One-dimensional reduction of viscous jets. I. Theory

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We build a general formalism to describe thin viscous jets as one-dimensional objects with an internal structure. We present in full generality the steps needed to describe the viscous jets around their central line, and we argue that the Taylor expansion of all fields around that line is conveniently expressed in terms of symmetric trace-free tensors living in the two dimensions of the fiber sections. We recover the standard results of axisymmetric jets and we report the first and second corrections to the lowest order description, also allowing for a rotational component around the axis of symmetry. When applied to generally curved fibers, the lowest order description corresponds to a viscous string model whose sections are circular. However, when including the first corrections, we find that curved jets generically develop elliptic sections. Several subtle effects imply that the first corrections cannot be described by a rod model since it amounts to selectively discard some corrections. However, in a fast rotating frame, we find that the dominant effects induced by inertial and Coriolis forces should be correctly described by rod models. For completeness, we also recover the constitutive relations for forces and torques in rod models and exhibit a missing term in the lowest order expression of viscous torque. Given that our method is based on tensors, the complexity of all computations has been beaten down by using an appropriate tensor algebra package such as xAct, allowing us to obtain a one-dimensional description of curved viscous jets with all the first order corrections consistently included. Finally, we find a description for straight fibers with elliptic sections as a special case of these results, and recover that ellipticity is dynamically damped by surface tension. An application to toroidal viscous fibers is presented in the companion paper [Pitrou, Phys. Rev. E **97**, 043116 (2018)].

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I. INTRODUCTION

Solving exactly the nonlinear fluid equations for long viscous jets is extremely complicated, and one needs to resort to an approximation scheme to study the dynamics of these systems. Due to the elongated shape, there is an obvious simplification which consists in considering a one-dimensional description. A body is considered as being slender if its radius R is typically much smaller than the inverse size of velocity gradients L, that is, if the velocity field changes on length scales which are larger than the fiber radius. Hence, the one-dimensional reduction induces naturally an expansion in the slenderness parameter $\epsilon_R \equiv R/L$. Given that at lowest order a solid is approximated by a point particle, then we expect that a slender jet is approximated at lowest order by some type of string. Furthermore, as extended objects are described as point particles with an internal structure encoded in various moments (e.g., in the moment of inertia), the internal structure of the one-dimensional object which approximates a viscous jet is encoded in some moments which vary continuously along the one-dimensional fiber. From the perturbative expansion in the small parameter ϵ_R, we show how this series of moments must be truncated at a given order of corrections around the string description. The various moments describing the viscous jet happen to separate naturally into moments which evolve dynamically and moments which are related by constraints to the former ones.

For simplicity, we restrict our analysis to incompressible Newtonian fluids whose internal forces are captured entirely by a constant viscosity parameter, and we allow for surface tension effects. These ingredients are sufficient to describe the dynamics of drop formation from the Rayleigh-Plateau instability [1–3]. However, concerning the global shape of the viscous jet, our aim is to remain as general as possible, allowing for curved fibers (that is curved central lines) with possibly noncircular cross sections. Indeed, there are a series of geometrical simplifications which are usually performed given the symmetries of specific problems. From the most restrictive to the most general, we find the axisymmetric case, the straight fiber case with noncircular sections, the curved fiber case with circular sections, and the curved fiber case with noncircular sections.

(i) *Axisymmetric fibers.* The fiber central line is a straight line and the cross sections around that central line are disks only. The one-dimensional reduction of viscous jets for this geometry has been extensively studied in previous literature with several nonequivalent methods. A first method consists in using the Cosserat theory [4], and it has been shown that this method is in fact equivalent to expanding the velocity fields along a suitable basis of functions [3,5]. The second method is based on a radial expansion (mathematically a Taylor expansion) of velocity fields and it has been developed in, e.g., García and Castellanos [6], Eggers and Dupont [7], or

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Bechtel *et al.* [8] when allowing for a possible angular rotation around the axis of symmetry. The validity of these methods has been studied in details in the subsequent literature, e.g., in Perales and Vega [9], Gañán-Calvo *et al.* [10], Montanero *et al.* [11], or Vincent *et al.* [12]. In Sec. VI, we recover the standard lowest order results plus first corrections using the radial expansion method. We also report a general method to obtain recursively its corrections up to any order and report the second set of corrections. In this geometry, once the constraints from the stress tensor on the fiber side have been used, the fundamental dynamical variables appear to be the velocity along the axis v, the local rotation rate around that axis $\dot{\phi}$, and the radius R.

(ii) *Straight fibers.* The fiber central line is still a straight line, but the cross sections can have more general shapes. We find that the section shape is most conveniently expanded into shape multipoles which are symmetric trace-free tensors. The lowest multipole describes for instance the elliptical modulation of the cross sections [13,14]. Under this description, the shape multipoles are additional fundamental variables.

(iii) *Curved fibers with circular sections.* The fiber central line can have any general shape as long as the curvature radius remains larger than the typical extension of the cross sections. A formalism was initially developed in Entov and Yarin [15] and further summarized in Yarin [16,17]. Curved fibers were considered with surface tension effects in Dewynne *et al.* [18,19] and also in Arne *et al.* [20,21] to study rotational spinning processes such as those used in the production of glass wool or candy floss. A similar viscous rod model, based on curvilinear coordinates adapted to the problem, has been developed by Ribe [22] and Ribe *et al.* [23,24] to study the coiling of viscous jets, and numerical methods were developed by, e.g., Audoly *et al.* [25,26] to obtain general solutions. In this article, we develop a formalism based on a $2+1$ splitting [27] of equations, that is, a separation between the two-dimensional fiber sections and its one-dimensional central line, to reduce curved viscous jets as one-dimensional object. We first describe in full generality the central line along which the jet is described by following essentially the method developed by Ribe. The tangential direction of this central line naturally determines a fiber direction and a fiber section which is orthogonal to it, along which our $2+1$ splitting of equations is performed. Then, using the irreducible representation of SO(2), we build an expansion of the velocity field. It is based on symmetric trace-free tensors which are lying in the fiber sections and we show that these tensors are the moments which naturally take into account the internal structure of the fiber. Eventually, the fundamental variables are the same as for straight fibers with noncircular sections (velocity along the axis v, local rotation rate around that axis $\dot{\phi}$, fiber radius R, and shape multipoles), since all other velocity moments can be obtained as constraints from these variables. These fundamental variables must also be supplemented by the fiber central line position and velocity. The difference with the straight case lies mainly in the fact that circular sections are only compatible with the lowest order description, that is, with the viscous string model. Indeed, as soon as corrections are included, the shape multipoles are necessarily sourced. For instance, terms which are quadratic in the central line curvature generically source the ellipticity.

Our consistent description of elongated but possibly curved viscous fibers allows to find a number of qualitative results on the structure of the one-dimensional models which contrast with past literature. Most importantly, we find that the first corrections for curved fiber geometries cannot be encompassed by the rod model of Arne *et al.* [20], Ribe [22], and Ribe *et al.* [23]. These methods are based on the observation that, when considering extended solid objects instead of point particles, we must supplement the momentum balance equation by an angular momentum equation. It is thus expected that to go beyond the string approximation which can also be obtained from a momentum balance equation, we should use some form of angular momentum balance equation. However, this method inspired from solids fails for viscous fluids for a number of reasons which are absent in nondeformable solids.

At lowest order in ϵ_R, the rotation of the fiber section follows the rotation of the fluid on the fiber central line. Since the dynamics of the central line is determined from the momentum balance equation, its local rotation rate is also derived from it, implying that the angular momentum method cannot bring any new dynamical information about fiber section rotation. In fact, it is precisely because the rotation of sections is determined from the rotation of the central line at lowest order that the angular momentum method is instead a constraint on the sectional component of the viscous forces which appear as corrections to the lowest order description. Eventually, the coupling between the momentum balance and angular momentum balance equation amounts to selecting only some corrections and discarding the other ones as several order ϵ_R^2 effects are missing. For instance, the sectional component of the velocity *on the central line* differs from the sectional component of the velocity *of the central line* by corrections of order ϵ_R^2. Additionally, the longitudinal velocity develops a Hagen-Poiseuille profile (that is a parabolic profile in terms of the radial distance), which blurs the notion of solid displacement of fiber sections.

For these various reasons, we find that we should not build a one-dimensional reduction of viscous fibers from the usual methods which have been developed to describe the continuous deformation of solids, but we should instead start from a Taylor expansion of the velocity field and find a consistent truncation at any given order. When deriving corrections to a viscous string model, this requires to abandon the hypothesis of circular sections and to derive the dynamical equations for the shape moments as well.

A. Outline

In Sec. II, we review the general formalism to describe the fiber central line and fiber sections. We introduce a coordinates system and a vector basis which are adapted to the description of curved fibers. In Sec. III, we review in details how scalar fields and vector field such as velocity can be expanded in multipoles using an adapted $2+1$ decomposition. This leads us to introduce irreducible representations of SO(2) according to which these multipoles are classified. With this formalism clearly established, it is then possible to explore in full generality all the kinematical relations of fluid fibers in Sec. IV and then the dynamical laws in Sec. V. The formalism is then applied for the two main geometries of interest. First, in Sec. VI we rederive the axisymmetric results up to first corrections, including the interplay between the velocity along the axis of symmetry, and rotation around that same axis. Second corrections of the axisymmetric case are also reported in Appendix E. In Sec. VII, we then tackle the problem of curved fibers, first deriving the lowest order string model and then discussing the general method to obtain corrections. We report the first corrections in the curved case and show that elliptical shapes are necessarily sourced at that order. We also discuss the special case of straight but noncircular sections in Sec. VII D. Finally, we compare our method with the rod models where dynamical equations are usually obtained from momentum and angular momentum balance and we discuss the range of validity of these methods. Several technical developments are gathered in the Appendices, among which the symmetric trace-free tensors in Appendix A which we use throughout the article. The formalism is applied to study toroidal viscous fibers in Pitrou [28].

B. Notation

Assuming incompressibility, the mass density ρ is constant. Hence, in the remainder of this article we will use the notation

$$\mu/\rho \to \mu, \quad \nu/\rho \to \nu, \quad P/\rho \to P, \tag{1.1}$$

where μ is the viscosity of the fluid, ν the surface tension parameter, and P the pressure. We use Einstein summation convention whenever indices are placed in pairs with one index up and one index down as, e.g., in $x^\mu x_\mu$ or $X^i Y_i$.

II. GEOMETRY

For axisymmetric fibers, it is natural to use cylindrical coordinates. The third coordinate (z) is naturally associated with the axis of symmetry, and the other coordinates (r,θ) parametrize the two-dimensional space which is orthogonal to this axis. However, for generally curved fibers, we must use fiber adapted coordinates. They are closely related to cylindrical coordinates in the sense that we choose a central line inside the fiber and we use the natural coordinate of this line as the third coordinate. The two other coordinates are then used to describe the planes which are orthogonal to this central line. In this section, we construct fiber adapted coordinates and the orthonormal basis which is naturally associated with it.

A. Description of the fiber central line

Throughout this article, we use Cartesian coordinates x^μ, with the corresponding canonical basis of vector and covectors (forms) $\boldsymbol{e}_\mu$ and $\boldsymbol{e}^\mu = dx^\mu$. Greek indices refer to components in this canonical basis. The scalar and wedge products of two vectors $\boldsymbol{X} = X^\mu \boldsymbol{e}_\mu$ and $\boldsymbol{Y} = Y^\mu \boldsymbol{e}_\mu$ are simply

$$\boldsymbol{X}\cdot\boldsymbol{Y} \equiv X^\mu Y_\mu\,, \quad [\boldsymbol{X}\times\boldsymbol{Y}]^\alpha \equiv \varepsilon^{\alpha\mu\nu} X_\mu Y_\nu, \tag{2.1}$$

where $\varepsilon^{\alpha\mu\nu}$ is the totally antisymmetric tensor with $\varepsilon^{123} = 1$.

We assume that it is possible to define a fiber central line (FCL) as an approximation to the fiber shape. We postpone to Sec. IV I the discussion about the geometrical construction of this line. The position of this FCL and its tangent vector $\boldsymbol{T} = T^\mu \boldsymbol{e}_\mu$ are given by

$$R^\mu(s,t), \quad T^\mu \equiv \partial_s R^\mu, \quad T^\mu T_\mu = 1. \tag{2.2}$$

The last condition ensures that the coordinate s can also be used to measure lengths along the FCL, and t is the absolute time. A parallel projector, also named longitudinal projector, and an orthogonal projector, also named sectional projector, can be defined as

$$P_{\parallel}{}^{\mu}_{\nu} = T^\mu T_\nu, \quad P_{\perp}{}^{\mu}_{\nu} = \delta^\mu_\nu - T^\mu T_\nu = \delta^\mu_\nu - P_{\parallel}{}^{\mu}_{\nu}, \tag{2.3}$$

and can be used to project any vectorial quantity along and orthogonally to the fiber tangential direction T^μ.

The velocity of the FCL is given by

$$\boldsymbol{U} \equiv \partial_t \boldsymbol{R} \tag{2.4}$$

from which we deduce that the evolution of the tangent vector obeys

$$\partial_s\partial_t \boldsymbol{R} = \partial_t\partial_s \boldsymbol{R} \Rightarrow \partial_t \boldsymbol{T} = \partial_s \boldsymbol{U}. \tag{2.5}$$

From the normalization condition (2.2) of T^μ, we deduce that

$$\boldsymbol{T}\cdot\partial_t\boldsymbol{T} = 0 \Rightarrow \boldsymbol{T}\cdot\partial_s\boldsymbol{U} = 0. \tag{2.6}$$

This last relation means that the FCL velocity can only rotate the FCL direction $\boldsymbol{T}$, but without stretching it. Indeed, since it is only a geometrical line there would be no physical information contained in such stretching and we thus chose the normalization (2.2). The FCL can be understood as a nonextensible but flexible wire that would be inside the bulk of the viscous jet, and the coordinate s can be thought of as the distance along that wire from a reference point $s = 0$.

The curvature of the FCL is defined as the rate of change of the tangential direction along the line for a fixed time, that is

$$\boldsymbol{\kappa} \equiv \boldsymbol{T}\times\partial_s\boldsymbol{T}, \quad \partial_s\boldsymbol{T} = \boldsymbol{\kappa}\times\boldsymbol{T}, \tag{2.7}$$

and it depends on (s,t). Note that we have chosen deliberately $\boldsymbol{\kappa}\cdot\boldsymbol{T} = 0$ and our convention thus differs from Arne *et al.* [20], Ribe [22], and Ribe *et al.* [23] where the longitudinal component of curvature does not necessarily vanish.

Finally, to alleviate the notation, for any vector $\boldsymbol{X}$ we will use the notation

$$\widetilde{\boldsymbol{X}} \equiv \boldsymbol{T}\times\boldsymbol{X}. \tag{2.8}$$

Note that the vector $\widetilde{\boldsymbol{\kappa}}$ points toward the exterior of the FCL curvature.

B. Local orthonormal basis

On each fiber section we can consider an orthonormal basis $(\boldsymbol{d}_{\underline{1}},\boldsymbol{d}_{\underline{2}},\boldsymbol{d}_{\underline{3}} \equiv \boldsymbol{T})$ where the vectors of the basis depend only on (s,t). We use coordinates $\underline{i},\underline{j},\underline{k},\ldots$ to refer to components along this basis. It is oriented such that

$$\boldsymbol{d}_{\underline{i}}\times\boldsymbol{d}_{\underline{j}} = \varepsilon_{\underline{i}\underline{j}}{}^{\underline{k}}\boldsymbol{d}_{\underline{k}}, \tag{2.9}$$

where $\varepsilon_{\underline{i}\underline{j}\underline{k}} = \varepsilon^{\underline{i}\underline{j}\underline{k}} = \varepsilon_{\underline{i}\underline{j}}{}^{\underline{k}}$ with $\varepsilon_{\underline{123}} = 1$ is the alternating symbol which is fully antisymmetric. This orthonormal basis is related to the canonical Cartesian basis by a change of basis

$$\boldsymbol{d}_{\underline{i}} = d^\mu_{\underline{i}}\boldsymbol{e}_\mu. \tag{2.10}$$

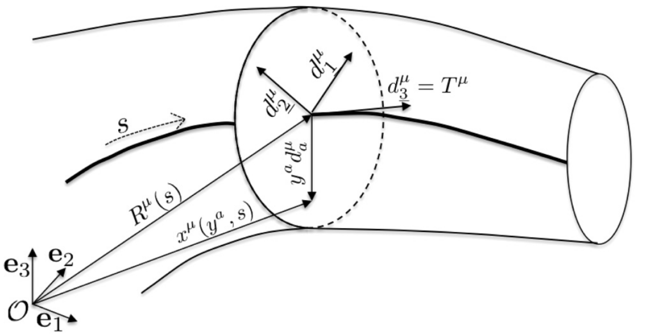

FIG. 1. Fiber central line position R^μ, and the associated orthonormal basis $d_{\underline{i}}^\mu$. A point lying in a fiber section is then labeled by x^μ defined in Eq. (2.29).

Similarly, the cobasis which depends also only on (s,t) is related to the canonical cobasis by a change of cobasis

$$\boldsymbol{d}^{\underline{i}} = d^{\underline{i}}_\mu \boldsymbol{e}^\mu. \tag{2.11}$$

The change of basis and cobasis satisfies the basic properties

$$d^{\underline{i}}_\mu = \delta^{\underline{i}\underline{j}}\delta_{\mu\nu} d^\nu_{\underline{j}} = d^\mu_{\underline{i}}, \tag{2.12a}$$

$$d^\mu_{\underline{3}} = T^\mu, \quad d^{\underline{3}}_\mu = T_\mu = \delta_{\mu\nu}T^\nu. \tag{2.12b}$$

Any vector on the central line is decomposed as

$$\boldsymbol{X}(s,t) = X^{\underline{i}}(s,t)\boldsymbol{d}_{\underline{i}}(s,t). \tag{2.13}$$

The fiber central line and its associated orthonormal basis are illustrated in Fig. 1. Note that for components in the orthonormal basis, the position of indices does not matter. Indeed, from (2.12), we find that for any vector $X^{\underline{i}} = X_{\underline{i}}$. In the orthonormal basis, the scalar products and wedge products of two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ are

$$\boldsymbol{X}\cdot\boldsymbol{Y} = X^{\underline{i}}Y_{\underline{i}}, \quad [\boldsymbol{X}\times\boldsymbol{Y}]^{\underline{i}} = \varepsilon^{\underline{i}\underline{j}\underline{k}}X_{\underline{j}}Y_{\underline{k}}. \tag{2.14}$$

Following the evolution of the tangent vector along the fiber (2.7), we choose to transport this basis along the FCL according to

$$\partial_s \boldsymbol{d}_{\underline{i}} = \boldsymbol{\kappa}\times\boldsymbol{d}_{\underline{i}} \Leftrightarrow \kappa^{\underline{i}} = \tfrac{1}{2}\epsilon^{\underline{i}\underline{j}\underline{k}}(\partial_s \boldsymbol{d}_{\underline{j}})\cdot\boldsymbol{d}_{\underline{k}}, \tag{2.15}$$

which is consistent with (2.7) since $\boldsymbol{T} = \boldsymbol{d}_{\underline{3}}$.

C. Rotation of the orthonormal basis

The rotation rate of the orthonormal basis is defined by

$$\partial_t \boldsymbol{d}_{\underline{i}} = \boldsymbol{\omega}\times\boldsymbol{d}_{\underline{i}} \quad\Leftrightarrow\quad \omega^{\underline{i}} = \tfrac{1}{2}\varepsilon^{\underline{i}\underline{j}\underline{k}}(\partial_t \boldsymbol{d}_{\underline{j}})\cdot\boldsymbol{d}_{\underline{k}}. \tag{2.16}$$

In particular $\partial_t \boldsymbol{T} = \boldsymbol{\omega}\times\boldsymbol{T}$ and the sectional projection of rotation satisfies

$$P_\perp(\boldsymbol{\omega}) = \boldsymbol{T}\times\partial_t\boldsymbol{T} = \boldsymbol{T}\times\partial_s\boldsymbol{U}. \tag{2.17}$$

From the definitions (2.7) and (2.16) of curvature and rotation, we get the structure relation

$$\partial_t\boldsymbol{\kappa} - \partial_s\boldsymbol{\omega} = \boldsymbol{\omega}\times\boldsymbol{\kappa}. \tag{2.18}$$

This relation is a consequence of the flatness of the classical structure of space-time as explained in Appendix D. When projected on $\boldsymbol{T}$, it also implies

$$(\partial_s\boldsymbol{\omega})\cdot\boldsymbol{T} = 0. \tag{2.19}$$

We use the indices $a,b,\ldots = 1,2$ for the sectional components, that is, which are orthogonal to the fiber tangential direction. For instance, the curvature $\boldsymbol{\kappa}$ is projected since $P_\perp(\boldsymbol{\kappa}) = \boldsymbol{\kappa}$, but the rotation is not a projected vector, so their decompositions in components read as

$$\boldsymbol{\kappa} = \kappa^{\underline{a}}\boldsymbol{d}_{\underline{a}}, \quad \boldsymbol{\omega} = \omega^{\underline{i}}\boldsymbol{d}_{\underline{i}}, \quad P_\perp(\boldsymbol{\omega}) = \omega^{\underline{a}}\boldsymbol{d}_{\underline{a}}. \tag{2.20}$$

The notation (2.8) in components is simply

$$\widetilde{X}^{\underline{a}} = (\boldsymbol{T}\times\boldsymbol{X})^{\underline{a}} = X_{\underline{b}}\varepsilon^{\underline{b}\underline{a}}, \tag{2.21}$$

where $\varepsilon_{\underline{a}\underline{b}} = \varepsilon^{\underline{a}\underline{b}} \equiv \varepsilon_{\underline{a}\underline{b}\underline{3}}$ is the two-dimensional alternating symbol ($\varepsilon_{\underline{12}} = 1$). Since $\partial_s\varepsilon^{\underline{a}\underline{b}} = \partial_t\varepsilon^{\underline{a}\underline{b}} = 0$ the tilde operation and the ∂_s or ∂_t derivatives commute.

