c)N^{1/\nu}$, all the distinct plots of Figs. 1(a) and 1(b) should collapse into their respective universal curves. Indeed, by tuning γ/ν and $1/\nu$ we find excellent data collapse [see Figs. 1(c) and 1(d)] if we use $\gamma/\nu = 0.478$ and $1/\nu = 0.535$ for PR and SR, both. Note that the t_c value also affects the data collapse and hence by tuning the initial estimates for t_c, we get the best data collapse if we use $t_c = 0.888\,50$ for PR and $t_c = 0.860\,18$ for SR. These values are very sensitive to data collapse. The quality of data collapse itself provides a clear testament to the extent of accuracy of these values. What is most noteworthy, however, is that both PR and SR share the same value for the exponents γ and ν. Using now the relation $N \sim (t-t_c)^{-\nu}$ in $\chi \sim N^{\gamma/\nu}$ we find that

$$\chi \sim (t-t_c)^{-\gamma}, \tag{3}$$

where $\gamma = 0.893$ for both PR and SR within the acceptable limit of error. It clearly shows that the susceptibility now diverges even without the exclusion of the largest cluster and that, too, with the same γ value for both the rules. Besides we find the value of γ far too less than what we find from existing other definitions for susceptibility.

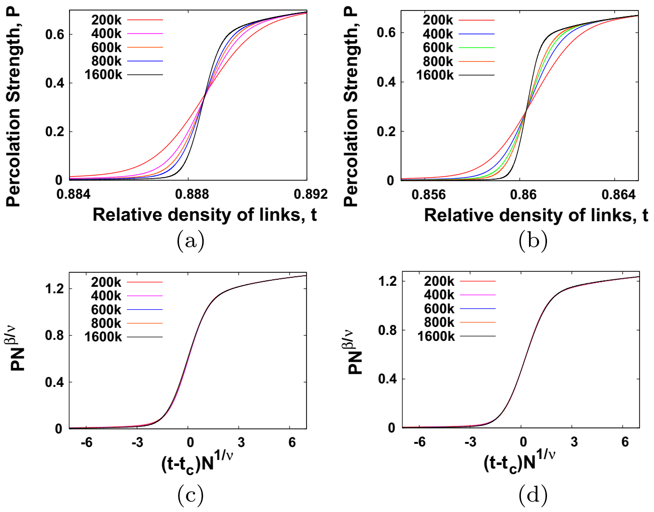
Now, we consider the order parameter P itself and plot it as a function of t in Figs. 2(a) and 2(b) for PR and SR, respectively. We follow the same standard procedure as in Refs. [28,29] and find $\beta/\nu = 0.045$ for both the variants. It is well known



(a) (b)

(c) (d)

FIG. 1. Susceptibility χ versus t for PR and SR on the ER network is shown in (a) and (b), respectively. We plot $\chi N^{-\gamma/\nu}$ vs $(t-t_c)N^{1/\nu}$ and find excellent data collapse of all the distinct curves of (a) in (c) and (b) in (d) for PR and SR, respectively, sharing the same critical exponents.



(a) (b) (c) (d)

FIG. 2. Plots of order parameter P vs t for PR in (a) and SR in (b). We use the same data to plot $PN^{\beta/\nu}$ vs $(t-t_c)N^{1/\nu}$ and find that all the distinct plots of (a) and (b) collapse superbly in (c) and (d), respectively, sharing the same exponents.

that $P(t,N)$ exhibits finite-size scaling. One way of testing it is to plot $PN^{\beta/\nu}$ vs $(t-t_c)N^{1/\nu}$ and check if all the distinct curves of P vs t curves collapse or not. Indeed, Figs. 2(c) and 2(d) suggest that they all collapse superbly with $\beta/\nu = 0.045$ and $1/\nu = 0.535$ values regardless of whether it is PR or SR. Substituting the relation $N \sim (t-t_c)^{-\nu}$ in $P \sim N^{-\beta/\nu}$ we get

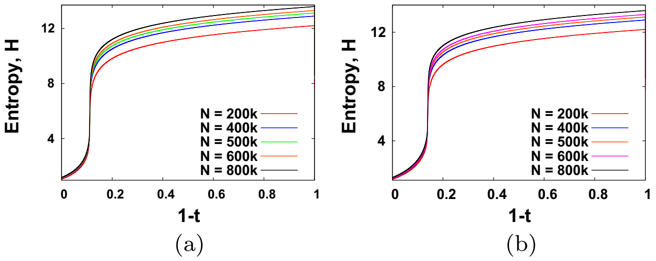
$$P(t) \sim (t-t_c)^{\beta}. \tag{4}$$

This is exactly how the order parameter behaves near critical point in the thermal CPT as well. We once again find that both PR and SR rules share the same exponent $\beta = 0.084$ within the acceptable limits of error. Such unusually low value of β compared to that of the RP on ER where $\beta = 1$ is the hallmark of EP transition [30]. Note also that Grassberger *et al.* obtained $1/\nu = 0.5$ and $\beta = 0.0861(5)$ for PR on ER [15]. Our values are quite close to their values; however, little differences are there which mark a significant improvement in the quality of data collapse.