From (2.18) the relations between the derivatives of components of curvature and rotation read as

$$\partial_t\kappa^{\underline{i}} - \partial_s\omega^{\underline{i}} = \varepsilon^{\underline{i}\underline{j}\underline{k}}\kappa_{\underline{j}}\omega_{\underline{k}}. \tag{2.22}$$

Separating the sectional and longitudinal components, these relations are just

$$\partial_t\kappa^{\underline{a}} - \partial_s\omega^{\underline{a}} = \varepsilon^{\underline{a}\underline{b}}\kappa_{\underline{b}}\omega^{\underline{3}} = -\widetilde{\kappa}^a\omega^{\underline{3}}, \tag{2.23a}$$

$$\partial_s\omega^{\underline{3}} = \varepsilon^{\underline{a}\underline{b}}\omega_{\underline{a}}\kappa_{\underline{b}} = -\omega_a\widetilde{\kappa}^a = \widetilde{\omega}_a\kappa^a. \tag{2.23b}$$

This last relation (2.23b), which is the component version of (2.19), states that once the curvature and the sectional projection of the rotation are fixed at a given time along the FCL, then the longitudinal component of rotation $\omega_{\underline{3}}$ is also determined at that time along the FCL, as a consequence of the structure relation (2.18). We call this type of relation a *constraint equation*.

D. Essential relations for components

For a vector $\boldsymbol{X}(s,t)$, the components of the derivatives [e.g., $(\partial_t X)^{\underline{i}} \equiv \boldsymbol{d}^{\underline{i}}\cdot\partial_t\boldsymbol{X}$] are not the derivatives of the components [e.g., $\partial_t X^{\underline{i}} \equiv \partial_t(\boldsymbol{d}^{\underline{i}}\cdot\boldsymbol{X})$]. Indeed, they are related by

$$\begin{aligned}(\partial_t X)^{\underline{i}} &= \partial_t X^{\underline{i}} + (\omega\times X)^{\underline{i}}\\ &= \partial_t X^{\underline{i}} + \varepsilon^{\underline{i}\underline{j}\underline{k}}\omega_{\underline{j}}X_{\underline{k}},\end{aligned} \tag{2.24a}$$

$$\begin{aligned}(\partial_s X)^{\underline{i}} &= \partial_s X^{\underline{i}} + (\kappa\times X)^{\underline{i}}\\ &= \partial_s X^{\underline{i}} + \varepsilon^{\underline{i}\underline{j}\underline{k}}\kappa_{\underline{j}}X_{\underline{k}}.\end{aligned} \tag{2.24b}$$

In particular, for the projected (or sectional) indices, this reads as

$$(\partial_t X)^{\underline{a}} = \partial_t X^{\underline{a}} - \widetilde{\omega}^{\underline{a}}X_{\underline{3}} + \widetilde{X}^{\underline{a}}\omega_{\underline{3}}, \tag{2.25a}$$

$$(\partial_s X)^{\underline{a}} = \partial_s X^{\underline{a}} - \widetilde{\kappa}^{\underline{a}}X_{\underline{3}}. \tag{2.25b}$$

If a vector $\boldsymbol{X}$ is projected, that is orthogonal to the tangential direction ($X_{\underline{3}} = 0$), then in that special case, $(\partial_s X)^{\underline{a}} = \partial_s X^{\underline{a}}$ but note that we still have $(\partial_t X)^{\underline{a}} \neq \partial_t X^{\underline{a}}$. From the derivatives of the components, that is, from $\partial_t X^{\underline{i}}$, $\partial_s X^{\underline{i}}$, we can conversely obtain the derivatives of the vector using (2.24) with the simple relations

$$\partial_s \boldsymbol{X} = (\partial_s X)^{\underline{i}} \boldsymbol{d}_{\underline{i}} = (\partial_s X^{\underline{i}})\boldsymbol{d}_{\underline{i}} + \boldsymbol{\kappa} \times \boldsymbol{X}, \tag{2.26a}$$

$$\partial_t \boldsymbol{X} = (\partial_t X)^{\underline{i}} \boldsymbol{d}_{\underline{i}} = (\partial_t X^{\underline{i}})\boldsymbol{d}_{\underline{i}} + \boldsymbol{\omega} \times \boldsymbol{X}. \tag{2.26b}$$

Let us report in coordinates some relations previously obtained in a covariant form. Equation (2.6) reads as in components

$$\partial_s U^{\underline{3}} + \varepsilon^{\underline{ab}} \kappa_{\underline{a}} U_{\underline{b}} = \partial_s U^{\underline{3}} + \widetilde{\kappa}^{\underline{b}} U_{\underline{b}} = 0. \tag{2.27}$$

As for the property (2.17) for the projection of the rotation, it reads as simply

$$\omega^{\underline{a}} = \partial_s \widetilde{U}^{\underline{a}} + \kappa^{\underline{a}} U^{\underline{3}}. \tag{2.28}$$

This relation which is also a constraint equation states that once the sectional projection of the central line velocity ($U^{\underline{a}}$) is known at a given time along the central line, then the projection of rotation ($\omega^{\underline{a}}$) is determined. Since the longitudinal component of rotation $\omega^{\underline{3}}$ is determined from (2.23b), we can then determine the time evolution of the components of curvature from (2.23a).

E. Fiber adapted coordinates

We use Cartesian coordinates $y^a = y^1, y^2$ inside the fiber section labeled by (s,t) so as to parametrize points which do not lie exactly on the FCL. With $y^3 \equiv s$, the fiber adapted (FA) coordinates are the set of $(y^i) = (y^a, s)$. The FA coordinates of a point in the fiber are related to the Cartesian coordinates x^μ by

$$x^\mu(y^i) = R^\mu(s,t) + y^a d^\mu_{\underline{a}}(s,t). \tag{2.29}$$

These coordinates are illustrated in Fig. 1. The canonical basis and cobasis associated with the FA coordinates are

$$\boldsymbol{e}_i = e^\mu_i \boldsymbol{e}_\mu, \quad \boldsymbol{e}^i = e^i_\mu \boldsymbol{e}^\mu, \quad e^\mu_i = \frac{\partial x^\mu}{\partial y^i}, \quad e^i_\mu = \frac{\partial y^i}{\partial x^\mu} \tag{2.30}$$

and they are related to the orthonormal basis by

$$\boldsymbol{e}_3 = h\boldsymbol{T} = h\boldsymbol{d}_{\underline{3}}, \quad \boldsymbol{e}^3 = \frac{1}{h}\boldsymbol{d}^{\underline{3}}, \tag{2.31a}$$

$$\boldsymbol{e}_a = \boldsymbol{d}_{\underline{a}}, \quad \boldsymbol{e}^a = \boldsymbol{d}^{\underline{a}}, \tag{2.31b}$$

where we defined

$$h \equiv 1 + \widetilde{\kappa}_a y^a. \tag{2.32}$$

Given the relations (2.31b), we will ignore in the rest of this article the difference between indices referring to $\boldsymbol{e}_a$ (resp. $\boldsymbol{e}^a$) and $\boldsymbol{d}_{\underline{a}}$ (resp. $\boldsymbol{d}^{\underline{a}}$) since they are equal and refer to unit vectors in both cases. The distinction is only meaningful for the third components, that is, the components along the tangential direction, since $\boldsymbol{d}_{\underline{3}}$ is normalized whereas $\boldsymbol{e}_3$ is not.

The relation between the FA coordinates canonical basis and the orthonormal basis can also be written in the form

$$\boldsymbol{d}_{\underline{i}} = d^j_{\underline{i}} \boldsymbol{e}_j, \quad \boldsymbol{d}^{\underline{i}} = d^{\underline{i}}_j \boldsymbol{e}^j \tag{2.33}$$

with

$$d^3_{\underline{3}} = h^{-1}, \quad d^a_{\underline{3}} = 0, \quad d^3_{\underline{a}} = 0, \quad d^b_{\underline{a}} = \delta^b_a, \tag{2.34a}$$

$$d^{\underline{3}}_3 = h, \quad d^{\underline{3}}_a = 0, \quad d^{\underline{a}}_3 = 0, \quad d^{\underline{a}}_b = \delta^a_b. \tag{2.34b}$$

The metric and its inverse in the canonical basis of the FA coordinates are simply

$$g_{ij} \equiv \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^2 \end{pmatrix}, \tag{2.35a}$$

$$g^{ij} \equiv \boldsymbol{e}^i \cdot \boldsymbol{e}^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^{-2} \end{pmatrix}, \tag{2.35b}$$

We notice that it looks very much like a metric of cylindrical coordinates. The main difference lies in the fact that h depends on all coordinates (y^1, y^2, s) whereas for cylindrical coordinates h is replaced by the radial coordinate [$h \to r$ in Eq. (2.35)].

Finally, we define the unit directional vector n^a and the unit orthoradial vector $\widetilde{n}^a$ in the section plane by

$$y^a \equiv r n^a, \quad \widetilde{n}^a \equiv -\varepsilon^{ab} n_b, \quad r^2 \equiv y_a y^a. \tag{2.36}$$

Since n^a and $\widetilde{n}^a$ are unit vectors which are mutually orthogonal and projected, then we find the identities

$$P_{\perp}{}^a_b = \delta^a_b = n^a n_b + \widetilde{n}^a \widetilde{n}_b, \tag{2.37}$$

$$\epsilon^{ab} = n^a \widetilde{n}^b - \widetilde{n}^a n^b. \tag{2.38}$$

Using (r, n^a) rather than (y^1, y^2) in the section plane amounts simply to using polar coordinates instead of Cartesian coordinates. With the variables (r, n^a, s), the FA coordinates can be understood as a cylindrical coordinates system that one would have deformed so that the z axis is curved and onto the FCL. The factor h defined in Eq. (2.32) which is larger that unity in the exterior of curvature and smaller than unity in the interior ($\widetilde{\kappa}^a$ points toward the exterior of curvature) is the local amount of stretching which had to be applied to perform this deformation.

F. 2 + 1 decomposition

As mentioned in Sec. II A, any vector can be decomposed into its part along $\boldsymbol{T}$ (its longitudinal part) and a sectional part according to

$$X^\mu = \overline{X} T^\mu + X^\mu_\perp, \quad X^\mu_\perp \equiv P^\mu_{\perp\nu} X^\nu, \quad \overline{X} \equiv \boldsymbol{X} \cdot \boldsymbol{T} \tag{2.39}$$

or in components

$$\overline{X} = X^{\underline{3}}, \quad X^a_\perp = X^a. \tag{2.40}$$

Given this last property, we will omit the $\perp$ subscript when dealing with the components of the projection of a tensor. This decomposition is easily extended to tensors as each index needs to be decomposed into a longitudinal and a sectional part.

From now on, we thus use the notation $\overline{U} = U^{\underline{3}}$ and $\overline{\varpi} = \omega^{\underline{3}}$ for the longitudinal components of the fiber velocity and of the rotation rate.

G. Spatial derivatives in fiber adapted coordinates

The components of the derivative of a vector in the canonical basis of the FA coordinates are given by[1]

$$e_i^\mu e_j^\nu(\partial_\mu v_\nu) = \partial_i v_j - \Gamma^k_{ij} v_k, \tag{2.41a}$$

$$e_i^\mu e^j_\nu(\partial_\mu v^\nu) = \partial_i v^j + \Gamma^j_{ik} v^k, \tag{2.41b}$$

where the Christoffel symbols are defined as

$$\Gamma^j_{ik} \equiv e_i^\mu e^j_\nu \left(\partial_\mu e_k^\nu\right). \tag{2.42}$$

They are related to the components of the metric in the canonical basis of the FA coordinates by

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk}). \tag{2.43}$$

The only nonvanishing Christoffel symbols are

$$\Gamma^3_{33} = \frac{\partial_s h}{h}, \quad \Gamma^3_{a3} = \frac{\widetilde{\kappa}_a}{h}, \quad \Gamma^a_{33} = -h\widetilde{\kappa}_a. \tag{2.44}$$

The derivative of the orthonormal basis is then deduced from (2.41) and the relations (2.31). We find

$$d^{\underline{j}}_\nu\left(\partial_\mu d^\nu_{\underline{i}}\right) = d^{\underline{j}}_\nu e^3_\mu \varepsilon^{\alpha\beta\nu}\kappa_\alpha (d^{\underline{i}})_\beta = e^3_\mu \varepsilon^{a\underline{i}\underline{j}}\kappa_a, \tag{2.45}$$

where from (2.31) we must use $e^3_\mu = T_\mu/h$. Finally, the divergence of a vector has a simple expression in FA coordinates. For a general vector X^μ it reads as

$$\begin{aligned}\partial_\mu X^\mu &= \frac{1}{h}\partial_i(hX^i) = \frac{1}{h}[\partial_a(hX^a) + \partial_s X^{\underline{3}}] \\ &= \frac{1}{h}[\widetilde{\kappa}_a X^a + \partial_s X^{\underline{3}}] + \partial_a X^a.\end{aligned} \tag{2.46}$$

In particular, this allows to obtain the Laplacian of a scalar function S by using $X^\mu = \partial^\mu S$, and we get

$$\Delta S = \partial^a\partial_a S + \frac{\partial_s^2 S}{h^2} - \frac{(\partial_s S) y^a \partial_s \widetilde{\kappa}_a}{h^3} + \frac{\widetilde{\kappa}^a \partial_a S}{h}. \tag{2.47}$$

III. FIELDS EXPANSION ON SECTIONS

With the FA coordinates we have an appropriate method to describe the position of a fluid particle inside the viscous jet. However, in order to describe the dynamics of the viscous fluid inside the fiber, we also need to find an appropriate description for the fluid velocity itself, and this is the goal of this section. Clearly, in the slender approximation the velocity field is necessarily very close to the velocity U^μ on the FCL. It is thus natural to Taylor expand the velocity field around the velocity on the FCL. However, we cannot keep all orders of this expansion and we need general principles to guide us in truncating such expansion. Since the fiber sections are two dimensional, it is natural to classify the various orders of the Taylor expansion according to their transformation property under SO(2), that is, the local group of rotations around the tangential direction. In this section, we give the essential steps to find the irreducible representations of SO(2) and their tensorial expressions. Further details for this method are reported in Appendix A.

[1] For simplicity, we have chosen to use the Cartesian coordinates x^μ in the ambient space. If we were to choose general curvilinear coordinates, e.g., spherical coordinates, then in all our equations we should promote the partial derivative to a covariant derivative and perform the replacement $\partial_\mu \to \nabla_\mu$, $\partial_i \to \nabla_i$ and $\delta^{\mu\nu} \to g^{\mu\nu}$, $\delta_{\mu\nu} \to g_{\mu\nu}$ everywhere. We would also have to use the rule $\partial_s \to e_3^\mu \nabla_\mu$. The indices $\mu,\nu\ldots$ could even be given an abstract meaning and not refer to a particular system of coordinates. This is a standard notation for general relativity [29] in which the background space is unavoidably curved, but for classical physics this extra layer of abstraction is not necessary. We have thus chosen the simplest and most transparent notation based on an ambient set of Cartesian coordinates with its associated partial derivative, but we must bear in mind that the results are more general.

A. Taylor expansion

Any 2-field, that is, a field on the fiber section with only sectional indices, can be Taylor expanded in the variables (y^1, y^2) for each (s,t). For a scalar field, this Taylor expansion is of the form

$$S(y^a,s,t) = \sum_{\ell=0}^{\infty} S_L(s,t) y^L, \tag{3.1}$$

where we use the multi-index notation $L = a_1 \ldots a_\ell$ on which the Einstein summation convention applies. The S_L are necessarily symmetric rank-ℓ tensors since the y^L are symmetric. They are obtained by successive applications of $\partial/\partial y^a$ on S. If there is no index, that is, for $\ell = 0$ in Eq. (3.1), then we use the notation $S_\varnothing$. For a 2-vector the expansion is instead of the form

$$V^a(y^i,t) = \sum_{\ell=0}^{\infty} V^a_L(s,t) y^L, \tag{3.2}$$

and this is obviously extended to higher order tensors. The tensors V^a_L are symmetric in the ℓ indices L but there is no particular symmetry involving the index a.

If we decompose the total fluid velocity V^μ into its longitudinal part $\overline{V}$ and its projected part V^a as in Sec. II F, then the former is a scalar function whereas the latter is a 2-vector and they are Taylor expanded, respectively, as in Eqs. (3.1) and (3.2).

B. Irreducible representations of SO(2)

It proves useful to decompose the y^a dependence in the Taylor expansion of (3.1) by separating the dependence in the radial coordinate r and the direction n^a. The dependence in the direction n^a can be further decomposed onto the irreducible representations (irreps hereafter) of SO(2). This method follows a method long known in three dimensions where the direction vector lies in the two-sphere, and the irreps considered are those of SO(3) [30–33].

The irreps of SO(2) are given by the functions $e^{in\theta}$. More precisely, the irreps are two dimensional as they are represented by $e^{\pm in\theta}$, except for D_0 which is one dimensional and which is just the set of constants. We note these irreps D_n. Any function depending on $n^a = (\cos\theta, \sin\theta)$ is indeed expanded in Fourier series as

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta} = f_0 + \sum_{n=1}^{\infty} f_n e^{in\theta} + f_{-n} e^{-in\theta}. \quad (3.3)$$

The *symmetric trace-free* (STF) 2-tensors are also irreps of the rotation group SO(2) just like STF 3-tensors are irreps of the group SO(3). The corresponding expansion of f reads as simply

$$f(n^i) = \sum_{\ell=0}^{\infty} f_L n^L, \quad (3.4)$$

where this time the f_L are symmetric but also traceless tensors of rank ℓ (we recall the multi-index notation $L = a_1 \dots a_\ell$). The two expansions are related thanks to (for $m > 0$)

$$\mathcal{Y}^m_{a_1\dots a_m} \equiv (d_1 + i d_2)^{a_1} \dots (d_1 + i d_2)^{a_m}, \quad (3.5a)$$

$$\mathcal{Y}^{-m}_{a_1\dots a_m} \equiv (d_1 - i d_2)^{a_1} \dots (d_1 - i d_2)^{a_m}. \quad (3.5b)$$

Indeed, it is easily found that

$$f_L = \sum_{m=\pm|\ell|} \mathcal{Y}^m_L f_m, \quad f_m = \frac{1}{2^\ell} f^L \mathcal{Y}^m_L. \quad (3.6)$$

The STF tensors of rank ℓ have indeed only two degrees of freedom. For instance, for a symmetric rank-4 tensor, the only independent degrees of freedom are f_{1111} and f_{1112} since from the traceless condition the other components are related through $f_{1222} = -f_{1112}$, $f_{2222} = -f_{1122} = f_{1111}$. We are led to decompose all tensors appearing in Taylor expansions in STF tensors so as to obtain a decomposition in terms of irreps.

C. Irreps of scalar functions

For scalar functions in the fiber section plane, the expansion (3.1) is already made in terms of symmetric tensors and we only need to remove the traces. For any symmetric tensor S_L, the traceless part can be extracted as

$$S_{\langle a_1\dots a_\ell\rangle} = \sum_{n=0}^{[\ell/2]} a^n_\ell \delta_{(a_1 a_2} \dots \delta_{a_{2n-1} a_{2n}} S_{a_{2n+1}\dots a_\ell) b_1 \dots b_n}{}^{b_1\dots b_n}, \quad (3.7)$$

$$a^n_\ell \equiv (-1)^n \frac{(2\ell - 2n - 2)!!(2n-1)!!}{(2\ell-2)!!} \binom{\ell}{2n}. \quad (3.8)$$

In the expression above, the symmetric part of a tensor $T_{a_1\dots a_\ell}$ which is not initially symmetric is denoted $T_{(a_1\dots a_\ell)}$, and the STF part is denoted with angle brackets $T_{\langle a_1\dots a_\ell\rangle}$. If the symmetrization ranges on the indices of a product of tensors, this product has to be considered as a single tensor for which the indices are symmetrized. We also use the notation $n!! = n(n-2)(n-4)\dots$.

From the Taylor expansion (3.1), we see that by removing the traces in the tensors S_L, we get factors of $r^2 \equiv y_a y^a$. The expansion in terms of STF tensors only is thus of the form

$$S(y^i, t) = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} S^{(n)}_L(s,t) y^L r^{2n}, \quad (3.9)$$

where the $S^{(n)}_L$ are STF. The dependence in the direction and the radial coordinates have now been clearly separated. The directional dependence is decomposed onto STF tensors, and the dependence in the radial distance is an even polynomial as it is a polynomial in r^2. If no ambiguity can arise, we can omit the sums over ℓ, n so as to alleviate the notation. The STF multipoles of the decomposition (3.9) can be obtained from angular integrals as explained in Appendix A.

Finally, in the case $\ell = 0$, that is, for the monopole of the directional dependence, the coefficients are noted $S^{(n)}_\varnothing$. For instance, if we consider the longitudinal part of the velocity field $\overline{V}$, then $\overline{V}^{(0)}_\varnothing$ corresponds to a uniform flow, and $\overline{V}^{(1)}_\varnothing$ corresponds to a parabolic velocity profile, also known as a Hagen-Poiseuille (HP) flow.

D. Irreps of 2-vector fields

It is shown in Appendix A 2 that the expansion in irreps of a 2-vector is necessarily of the form

$$V_a = \widehat{V}^{(n)}_{aL} y^L r^{2n} + \frac{y_a}{2} \widehat{V}^{(n)}_L y^L r^{2n} - \varepsilon_{ab} y^b \mathring{V}^{(n)} r^{2n}, \quad (3.10)$$

where the $\widehat{V}^{(n)}_L$ and $\widehat{V}^{(n)}_L$ are STF tensors and where we recall that there is an implicit sum on ℓ and n. At first sight, this form is very cumbersome if we are familiar with the irreps of 3-vectors [that is, irreps of SO(3)]. Indeed for 3-vectors, the directional dependence can always be expanded in terms of electric type and magnetic type multipoles [31,34]. The result for 2-vectors is necessarily very different because the antisymmetric tensor has only two indices in two dimensions, hence the expressions from the three-dimensional results, which also involve the antisymmetric tensor in three dimensions, cannot be exported directly to two dimensions.

Note also that the tensors $\widehat{V}^{(n)}_L$ have no monopole since they must have at least one index. Conversely, the functions $\mathring{V}^{(n)}$ are purely monopolar since they do not have any tensorial index. So, we can interpret (3.10) by saying that we have also two sets of STF tensors. The first set is made of the $\widehat{V}^{(n)}_L$, $\ell \geqslant 0$, and the second set consists in the $\mathring{V}^{(n)}$ and the $\widehat{V}^{(n)}_L$, $\ell \geqslant 1$. Physically, $\mathring{V}^{(0)}$ corresponds to a solid rotation around the fiber tangential axis (axial rotation hereafter) and $\widehat{V}^{(0)}_\varnothing$ corresponds to a radial infall of the fluid.

Finally, note that we could also decompose a vector field by projecting its free index so as to obtain a scalar function. For instance, we can select the radial and orthoradial components by projecting along n^a and $\tilde{n}^a$. Then, each scalar function can be expanded as in (3.9). The relation between the two methods is (with an implied sum on ℓ and n)

$$r V_{\underline{r}} \equiv y^a V_a = \widehat{V}^{(n)}_L y^L r^{2n} + \tfrac{1}{2} \widehat{V}^{(n)}_L y^L r^{2(n+1)}, \quad (3.11a)$$

$$r V_{\underline{\theta}} \equiv \tilde{y}^b V_b = \varepsilon^b_{a_1} \widehat{V}^{(n)}_{a_{L-1} b} y^{a_L} r^{2n} + \mathring{V}^{(n)} r^{2(n+1)}. \quad (3.11b)$$

This points to a simple method to extract the STF components of a vector. We first extract the STF components of its radial and orthoradial projections as for scalar functions (see Appendix A) and then deduce the STF multipoles of the decomposition (3.10) from the relations

$$[rV_{\underline{r}}]^{(n)}_L = \widehat{V}^{(n)}_L + \tfrac{1}{2}\widehat{V}^{(n-1)}_L, \tag{3.12a}$$

$$[rV_{\underline{\theta}}]^{(n)}_L = \varepsilon^b_{\langle a_1}\widehat{V}^{(n)}_{a_{L-1}\rangle b} + \delta^0_\ell \mathring{V}^{(n-1)}, \tag{3.12b}$$

which are inverted as

$$\mathring{V}^{(n)} = [rV_{\underline{\theta}}]^{(n+1)}_{\varnothing}, \tag{3.13a}$$

$$\widehat{V}^{(n)}_L = -\varepsilon^b_{\langle a_1}[rV_{\underline{\theta}}]^{(n)}_{a_{L-1}\rangle b}, \tag{3.13b}$$

$$\widehat{V}^{(n)}_L = -2\widehat{V}^{(n+1)}_L + 2[rV_{\underline{r}}]^{(n+1)}_L. \tag{3.13c}$$

As a final comment on STF tensors, we must stress that these are much more adapted to abstract tensor manipulation than the usual representation (3.3), and we chose to perform all the tensor manipulations of this article with xAct [35].

IV. KINEMATICS

We are now equipped with all the necessary formalism to study in details the kinematics and dynamics of viscous fibers. In this section we present all the relations needed for the kinematics, and in the next section we shall focus on the description of dynamics, so as to obtain the one-dimensional reduction of viscous fibers from its intrinsic physical laws.

A. Velocity parametrization

We separate the total fluid velocity $\mathcal{V}^\mu$ into the velocity of the FCL (U^μ) and the small difference V^μ as

$$\mathcal{V}^\mu = U^\mu + V^\mu. \tag{4.1}$$

$\boldsymbol{V}$ is decomposed into a longitudinal part $\overline{V}$ and a sectional part V^a which are decomposed as in Eqs. (3.9) and (3.10). In order to avoid cluttering of indices when the decomposition (3.10) is used for the velocity field, we shall use the short notation

$$v^{(n)}_L \equiv \overline{V}^{(n)}_L, \quad v^{(n)} \equiv v^{(n)}_{\varnothing}, \quad v \equiv v^{(0)}_{\varnothing}, \tag{4.2a}$$

$$\dot{\phi}^{(n)} \equiv \mathring{V}^{(n)}, \quad \dot{\phi} \equiv \mathring{V}^{(0)}, \tag{4.2b}$$

$$u^{(n)}_L \equiv \widehat{V}^{(n)}_L, \quad u^{(n)} \equiv u^{(n)}_{\varnothing}, \quad u \equiv u^{(0)}_{\varnothing}. \tag{4.2c}$$

B. Incompressibility

The incompressibility translates into a condition on the velocity field as it implies that its divergence vanishes. This incompressibility condition is a scalar equation

$${}^C\mathcal{E} \equiv [\partial_\mu \mathcal{V}^\mu = 0], \tag{4.3}$$

which is a constraint for the velocity field. Since (2.6) or (2.27) imply that the fiber central line velocity U^μ is also divergenceless ($\partial_\mu U^\mu = 0$), we deduce that $\partial_\mu V^\mu = 0$ and from (2.46) it reads as, in terms of the FA coordinates,

$$h\partial_a V^a + \widetilde{\kappa}_a V^a + \partial_s \overline{V} = 0. \tag{4.4}$$

As this constraint is scalar, it can be expanded into irreps just like (3.9) and each STF tensor of this expansion must vanish identically. Using the property (A13), we get

$$\begin{aligned} 0 = [{}^C\mathcal{E}]^{(n)}_L &= \partial_s v^{(n)}_L + 2(n+1)\widehat{V}^{(n+1)}_L + \delta^1_\ell \kappa_a \dot{\phi}^{(n)} \\ &+ 2(n+1)\widehat{V}^{(n+1)}_{\langle L-1}\kappa_{a_\ell\rangle} + (n+1)\widetilde{\kappa}^a \widehat{V}^{(n)}_{aL} \\ &+ \left(\frac{\ell}{2}+n+1\right)\left(\frac{1}{2}\widetilde{\kappa}^a u^{(n-1)}_{aL} + u^{(n)}_L + u^{(n)}_{\langle L-1}\widetilde{\kappa}_{a_\ell\rangle}\right). \end{aligned} \tag{4.5}$$

In general, the tensors $u^{(n)}_L$ can always be expressed in terms of the other tensors using the incompressibility conditions (4.5). In particular, from $[{}^C\mathcal{E}]^{(0)}_{\varnothing}$ we get

$$u^{(0)}_{\varnothing} = -\widehat{V}^{(0)a}\widetilde{\kappa}_a - \partial_s v. \tag{4.6}$$

This relation has a simple physical interpretation. Indeed, the right-hand side is (minus) the stretching rate of the velocity field on the central line $\boldsymbol{T}\cdot\partial_s\boldsymbol{V}$ given that on the central line the velocity is just $\boldsymbol{V} = \widehat{V}^{(0)a}\boldsymbol{d}_a + v^{(0)}\boldsymbol{T}$. It thus states that a radial infall necessarily appears when the fiber is stretching, so as to ensure volume conservation.

C. Coordinate velocity

Initially, the total velocity $\mathcal{V}^\mu$ is defined as the rate of coordinate change, that is, as dx^μ/dt for the fluid elementary particles. However, if we take the components of the velocity in the basis associated with FA coordinates, that is the $\mathcal{V}^i$, we do not get the rate of change of the FA coordinates dy^i/dt. We thus need to infer the nontrivial relation between $\mathcal{V}^i$ and dy^i/dt. For this we define the speed of the coincident point as

$$\mathcal{V}^\mu_C \equiv \left.\frac{\partial x^\mu}{\partial t}\right|_{y^i}, \quad \mathcal{V}^i_C \equiv -\left.\frac{\partial y^i}{\partial t}\right|_{x^\mu} = e^i_\mu \mathcal{V}^\mu_C, \tag{4.7}$$

that we gave in both the Cartesian and the FA system of coordinates (see Appendix C for details). The speed of the coincident point is the speed of a point that would have constant FA coordinates y^1, y^2, s. The total velocity is related by

$$\mathcal{V}^\mu = \frac{dx^\mu}{dt} = \frac{\partial x^\mu}{\partial y^i}\frac{dy^i}{dt} + \mathcal{V}^\mu_C, \tag{4.8}$$

that is, in FA coordinates by

$$\mathcal{V}^i = \mathcal{V}^i_R + \mathcal{V}^i_C, \quad \mathcal{V}^i_R \equiv \frac{dy^i}{dt}. \tag{4.9}$$

$\mathcal{V}^i_R$ is thus the coordinate velocity for the FA coordinates since it equals the rate of change of FA coordinates. From the above definition of $\mathcal{V}^\mu_C$ and using the parametrization (2.29) together with the property of the rotation (2.16), we get the expressions

$$\begin{aligned} \mathcal{V}^\mu_C &= U^\mu + [\omega \times (y^a d_a)]^\mu \\ &= U^\mu + (\widetilde{\omega}_a y^a)T^\mu + \overline{\omega}\widetilde{y}^a d^\mu_a, \end{aligned} \tag{4.10a}$$

$$\begin{aligned} \mathcal{V}^\mu_R &= V^\mu - [\omega \times (y^a d_a)]^\mu \\ &= V^\mu - (\widetilde{\omega}_a y^a)T^\mu - \overline{\omega}\widetilde{y}^a d^\mu_a. \end{aligned} \tag{4.10b}$$

On the expression (4.10a) it appears that the coincident point velocity is the velocity of the FCL, on which is added the solid rotation of the system of FA coordinates which does not vanish when the point considered is not lying exactly on the FCL. Part of the rotation is due to the projected part of the rotation rate ω_a and corresponds to the rotation of section planes, and the rest of the rotation is due to the longitudinal part of rotation $\overline{\omega}$ and corresponds to a rotation of the basis vectors d_1 and d_2 around the fiber tangential direction, that is, to a rotation inside the section plane itself.

D. Section shape description

The section shape can be characterized by its radius as a function of the direction in the section, that is, by its curve in polar coordinates

$$\mathcal{R}(s,t,n^a). \tag{4.11}$$

As any function depending on the direction inside the fiber section, it can be decomposed in STF multipoles $\mathcal{R}_L$:

$$\mathcal{R}(s,t,n^a) = R(s,t)\left(1 + \sum_{\ell=1}^{\infty} \mathcal{R}_L(s,t)R^\ell n^L\right). \tag{4.12}$$

We call $R(s,t)$ the radius of the fiber and the lowest multipole $\mathcal{R}_{ab}$ can be interpreted as an elliptic elongation of the fiber section shape. Instead of working with the multipoles $\mathcal{R}_L$ which have a dimension $1/L^\ell$, we can define dimensionless STF moments as

$$\widehat{\mathcal{R}}_L \equiv \mathcal{R}_L R^\ell. \tag{4.13}$$

There exist alternate ways for describing the shape of the fiber section, and we give an example of another method in Appendix B and relate it to the description (4.12).

E. Normal vector

For a function depending on the direction $f(n^a)$, we can define an orthoradial derivative by

$$\frac{D}{Dn^a} f(n^c) \equiv \perp_a^b \frac{\partial f(y^c)}{\partial y^b}\bigg|_{y^c=n^c}, \tag{4.14}$$

where the orthoradial projector is defined by

$$\perp_b^a \equiv \delta_b^a - n^a n_b. \tag{4.15}$$

This projector satisfies $\perp_b^a n^b = 0$, $\perp_b^a \widetilde{n}^b = \widetilde{n}^a$, and $\perp_a^a = 1$. Note that it can also be written as

$$\perp_a^b = \widetilde{n}_a \widetilde{n}^b = r\partial_b n^a. \tag{4.16}$$

When applied on the fiber radius, this yields

$$\frac{D\mathcal{R}}{Dn^a} = r\frac{\partial \mathcal{R}}{\partial y^a} = R\sum_{\ell=1}^{\infty}(\ell+1)\perp_a^b \widehat{\mathcal{R}}_{bL} n^L \tag{4.17}$$

$$= R\sum_{\ell=1}^{\infty}(\ell+1)\widetilde{n}_a(\widehat{\mathcal{R}}_{bL}\widetilde{n}^b n^L). \tag{4.18}$$

A normal covector (not necessarily unity) to the boundary surface is $\boldsymbol{N} = N_\mu dx^\mu$ with

$$N_\mu = \partial_\mu \Phi|_{\Phi=0}, \tag{4.19}$$

where the boundary function Φ is defined as

$$\Phi(s,t,y^a) = r - \mathcal{R}(s,t,n^a) = \sqrt{y^a y_a} - \mathcal{R}(s,t,n^a). \tag{4.20}$$

The components of the normal vector are in our case

$$N_a = n_a - \frac{1}{\mathcal{R}}\frac{D\mathcal{R}}{Dn^a} \tag{4.21a}$$

$$= n_a - \widetilde{n}_a \frac{(\ell+1)\widehat{\mathcal{R}}_{bL}\widetilde{n}^b n^L}{1+\widehat{\mathcal{R}}_L n^L}, \tag{4.21b}$$

$$N_3 = -\partial_s \mathcal{R}, \quad N_{\underline{3}} = \frac{1}{h}N_3, \tag{4.21c}$$

where sums over ℓ are implied. This vector can be normalized and the unit normal vector is just

$$\widehat{N}^\mu \equiv \frac{N^\mu}{\sqrt{N_a N^a + (N_{\underline{3}})^2}}. \tag{4.22}$$

F. Scalar extrinsic curvature

In order to use Young-Laplace law for surface tension, we need the general expression for the scalar part of the extrinsic curvature of the fiber boundary, which by definition is the divergence of the unit normal vector. Using (2.46), it is thus obtained as

$$\mathcal{K} \equiv \partial_\mu \widehat{N}^\mu = \frac{\widetilde{\kappa}_a \widehat{N}^a}{h} + \partial_a \widehat{N}^a + \frac{\partial_s \widehat{N}^{\underline{3}}}{h}. \tag{4.23}$$

G. Boundary kinematics

Since the boundary must follow the velocity field, the constraint $\Phi = 0$ must propagate with the velocity, that is,

$$\left[\frac{d\Phi}{dt}\right]_{\Phi=0} = \left[\frac{\partial\Phi}{\partial t} + \frac{dy^i}{dt}\frac{\partial\Phi}{\partial y^i}\right]_{\Phi=0} = 0. \tag{4.24}$$

Since $\partial_s r = \partial_t r = 0$, then $\partial_t \Phi = -\partial_t \mathcal{R}$ and given the definition (4.19) for the nonunit normal vector and the definition (4.9) of the coordinate velocity, this constraint is simply rewritten as

$$^{\mathcal{R}}\mathcal{E} \equiv \left[\partial_t \mathcal{R} = \mathcal{V}_R^i N_i = \mathcal{V}_R^3 N_3 + \mathcal{V}_R^a N_a\right]_{\Phi=0}. \tag{4.25}$$

We must be careful with the fact that it is $\mathcal{V}_R^3 = ds/dt$ which appears and not $\mathcal{V}_R^{\underline{3}}$, and we must thus use $\mathcal{V}_R^3 = d_{\underline{3}}^3 \mathcal{V}_R^{\underline{3}} = h^{-1}\mathcal{V}_R^{\underline{3}}$ to relate them.

Despite its apparent simplicity, this equation is actually rather complicated. First, we stress that it is the coordinate velocity which appears since it is the velocity which gives the rate of change for FA coordinates. But, more importantly, all quantities must be evaluated on the fiber side, meaning that every occurrence of y^a must be replaced by $\mathcal{R}n^a$ where $\mathcal{R}$ itself has a directional dependence given by (4.12). This equation is thus in general extremely nonlinear in the STF multipoles $\mathcal{R}_L$. We are forced to realize that it is hopeless to solve the general problem of curved viscous jets without major simplifications, which leads to consider a perturbative scheme on which we should perform a consistent truncation.

H. Slenderness perturbative expansion

If the typical length for variations in the velocity field is L and is much larger than the radius R, then the approximation that the body considered is elongated holds and we can hope to find a coherent one-dimensional reduction. The small parameter of our perturbative expansion is thus

$$\epsilon_R \equiv R/L. \tag{4.26}$$

For instance, the moment $v^{(1)}$ comes originally from a Taylor expansion of the velocity, as can be checked from the dimension $[v^{(1)}] = [v^{(0)}]/L^2$. A term like $v^{(1)}R^2$ is thus of order ϵ_R^2 compared to the lowest order velocity $v^{(0)}$, meaning that the former is really a correction to the latter. In general, higher order multipoles correspond to higher orders in ϵ_R because they primarily come from gradients of the velocity fields which bring inverse powers of L. As an example, $v^{(0)}_{ab}y^a y^b$ is of order ϵ_R^2 compared to $v = v^{(0)}_{\varnothing}$.

When including the effect of curvature, we also assume that the scale $1/|\boldsymbol{\kappa}|$ (the curvature radius) is also of the order of L at most. Indeed, if there is curvature, the velocity flow must adapt on scales which are commensurate with the curvature radius. As a consequence, terms of the type $\kappa_a y^a$ must also be of order ϵ_R. In any case, it would be impossible to consider sections which have a section radius larger than the fiber curvature radius. Indeed, in that case fiber sections would intersect, hence, $|\boldsymbol{\kappa}|R \propto \epsilon_R$ must be small to obtain a satisfactory one-dimensional approximation.

I. Gauge fixing

Since there are three degrees of freedom in the position of the FCL inside the viscous jet, we can fix two of these by asking that there is no dipole in $\mathcal{R}$. This corresponds to the intuitive requirement that the fiber central line should be *in the middle* of the fiber section. Formally, this means that we fix the gauge by setting

$$\mathcal{R}_a = 0, \tag{4.27}$$

and it leads to a constraint equation when considering the dipole of (4.25).

The gauge restriction (4.27) is an order ϵ_R^2 expression as it is automatically satisfied at lowest order. It reads as indeed

$$\widehat{V}^{(0)}_a = \tfrac{1}{6}R^2\big(-2\widehat{V}^{(1)}_a + 6\mathcal{H}v^{(0)}_a - 6\mathcal{H}v\widetilde{\kappa}_a + 2\kappa_a\dot{\phi} - 6\mathcal{H}\widetilde{\omega}_a - 3\widetilde{\kappa}_a\partial_s v + 2\partial_s v^{(0)}_a\big) + O\big(\epsilon_R^4\big), \tag{4.28}$$

where we have defined the stretching factor

$$\mathcal{H} \equiv \partial_s \ln \mathcal{R}. \tag{4.29}$$

The gauge restriction fixes the global velocity shift with respect to the fiber central line velocity U^a which is encoded by the velocity moment $\widehat{V}^{(0)}_a$. And, since this shift is of order ϵ_R^2, we can state that the projected component of the velocity on the central line is nearly the projected velocity of the central line itself, that is, $[\mathcal{V}^a \simeq U^a]_{y^1=y^2=0}$.

With (4.27), we have fixed only two of the three gauge degrees of freedom which arise from the fact that we are free to choose any curve as the FCL. The third degree of freedom corresponds to the possible reparametrization of the fiber inside the same curve, that is, to the replacement $s \to s + f(s,t)$. Given the choice of normalization for the tangent vector in Eq. (2.2), this freedom is only a global but time-dependent reparametrization freedom $s \to s + f(t)$. For every problem considered, there is a natural way to fix unambiguously the affine parameter s, for instance, setting it to $s = 0$ at the boundary.

J. Shape restriction

With the gauge choice of the previous section, we have managed to cancel the shape dipole. However, we cannot assume that in general higher order shape multipoles vanish. Indeed, if we have curvature, then terms of the type $\propto R^2\kappa_{\langle a}\kappa_{b\rangle}$ which are of order ϵ_R^2 would source the section shape quadrupole $\mathcal{R}_{ab}$. Typically, we expect to find that shape terms like $\mathcal{R}_L y^L$ or $R^\ell \mathcal{R}_L$ are of order ϵ_R^ℓ. In Sec. VII, we discuss this scaling and show that circular sections are compatible with the string description of curved fibers, which is the lowest order description in which order ϵ_R^2 effects are ignored.

Of course, if we restrict to axisymmetric jets as we shall do in Sec. VI, then $\kappa_a = 0$ and circular sections are consistent throughout even though one might still want to consider straight jets with noncircular sections.

K. Velocity shear rate

In order to describe the dynamics of the viscous fluid inside the fiber, we will need to consider the gradient of the velocity field. Let us define the nonsymmetric tensor

$$S_{\mu\nu} \equiv \partial_\mu \mathcal{V}_\nu. \tag{4.30}$$

Its components are given by

$$S_{ab} = \partial_a \mathcal{V}_b, \tag{4.31a}$$

$$S_{a\underline{3}} = \partial_a \mathcal{V}_{\underline{3}}, \tag{4.31b}$$

$$hS_{\underline{3}a} = \partial_s \mathcal{V}_a - \widetilde{\kappa}_a \mathcal{V}_{\underline{3}}, \tag{4.31c}$$

$$hS_{\underline{33}} = \partial_s \mathcal{V}_{\underline{3}} + \widetilde{\kappa}^a \mathcal{V}_a. \tag{4.31d}$$

These components are obtained either from the components in the canonical basis of the FA coordinates, that is, the S_{ab}, S_{3a}, S_{a3}, and S_{33} which we compute from (2.41), that we then project on the orthonormal basis, or using directly (2.45) in the velocity decomposition $\boldsymbol{\mathcal{V}} = \mathcal{V}^{\underline{i}}\boldsymbol{d}_{\underline{i}}$. This tensor can be decomposed as

$$S_{\mu\nu} = \tfrac{1}{2}\sigma_{\mu\nu} + \omega_{\mu\nu}, \tag{4.32}$$

where we used that the velocity shear rate is (twice) the symmetric part

$$\sigma_{\mu\nu} \equiv 2S_{(\mu\nu)} = S_{\mu\nu} + S_{\nu\mu}, \tag{4.33}$$

and the vorticity is the antisymmetric part

$$\varpi_{\mu\nu} \equiv S_{[\mu\nu]} = \tfrac{1}{2}(S_{\mu\nu} - S_{\nu\mu}). \tag{4.34}$$

We also define the vorticity (Hodge) dual vector by

$$\varpi_\alpha \equiv \tfrac{1}{2}\varepsilon_{\alpha\mu\nu}\varpi^{\mu\nu} \Rightarrow \varpi_{\mu\nu} = \varepsilon_{\mu\nu\alpha}\varpi^\alpha. \tag{4.35}$$

From (4.31), we find that the components of the shear in the orthonormal basis are then given by

$$\sigma_{ab} = \partial_a \mathcal{V}_b + \partial_b \mathcal{V}_a, \tag{4.36a}$$

$$h\sigma_{a\underline{3}} = h\partial_a \mathcal{V}_{\underline{3}} + \partial_s \mathcal{V}_a - \widetilde{\kappa}_a \mathcal{V}_{\underline{3}}, \tag{4.36b}$$

$$h\sigma_{\underline{33}} = 2(\partial_s \mathcal{V}_{\underline{3}} + \widetilde{\kappa}^a \mathcal{V}_a). \tag{4.36c}$$

Similarly, the components of the vorticity vector are

$$\varpi^a = \tfrac{1}{2}\varepsilon^{ab}(S_{b\underline{3}} - S_{\underline{3}b}) = \frac{1}{2}\varepsilon^{ab}\left[\partial_b \mathcal{V}_{\underline{3}} - \frac{1}{h}(\partial_s \mathcal{V}_b - \widetilde{\kappa}_b \mathcal{V}_{\underline{3}})\right] \tag{4.37a}$$

$$\varpi^{\underline{3}} = \frac{1}{2}\varepsilon^{ab}\partial_{[a}\mathcal{V}_{b]} = \frac{1}{4}\varepsilon^{ab}(\partial_a \mathcal{V}_b - \partial_b \mathcal{V}_a)\,. \tag{4.37b}$$

V. DYNAMICS

The dynamics of viscous fluids is well known and arises from the Navier-Stokes equation. However, in order to achieve a one-dimensional reduction for viscous jets, we must find a way to get rid of the physics on the fiber boundary. Enforcing the boundary conditions on the stress tensor on the fiber boundary leads to a set of three constraints which can be conveniently used to reduce the number of free fields in our one-dimensional reduction. This section is dedicated to the general construction of this method and we then apply it in the subsequent sections for axisymmetric and curved fibers.