Phase transitions always entail a change in entropy and hence no model for phase transition is complete without a proper definition for it. To this end, we find that the most suitable choice for entropy in percolation is the Shannon entropy, which is defined as

$$H(t) = -K \sum_{i}^{m} \mu_i \log \mu_i, \tag{5}$$

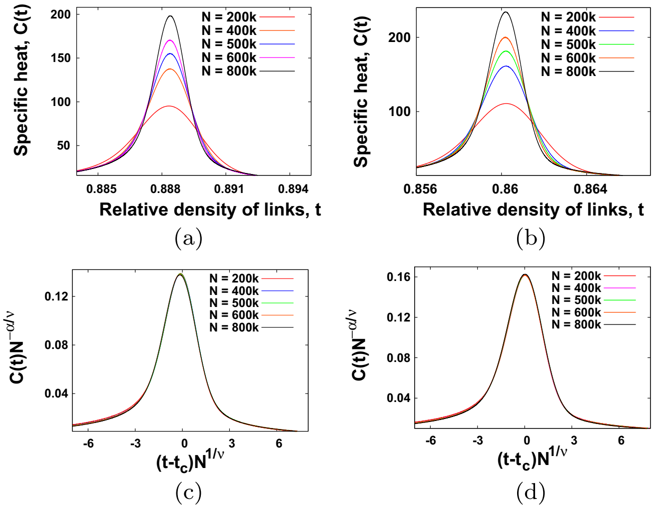
where we choose $K = 1$ since it merely amounts to a choice of a unit of measure of entropy [31]. Although there is no explicit restriction per se on the choice of μ_i, there exist some implicit restrictions. The textbook definitions of thermal entropy S and the specific heat C suggest that the S vs T plot must always have a sigmoidal shape with positive slope [22]. Recently, Vieira *et al.* used the cluster size distribution function n_s in place of μ_i to measure entropy for EP and Tsang *et al.* used the probability $w_s = (sn_s)/\sum sn_s$ to measure the entropy for RP [32,33]. Both groups found that the entropy is maximum at the critical point and very low at either end of the control parameter. However, we know that the order parameter measures the extent of order and entropy measures the degree of disorder and hence both cannot be zero at $t = 0$ since the system cannot be in the ordered and in the disordered state at the same t value. Besides, bell-shape or concave curvelike entropy violates the second law of thermodynamics since that would mean in the high q regime entropy decreases with increasing q which is equivalent to temperature. The problem lies in the fact that the sum in Eq. (5) is not over a class of cluster size s, rather it is over each individual labeled cluster and hence one can neither use w_s nor n_s to measure entropy. To find the appropriate probability μ_i for Eq. (5), we assume that for a given t there are m distinct and disjoint labeled clusters $i = 1,2,\ldots,m$ of size $s_1,s_2,\ldots,s_m$, respectively. We then propose a labeled cluster picking probability) μ_i, that a node picked at random belongs to cluster i, and assume that it depends on the size s_i of the cluster i itself, so that $\mu_i = s_i/\sum_j s_j$ where $\sum_j s_j = N$.



(a) (b)

FIG. 3. Entropy H vs $1-t$ for PR in (a) and SR in (b).

Incorporating $\mu_i = s_i/N$ in Eq. (5) we obtain entropy for explosive percolation. To visualize it, we plot it in Figs. 3(a) and 3(b) as a function of $q = 1-t$ for PR and SR, respectively. We observe that the maximum entropy occurs at $q = 1$ where $\mu_i = 1/N \ \forall \ i$. It means means that every node has the same probability to be picked if we hit one at random since initially every node is a cluster of size one. This is exactly like the state of the isolated ideal gas since here, too, all accessible microstates are equally probable. The $q = 1$ state is thus the most confused or disordered state. Now as we lower the q value, we see that entropy decreases slowly but as we approach toward $q_c = 1-t_c$ we observe a dramatic decrease in entropy. This is because as we approach q_c from higher q value, we find that many moderately large sized clusters get accumulated and eventually reach a critical state which is termed as "powder keg" by Friedman and Landsberg [7]. Thereafter, the addition of even a few links may trigger the growth of the largest cluster in an explosive fashion. We find that at $q = 0$ the entropy H is minimally low but the order parameter P is maximally high and hence it is clearly the ordered phase. We thus see that at $q = 1$ the entropy is maximally high but the order parameter $P = 0$ and hence it corresponds to the disordered phase. The term percolation therefore refers to the transition from ordered phase characterized by vanishingly small entropy at $q < q_c$ to disordered phase characterized by $P = 0$ at $q > q_c$ as one tunes the control parameter q. We thus find that in percolation too, like in the thermal CPT, entropy H and order parameter P complement each other.



(a) (b)

(c) (d)

FIG. 4. Plots of specific heat C vs t for PR and SR are shown in (a) and (b), respectively, for different network size N. In (c) and (d) we plot $CN^{-\alpha/\nu}$ vs $(t-t_c)N^{1/\nu}$ and find that all the distinct plots of (a) and (b) collapse superbly sharing the same α and ν values.