A. Total stress tensor and viscous forces

The total stress tensor is decomposed as

$$\tau_{\mu\nu} = \tau^{(P)}_{\mu\nu} + \tau^{(\mu)}_{\mu\nu}, \tag{5.1}$$

where the two components arise from pressure forces and shear viscosity. For a Newtonian fluid, they are simply given by

$$\tau^{(P)}_{\mu\nu} = -P g_{\mu\nu}, \quad \tau^{(\mu)}_{\mu\nu} = \mu\sigma_{\mu\nu}. \tag{5.2}$$

The pressure is a scalar and it is thus decomposed as in Eq. (3.9):

$$P = \sum_{\ell=0}^{\infty}\sum_{n=0}^{\infty} P_L^{(n)} y^L r^{2n}, \tag{5.3}$$

where the $P_L^{(n)}$ are STF tensors.

From the total stress tensor (5.1), we can define a viscous force per unit area on the fiber sections

$$\mathcal{F}_\mu \equiv \tau_{\mu\underline{3}}. \tag{5.4}$$

As any vector it has a longitudinal part $\overline{\mathcal{F}}$ which corresponds to viscous traction or compression on fiber sections, and a sectional part $\mathcal{F}^a$.

B. Boundary conditions

The boundary condition for the stress tensor is the vector constraint

$$\mathcal{C}_\mu \equiv [\tau_{\mu\nu}\widehat{N}^\nu + \nu\mathcal{K}\widehat{N}_\mu = 0] \equiv \left[\tau^{(\mu)}_{\mu\nu}\widehat{N}^\nu + \nu\mathcal{K}\widehat{N}_\mu - P\widehat{N}_\mu = 0\right]. \tag{5.5}$$

As any vector, it can be decomposed into its longitudinal component $\overline{\mathcal{C}}$ and its sectional component $\mathcal{C}^a$. The latter can be further decomposed into a radial contribution and an orthoradial contribution as

$$^{r}\mathcal{C} \equiv \mathcal{C}_a n^a, \quad {}^{\theta}\mathcal{C} \equiv \mathcal{C}_a \widetilde{n}^a. \tag{5.6}$$

However, these are not fields on the fiber sections since they are defined only at the boundary. They depend on the position on the FCL s, on time t, but their sectional dependence is only a dependence in the direction vector n^a. They also depend on the fiber radius R and on the shape multipoles R_L. So, in general they can be expanded in STF components as

$$\overline{\mathcal{C}}(s,t,n^a) = \sum_{\ell=0}^{\infty}\overline{\mathcal{C}}_L(s,t,R)R^\ell n^L \tag{5.7a}$$

$$= \sum_{\ell=0}^{\infty}\sum_{n}\overline{\mathcal{C}}_L^{(n)}(s,t)R^{2n+\ell}n^L, \tag{5.7b}$$

where in the second line we have also expanded the dependence of the STF multipoles in powers of R, and with similar expansions for the radial and orthoradial boundary constraints $^{\theta}\mathcal{C}$ and $^{r}\mathcal{C}$. To be precise, for the latter, the expansion takes the form (5.7) for $^{r}\mathcal{C}/R$. Note again the difference with (3.9) as the powers of r are replaced by powers of R because constraints are only defined on the boundary. In practice, it is simpler to consider the constraint (5.5) with the non-normalized normal vector N^μ instead of $\widehat{N}^\mu$ since they are equivalent.

We realize that the total sum involves all orders of the form ϵ_R^{2m}. Since we are eventually interested in results which are valid up to a given order in ϵ_R, we define the moments of the constraints *up to a given order* by

$$\overline{\mathcal{C}}_L^{(\leqslant n)} = \sum_{m=0}^{m=n}\overline{\mathcal{C}}_L^{(m)}R^{2m}, \tag{5.8}$$

with similar definitions for the radial and orthoradial constraints.

As we shall detail in two examples in Secs. VI and VII, we will deduce general relations from the vector constraint (5.5) or, more precisely, from its three scalar components (longitudinal, radial, and orthoradial).

C. Volumic forces

Once the stress tensor is computed, it is straightforward to get the volumic forces f^μ since they are expressed as

$$f^\mu = \partial_\nu\tau^{\nu\mu} + g^\mu = \mu\Delta\mathcal{V}^\mu - \partial^\mu P + g^\mu, \tag{5.9}$$

where g^μ are long range volumic forces such as gravity. Its components are related to the components of the stress tensor thanks to

$$f^{\underline{3}} = \partial_a\tau^{a\underline{3}} + \frac{1}{h}(\partial_s\tau^{\underline{33}} + 2\widetilde{\kappa}_a\tau^{a\underline{3}}) + g^{\underline{3}},$$

$$f^a = \partial_b\tau^{ba} + \frac{1}{h}(\partial_s\tau^{\underline{3}a} + \widetilde{\kappa}_b\tau^{ba} - \widetilde{\kappa}^a\tau^{\underline{33}}) + g^a. \tag{5.10}$$

D. Navier-Stokes equation

The dynamics of a viscous fluid is governed by the conservation equation and the Navier-Stokes equation. The conservation equation has already been used since from incompressibility it implied the divergenceless condition (4.3). Having developed all the tools to express the volumic forces, we are now in position to write the Navier-Stokes equation. It is of the form

$$\mathcal{D}^\mu \equiv [A^\mu = f^\mu], \tag{5.11}$$

where the acceleration vector is

$$A_\mu \equiv \partial_t \mathcal{V}_\mu|_{y^i} + \mathcal{V}_R^\nu S_{\nu\mu} + G_\mu. \tag{5.12}$$

If the dynamics is considered in a (constantly) rotating frame, the fictitious or geometrized forces (inertial and Coriolis forces) gathered in G^μ are expressed as

$$G^\mu \equiv 2\varepsilon^{\mu\alpha\beta}\Omega_\alpha \mathcal{V}_\beta + (\Omega_\nu x^\nu)\Omega^\mu - \Omega^2 x^\mu, \tag{5.13}$$

where Ω^μ is the rotation of the frame with respect to a Galilean (inertial) frame.

Just as any vector, the Navier-Stokes equation (5.11) is decomposed into a longitudinal part $\overline{\mathcal{D}}$ and a sectional part $\mathcal{D}^a$ which we can further decompose in irreps as in Sec. III D. The components of the first term of (5.12) are simply obtained from

$$(\partial_t \mathcal{V})^{\underline{3}} \equiv d^{\underline{3}}_\mu \partial_t \mathcal{V}^\mu = \partial_t \overline{\mathcal{V}} + \widetilde{\omega}_a \mathcal{V}^a, \tag{5.14a}$$

$$(\partial_t \mathcal{V})^a \equiv d^a_\mu \partial_t \mathcal{V}^\mu = \partial_t \mathcal{V}^a - \overline{\mathcal{V}}\widetilde{\omega}^a + \widetilde{\mathcal{V}}^a \overline{\omega}, \tag{5.14b}$$

where we recall the notation $\overline{\mathcal{V}} \equiv \mathcal{V}^{\underline{3}}$ for the longitudinal component of the velocity.

E. Secondary incompressibility constraint

The incompressibility implies the divergenceless condition (4.3), and as such it can be considered as a primary constraint. Then, by ensuring that this constraint remains satisfied when time evolves, we obtain a secondary constraint. Indeed, taking the divergence of the Navier-Stokes equation (5.11) we obtain an incompressibility *constraint*

$$\partial_\mu A^\mu = \partial_\mu f^\mu = -\Delta P, \tag{5.15}$$

where we have assumed that the long range forces are constant (e.g., for gravity) or have at least no divergence ($\partial_\mu g^\mu = 0$). If the long range forces have a divergence, then this should be included in Eq. (5.15).

From the expression (5.12) of the acceleration, this constraint is

$${}^D\mathcal{C} \equiv [S_{\mu\nu}S^{\nu\mu} + 2\varepsilon^{\alpha\beta\mu}\Omega_\beta S_{\alpha\mu} - 2\Omega^2 = -\Delta P]. \tag{5.16}$$

Using the decomposition (4.32) of the velocity gradient tensor into velocity shear rate and vorticity, it can be recast nicely as

$$\tfrac{1}{4}\sigma_{\mu\nu}\sigma^{\nu\mu} - 2(\Omega_\mu + \varpi_\mu)(\Omega^\mu + \varpi^\mu) = -\Delta P. \tag{5.17}$$

The right-hand side of this equation can be computed by using (2.47). We will use this secondary incompressibility constraint to get constraints on the various moment of the pressure field.

F. Dimensionless reduction

So far, all physical quantities have a physical dimension. It is, however, possible to build dimensionless quantities. Usually for viscous fluids, this is done by noting that the dimensions of viscosity and surface tension [recalling that we have divided them by the mass density in Eq. (1.1)] are

$$[\mu] = L^2T^{-1}, \quad [\nu] = L^3T^{-2}. \tag{5.18}$$

It is thus possible to define a length scale μ^2/ν and a timescale μ^3/ν^2 from which we can define dimensionless quantities for all physical quantities in the problem at hand. However, we want to allow for the possibility of having no surface tension, and we do not wish to use ν to define dimensionless quantities. Instead, we decide that there is a natural length scale L in our problem which corresponds to the typical length of velocity variations. The timescale is then obtained from L^2/μ. The main quantities in the problem are simply adimensionalized using these length scales and timescales. For instance, we define dimensionless quantities with

$$\begin{aligned} &t = \frac{L^2}{\mu}\underline{t}, \quad s = L\,\underline{s}, \quad v = \frac{\mu}{L}\,\underline{v}, \quad g = \frac{\mu^2}{L^3}\underline{g},\\ &\dot{\phi} = \frac{\mu}{L^2}\,\underline{\dot{\phi}}, \quad R = L\underline{R}, \quad \nu = \frac{\mu^2}{L}\underline{\nu}, \\ &\kappa^a = \frac{1}{L}\underline{\kappa}^a, \quad \omega^a = \frac{\mu}{L^2}\,\underline{\omega}^a, \end{aligned} \tag{5.19}$$

where we recall the notation $v = v^{(0)}_\varnothing$ and $\dot{\phi} = \dot{\phi}^{(0)}_\varnothing$. For higher order multipoles, this construction of dimensionless variables is straightforwardly performed as the dimension of the multipoles is read from the expansion from which it is defined. For instance, from (3.9) we get $v^{(n)}_L = \mu/L^{1+\ell+2n}\underline{v}^{(n)}_L$. Note also that by construction $\epsilon_R = \underline{R}$ so that in practice it is by expanding in powers of $\underline{R}$ that we identify for any expression the various orders in powers of ϵ_R.

The dimensionless ratios of fluid mechanics which are relevant for viscous jets in a rotating frame are the Reynolds number, the Froude number, the Rossby number, and the Weber number. If we consider a typical reference reduced velocity $\underline{v}^{\mathrm{r}}$, then they are simply expressed as

$$\mathrm{Re} = \underline{v}^{\mathrm{r}}, \quad \mathrm{Fr}^2 = \frac{(\underline{v}^{\mathrm{r}})^2}{\underline{g}}, \quad \mathrm{Rb} = \frac{\underline{v}^{\mathrm{r}}}{\underline{\Omega}}, \quad \mathrm{We} = \frac{(\underline{v}^{\mathrm{r}})^2}{\underline{\nu}}. \tag{5.20}$$

In the remainder of this article, we use the dimensionless physical quantities rather than the dimensionless numbers, but using the dictionary (5.20), all expressions can be recast with these dimensionless numbers.

In the next section, we present two main applications of our formalism and we shall assume that we have performed such a dimensionless reduction for all quantities. In order to avoid cluttering of notation, we will omit in the remainder of this article to specify that the quantities are dimensionless. In practice, the dimensionless reduction amounts simply to replacing $\mu \to 1$ in all our equations but keeping ν, and it is thus used as a consistency check. Note that several other schemes would have been possible to build dimensionless quantities, just by using other physical quantities that might be present in the problem. For instance if we use gravity, then we can build a length scale and a timescale without resorting to a choice of L simply by

$$L = \mu^{2/5}g^{1/5}, \quad T = \mu^{2/5}g^{-4/5}. \tag{5.21}$$

After building dimensionless quantities with these scales, the gravity vector g^μ is replaced by a unit vector while $\mu \to 1$ in all equations.

Similarly, if we are working in a rotating frame we can use the rotation vector magnitude to build a timescale and then from the viscosity we can build a length scale as

$$T = \Omega^{-1}, \quad L = \sqrt{\mu/\Omega}. \tag{5.22}$$

After building dimensionless quantities, the rotation vector Ω^μ is replaced by a unit vector while $\mu \to 1$ in all equations.

VI. APPLICATION TO AXISYMMETRIC JETS

Using the formalism developed so far in the particular case of a straight viscous jet ($\kappa^a = 0$) would be equivalent to kill a fly with a sledgehammer, especially if we also require axisymmetry of the fiber around the FCL. Indeed, in that case the FA coordinates are just Cartesian coordinates, with the third coordinate s corresponding to the axis of symmetry of the problem. This problem has been studied already in the literature [6,7] and we shall rederive, recover, and extend these standard results to include higher order corrections, and a possible rotation of the viscous fluid around the axis of symmetry (called torsion by Bechtel *et al.* [8]).

From the assumed rotational symmetry of the problem, only the monopolar moments are nonvanishing and we need only to consider the $v^{(n)} = v^{(n)}_\varnothing$, $\dot\phi^{(n)} = \dot\phi^{(n)}_\varnothing$, and $u^{(n)} = u^{(n)}_\varnothing$. We recall that we use the short notation introduced in the definitions (4.2), and in particular we emphasize the notation $v = v^{(0)}$ for the fundamental lowest order component of the longitudinal velocity. It is also possible to further restrict the problem to nonrotational flows, and require that all the $\dot\phi^{(n)}$ vanish, as done in, e.g. García and Castellanos [6], Eggers and Dupont [7], but we will not perform such simplification and allow for a rotation of the fluid around the axis of symmetry as in Bechtel *et al.* [8].

From the symmetry of the problem we also deduce that $U^a = 0$ and the rotation rate of the central line satisfies necessarily $\omega^a = 0$. Then, from (2.23b), the longitudinal component of rotation $\overline{\varpi}$ is constant along the fiber so it is reasonable to choose $\overline{\varpi} = 0$. Eventually, the only possible velocity component for the central line is $\overline{U} = U_{\underline{3}}$. However, it is natural to also set $U_{\underline{3}} = 0$ which amounts to taking a nonmoving FCL. As a consequence from (4.10) the total velocity $\mathcal{V}^\mu$ is also the coordinate velocity $\mathcal{V}^\mu_R$. The sledgehammer comes from the fact that the FA coordinates $(y^1, y^2, y^3 = s)$ are just Cartesian coordinates so they can be chosen to be the Cartesian coordinates (x^1, x^2, x^3), and all the machinery of FA coordinates is not used in this case.

Finally, the long range volumic forces need also to respect the symmetry and $g^a = 0$, so if we consider the effect of gravity, the fiber needs to be vertical and we will write simply $\overline{g} = g^{\underline{3}} = g$. Let us also mention that we discard the possibility of considering a rotating frame ($\Omega^a = \overline{\Omega} = \Omega^\mu = 0$). Indeed, even though it is possible in principle to also consider the problem in a frame rotating around the axis of symmetry (that is $\overline{\Omega} \neq 0$), this would be of very limited interest as it can be obtained from the replacement $\dot\phi^{(0)} \to \dot\phi^{(0)} + \overline{\Omega}$ (see the discussion in Sec. VII E).

A. Lowest order viscous string model

The divergenceless condition (4.5) leads simply to the set of relations

$$u^{(n)} = -\frac{1}{n+1}\partial_s v^{(n)}, \tag{6.1}$$

so that we need only to consider the $v^{(n)}$ and the $\dot\phi^{(n)}$. At lowest order ($n = 0$), (6.1) has a very simple interpretation. A gradient in the longitudinal velocity $\partial_s v^{(0)}$ leads to a radial inflow $u^{(0)}$ because of incompressibility. Indeed, if the flow stretches, the radius shrinks to ensure volume conservation and thus incompressibility.

The lowest contributions of the Navier-Stokes equation are $\overline{\mathcal{D}}^{(0)}$ for the longitudinal part and $\mathring{\mathcal{D}}^{(0)}$ for the rotational part. At lowest order they lead to

$$\partial_t v + v\partial_s v = -\partial_s P^{(0)} + 4v^{(1)} + \partial_s^2 v + g, \tag{6.2a}$$

$$\partial_t\dot\phi + v\partial_s\dot\phi - \dot\phi\partial_s v = \partial_s^2\dot\phi + \dot\phi^{(1)}. \tag{6.2b}$$

The evolution of the radius is easily obtained from (4.25) and it reads as, at lowest order,

$$\partial_t \ln R = -\mathcal{H}v - \tfrac{1}{2}\partial_s v + O\left(\epsilon_R^2\right). \tag{6.3}$$

In order to find a closed system of equations from (6.2), we need to find $P^{(0)}$, $v^{(1)}$, and $\dot\phi^{(1)}$ from the boundary constraint equation (5.5). First, it turns out that in the axisymmetric case, the contribution of surface tension can only be in $P^{(0)}$, and we can separate the pressure field as

$$P \equiv P_v + p, \quad P_v \equiv v\mathcal{K}, \tag{6.4}$$

where p is the contribution coming from viscous forces and where the extrinsic scalar curvature $\mathcal{K}$ does not depend on r. It is also more convenient to combine the longitudinal part and the radial part of the boundary constraint (5.5) to obtain a constraint which gives directly p and remove the pressure dependence in the longitudinal constraint. Indeed, the components of the normal vector take exactly the form

$$N^a = n^a, \quad N^{\underline{3}} = -\mathcal{H}R, \tag{6.5}$$

and we find that the three scalar constraints can be expressed in the form

$${}^P\mathcal{C} \equiv [(1-\alpha^2)p = \mu(\sigma_{ab}n^a n^b - \alpha^2\sigma_{\underline{33}})], \tag{6.6a}$$

$$\overline{\mathcal{C}} \equiv [(1-\alpha^2)\sigma_{\underline{3}a}n^a = \alpha(\sigma_{\underline{33}} - \sigma_{ab}n^a n^b)], \tag{6.6b}$$

$${}^\theta\mathcal{C} \equiv [\sigma_{ab}n^a\tilde{n}^b = \alpha\sigma_{\underline{3}a}\tilde{n}^a], \tag{6.6c}$$

where $\alpha \equiv \mathcal{H}R$.

Using that constraints are expanded according to (5.7), then from the lowest order of the pressure constraint (${}^P\mathcal{C}^{(0)}_\varnothing$) we obtain the pressure monopole at lowest order

$$p^{(0)} = u^{(0)} + O\left(\epsilon_R^2\right) = -\partial_s v + O\left(\epsilon_R^2\right). \tag{6.7}$$

The longitudinal constraint $\overline{\mathcal{C}}$ at lowest order (that is $\overline{\mathcal{C}}^{(0)}_\varnothing$) gives the HP profile encoded by $v^{(1)}$:

$$v^{(1)} = \tfrac{3}{2}\mathcal{H}\partial_s v + \tfrac{1}{4}\partial_s^2 v + O\left(\epsilon_R^2\right). \tag{6.8}$$

Finally, the orthoradial constraint ${}^\theta\mathcal{C}$ does not vanish identically if we have rotation. Instead, if we consider ${}^\theta\mathcal{C}^{(1)}_\varnothing$, we get

$$\dot\phi^{(1)} = \tfrac{1}{2}\mathcal{H}\partial_s\dot\phi + O\left(\epsilon_R^2\right). \tag{6.9}$$

The lowest order of the incompressibility constraint ${}^D\mathcal{C}$ is not needed for the lowest order dynamics and is only required when considering higher order corrections as we shall see in Sec. VI B.

Using (6.7)–(6.9) replaced in Eq. (6.2), we are now able to obtain

$$\partial_t v = g - \nu \partial_s \mathcal{K} - v \partial_s v + 6\mathcal{H}\partial_s v + 3\partial_s^2 v + O\left(\epsilon_R^2\right), \quad (6.10a)$$

$$\partial_t \dot{\phi} = \dot{\phi}\partial_s v + 4\mathcal{H}\partial_s \dot{\phi} - v\partial_s \dot{\phi} + \partial_s^2 \dot{\phi} + O\left(\epsilon_R^2\right). \quad (6.10b)$$

Together with (6.3), it forms a closed set of equations. Since rotation does not couple to the longitudinal velocity in Eq. (6.10a), it is reasonable to consider that the dynamical equation (6.10b) should be considered only when including the first corrections. In fact, in order to obtain it we had to consider ${}^{\theta}\mathcal{C}^{(1)}_{\varnothing}$ and not ${}^{\theta}\mathcal{C}^{(0)}_{\varnothing}$ which vanishes identically, so we realize that we have been using a higher order constraint to be able to close (6.2b).

We note finally that if surface tension is ignored, the evolution of velocity and rotation does not depend on the radius. However, as soon as we consider surface tension, the dependence in R appears of course through $\mathcal{K}$, and we thus need (6.3) to complement the dynamical equations for v. We intentionally did not replace in Eq. (6.10) the expression of the extrinsic scalar curvature $\mathcal{K}$ since even though we might use our perturbative expansion in powers of R, it proves often useful to keep its most general expression when considering the axisymmetric geometry. Instead of using $\mathcal{K} = 1/R$, which is the lowest order expression obtained from (4.23), we obtain a much better description if we use instead the exact expression [5]

$$\mathcal{K} = \frac{1}{R\sqrt{1 + (\partial_s R)^2}} - \frac{\partial_s^2 R}{[1 + (\partial_s R)^2]^{3/2}}, \quad (6.11)$$

which is easily obtained from the general expression of the extrinsic scalar curvature (4.23). In Eq. (6.10a), one should thus use (6.11). It amounts to resumming all higher order contributions from surface tension effects, and this is made possible thanks to the decoupling property (6.4).

Equation (6.10a) together with (6.11) for surface tension induced pressure and the boundary kinematic relation (6.3) constitute the lowest order model for an axisymmetric jet. Note that the expression for $\partial_t v$ involves second order derivative with respect to the affine parameter s. The factor 3 in front of the last term of (6.10a) is the famous Trouton factor [36] and we review its origin in Sec. VII F 1. As for the factor $6\mathcal{H}$ in the previous term, it is easily understood from the longitudinal component of viscous forces per unit area (5.4) $\overline{\mathcal{F}}^{(0)}_{\varnothing} = 3\mu\partial_s v$. Indeed, this implies that the longitudinal viscous forces integrated on a circular section are

$$\overline{F} \simeq 3\mu\pi R^2 \partial_s v, \quad (6.12)$$

which implies that the lineic density of longitudinal forces is $\mu\partial_s(3\pi R^2 \partial_s v)$.

In the next section we detail the general method to obtain corrections up to any order in the small parameter ϵ_R and report the detailed expressions of the first set of corrections (corrections up to order ϵ_R^2). The second set of corrections (that is up to order ϵ_R^4) is reported in Appendix E.

TABLE I. Structure of dependencies from the pressure, radial, and orthoradial boundary constraints.

Equation	Variable	Dependence
${}^{p}\mathcal{C}$ (6.13)	$p^{(0)}$	$v^{(0)}$ and $p^{(1)}R^2$, $p^{(2)}R^4 \ldots$
$\overline{\mathcal{C}}$ (6.14)	$v^{(1)}$	$v^{(0)}$ and $v^{(2)}R^2$, $v^{(3)}R^4 \ldots$
${}^{\theta}\mathcal{C}$ (6.15)	$\dot{\phi}^{(1)}$	$\dot{\phi}^{(0)}$ and $\dot{\phi}^{(2)}R^2$, $\dot{\phi}^{(3)}R^4 \ldots$

B. General method for higher order corrections

The equations (6.2) are formally unchanged when considering higher orders because they are exact. However, they involve quantities that we have determined from the side constraints up to order ϵ_R^0 contributions. We thus need to determine these quantities ($v^{(1)}$, $\dot{\phi}^{(1)}$, $p^{(0)}$) with greater precision, that is, also taking into account contributions of order ϵ_R^2. To this end, we need to consider ${}^{p}\mathcal{C}^{(\leqslant 1)}$, $\overline{\mathcal{C}}^{(\leqslant 1)}$, and ${}^{\theta}\mathcal{C}^{(\leqslant 1)}$. After a straightforward computation, we can show that the longitudinal constraint $\overline{\mathcal{C}}$ reads as

$$\sum_{n=0}^{\infty} R^{2n}[\partial_s u^{(n)} + 4(n+1)v^{(n+1)} - \mathcal{H}^2\partial_s u^{(n-1)} - 4n\mathcal{H}^2 v^{(n)} + (8n+6)\mathcal{H}u^{(n)}] = 0, \quad (6.13)$$

and the truncated constraint $\overline{\mathcal{C}}^{(\leqslant 1)}$ is found by keeping only $n = 0$ and 1 in this sum. However, we notice that if we want to deduce $v^{(1)}$ from $\overline{\mathcal{C}}^{(\leqslant 1)}$, that is, keeping corrections of order ϵ_R^2, then we need an expression for $v^{(2)}$ at lowest order. Similarly, if we want corrections up to order ϵ_R^6, we need $v^{(3)}$ at lowest order and $v^{(2)}$ up to corrections of order ϵ_R^2 and $v^{(1)}$ up to corrections of order ϵ_R^4.

It is straightforward to show that the pressure constraint ${}^{p}\mathcal{C}$ has the general form

$$\sum_n p^{(n)}R^{2n} = \sum_n R^{2n}\left[\frac{(2n+1)(1+\mathcal{H}^2R^2)}{1-\mathcal{H}^2R^2}\right]u^{(n)} \quad (6.14)$$

and if we want to deduce $p^{(0)}$ up to order corrections of order ϵ_R^2 we need the truncation ${}^{p}\mathcal{C}^{(\leqslant 1)}$ in which we need to replace an expression for $p^{(1)}$ at lowest order. Similarly, if we want $p^{(0)}$ up to corrections ϵ_R^4, then from ${}^{p}\mathcal{C}^{(\leqslant 2)}$ we need $p^{(2)}$ at lowest order and $p^{(1)}$ up to corrections ϵ_R^2, and so on.

Finally, the orthoradial constraint ${}^{\theta}\mathcal{C}$ reads as in full generality

$$\sum_{n=1}^{\infty} R^{2n}[2n\dot{\phi}^{(n)} - \mathcal{H}\partial_s\dot{\phi}^{(n-1)}] = 0, \quad (6.15)$$

and in particular, from ${}^{\theta}\mathcal{C}^{(\leqslant 2)}$ we deduce that if we need $\dot{\phi}^{(1)}$ up to corrections of order ϵ_R^2, then we need an expression for $\dot{\phi}^{(2)}$ at lowest order. The structure of these recursive dependencies in the three boundary constraints is summarized in Table I.

This problem is solved if we now also consider higher moments of the Navier-Stokes equation (5.11) together with the incompressibility constraint (5.15) so as to find expressions for the missing moments. The key is to notice that these equations contain a Laplacian, e.g., ΔP for the incompressibility constraint or $\Delta\mathcal{V}^{\mu}$ for the Navier-Stokes equation. Since for any scalar S expanded as (3.9), the coefficients of the expansion of ΔS are

$$[\Delta S]_L^{(n)} = 4(n+1)(n+1+\ell)S_L^{(n+1)} + \partial_s^2 S_L^{(n)}, \qquad (6.16)$$

then in the axisymmetric case we can use this property for $\ell = 0$, and from the incompressibility constraint (5.15) we can express $P^{(n+1)}$ in terms of $\partial_s^2 P^{(n)}$. Similarly from the longitudinal part of the Navier-Stokes equation (5.11) we can express $v^{(n+1)}$ in terms of $\partial_s^2 v^{(n)}$ since it contains $\Delta \overline{V}$.

Indeed, the general expansion of the longitudinal part of the Navier-Stokes equation in powers of r^{2n} (the $\overline{\mathcal{D}}^{(n)}$) is

$$\partial_t v^{(n)} + \sum_m u^{(n-m)}\partial_s v^{(m)} + \sum_m m v^{(m)} u^{(n-m)} = 4(n+1)^2 v^{(n+1)} + \partial_s^2 v^{(n)} - \partial_s p^{(n)} + \delta_n^0(\overline{g} - \partial_s P_\nu). \qquad (6.17)$$

For instance, using $\overline{\mathcal{D}}^{(1)}$, we can obtain $v^{(2)}$ at lowest order, in function of $\partial_t v^{(1)}$ and also $\partial_s p^{(1)}$. Let us ignore this latter dependence for the sake of simplicity. Given that we already know the lowest order expression of $v^{(1)}$ in terms of $v = v^{(0)}$ from (6.8), then using it we obtain $v^{(2)}$ as a function of $v^{(0)}$. The time derivatives on $v^{(0)}$ can be further replaced with the lowest order dynamical equation of $v^{(0)}$ (6.10a). In the end, we have obtained $v^{(2)}$ in terms of $v^{(0)}$ and its derivatives with respect to s, thus having the structure of a constraint equation.

As for the rotational part, its expansion in powers of r^{2n}, that is $\mathring{\mathcal{D}}^{(n)}$, leads to

$$\partial_t \dot{\phi}^{(n)} + \sum_m \left(v^{(n-m)}\partial_s \dot{\phi}^{(m)} + (m+1)\dot{\phi}^{(m)} u^{(n-m)}\right) = \partial_s^2 \dot{\phi}^{(n)} + 4(n+1)(n+2)\dot{\phi}^{(n+1)}. \qquad (6.18)$$

Similarly, from $\mathring{\mathcal{D}}^{(n)}$ we see that we can obtain $\dot{\phi}^{(n+1)}$ as a function of $\partial_s^2 \dot{\phi}^{(n)}$, and the time derivatives are eventually eliminated in the same manner once all replacements with lowest order relations are performed. Finally, using the expansion in powers of r^{2n} of the incompressibility constraint, that is, using the ${}^D\mathcal{C}^{(n)}$ we get

$$\sum_m \left[\frac{3n+5}{2} + 2m(n-m)\right] u^{(m)} u^{(n-m)} + \sum_m [-2(n+1)\dot{\phi}^{(m)}\dot{\phi}^{(n-m)} + 2m v^{(m)}\partial_s u^{(n-m)}] = -\partial_s^2 p^{(n)} - 4(n+1)^2 p^{(n+1)}. \qquad (6.19)$$

We see that from ${}^D\mathcal{C}^{(n)}$ we can obtain $p^{(n+1)}$ in terms of $\partial_s^2 p^{(n)}$ and this time it is directly in form of a constraint since it does not involve any time derivatives. The structure of these recursive dependencies deduced from the Navier-Stokes equation and the incompressibility constraint is summarized in Table II.

We understand that the general procedure is very recursive but simple, and this motivates the use of abstract calculus packages (such as *Mathematica*) to circumvent the complexity of these tedious abstract computations.

(1) Initially from the lowest order of the constraints ${}^p\mathcal{C}$, $\overline{\mathcal{C}}$, and ${}^\theta\mathcal{C}$ [Eqs. (6.13)–(6.15)] we get $p^{(0)}$, $v^{(1)}$, and $\dot{\phi}^{(1)}$ at lowest order in function of $v^{(0)}$ and $\dot{\phi}^{(0)}$, namely, (6.7)–(6.9).

TABLE II. Structure of dependencies for the constraints deduced from the Navier-Stokes equation and the incompressibility constraint.

Equation	Variable	Dependence
$\overline{\mathcal{D}}^{(n)}$ (6.17)	$v^{(n+1)}$	$p^{(n)}$, $v^{(m)}$, $m \leqslant n$
$\mathring{\mathcal{D}}^{(n)}$ (6.18)	$\dot{\phi}^{(n+1)}$	$\dot{\phi}^{(m)}$, $v^{(m)}$, $m \leqslant n$
${}^D\mathcal{C}^{(n)}$ (6.19)	$p^{(n+1)}$	$p^{(n)}$, $v^{(m)}$, $\dot{\phi}^{(m)}$, $m \leqslant n$

(2) Then, if we know the $p^{(q)}$, $v^{(q)}$, and $\dot{\phi}^{(q)}$ with $0 \leqslant q \leqslant n$ up to order ϵ_R^{2m}, then from (6.17)–(6.19) we can deduce $p^{(n+1)}$, $v^{(n+1)}$, and $\dot{\phi}^{(n+1)}$ up to order ϵ_R^{2m} as summarized in Table II.

(3) By using the constraints ${}^p\mathcal{C}$, $\overline{\mathcal{C}}$, and ${}^\theta\mathcal{C}$ [Eqs. (6.13)–(6.15)] we see that we can find $p^{(0)}$, $v^{(1)}$, and $\dot{\phi}^{(1)}$ up to order ϵ_R^{2m} if we know the $p^{(q)}$, $v^{(q+1)}$, and $\dot{\phi}^{(q+1)}$ up to order $\epsilon_R^{2(m-q)}$, and this is summarized in Table I.

(4) From the second point, we see that if we know the $p^{(0)}$, $v^{(1)}$, and $\dot{\phi}^{(1)}$ up to order ϵ_R^{2m}, then we also know all $p^{(q)}$, $v^{(q+1)}$, and $\dot{\phi}^{(q+1)}$ to the same order ϵ_R^{2m} and from the third point this is what we need to know the $p^{(0)}$, $v^{(1)}$, and $\dot{\phi}^{(1)}$ up to order $\epsilon_R^{2(m+1)}$, which validates this recursive method.

(5) Eventually, time derivatives on the fundamental variables v and $\dot{\phi}$ which appear in corrective terms can be replaced by using their dynamical equations at a lower order and it is thus possible to obtain the corrections to the lowest order model up to any order only in terms of derivatives with respect to s.

Finally, the evolution of the radius is easily found up to any given order in ϵ_R^{2n}. Indeed, given that

$$\partial_t \ln R = \sum_m \left(\frac{1}{2}u^{(m)} - \mathcal{H} v^{(m)}\right) R^{2m}, \qquad (6.20)$$

then in order to obtain the evolution up to ϵ_R^{2n} corrections, we need the $u^{(m)}$ and $v^{(m)}$ (with $m \leqslant n$) up to $\epsilon_R^{2(n-m)}$ corrections.

C. First corrections

Implementing the procedure described in the previous section, we first get the constraints

$$p^{(1)} = \tfrac{1}{2}\dot{\phi}^2 - \tfrac{3}{8}(\partial_s v)^2 - v\tfrac{1}{4}\partial_s^2\mathcal{K} + \tfrac{1}{4}\partial_s^3 v + O\left(\epsilon_R^2\right), \qquad (6.21a)$$

$$v^{(2)} = \tfrac{9}{16}\mathcal{H}\partial_s\mathcal{H}\partial_s v - \tfrac{3}{16}\mathcal{H}(\partial_s v)^2 + \tfrac{1}{16}\dot{\phi}\partial_s\dot{\phi} - v\tfrac{3}{32}\mathcal{H}\partial_s^2\mathcal{K} + \tfrac{9}{16}\mathcal{H}^2\partial_s^2 v - \tfrac{9}{64}\partial_s v\partial_s^2 v - v\tfrac{1}{32}\partial_s^3\mathcal{K} + \tfrac{9}{32}\mathcal{H}\partial_s^3 v + \tfrac{3}{64}\partial_s^4 v + O\left(\epsilon_R^2\right), \qquad (6.21b)$$

$$\dot{\phi}^{(2)} = -\tfrac{1}{32}\dot{\phi}\partial_s\mathcal{H}\partial_s v + \tfrac{1}{12}\mathcal{H}\partial_s\mathcal{H}\partial_s\dot{\phi} - \tfrac{1}{48}\partial_s\dot{\phi}\partial_s^2\mathcal{H} - \tfrac{1}{96}\mathcal{H}\dot{\phi}\partial_s^2 v + \tfrac{1}{12}\mathcal{H}^2\partial_s^2\dot{\phi} - \tfrac{1}{24}\partial_s\mathcal{H}\partial_s^2\dot{\phi} - \tfrac{1}{192}\dot{\phi}\partial_s^3 v + O\left(\epsilon_R^2\right). \qquad (6.21c)$$

We also find the corrections to the constrained quantities which were already computed at lowest order when deriving the lowest order string model. Indeed, the expressions

(6.7)–(6.9) need to be corrected with the terms

$$p^{(0)} \supset R^2\big[-\tfrac{1}{2}\dot\phi^2 - 3\mathcal{H}^2\partial_s v - \tfrac{9}{4}\partial_s\mathcal{H}\partial_s v + \tfrac{3}{8}(\partial_s v)^2 + v\tfrac{1}{4}\partial_s^2\mathcal{K} - \tfrac{9}{4}\mathcal{H}\partial_s^2 v - \tfrac{5}{8}\partial_s^3 v\big], \tag{6.22a}$$

$$v^{(1)} \supset R^2\big[\tfrac{3}{2}\mathcal{H}^3\partial_s v + \tfrac{3}{2}\mathcal{H}\partial_s\mathcal{H}\partial_s v + \tfrac{3}{8}\mathcal{H}(\partial_s v)^2 - \tfrac{1}{8}\dot\phi\partial_s\dot\phi + \tfrac{3}{16}\partial_s v\partial_s^2\mathcal{H} + v\tfrac{3}{16}\mathcal{H}\partial_s^2\mathcal{K} + \tfrac{3}{2}\mathcal{H}^2\partial_s^2 v + \tfrac{3}{8}\partial_s\mathcal{H}\partial_s^2 v + \tfrac{9}{32}\partial_s v\partial_s^2 v + v\tfrac{1}{16}\partial_s^3\mathcal{K} + \tfrac{1}{16}\mathcal{H}\partial_s^3 v - \tfrac{1}{16}\partial_s^4 v\big], \tag{6.22b}$$

$$\dot\phi^{(1)} \supset R^2\big[\tfrac{1}{16}\dot\phi\partial_s\mathcal{H}\partial_s v + \tfrac{1}{12}\mathcal{H}\partial_s\mathcal{H}\partial_s\dot\phi + \tfrac{1}{24}\partial_s\dot\phi\partial_s^2\mathcal{H} + \tfrac{1}{48}\mathcal{H}\dot\phi\partial_s^2 v + \tfrac{1}{12}\mathcal{H}^2\partial_s^2\dot\phi + \tfrac{1}{12}\partial_s\mathcal{H}\partial_s^2\dot\phi + \tfrac{1}{96}\dot\phi\partial_s^3 v\big]. \tag{6.22c}$$

Once replaced in Eq. (6.2), we finally get the corrections to (6.10) which read as

$$\partial_t v \supset R^2\big[\mathcal{H}\dot\phi^2 + 12\mathcal{H}^3\partial_s v + \tfrac{33}{2}\mathcal{H}\partial_s\mathcal{H}\partial_s v + \tfrac{3}{4}\mathcal{H}(\partial_s v)^2 + \tfrac{1}{2}\dot\phi\partial_s\dot\phi + 3\partial_s v\partial_s^2\mathcal{H} + v\tfrac{1}{4}\mathcal{H}\partial_s^2\mathcal{K} + \tfrac{27}{2}\mathcal{H}^2\partial_s^2 v + 6\partial_s\mathcal{H}\partial_s^2 v + \tfrac{3}{8}\partial_s v\partial_s^2 v + \tfrac{15}{4}\mathcal{H}\partial_s^3 v + \tfrac{3}{8}\partial_s^4 v\big], \tag{6.23a}$$

$$\partial_t\dot\phi \supset R^2\big[\tfrac{1}{2}\dot\phi\partial_s\mathcal{H}\partial_s v + \tfrac{2}{3}\mathcal{H}\partial_s\mathcal{H}\partial_s\dot\phi + \tfrac{1}{3}\partial_s\dot\phi\partial_s^2\mathcal{H} + \tfrac{1}{6}\mathcal{H}\dot\phi\partial_s^2 v + \tfrac{2}{3}\mathcal{H}^2\partial_s^2\dot\phi + \tfrac{2}{3}\partial_s\mathcal{H}\partial_s^2\dot\phi + \tfrac{1}{12}\dot\phi\partial_s^3 v\big], \tag{6.23b}$$

$$\partial_t \ln R \supset R^2\big(-\tfrac{3}{2}\mathcal{H}^2\partial_s v - \tfrac{3}{8}\partial_s\mathcal{H}\partial_s v - \tfrac{5}{8}\mathcal{H}\partial_s^2 v - \tfrac{1}{16}\partial_s^3 v\big). \tag{6.23c}$$

We report in Appendix E the next order corrections which are of order ϵ_R^4. Note also that surface tension effects do not enter explicitly the dynamical equation for axial rotation (6.23b) nor the dynamical equation for the radius (6.23c), but they still matter due to the couplings between v, $\dot\phi$, and R.

In general, if we keep terms of order ϵ_R^{2n} in the expression giving $\partial_t v$, that is, if we consider the nth correction, then it involves terms which have $2(n+1)$ order derivatives in the affine parameter s, typically from terms of the form $R^{2n}\partial_s^{2(n+1)}v$, and the differential complexity is increased. If we consider for instance a steady regime in which all time derivatives vanish, this can lead to a rather stiff differential system as the coefficients in front of the highest derivatives are typically the smallest.

D. Comparison with the Cosserat model

As shown by García and Castellanos [6], the Cosserat model can be viewed as an averaged model. Indeed, given that the longitudinal velocity is given by $v^{(0)} + r^2 v^{(1)} + r^4 v^{(2)} + \cdots$, it could be natural to consider an averaged velocity and derive the dynamical equation for this variable and not for $v^{(0)}$ which should be considered as a derived variable. The average longitudinal velocity is simply defined as

$$\overline{v} \equiv \frac{2}{R^2}\int_0^R \overline{V} r\,dr = \frac{2}{R^2}\int_0^R \sum_n v^{(n)} r^{2n} r\,dr = \sum_n \frac{R^{2n}}{n+1} v^{(n)}. \tag{6.24}$$

We obtain that from (6.20) and (6.1) the kinematic equation for the time evolution of the fiber radius reads as exactly [6]

$$\partial_t \ln R = -\mathcal{H}\overline{v} - \tfrac{1}{2}\partial_s\overline{v}. \tag{6.25}$$

Obviously, the expression (6.24) needs to be truncated at a given power of R which is the order at which the equations are considered. Once truncated, we can invert it because from our algorithm, the $v^{(n)}$ are expressed in terms of (the derivatives of) $v^{(0)}$ and $\dot\phi^{(0)}$. After inverting, we obtain $v = v^{(0)}$ as a power series in $\overline{v}$ and its derivatives. For instance, up to the first corrections we get

$$\overline{v} = v + \frac{R^2}{2}\left(\frac{3}{2}\mathcal{H}\partial_s v + \frac{1}{4}\partial_s^2 v\right) + O\left(\epsilon_R^4\right), \tag{6.26a}$$

$$v = \overline{v} - \frac{R^2}{2}\left(\frac{3}{2}\mathcal{H}\partial_s\overline{v} + \frac{1}{4}\partial_s^2\overline{v}\right) + O\left(\epsilon_R^4\right). \tag{6.26b}$$

At lowest order, the form of (6.10a) is the same if it is expressed with $\overline{v}$ or with v because both velocities are equal at lowest order. Differences appear only when including the first set of corrections. Eventually, we find

$$\partial_t\overline{v} + R^2\big(-\tfrac{1}{2}\mathcal{H}\partial_t\partial_s\overline{v} - \tfrac{1}{8}\partial_t\partial_s^2\overline{v}\big) = g - \partial_s P_v^{(0)} + 6\mathcal{H}\partial_s\overline{v} - \overline{v}\partial_s\overline{v} + 3\partial_s^2\overline{v} + R^2\big(\mathcal{H}\dot\phi^2 - 6\mathcal{H}^3\partial_s\overline{v} - \tfrac{1}{4}\mathcal{H}(\partial_s\overline{v})^2 + \tfrac{1}{2}\dot\phi\partial_s\dot\phi + \tfrac{3}{4}\partial_s\overline{v}\partial_s^2\mathcal{H} - \tfrac{3}{2}\mathcal{H}^2\partial_s^2\overline{v} + \tfrac{1}{2}\mathcal{H}\overline{v}\partial_s^2\overline{v} + \tfrac{3}{4}\partial_s\mathcal{H}\partial_s^2\overline{v} + \tfrac{1}{8}\overline{v}\partial_s^3\overline{v}\big), \tag{6.27}$$

which we have written in a form which matches Eq. (55) of García and Castellanos [6]. However, it does not match the Cosserat equation (53) of García and Castellanos [6] [which is also Eq. (74) of Eggers [3] derived with a Galerkin approximation method]. As explained in details by García and Castellanos [6], this is because some terms involving $v^{(1)}$ are removed. In the case of the Cosserat model of Eggers [3], these terms would probably be recovered when including higher orders of the Galerkin method, given that at lowest order in the Galerkin approximation the information that the next corrections are parabolic in nature has not been put in. In a sense, the lowest order of the Galerkin approximation also amounts to ignoring some parabolic terms of the type $v^{(1)}r^2$. In our method, the contributions $v^{(1)}$ and $v^{(2)}$ are taken into account from the constraints (6.8) [corrected by (6.22b) and (6.21b)].