Once entropy is known, we can use the standard definition of specific heat $C = TdS/dT$ where S is the thermal entropy. If we now replace T by $1-t$ and S by Shannon entropy H, we get the desired definition of specific heat for percolation

$$C(t) = (1-t)\frac{dH}{d(1-t)}. \quad (6)$$

However, we can also define it as $C = -(1-t)dH/dt$ since $dH/dt = -dH/d(1-t)$. Taking differentiation of H with respect to $1-t$ from first principles and multiplying that value with the corresponding value of $(1-t)$, we can immediately obtain $C(t)$. We then plot $C(t)$ in Figs. 4(a) and 4(b) as a function of t for PR and SR, respectively. To compute the corresponding critical exponent α, once again we use the FSS hypothesis and find $\alpha/\nu = 0.535$ for both PR and SR. Finally, we plot $CN^{-\alpha/\nu}$ vs $(t-t_c)N^{1/\nu}$ and obtain a perfect data collapse with $\alpha/\nu = 0.535$ and $1/\nu = 0.535$ for both PR and SR as shown in Figs. 4(c) and 4(d). We then use the relation $N \sim (t-t_c)^{-\nu}$ in $C(t) \sim N^{\alpha/\nu}$ and immediately find that the specific heat diverges like

$$C(t) \sim (t-t_c)^{-\alpha}, \quad (7)$$

where $\alpha = 1$ for both PR and SR. The quality of data collapse is a clear testament of the accuracy of the α value.

Since 2009, much of the time has been spent in resolving the issue of whether EP describes the first order or second order phase transition. In addition, the classifications of any system into universality classes are always an interesting proposition in phase transition and critical phenomena. As far as random percolation is concerned, it is well known that the site and bond percolation on all lattices having the same spatial dimensions belong to the same universality class. However, there has hardly been any work on checking whether the explosive percolation, too, can be classified into universality classes. Bastas *et al.* reported that site and bond type explosive percolation on the same lattice do not belong to the same universality class [16]. Radicchi and Castellano, on the other hand, reported that the site-bond universality is violated even in the case of random percolation if the substrate is a network with null percolation thresholds [34]. In the case of EP, we do not know to date whether any two processes or types of explosive percolation belong to the same universality class. Finding that PR and SR of explosive percolation belong to the same universality class is therefore a significant development.

Yet another interesting thing to check is whether the Rushbrooke inequality (RI) holds in EP or not. To this end, we substitute our values of $\alpha = 1$, $\gamma = 0.893$, and $\beta = 0.084$ in the Rushbrooke relation and find $\alpha + 2\beta + \gamma = 2.061$. Recently, we applied the same approach to the random percolation on square and weighted planar stochastic (WPS) lattices and found the same result albeit they belong to different universality classes [25]. In all three cases investigated so far, including the one in question, we find that RI holds almost as an equality. Many experiments and exactly solvable models of its thermal counterpart too suggest that the RI actually holds as an equality [35]. In fact, the static scaling hypothesis based on the assumption that Gibb's free energy is a generalized homogeneous function allows one to show that RI is indeed an exact equality. All these results, namely, finite-size scaling, universality, and Rushbrooke inequality, clearly suggest that explosive percolation is actually a model for continuous phase transition. However, unlike random percolation, it exhibits some unusual finite-size behaviors. For instance, we find hysteresis loops in its forward and reverse processes, double hump in the distributions of the order parameter P, the time difference $\Delta = t_2 - t_1$ between the last step t_1 for which the largest cluster $C < N^{1/2}$ and the first step t_2 for which $C > 0.5N$ is not extensive [15–19].

To summarize, we have given a thermodynamic formulation of explosive percolation. First, we found that entropy H undergoes a sudden rise across q_c so that in the supercritical regime (i.e., $q < q_c$ phase) the order parameter P is high but the entropy H is vanishingly low and vice versa in the subcritical regime (i.e., $q > q_c$ phase). Note that P quantifies the extent order and H measures the degree of disorder. It implies that we can regard the subcritical phase as more disordered, i.e., has a higher symmetry than the supercritical phase revealing that explosive percolation is an order-disorder transition like ferromagnetic transition in the thermal CPT. Second, we have shown that the specific heat and susceptibility diverge at the critical point without having to exclude the largest cluster. Third, we obtained the critical exponents α, β, γ, and ν numerically and found that their values for PR and SR are the same, revealing that they belong to the same universality class. Such PR-SR universality is highly intriguing and unexpected, especially against the background of the breakdown of the usual site-bond universality even in the lattice. Finally, we have also shown that the value of the critical exponents α, β, and γ obey the Rushbrooke inequality. Our work confirms that the explosive percolation is a model for continuous phase transition since we now know that entropy, order parameter, specific heat, susceptibility, and their critical exponents behave exactly in the same way as in the thermal CPT. We believe that the present work will be of great interest to the scientific community.

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