Furthermore, our results are extended to include a possible axial rotation from the inclusion of the $\dot\phi^{(n)}$. We find that when including the first corrections to the viscous string approximations, the corrected dynamics of v couples with the lowest order axial rotation $\dot\phi = \dot\phi^{(0)}$ whose evolution needs then to be computed from the lowest order dynamical equation (6.10b). This interplay between longitudinal velocity and axial rotation which takes place when the first corrections are included has already been described in Bechtel *et al.* [8], but our formalism allows already for more compact and geometrically more meaningful expressions. However, it is only when considering curved fibers that our formalism based on STF multipoles appears to be powerful as we shall now see in details in the following section.

VII. APPLICATION TO CURVED FIBERS

A. Overview of curved fiber specificities

Whenever we consider curved fibers, we must take into account the property that the FCL curvature κ^a does not vanish anymore, nor does the rotation rate $\omega^{\underline{i}}$ of the orthonormal basis. As stressed in Sec. IV J, even if we restrict to a stationary regime for which all time derivatives vanish and thus $\omega^a = 0$, we can still generically form STF products of the type

$$\kappa_{\langle a_1} \dots \kappa_{a_\ell \rangle} \tag{7.1}$$

which might source the STF moments of order ℓ. For instance, terms proportional to $\kappa_{\langle a}\kappa_{b\rangle}$ would source the shear part of the velocity field $\widehat{V}_{ab}$ from the Navier-Stokes equation, and this would in turn induce a deformation of the fiber shape of the type $\mathcal{R}_{ab}$ from the kinetic condition on the fiber side (4.25). Since the lowest order description corresponds to a string approximation for which the section size and shape are irrelevant, these terms are expected to arise only when including the first corrections. For instance, the combinations

$$\kappa_{\langle a}\kappa_{b\rangle} y^a y^b \quad \text{or} \quad \kappa_{\langle a}\kappa_{b\rangle} R^2 \tag{7.2}$$

are of order ϵ_R^2. For the lowest order description, one might consider circular sections, but as soon as we consider refinements to this description, we must abandon the circular shape assumption. As we shall find in the remainder of this section, the fundamental dynamical variables are the same as for the axisymmetric case ($v = v_\varnothing^{(0)}$, $\dot\phi = \dot\phi_\varnothing^{(0)}$, and R) on which we add the various shape multipoles $\mathcal{R}_L$ and also the position and velocity of the FCL. If we consider a description up to order n in powers of ϵ_R, then we shall find that we must include at least the multipoles $\mathcal{R}_L$ with $\ell \leqslant n$. As soon as we leave the realm of straight fibers, we open Pandora's box and we cannot obtain all STF devils from constraints, as the shape multipoles become dynamical.

This departure from circular fiber sections is even more obvious when considering the motion in a steady rotating frame. Indeed, the inertial forces will bring typical contributions of the form $\Omega_{\langle a}\Omega_{b\rangle}$, which are similar to tidal forces and induce an elliptic elongation. Such contribution arises naturally in the Navier-Stokes equation (5.11) as can be seen from the general expression of the inertial forces (5.13), and they typically source the velocity shear moments $\widehat{V}_{ab}^{(n)}$, which in turn source the elliptic deformation $\mathcal{R}_{ab}$ from the boundary kinematics (4.25).

We first derive the lowest order viscous string model in the curved case in Sec. VII B and discuss corrections in Sec. VII C. In Sec. VII D we also consider the special case of straight but nonaxisymmetric fibers with mild elliptic shapes.

B. Viscous string model

1. Incompressibility conditions

The incompressibility conditions on moments (4.5) are used to replace the moments $u_L^{(n)}$. In particular, from $[{}^C\mathcal{E}]_\varnothing^{(0)}$, $[{}^C\mathcal{E}]_\varnothing^{(1)}$, $[{}^C\mathcal{E}]_a^{(0)}$, $[{}^C\mathcal{E}]_a^{(1)}$, and $[{}^C\mathcal{E}]_{ab}^{(0)}$ we obtain the general conditions

$$u_\varnothing^{(1)} = -\widehat{V}^{(1)a}\widetilde{\kappa}_a - \tfrac{1}{2}\widetilde{\kappa}^a\widehat{V}_a^{(0)} - \tfrac{1}{2}\partial_s v_\varnothing^{(1)}, \tag{7.3a}$$

$$u_a^{(0)} = -\tfrac{4}{3}\widehat{V}_a^{(1)} - \tfrac{2}{3}\widehat{V}_{ab}^{(0)}\widetilde{\kappa}^b - \widetilde{\kappa}_a u_\varnothing^{(0)} - \tfrac{2}{3}\kappa_a\dot\phi - \tfrac{2}{3}\partial_s v_a^{(0)}, \tag{7.3b}$$

$$u_a^{(1)} = -\tfrac{8}{5}\widehat{V}_a^{(2)} - \tfrac{4}{5}\widehat{V}_{ab}^{(1)}\widetilde{\kappa}^b - \widetilde{\kappa}_a u_\varnothing^{(1)} - \tfrac{1}{2}\widetilde{\kappa}^b\widehat{V}_{ab}^{(0)} - \tfrac{2}{5}\kappa_a\dot\phi^{(1)} - \tfrac{2}{5}\partial_s v_a^{(1)}, \tag{7.3c}$$

$$u_{ab}^{(0)} = -\widehat{V}_{ab}^{(1)} - \widehat{V}_{\langle a}^{(1)}\widetilde{\kappa}_{b\rangle} - \widetilde{\kappa}_{\langle a}\widehat{V}_{b\rangle}^{(0)} - \tfrac{1}{2}\partial_s v_{ab}^{(0)} - \tfrac{1}{2}\widehat{V}_{abc}^{(0)}\widetilde{\kappa}^c, \tag{7.3d}$$

that are used extensively together with (4.6) throughout Sec. VII.

2. Normal vector and curvature

At lowest order, the unit normal vector components are simply

$$\widehat{N}^{\underline{3}} = -\mathcal{H}R + O(\epsilon_R^2), \tag{7.4a}$$

$$\widehat{N}^a = n^a + O(\epsilon_R^2), \tag{7.4b}$$

and from (4.23) the scalar extrinsic curvature reads as

$$\mathcal{K} = \frac{1}{R} + \widetilde{\kappa}_a n^a + O(\epsilon_R). \tag{7.5}$$

If we expand $R\mathcal{H}$ as in Eq. (5.7b), then $[R\mathcal{K}]_\varnothing^{(0)} = 1$ and $[R\mathcal{K}]_a^{(0)} = \widetilde{\kappa}_a$. Note that we cannot separate the pressure contribution from viscous and surface tension effects as in Eq. (6.4). The scalar extrinsic curvature must be used directly inside the boundary constraint (5.5).

3. Navier-Stokes equation

Using the components expression for the velocity gradient (4.31), the velocity shear (4.36), and the velocity derivatives (5.14), the lowest multipoles of the longitudinal and sectional components of acceleration and of the volumic forces are [reminding that we are using the notation (4.2) throughout]

$$\overline{A}_\varnothing^{(0)} = (\partial_t + v\partial_s)v + \partial_t\bar{U} + U^a\widetilde{\omega}_a + \overline{I} + 2U^a\widetilde{\Omega}_a, \tag{7.6a}$$

$$\overline{f}_\varnothing^{(0)} = \overline{g} + 4v^{(1)} + v_a^{(0)}\widetilde{\kappa}^a - \kappa^a\omega_a - \partial_s P_\varnothing^{(0)} - v\kappa_a\kappa^a + 2\widetilde{\kappa}^a\partial_s\widehat{V}_a^{(0)} + \widehat{V}_a^{(0)}\partial_s\widetilde{\kappa}^a + \partial_s^2 v, \tag{7.6b}$$

$$\widehat{A}_a^{(0)} = -v^2\widetilde{\kappa}_a + 2\widetilde{U}_a\overline{\Omega} + I_a - 2\bar{U}\widetilde{\Omega}_a - 2v\widetilde{\Omega}_a - \bar{U}\widetilde{\omega}_a - 2v\widetilde{\omega}_a + \partial_t U_a + \widetilde{U}_a\overline{\omega}, \tag{7.7a}$$

$$\widehat{f}_a^{(0)} = g_a - P_a^{(0)} + \tfrac{8}{3}\widehat{V}_a^{(1)} - \tfrac{5}{3}\kappa_a\dot\phi - \tfrac{3}{2}\widetilde{\kappa}_a\partial_s v - \tfrac{2}{3}\partial_s v_a^{(0)} - v\partial_s\widetilde{\kappa}_a - \partial_s\widetilde{\omega}_a, \tag{7.7b}$$

$$\mathring{A}^{(0)} = -v_a^{(0)}(\Omega^a + \omega^a + \kappa^a v) - (\dot\phi + \overline{\Omega})\partial_s v + (\partial_t + v\partial_s)\dot\phi, \tag{7.8a}$$

$$\begin{aligned}\overset{\circ}{f}{}^{(0)} = &-\tfrac{4}{3}\widehat{V}^{(1)}_a\kappa^a - \tfrac{7}{6}\kappa_a\kappa^a\dot{\phi} + 8\dot{\phi}^{(1)}\\ &-\tfrac{7}{6}\kappa^a\partial_s v^{(0)}_a - \tfrac{1}{2}v^{(0)a}\partial_s\kappa_a + \tfrac{1}{2}v\widetilde{\kappa}_a\partial_s\kappa^a\\ &+\tfrac{1}{2}\omega^a\partial_s\widetilde{\kappa}_a + \widetilde{\kappa}_a\partial_s\omega^a + \partial^2_s\dot{\phi}.\end{aligned} \tag{7.8b}$$

In these expressions, we have defined the inertial force on the FCL by

$$\boldsymbol{I} \equiv \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{R}). \tag{7.9}$$

$\overline{I}$ and I^a are as usual its longitudinal and sectional components. Similarly, we recall that Ω^a and g^a are the sectional components of the steady frame rotation (if any) and of long range forces, whereas $\overline{\Omega}$ and $\overline{g}$ are their longitudinal projections. Since $\boldsymbol{\Omega}$ and $\boldsymbol{g}$ are constant vectors, then from (2.15) their components vary along the FCL according to

$$\partial_s\overline{g} = -\widetilde{\kappa}_a g^a, \quad \partial_s g^a = \kappa^a\overline{g}, \tag{7.10}$$

with similar expressions for the components of $\boldsymbol{\Omega}$. However, $\boldsymbol{I}$ is not constant and we must use

$$\partial_s\overline{I} = -\Omega^2 - \widetilde{\kappa}_a I^a, \quad \partial_s I^a = \overline{\Omega}\Omega^a + \widetilde{\kappa}_a\overline{I}. \tag{7.11}$$

In reality, the expressions (7.6)–(7.8a) are formally more complex, as they also involve terms which contain $\widehat{V}^{(0)}_a$ and $\widehat{V}^{(0)}_{ab}$. The former vanishes at lowest order from the gauge condition (4.28), and when studying the structure of corrections in Sec. VII C, we will show that from the orthoradial boundary condition the latter vanishes as well at lowest order. The missing terms are gathered in Sec. VII C 3.

From (7.6) we can infer the lowest moment of the longitudinal part of Navier-Stokes equation, that is $\overline{\mathcal{D}}^{(0)}_\varnothing$, which could also be obtained from the longitudinal projection of a momentum balance equation. From (7.7) we can infer the lowest order multipoles of the sectional part of the Navier-Stokes equations $\mathcal{D}^{(0)}_a$. It could also be obtained from the sectional projection of the momentum balance equation. Finally, from (7.8), we infer $\overset{\circ}{\mathcal{D}}{}^{(0)}$ which governs the dynamics of the axial rotation rate $\dot{\phi} = \dot{\phi}^{(0)}$. In the axisymmetric case, it was not strictly speaking part of the lowest order string description since axial rotation decoupled at lowest order. For curved fibers, it seems at first sight when examining (7.7) that axial rotation retroacts on the dynamical equations for U_a. However, when including the boundary constraints, it will appear that this is not the case. As in the axisymmetric case, the dynamical equation for the axial rotation is in fact already a first correction and not part of the viscous string model.

In order to obtain closed dynamical equations from (7.6)–(7.8), we need to find the lowest order expressions for $v^{(0)}_a$, $P^{(0)} = P^{(0)}_\varnothing$, $v^{(1)} = v^{(1)}_\varnothing$, $\widehat{V}^{(1)}_a$, and $\dot{\phi}^{(1)}$, which as in the axisymmetric case are going to be obtained from the lowest order of the boundary condition (5.5).

4. *Constraint equations*

We recall the notation introduced in Sec. V B for the various constraints obtained on the boundary and their multipole expansions.

(i) First, from the lowest order monopole and dipole of the longitudinal constraint, that is $\overline{\mathcal{C}}^{(0)}_\varnothing$ and $\overline{\mathcal{C}}^{(0)}_a$, we get

$$v^{(1)} = \tfrac{3}{2}\mathcal{H}\partial_s v + \tfrac{1}{4}\partial^2_s v + O\left(\epsilon^2_R\right), \tag{7.12a}$$

$$v^{(0)}_a = v\widetilde{\kappa}_a + \widetilde{\omega}_a + O\left(\epsilon^2_R\right). \tag{7.12b}$$

The constraint (7.12a) is exactly the same as the one obtained in the axisymmetric case (6.8) and we will find that the effect of curvature appears only at higher orders. However, the latter constraint (7.12b) deserves a thorough comment. From the expression of the total velocity (4.1) and the decomposition (3.10) for the relative velocity $\boldsymbol{V}$, the velocity field on the FCL (that is, when $y^1 = y^2 = 0$) is

$$\boldsymbol{\mathcal{V}}_{\rm Cen} = \boldsymbol{U} + v\boldsymbol{T} + \widehat{V}^{(0)}_a\boldsymbol{d}_a = \boldsymbol{U} + v\boldsymbol{T} + O\left(\epsilon^2_R\right) \tag{7.13}$$

since from the gauge condition (4.28), $\widehat{V}^{(0)}_a$ is only an order ϵ^2_R quantity. The rotation rate of the fluid on the central line is thus approximately

$$\omega^a_{\rm Cen} \equiv [\boldsymbol{T}\times\partial_s(\boldsymbol{U} + v\boldsymbol{T})]^a = \omega^a + \kappa^a v. \tag{7.14}$$

Furthermore, the dipolar component $v^{(0)}_a$ corresponds to the sectional part of the solid rotation of the fluid contained in a fiber section. Indeed, if we focus on terms which are linear in y^a in the decomposition (3.10), and if we ignore radial infall ($u^{(0)}_\varnothing$) and axial rotation ($\dot{\phi}^{(0)}$), we realize that the relative velocity $\boldsymbol{V}$ contains

$$\boldsymbol{V} \supset v^{(0)}_a y^a\boldsymbol{T} = \left[\omega^a_{\rm Sec}\boldsymbol{d}_a \times y^b\boldsymbol{d}_b\right], \quad \omega^{\rm Sec}_a \equiv -\widetilde{v}^{(0)}_a.$$

The constraint (7.12b) thus states that the solid rotation of the fiber section $\omega^a_{\rm Sec}$ is equal to the rotation of the fluid on the FCL $\omega^a_{\rm Cen}$. It means that once the central line velocity U^a is determined (together with the curvature and the longitudinal lowest order velocity v), then the rotation of the fluid on the FCL is determined, and the fiber section rotation must follow exactly the same rotation rate. This fact can be rephrased more rigorously by replacing the constraint (7.12b) in the expression (4.37) for the vorticity, as we obtain

$$\varpi^a = \omega^a_{\rm Cen} + O\left(\epsilon^2_R\right) = \omega^a + \kappa^a v + O\left(\epsilon^2_R\right). \tag{7.15}$$

It is yet another way to see that the rotation of the fluid in fiber sections (vorticity) is guided by the rotation of the fluid on the FCL.

The consequence is that the fluid contained in a given section, which by construction is orthogonal to the tangential direction $\boldsymbol{T}$, remains always in a geometrically defined fiber section. Or, said differently, the fluid particles belonging to different fiber sections are not mixed by the fluid velocity. We must stress again that this result is only valid at lowest order in ϵ_R.

Hence, just by contemplating of (7.12b) we can understand why the lowest order approximation is called a string approximation. It is because the fiber sections are slaves of the FCL, as they are determined from it without retroaction at lowest order. Furthermore, and this is even more important, this type of behavior also corresponds to a form of flexible rod model since the fiber sections are not mixed and remain orthogonal to the FCL. If there was no longitudinal velocity ($v = 0$), a good analogy would be the spinal column with the vertebra being the sections. If we want to consider a longitudinal velocity, a good analogy would be a collar made of beads. The beads have a cylindrical hole through which the collar string is passed, and if the beads can slide along the string of the collar, their orientation with respect to the string tangential direction is necessarily fixed thanks to the cylindrical hole. We understand already at that point that it is hopeless to try to find consistently the corrections of the string model if we start from a flexible rod model since the lowest order description is already a form of flexible rod model.

(ii) Second, from the lowest order monopole and dipole of the orthoradial constraint, that is ${}^{\theta}\mathcal{C}^{(1)}_{\varnothing}$ and ${}^{\theta}\mathcal{C}^{(0)}_{a}$, we obtain

$$\dot{\phi}^{(1)} = \tfrac{1}{2}(\mathcal{H}\widetilde{\kappa}_a\omega^a + \mathcal{H}\partial_s\dot{\phi}) + O\left(\epsilon_R^2\right), \tag{7.16a}$$

$$\widehat{V}^{(1)}_a = \tfrac{1}{8}(2\kappa_a\dot{\phi} - \widetilde{\kappa}_a\partial_s v + 2v\partial_s\widetilde{\kappa}_a + 2\partial_s\widetilde{\omega}_a) + O\left(\epsilon_R^2\right). \tag{7.16b}$$

(iii)And, third, from the lowest order monopole and dipole of the radial boundary constraint, that is from ${}^{r}\mathcal{C}^{(0)}_{\varnothing}$ and ${}^{r}\mathcal{C}^{(0)}_{a}$, and using the gauge condition (4.28) and the previous constraints we get

$$P^{(0)}_{\varnothing} = \frac{v}{R} - \partial_s v + O(\epsilon_R), \tag{7.17a}$$

$$P^{(0)}_{a} = -\kappa_a\dot{\phi} + \frac{1}{2}\widetilde{\kappa}_a\partial_s v - v\partial_s\widetilde{\kappa}_a - \partial_s\widetilde{\omega}_a + \frac{v\widetilde{\kappa}_a}{R} + O(\epsilon_R). \tag{7.17b}$$

5. *Dynamics of the string model*

Inserting the constraints of the previous section in the expressions (7.6)–(7.8), we finally obtain the system of equations

$$\partial_t v = -\partial_t\bar{U} - U^a\widetilde{\omega}_a + \overline{g} + \frac{v\mathcal{H}}{R} - v\partial_s v + 6\mathcal{H}\partial_s v + 3\partial_s^2 v + O(\epsilon_R), \tag{7.18a}$$

$$(\partial_t U)_a = g_a + v^2\widetilde{\kappa}_a + 2v\widetilde{\omega}_a - 3\widetilde{\kappa}_a\partial_s v - \frac{v\widetilde{\kappa}_a}{R} + O(\epsilon_R), \tag{7.18b}$$

$$\partial_t\dot{\phi} = 4\mathcal{H}\widetilde{\kappa}^a\omega_a + \dot{\phi}\partial_s v + \omega^a\partial_s\widetilde{\kappa}_a + 4\mathcal{H}\partial_s\dot{\phi} - v\partial_s\dot{\phi} + \partial_s^2\dot{\phi} + O\left(\epsilon_R^2\right), \tag{7.18c}$$

where using (2.25a) we used the compact expression

$$(\partial_t U)_a = \partial_t U_a + \overline{\omega}\widetilde{U}_a - \bar{U}\widetilde{\omega}_a. \tag{7.19}$$

Note also that from (2.24a) we also get $(\partial_t U)^{\underline{3}} = \partial_t\overline{U} + \widetilde{\omega}_a U^a$ allowing to rewrite (7.18a) in a slightly more compact form if desired. Finally, when surface tension effects are included, we also need to determine the dynamics of the fiber radius as it couples to (7.18b) and (7.18a), and at lowest order it reads exactly as in the axisymmetric case, that is,

$$\partial_t \ln R = -\mathcal{H}v - \tfrac{1}{2}\partial_s v. \tag{7.20}$$

Several comments are in order for this viscous string model.

(1) We can check that there is no retroaction of $\dot{\phi}$ on the lowest order dynamical equations for v and U_a. The axial rotation dynamical equation (7.18c) is in fact part of the first corrections and not part of the lowest order string model.

(2) The equations are at most linear in the curvature κ_a even though we have not linearized in this variable.

(3) To compare (7.18a) and (7.18b) with the result obtained from a momentum balance equation by Arne *et al.* [20,21], Ribe [22], and Ribe *et al.* [23], we must first use that the average velocity inside a fiber is approximately the velocity on the FCL given at lowest order by (7.13). Then, from the properties

$$(\partial_t\mathcal{V}_{\rm Cen})^{\underline{3}} = \partial_t v + (\partial_t U)^{\underline{3}}, \tag{7.21a}$$

$$(\partial_t\mathcal{V}_{\rm Cen})_a = (\partial_t U)_a - v\widetilde{\omega}_a, \tag{7.21b}$$

$$v(\partial_s\mathcal{V}_{\rm Cen})^{\underline{3}} = v\partial_s v, \tag{7.21c}$$

$$v(\partial_s\mathcal{V}_{\rm Cen})_a = -v^2\widetilde{\kappa}_a - v\widetilde{\omega}_a, \tag{7.21d}$$

and defining a convective derivative by $\mathcal{D}_t \equiv \partial_t + v\partial_s$, the string equations are simply recast as

$$(\mathcal{D}_t\mathcal{V}_{\rm Cen})^{\underline{3}} \simeq \overline{g} + \frac{v\mathcal{H}}{R} + 6\mathcal{H}\partial_s v + 3\partial_s^2 v, \tag{7.22a}$$

$$(\mathcal{D}_t\mathcal{V}_{\rm Cen})_a \simeq g_a - 3\widetilde{\kappa}_a\partial_s v - \frac{v\widetilde{\kappa}_a}{R}. \tag{7.22b}$$

(4) We can in particular check that the contributions from the surface tension are exactly those found in Eqs. (37), (49), and (57) of Ribe *et al.* [23]. Since the vector $\widetilde{\kappa}^a$ points toward the exterior of the FCL curvature, the contribution $-v\widetilde{\kappa}_a/R$ tends to unfold the viscous jet as strongly curved region will be accelerated toward the center of curvature.

(5) If we ignore the surface tension contributions, and using the covariantization relations (2.26), Eq. (7.22) can be recast in the compact form

$$\mathcal{D}_t\boldsymbol{\mathcal{V}}_{\rm Cen} \simeq \boldsymbol{g} + \frac{1}{\pi R^2}\partial_s\boldsymbol{F}, \tag{7.23}$$

$$\boldsymbol{F} \equiv \pi R^2\overline{\mathcal{F}}^{(0)}_{\varnothing}\boldsymbol{T}, \quad \overline{\mathcal{F}}^{(0)}_{\varnothing} = 3(\partial_s v), \tag{7.24}$$

where the link with the momentum balance has now been made obvious. The expression of the viscous force $\boldsymbol{F}$ which is purely longitudinal has the same origin as in the axisymmetric case [see Eq. (6.12)].

6. *Rotating frame*

If we now consider that the problem is studied in a steady rotating frame, then Eqs. (7.18) need to be supplemented with the contributions

$$\partial_t v \supset -\overline{I} - 2U^a\widetilde{\Omega}_a, \tag{7.25a}$$

$$(\partial_t U)_a \supset -2\widetilde{U}_a\overline{\Omega} - I_a + 2(\overline{U} + v)\widetilde{\Omega}_a, \tag{7.25b}$$

$$\partial_t\dot{\phi} \supset -\widetilde{\Omega}^a(\omega_a + v\kappa_a) + \overline{\Omega}\partial_s v. \tag{7.25c}$$

The first two equations arise naturally from the longitudinal and sectional components of the inertial and Coriolis forces.

7. *Longitudinal central line velocity*

We notice that the dynamical equation for v [Eq. (7.18a)] contains $\partial_t\overline{U}$. However, we must remember that the longitudinal velocity of the central line contains a remaining gauge degree of freedom. Indeed, we have fixed the position of the central line inside the fiber by asking that there should be no shape dipole. However, as argued in Sec. IV I, we can still displace the fiber inside that curved central line, equivalent to a reparametrization $s \to s + f(t)$ which changes the velocities as $\overline{U} \to \overline{U} - \partial_t f$ and $v \to v + \partial_t f$.

Equation (7.18a) is in fact an equation for $\partial_t(v + \overline{U})$ and we must find a unique way to determine $\overline{U}$ independently. Let us fix the value of the central line longitudinal velocity $\overline{U}$ for a given affine parameter (say $s = 0$ at a fiber boundary) at all times. Typically, the fiber is attached at the boundary so we choose simply $\overline{U}(s = 0) = 0$. Then, from the condition (2.6) we obtain

$$\partial_s \overline{U} = \widetilde{\kappa}_a U^a \tag{7.26}$$

and thus $\overline{U}$ must be determined everywhere on the fiber at all times, effectively breaking the degeneracy in Eq. (7.18a).

8. Full set of equations for the string model

We are now in position to gather all the equations which are required for the curved viscous string model. First, there are a set of equations which are constraint equations and which must be solved at a given initial time since they are ordinary differential equations in s:

(1) Once $\boldsymbol{R}(s,t)$ is known at a given time, e.g. at initial time, then the tangential direction of the FCL is obtained from (2.2).

(2) The curvature $\boldsymbol{\kappa}$ of the FCL is then determined at that same given time from its definition (2.7).

(3) The orthonormal basis $\boldsymbol{d}_{\underline{i}}$ can be determined everywhere on the fiber at that same given time from (2.15), provided some choice is made on a fiber boundary.

(4) With this orthonormal basis we can extract the components κ^a of the curvature.

(5) When the FCL velocity components U^a and $\overline{U}$ are known at a given time, and the curvature components κ^a as well, then the rotation components $\omega^{\underline{i}}$ can be found from (2.28) for the sectional component and from (2.23b) for the longitudinal component at the same given time.

(6) The longitudinal part of the FCL velocity $\overline{U}$ is not dynamical but it is instead constrained by (7.26).

Then, we have dynamical equations which give the time evolution of variables from initial conditions, but they depend on partial derivatives ∂_s so they are partial differential equations:

(1) The position in space of the FCL $\boldsymbol{R}(s,t)$ is modified due to the FCL velocity $\boldsymbol{U}$, as seen in Eq. (2.4).

(2) The curvature components evolve in time thanks to (2.23a).

(3) The orthonormal basis evolves in time with (2.16).

(4) The longitudinal velocity v of the fluid inside the fiber evolves in time according to Eq. (7.18a).

(5) The sectional part of the FCL velocity U^a evolves according to Eq. (7.18b).

(6) The fiber radius R evolves according to (7.20).

When considering corrections to the viscous string model, this structure between constraints and dynamical equations is preserved.

9. Covariant expressions

If we prefer to work fully in the Cartesian canonical basis, that is, if for a vector we prefer using the covariant form $\boldsymbol{X}$ than the $X^{\underline{i}}$, then from (2.13) this is immediate. However, we need to use the covariantization relations (2.26) to recast the derivatives in the desired form $\partial_t \boldsymbol{X}$ or $\partial_s \boldsymbol{X}$. The resulting equations take a more transparent form if we separate all vectors in sectional and longitudinal parts according to (2.39) and write equations for these components. Since the longitudinal and sectional projections involve only the tangential direction $\boldsymbol{T}$, then the dependence in the section basis $\boldsymbol{d}_a$ disappears. The FA coordinates and the orthonormal basis were introduced to perform all intermediary computations, but the final results need not be expressed in this language. This justifies *a posteriori* why we have chosen the special choice (2.7) for the curvature and discarded the possibility of having a nonvanishing longitudinal component for the FCL curvature. Indeed, this would lead to the same final equations when expressed in a covariant form. However, all intermediate computations would be more involved because the expression of the metric (2.35) would be much more complicated and nondiagonal, and the Christoffels (2.44) would also be more complex. In particular, as a consequence of this choice for curvature, the partial derivatives with respect to s are easily written in a covariant form since for any vector $\boldsymbol{X}$ we deduce from (2.25b) the property

$$P_\perp(\partial_s \boldsymbol{X}_\perp) = (\partial_s X^a)\boldsymbol{d}_a. \tag{7.27}$$

This allows to read the covariant form of a given equation written in terms of sectional components nearly instantly.

We gather in covariant form the complete set of equations described in Sec. VII B 8. First, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{U}$ are decomposed in sectional parts and longitudinal parts as in (2.39). The constraint equations are

$$\partial_s \boldsymbol{R} = \boldsymbol{T}, \tag{7.28a}$$

$$\partial_s \boldsymbol{T} = \boldsymbol{\kappa} \times \boldsymbol{T} = -\widetilde{\boldsymbol{\kappa}} \Leftrightarrow \boldsymbol{\kappa} = \boldsymbol{T} \times \partial_s \boldsymbol{T}, \tag{7.28b}$$

$$\boldsymbol{\omega}_\perp = \boldsymbol{T} \times \partial_s \boldsymbol{U} = \partial_s \widetilde{\boldsymbol{U}} + \boldsymbol{\kappa}\overline{U}, \tag{7.28c}$$

$$0 = \boldsymbol{T} \cdot \partial_s \boldsymbol{\omega} \Leftrightarrow \partial_s \overline{\omega} = -\boldsymbol{\omega} \cdot \widetilde{\boldsymbol{\kappa}}, \tag{7.28d}$$

$$0 = \boldsymbol{T} \cdot \partial_s \boldsymbol{U} \Leftrightarrow \partial_s \overline{U} = -\boldsymbol{U} \cdot \widetilde{\boldsymbol{\kappa}}. \tag{7.28e}$$

As for the dynamical equations, they are recast as

$$\partial_t \boldsymbol{R} = \boldsymbol{U}, \tag{7.29a}$$

$$\partial_t \boldsymbol{\kappa} = \partial_s \boldsymbol{\omega} + \boldsymbol{\omega} \times \boldsymbol{\kappa} \Leftrightarrow P_\perp(\partial_t \boldsymbol{\kappa}) = P_\perp(\partial_s \boldsymbol{\omega}_\perp),$$

$$\partial_t \boldsymbol{T} = \boldsymbol{\omega} \times \boldsymbol{T}, \tag{7.29b}$$

$$\partial_t \ln R = -\mathcal{H}v - \tfrac{1}{2}\partial_s v, \tag{7.29c}$$

$$\partial_t v = -\partial_t \overline{U} - \boldsymbol{U} \cdot \widetilde{\boldsymbol{\omega}} + \overline{g} + \nu \frac{\mathcal{H}}{R} - v\partial_s v + 6\mathcal{H}\partial_s v + 3\partial_s^2 v - \overline{I} - 2(\boldsymbol{\Omega} \times \boldsymbol{U}) \cdot \boldsymbol{T}, \tag{7.29d}$$

$$P_\perp[\partial_t \boldsymbol{U}_\perp] = \boldsymbol{T} \times \left[(\overline{U} + 2v)\boldsymbol{\omega} + \left(v^2 - 3\partial_s v - \frac{\nu}{R}\right)\boldsymbol{\kappa}\right] + [\boldsymbol{g} - \boldsymbol{I}]_\perp + 2\boldsymbol{T} \times [(v + \overline{U})\boldsymbol{\Omega} - \overline{\Omega}\boldsymbol{U}], \tag{7.29e}$$

where we should use that for any vector

$$\partial_t \boldsymbol{X}_\perp = P_\perp[\partial_t \boldsymbol{X}_\perp] + (\boldsymbol{X}_\perp \cdot \widetilde{\boldsymbol{\omega}})\boldsymbol{T}, \quad (7.30a)$$
$$\partial_s \boldsymbol{X}_\perp = P_\perp[\partial_s \boldsymbol{X}_\perp] + (\boldsymbol{X}_\perp \cdot \widetilde{\boldsymbol{\kappa}})\boldsymbol{T}. \quad (7.30b)$$

10. Stationary regime

In the stationary regime, all partial time derivatives vanish and the viscous fiber model takes a simpler form. The velocity of the fiber center also necessarily vanishes and $U^a = \overline{U} = 0$ as well as the rotation ($\omega^{\underline{i}} = 0$). In that case, it becomes an ordinary differential equation in the FCL parameter s. For completeness, we report here the set of stationary equations, and they read as

$$\partial_s \boldsymbol{R} = \boldsymbol{T}, \quad (7.31a)$$
$$\partial_s \ln R \simeq -\frac{1}{2}\partial_s \ln v, \quad (7.31b)$$
$$3\partial_s^2 v + 6\mathcal{H}\partial_s v - v\partial_s v \simeq -\frac{\nu\mathcal{H}}{R} + \overline{I} - \overline{g}, \quad (7.31c)$$
$$\widetilde{\kappa}_a\left(v^2 - \frac{\nu}{R} - 3\partial_s v\right) \simeq I_a - g_a - 2v\widetilde{\Omega}_a, \quad (7.31d)$$
$$\partial_s \boldsymbol{d}_{\underline{i}} = \boldsymbol{\kappa} \times \boldsymbol{d}_{\underline{i}}. \quad (7.31e)$$

Equation (7.31d) is used to determine the curvature κ^a (but it can become singular) and (7.31c) is used to integrate v along the FCL. Equation (7.31b) is the statement that vR^2 is constant in a stationary regime, due to incompressibility.

If written in covariant form, the last three equations of (7.31) read as

$$3\partial_s^2 v + 6\mathcal{H}\partial_s v - v\partial_s v \simeq -\frac{\nu\mathcal{H}}{R} + \boldsymbol{T}\cdot(\boldsymbol{I} - \boldsymbol{g}), \quad (7.32a)$$
$$\boldsymbol{\kappa}\left(v^2 - \frac{\nu}{R} - 3\partial_s v\right) \simeq \boldsymbol{T} \times [\boldsymbol{g} - \boldsymbol{I}] - 2v\boldsymbol{\Omega}, \quad (7.32b)$$
$$\partial_s \boldsymbol{T} = \boldsymbol{\kappa} \times \boldsymbol{T} = -\widetilde{\boldsymbol{\kappa}}, \quad (7.32c)$$

which is the standard form for the stationary curved string model in the literature.

C. Beyond the string model

1. Limitations of the string model

Apart for surface tension effects, the spatial extension of sections, that is the radius R, does not play any role in the dynamical equations, meaning that the internal structure of the fiber has no impact on the dynamics. Furthermore, at lowest order the sectional part of the viscous forces which reduces to the components $\widehat{\mathcal{F}}_a^{(0)}$ vanishes. The viscous forces are purely longitudinal, as would be the case in a string, thus justifying the name of the approximation. If we want to consider a model for viscous fibers in which the size of spatial sections plays a role, we must necessarily consider higher order terms in the parameter ϵ_R.

Since fiber sections rotate at the same angular velocity as the fluid located on the FCL [see Eq. (7.15)], it means also that a rod model in which sections are not mixed and remain orthogonal to the fiber tangential direction cannot be part of this higher order model. Hence, when considering the lowest order of the angular momentum balance equation, we do not obtain a dynamical equation which gives the time evolution of the fiber section rotation as a function of viscous forces, but rather obtain a constraint on the viscous forces (more precisely on their sectional component) given that the fiber section rotation is already determined by the string model.

For higher order models, the corrections of order ϵ_R^2 in Eq. (7.15) will imply that fluid particles of different sections will be mixed as a result of time evolution. Higher order models must also necessarily take into account that the velocity of the fluid *on the central line* is not exactly the velocity *of the central line*. Indeed, Eq. (4.28) implies that there is a tiny shift between the two, which is an order ϵ_R^2 correction.

Finally, terms of the type (7.2) typically source the shape quadrupole $\mathcal{R}_{ab}$ and a model restricted to circular sections cannot be sufficient when considering corrections to the string limit. As we shall explain in this section, when including order ϵ_R^n effect, we must include all multipoles $\mathcal{R}_L$ with $\ell \leqslant n$. Since we are interested in the first corrections which are of order ϵ_R^2, we consider quadrupolar shape moments thereafter.

2. Normal vector and extrinsic curvature

When including a first set of corrections, the components of the normal vector and the unit normal vector are approximately given by

$$N_{\underline{3}} = -\mathcal{H}R + \mathcal{H}R^2(n_a\widetilde{\kappa}^a) + O\left(\epsilon_R^3\right), \quad (7.33a)$$
$$N_a = n_a - 2R^2 \perp_a^b \mathcal{R}_{bc}n^c - 3R^3 \perp_a^b \mathcal{R}_{bcd}n^c n^d + O\left(\epsilon_R^4\right), \quad (7.33b)$$
$$\widehat{N}_{\underline{3}} = -\mathcal{H}R + \mathcal{H}R^2(n_a\widetilde{\kappa}^a) + O\left(\epsilon_R^3\right), \quad (7.33c)$$
$$\widehat{N}_a = n_a - R^2\left(\frac{\mathcal{H}^2}{2}n_a - 2\perp_a^b \mathcal{R}_{bc}n^c\right) + R^3\left(-3\perp_a^b \mathcal{R}_{bcd}n^c n^d + \mathcal{H}^2 n_a n_b \widetilde{\kappa}^b\right) + O\left(\epsilon_R^4\right). \quad (7.33d)$$

From (4.23), the extrinsic curvature can then be obtained. Expanding $R\mathcal{K}$ in moments as in Eq. (5.7a), the first moments which are used to compute the first set of corrections to the string model are

$$[R\mathcal{K}]_\varnothing = 1 - R^2\left(\tfrac{3}{2}\mathcal{H}^2 + \tfrac{1}{2}\kappa_a\kappa^a + \partial_s\mathcal{H}\right) + O\left(\epsilon_R^4\right), \quad (7.34a)$$
$$[R\mathcal{K}]_a = \widetilde{\kappa}_a\left[1 + R^2\left(\tfrac{7}{2}\mathcal{H}^2 + \tfrac{3}{4}\kappa_b\kappa^b + 2\partial_s\mathcal{H}\right)\right] + R^2(\mathcal{H}\partial_s\widetilde{\kappa}_a - \mathcal{R}_{ab}\widetilde{\kappa}^b) + O\left(\epsilon_R^4\right), \quad (7.34b)$$
$$[R\mathcal{K}]_{ab} = 3\mathcal{R}_{ab} + \kappa_{\langle a}\kappa_{b\rangle} + O\left(\epsilon_R^2\right). \quad (7.34c)$$

3. Fundamental dynamical equations

The general method to build higher order corrections is similar to the axisymmetric case. The only difference is that now a given equation will not just give a constraint on the monopole but also on other moments. As we restrict to second order, we shall need to consider the monopole, dipole, and quadrupole of equations only. The incompressibility conditions (7.3) are also used throughout to express any dependence in multipoles $u_L^{(n)}$ in terms of other types of multipoles.

TABLE III. Fundamental dynamical equations, with the corresponding variables and the main variables on which their evolution depends. The dependencies for the last two equations giving the evolution of radius and shape quadrupole are given only when order ϵ_R^2 terms are included.

Equation	Variable	Essential dependence
$\overline{\mathcal{D}}^{(0)}_{\varnothing}$ (7.6) and (7.35)	$\partial_t v^{(0)}$	$v^{(1)}_{\varnothing}$, $P^{(0)}_{\varnothing}$, $v^{(0)}_a$, $\widehat{V}^{(0)}_a$
$\widehat{\mathcal{D}}^{(0)}_a$ (7.7) and (7.36)	$\partial_t U_a$	$\widehat{V}^{(1)}_a$, $P^{(0)}_a$, $v^{(0)}_a$, $\widehat{V}^{(0)}_a$, $\widehat{V}^{(0)}_{ab}$
${}^{\mathcal{R}}\mathcal{E}^{(\leqslant 2)}_{\varnothing}$ (4.25)	$\partial_t R$	$v^{(0)}_a$, $\widehat{V}^{(1)}_a$, $v^{(0)}_{\varnothing}$, $v^{(1)}_{\varnothing} R^2$
${}^{\mathcal{R}}\mathcal{E}^{(\leqslant 2)}_{ab}$ (4.25)	$\partial_t \mathcal{R}_{ab}$	$\widehat{V}^{(0)}_{ab}$, $\widehat{V}^{(1)}_{ab}$

We start from lowest moments of the Navier-Stokes equation components ($\overline{\mathcal{D}}^{(0)}_{\varnothing}$, $\widehat{\mathcal{D}}^{(0)}_a$), which give the fundamental dynamical equations for the longitudinal velocity $v = v^{(0)}_{\varnothing}$, and the FCL sectional velocity U^a. The evolution of the axial rotation $\dot{\phi} = \dot{\phi}^{(0)}$ has already been found in Eq. (7.18c) and we have argued that it should be considered as part of the first corrections. Equations (7.6) and (7.7) were not given in full generality as we had removed the contributions from $\widehat{V}^{(0)}_a$ and $\widehat{V}^{(0)}_{ab}$ which are order ϵ_R^2 quantities. The Navier-Stokes components (7.6) and (7.7) must be supplemented by the contributions

$$\overline{A}^{(0)}_{\varnothing} \supset \widehat{V}^{(0)}_a (2\widetilde{\Omega}^a + \widetilde{\omega}^a + v\widetilde{\kappa}^a + v^{(0)a}), \tag{7.35a}$$

$$\overline{f}^{(0)}_{\varnothing} \supset 2\widetilde{\kappa}^a \partial_s \widehat{V}^{(0)}_a + \widehat{V}^{(0)}_a \partial_s \widetilde{\kappa}^a, \tag{7.35b}$$

$$\widehat{A}^{(0)}_a \supset \widehat{V}^{(0)b} \widehat{V}^{(0)}_{ab} - \tfrac{1}{2}\widehat{V}^{(0)}_a \big(\partial_s v + \widehat{V}^{(0)}_b \widetilde{\kappa}^b\big) + v\partial_s \widehat{V}^{(0)}_a - \varepsilon_a{}^b \widehat{V}^{(0)}_b (\dot{\phi} + \bar{\omega} - 2\overline{\Omega}) + \partial_t \widehat{V}^{(0)}_a, \tag{7.36a}$$

$$\widehat{f}^{(0)}_a \supset \tfrac{1}{2}\widehat{V}^{(0)}_b \kappa_a \kappa^b - \tfrac{1}{2}\widehat{V}^{(0)}_a \kappa_b \kappa^b + \tfrac{1}{3}\widehat{V}^{(0)}_{ab}\widetilde{\kappa}^b + \partial_s^2 \widehat{V}^{(0)}_a \tag{7.36b}$$

to be fully general.

We also need to consider the dynamics of the shape. As in the string model, we need to consider the monopole of the boundary kinematics (4.25) to determine the evolution of the radius. The dipole of this equation has already been considered in Eq. (4.28) to fix the gauge and determine $\widehat{V}^{(0)}_a$. Furthermore, we also need to consider the quadrupole of the boundary kinematics equation (4.25) so as to determine the evolution of $\mathcal{R}_{ab}$. These dynamical equations need to be truncated at the required order. In Table III, we summarize the essential dependencies of the fundamental dynamical equations which need to be determined from constraints.

4. General structure of constraint equations

The method follows essentially the same steps as in the axisymmetric case. The constraints which were obtained at lowest order in Sec. VII B 4 need to be extended to include order ϵ_R^2 corrections, so as to be replaced in the fundamental dynamical equations. We follow the same procedure except that all constraints are considered up to a higher order. For instance, when deriving the string model we considered the monopole of the radial constraint at lowest order ${}^r\mathcal{C}^{(0)}_{\varnothing}$ to constrain $P^{(0)}_{\varnothing}$, and now we must consider ${}^r\mathcal{C}^{(1)}_{\varnothing}$. However, just like when finding the corrections of the axisymmetric case, the price to pay is that we introduce new dependencies. For instance, from ${}^r\mathcal{C}^{(1)}_{\varnothing}$ we can obtain corrections for the constraint which determines $P^{(0)}_{\varnothing}$, but it involves $P^{(0)}_{\varnothing} R^2$. This set of dependencies is summarized in Table IV.

TABLE IV. Structure of boundary constraints. Each multipole of each constraint is used to determine a variable as a function of other variables. In each case, we specify which variable is determined and report the essential variables on which it depends to emphasize the structure of the recursive method. All these constraints need to be truncated at a given power of R.

Equation	Variable	Essential dependence
${}^r\mathcal{C}_{\varnothing}$	$P^{(0)}$	$v^{(0)}$ and $P^{(1)}R^2$, $P^{(2)}R^4 \dots$
${}^r\mathcal{C}_a$	$P^{(0)}_a$	$v^{(0)}_a$ and $P^{(1)}_a R^2$, $P^{(2)}_a R^4 \dots$
${}^r\mathcal{C}_{ab}$	$P^{(0)}_{ab}$	$v^{(0)}_{ab}$ and $P^{(1)}_{ab} R^2$, $P^{(2)}_{ab} R^4 \dots$
$\overline{\mathcal{C}}_{\varnothing}$	$v^{(1)}$	$v^{(0)}$ and $v^{(2)}R^2$, $v^{(3)}R^4 \dots$
$\overline{\mathcal{C}}_a$	$v^{(0)}_a$	$v^{(1)}_a R^2$, $v^{(2)}_a R^4 \dots$
$\overline{\mathcal{C}}_{ab}$	$v^{(0)}_{ab}$	$v^{(1)}_{ab} R^2$, $v^{(2)}_{ab} R^4 \dots$
${}^\theta\mathcal{C}_{\varnothing}$	$\dot{\phi}^{(1)}$	$\dot{\phi}^{(0)}$ and $\dot{\phi}^{(2)}R^2$, $\dot{\phi}^{(3)}R^4 \dots$
${}^\theta\mathcal{C}_a$	$\widehat{V}^{(1)}_a$	$\widehat{V}^{(2)}_a R^2$, $\widehat{V}^{(3)}_a R^4 \dots$
${}^\theta\mathcal{C}_{ab}$	$\widehat{V}^{(0)}_{ab}$	$\widehat{V}^{(1)}_{ab} R^2$ and $\widehat{V}^{(2)}_{ab} R^4 \dots$
Gauge fixing	$\widehat{V}^{(0)}_a$	$v^{(0)}_a$, $\widehat{V}^{(1)}_a$, $v^{(0)} \dots$

The solution to this problem follows exactly the method used in the axisymmetric case. We use the higher order moments of the Navier-Stokes, which we consider as constraints and not dynamical equations, together with the incompressibility constraint (5.15) to remove these newly introduced variables. This is made possible since, as in the axisymmetric case, these equations have Laplacians which allow us to use the property (6.16).

We thus need to follow a recursive algorithm which is very similar to the one used in the axisymmetric case, but which is more involved since it involves the ℓth order STF moments when considering order ϵ_R^ℓ corrections. Furthermore, the structure of the recursive algorithm is slightly different for the dipole components since (i) the gauge constraint already determines one dipolar moment ($\widehat{V}^{(0)}_a$) and (ii) one equation is used to determine the external variable U^a. The corresponding set of dependencies is summarized in Table V.

5. Quadrupoles of constraints

Let us first examine the quadrupoles of the constraints for which we need only the lowest order expressions. From the quadrupole of the orthoradial constraint at lowest order, that is, from ${}^\theta\mathcal{C}^{(0)}_{ab}$ we get that

$$\widehat{V}^{(0)}_{ab} = R^2\Big[-\tfrac{3}{2}\widehat{V}^{(1)}_{ab} - \tfrac{3}{8}\kappa_{\langle a}\widetilde{\kappa}_{b\rangle}\dot{\phi} - \tfrac{3}{16}\kappa_{\langle a}\kappa_{b\rangle}\partial_s v + \tfrac{3}{8}\kappa_{\langle a}(v\partial_s\kappa_{b\rangle} + \partial_s\omega_{b\rangle})\Big] + O(\epsilon_R), \tag{7.37}$$

where we recall that the notation $\langle a_1 \dots a_n\rangle$ means the STF part and the notation $(a_1 \dots a_n)$ means the symmetric part. Actually,

TABLE V. Structure of dependence for the constraints obtained either from the higher moments of the Navier-Stokes equation or from the incompressibility constraint. We have indicated only the essential dependence, so as to emphasize clearly the structure of the recursive method, but it depends in general on the full set of lower order variables.

Equation	Variable	Essential dependence
$\overline{\mathcal{D}}^{(n+1)}_{\varnothing}$	$v^{(n+2)}_{\varnothing}$	$P^{(n+1)}_{\varnothing}$, $v^{(n+1)}_{\varnothing}$
$\overline{\mathcal{D}}^{(n)}_{a}$	$v^{(n+1)}_{a}$	$v^{(n)}_{a}$, $P^{(n)}_{a}$
$\overline{\mathcal{D}}^{(n)}_{ab}$	$v^{(n+1)}_{ab}$	$v^{(n)}_{ab}$, $P^{(n)}_{ab}$
$\mathring{\mathcal{D}}^{(n+1)}$	$\dot{\phi}^{(n+1)}$	$P^{(n+1)}_{\varnothing}$, $\dot{\phi}^{(n)}$
$\widehat{\mathcal{D}}^{(n+1)}_{a}$	$\widehat{V}^{(n+2)}_{a}$	$\widehat{V}^{(n+1)}_{a}$
$\widehat{\mathcal{D}}^{(n)}_{ab}$	$\widehat{V}^{(n+1)}_{ab}$	$\widehat{V}^{(n)}_{ab}$
${}^{D}\mathcal{C}^{(n)}_{\varnothing}$	$P^{(n+1)}_{\varnothing}$	$P^{(n)}_{\varnothing}$
${}^{D}\mathcal{C}^{(n)}_{a}$	$P^{(n+1)}_{a}$	$P^{(n)}_{a}$
${}^{D}\mathcal{C}^{(n)}_{ab}$	$P^{(n+1)}_{ab}$	$P^{(n)}_{ab}$

the property that surface tension terms start at order ϵ_R and viscous terms start at order ϵ_R^2 arises for all $\widehat{V}^{(0)}_L$ with $\ell \geqslant 2$.

This property is fortunate since conversely from the lowest order of the quadrupole of the radial constraint ${}^{r}\mathcal{C}^{(0)}_{ab}$ we find that $P^{(0)}_{ab}$ depends on $\widehat{V}^{(0)}_{ab}/R^2$. Once this dependence is replaced, we get

$$P^{(0)}_{ab} = \frac{\nu}{R}\left(3\mathcal{R}_{ab} + \kappa_{\langle a}\kappa_{b\rangle}\right) + \frac{3}{2}\kappa_{\langle a}\widetilde{\kappa}_{b\rangle}\dot{\phi} + \frac{3}{4}\kappa_{\langle a}\kappa_{b\rangle}\partial_s v - \frac{3}{2}\kappa_{\langle a}(v\partial_s\kappa_{b\rangle} + \partial_s\omega_{b\rangle}) + O(\epsilon_R). \tag{7.38}$$

Finally, from the quadrupole of the longitudinal constraint $\overline{\mathcal{C}}^{(0)}_{ab}$ we get

$$v^{(0)}_{ab} = 0 + O\left(\epsilon_R^2\right). \tag{7.39}$$

6. Monopoles and dipoles of constraints

We now examine the monopoles and dipoles of the constraints. The lowest orders have been found already in Sec. VII B 4 when deriving the viscous string model, and they need to be used to replace variables appearing in the order ϵ_R^2 terms. Once this is done, then from the constraints $\overline{\mathcal{C}}^{(\leqslant 1)}_{\varnothing}$ and $\overline{\mathcal{C}}^{(\leqslant 1)}_{a}$ we get that the constraints (7.12) should be supplemented by

$$v^{(1)} \supset \tfrac{1}{32}R^2\big[-64v^{(2)} - 16v^{(1)}_a\widetilde{\kappa}^a + 6\mathcal{H}\kappa_a\kappa^a\partial_s v + 84\mathcal{H}\partial_s\mathcal{H}\partial_s v - 12\mathcal{H}v\kappa^a\partial_s\kappa_a - 12\mathcal{H}\kappa^a\partial_s\omega_a + 6\partial_s v\partial_s^2\mathcal{H} + 84\mathcal{H}^2\partial_s^2 v + 12\partial_s\mathcal{H}\partial_s^2 v + 48\mathcal{H}^3\partial_s v + 20\mathcal{H}\partial_s^3 v + \partial_s^4 v\big], \tag{7.40a}$$

$$v^{(0)}_a \supset \tfrac{1}{4}R^2\big[-12v^{(1)}_a + 10\mathcal{H}\kappa_a\dot{\phi} - 11\mathcal{H}\widetilde{\kappa}_a\partial_s v + 10\mathcal{H}v\partial_s\widetilde{\kappa}_a + 10\mathcal{H}\partial_s\widetilde{\omega}_a - \widetilde{\kappa}_a\partial_s^2 v\big]. \tag{7.40b}$$

From the constraint ${}^{\theta}\mathcal{C}^{(\leqslant 1)}_{\varnothing}$ we get an expression for $\dot{\phi}^{(1)}$ but we do not need any higher order contribution since the dynamical equation for $\dot{\phi}$ [Eq. (7.18c)] is already part of the first corrections. However, from ${}^{\theta}\mathcal{C}^{(\leqslant 1)}_{a}$ we get the first corrections for the constraint on $\widehat{V}^{(1)}_a$. Finally, from the constraints ${}^{r}\mathcal{C}^{(\leqslant 1)}_{\varnothing}$ and ${}^{r}\mathcal{C}^{(\leqslant 1)}_{a}$, we obtain the corrections on the constraints for $P^{(0)}_{\varnothing}$ and $P^{(0)}_a$. These are rather large expressions, and we gathered them in Appendix F.

As in the symmetric case, the constraint for $v^{(1)}$ now depends on $v^{(2)}$ when including order ϵ_R^2 corrections, and we shall thus need another constraint to replace it. In fact, we also need constraints for $v^{(1)}_a$, $\widehat{V}^{(2)}_a$, and $\widehat{V}^{(1)}_{ab}$. As in the axisymmetric case, these additional constraints will come from higher moments of the Navier-Stokes equations and the incompressibility constraint (see Table V). Since the expressions of these constraints can be rather large, we report them in Appendix F.

7. Corrections for dynamical equations

The evolution of the radius with the first corrections included is given by

$$\partial_t \ln R \supset R^2\left(-\tfrac{3}{2}\mathcal{H}^2\partial_s v - \tfrac{3}{8}\partial_s\mathcal{H}\partial_s v - \tfrac{5}{8}\mathcal{H}\partial_s^2 v - \tfrac{1}{16}\partial_s^3 v\right). \tag{7.41}$$

It is striking that once all the constraints are properly replaced, we reach the same expression as in the axisymmetric case (6.23c), even though the original expression deduced from the monopole of (4.25) is formally more complex. In principle, we could apply the same method as in Sec. VI D and use the average longitudinal velocity instead of $v^{(0)}$, thus changing the fundamental variable.

Concerning the corrections of the dynamical equations for v and U_a, we report them in Appendix G. They are obtained from the replacement of all constraints, and then the replacement of the lowest order dynamical equation to remove the time derivatives appearing in the corrective terms. Several comments are in order here.

(i) As for the lowest order string model, these equations could be recast in a covariant form, that is, using the canonical Cartesian basis, following the method described in Sec. VII B 9. This is especially straightforward when using the property (7.27), so we do not write it explicitly.

(ii) Odd powers of R correspond to surface tension effects (R^{-1} for the lowest order and R for the corrections), whereas even powers of R correspond to inertial and viscous effects (R^0 for the lowest order and R^2 for the first corrections).

(iii) We remark that the quadrupoles of shape $\mathcal{R}_{ab}$ which appear from surface tension effects do not retroact on the dynamical equations for v and U_a.

(iv) In practice, it proves easier to multiply the equations considered [e.g., boundary constraint (5.5) or Navier-Stokes (5.11)] by h or h^2, so as to avoid unnecessary factors $h^{-1} = 1/(1+\widetilde{\kappa}_a y^a)$ or h^{-2} which would need to be expanded in an infinite series in $\widetilde{\kappa}_a y^a$. For instance, $\mathcal{K}$ given by (4.23) involves h^{-1}, but it is not the case for $h\mathcal{K}$. All products of tensors fully contracted with vectors y^a are then handled thanks to (A13). Hence, it is in principle possible to find very general relations between multipoles, as we did for instance with the incompressibility condition (4.5).

Note that the dynamical evolution of $\dot{\phi}$ needs to be obtained from Eq. (7.18c). Similarly, the dynamical evolution of the quadrupole now needs to be determined independently. From the quadrupole of (4.25) and using the constraint (7.37), we find the simple lowest order dynamical equation

$$(\partial_t + v\partial_s)\mathcal{R}_{ab} = -\widehat{V}^{(1)}_{ab} + \mathcal{R}_{ab}\partial_s v - 2\varepsilon_{(a}^{\ \ c}\mathcal{R}_{b)c}(\dot{\phi} - \overline{\varpi}), \tag{7.42}$$

in which one should replace the constraint (F7).

Note that since the evolution of $\mathcal{R}_{ab}$ is sourced by $\widehat{V}^{(1)}_{ab}$, then from (F7) we see that it contains typical terms of the type $\kappa_{\langle a}\kappa_{b\rangle}\partial_s v$. If we were to consider the dynamics of higher multipoles such as $\mathcal{R}_L$, it would be sourced by terms of the type $\kappa_{\langle a_1}\ldots\kappa_{a_\ell\rangle}\partial_s v$ among other terms. For comparison, from (7.20) we see that $\ln R$ is typically sourced by $\partial_s v$. Hence, the higher the shape multipole is, the more spatial derivatives are involved in its dynamics. This justifies why when considering corrections of order ϵ_R^n we need only to consider the multipoles $\mathcal{R}_L$ with $\ell \leqslant n$. It also justifies *a posteriori* why we are working with the shape multipoles $\mathcal{R}_L$ (more precisely their dimensionally reduced variables $\underline{\mathcal{R}}_L$) as defined in Eq. (4.12) and not the $\widehat{\mathcal{R}}_L$ of (4.13), which are better suited to describe relative shape perturbations.

D. Straight fibers with elliptic sections

It is now easy to consider the case of straight fibers but with noncircular sections. We need only to consider the special case $\kappa^a = \omega^a = \overline{\varpi} = U^a = \overline{U} = 0$. We recover immediately the dynamical equations found in the axisymmetric case with the first corrections included [(6.10), (6.23c), and corrections (6.23a)]. However, we also obtain the dynamical evolution of the shape quadrupole, which evolves as an independent equation. Indeed, to retroact on the dynamics of v we would need terms of the type $\mathcal{R}_{ab}\mathcal{R}^{ab}$ which would appear only when corrections of order ϵ_R^4 are included.

Equation (7.42) restricted to straight fibers takes formally the same form, except that the constraint (F7) now needs to be also considered in that restriction when replaced. The last term of Eq. (7.42) is expected, as it just states that axial rotation $\dot{\phi}$ will rotate the ellipticity, but the sectional rotation of the orthonormal basis ($\overline{\varpi}$) must also be taken into account and subtracted. However, it is only really an effect of the choice of basis to measure components. Indeed, we can rewrite it in a manifestly covariant form following the method of Sec. VII B 9. Defining $\mathcal{R}_{\mu\nu} \equiv \mathcal{R}_{ab}d^a_\mu d^b_\nu$, the shape quadrupole evolution is given by

$$\begin{aligned}&P_{\perp\mu}^{\ \alpha}P_{\perp\nu}^{\ \beta}[(\partial_t + v\partial_s)\mathcal{R}_{\alpha\beta}]\\ &\quad = -\widehat{V}^{(1)}_{ab}d^a_\mu d^b_\nu + \mathcal{R}_{\mu\nu}\partial_s v - 2\dot{\phi}T^\beta\varepsilon^{\alpha}_{\beta(\mu}\mathcal{R}_{\nu)\alpha},\end{aligned} \tag{7.43}$$

and we can check that the contribution from the orthonormal basis rotation $\overline{\varpi}$ has disappeared.

Let us now compute the evolution of the relative ellipticity, that is, the evolution of the dimensionless moments $\widehat{\mathcal{R}}_{ab} = R^2\mathcal{R}_{ab}$. By using the lowest order of the fiber radius evolution (7.41), we finally find that the dimensionless quadrupole evolves according to

$$(\partial_t + v\partial_s)\widehat{\mathcal{R}}_{ab} = -\frac{\nu}{R}\widehat{\mathcal{R}}_{ab} - 2\mathcal{H}v\widehat{\mathcal{R}}_{ab} - 2\varepsilon_{(a}^{\ \ c}\widehat{\mathcal{R}}_{b)c}(\dot{\phi} - \overline{\varpi}). \tag{7.44}$$

In a stationary regime ($\partial_t\widehat{\mathcal{R}}_{ab} = 0$) we see that there is a competition between surface tension effects which tend to decrease ellipticity and stretching ($\mathcal{H} \leqslant 0$) which tends to increase the relative contribution of ellipticity. Indeed, if we assess the evolution of $Q^2 \equiv \widehat{\mathcal{R}}_{ab}\widehat{\mathcal{R}}^{ab}$ from the previous equation, its second line which is a purely rotational effect does not contribute since $\varepsilon_a^{\ c}\widehat{\mathcal{R}}_{cb}\widehat{\mathcal{R}}^{ba} = 0$, and we get simply

$$(\partial_t + v\partial_s)Q = -\frac{\nu}{R}Q - 2\mathcal{H}vQ. \tag{7.45}$$

If it is true that stretching increases the ellipticity, the surface tension effects encompassed in the first term on the right-hand side would eventually dominate and damp ellipticity, as R is reduced by stretching.

E. Alternative method for rotating frames

In our formalism we have allowed for the possibility to be working in a rotating frame. This was taken into account in the Navier-Stokes equation, by adding the fictitious forces term (5.13). There is, however, a simpler method to recover all expressions in a rotating frame. First, we derive the results in a nonrotating frame, allowing to cut by approximately half the number of terms in the final results, and then we relate all variables in the nonrotating frame with their counterparts in the rotating frame. We denote by ${}^{\mathrm{G}}\omega^{\underline{i}}$ the components of the rotation rate of the orthonormal basis [defined by (2.16)] in the nonrotating frame, and by ${}^{\mathrm{R}}\omega^{\underline{i}}$ its counterpart in the rotating frame. Similarly, we define ${}^{\mathrm{G}}\mathcal{V}^{\underline{i}}$ and ${}^{\mathrm{R}}\mathcal{V}^{\underline{i}}$ as the velocity in the nonrotating frame and rotating frame, respectively, and adopt a similar notation for the FCL velocity components $U^{\underline{i}}$. These quantities are related by

$${}^{\mathrm{G}}\mathcal{V}^{\underline{i}} = {}^{\mathrm{R}}\mathcal{V}^{\underline{i}} + [\boldsymbol{\Omega} \times \boldsymbol{x}]^{\underline{i}}, \tag{7.46a}$$

$${}^{\mathrm{G}}U^{\underline{i}} = {}^{\mathrm{R}}U^{\underline{i}} + [\boldsymbol{\Omega} \times \boldsymbol{R}]^{\underline{i}}, \tag{7.46b}$$

$${}^{\mathrm{G}}\omega^{\underline{i}} = {}^{\mathrm{R}}\omega^{\underline{i}} + \Omega^{\underline{i}}, \tag{7.46c}$$

where we recall that the wedge products are performed according to (2.14). In particular, when considering the relative velocity with respect to the FCL [V^μ defined in Eq. (4.1)], we note that the previous relations imply that the moments of its longitudinal and sectional parts are unchanged except for

$${}^{\mathrm{G}}\dot{\phi}^{(0)} = {}^{\mathrm{R}}\dot{\phi}^{(0)} + \overline{\Omega}, \quad {}^{\mathrm{G}}v_a^{(0)} = {}^{\mathrm{R}}v_a^{(0)} + \widetilde{\Omega}_a. \tag{7.47}$$

In order to replace all variables referring to the nonrotating frame in terms of the variables referring to the rotating frame, we also need to be able to relate the time derivatives of the FCL velocity components. Specifically, we need

$$\partial_t {}^{\mathrm{G}}U^{\underline{i}} = \partial_t {}^{\mathrm{R}}U^{\underline{i}} - [{}^{\mathrm{R}}\omega \times (\boldsymbol{\Omega} \times \boldsymbol{R})]^{\underline{i}} + [\boldsymbol{\Omega} \times {}^{\mathrm{R}}U]^{\underline{i}}, \tag{7.48a}$$

$$\partial_t\Omega^{\underline{i}} = [\boldsymbol{\Omega} \times {}^{\mathrm{R}}\omega]^{\underline{i}}, \tag{7.48b}$$

where we emphasize that in these expressions, we are considering time derivatives of components. The relation (7.48a) is obtained from the definition (2.4) (which reads as here $\partial_t R^\mu = {}^{\mathrm{G}}U^\mu$) and (7.46b), by using the definition $[\boldsymbol{\Omega} \times \boldsymbol{R}]^{\underline{i}} \equiv d^{\underline{i}}_\mu(\boldsymbol{\Omega} \times \boldsymbol{R})^\mu$ and the property (2.16). The relation (7.48b) is obtained from the definition $\Omega^{\underline{i}} \equiv d^{\underline{i}}_\mu \Omega^\mu$ and the property (2.16). Finally, we also need the derivatives

$$\partial_s([\boldsymbol{\Omega} \times \boldsymbol{R}]^{\underline{i}}) = -\widetilde{\Omega}^{\underline{i}} - [\boldsymbol{\kappa} \times (\boldsymbol{\Omega} \times \boldsymbol{R})]^{\underline{i}}. \tag{7.49}$$

Using (7.46)–(7.48), we were able to check that starting from the expressions found in a nonrotating frame ($\Omega_a = \overline{\Omega} = 0$), we recover the expressions valid in a rotating frame, for all constraints and all dynamical equations. It is thus a healthy consistency check for the validity and correctness of the method.

F. Physical insights on constraints

In this section we give physical interpretations for the various velocity components. For simplicity, we neglect the effect of curvature, so as to emphasize the physical effects more clearly.

1. Pressure constraint and the Trouton ratio

Let us first analyze how the constraint for $v^{(1)}$ [Eq. (7.12)] arises at lowest order. If we have a longitudinal velocity gradient ($\partial_s v \neq 0$), then we have a radial infall which is constrained by incompressibility as we get $u = -\partial_s v$ from (4.6). As the radial infall is proportional to the radial distance $V^a \supset u y^a/2$, there is a radial gradient in this radial velocity. Thanks to viscous forces, this creates a component $\tau^{(\mu)}_{ab} \propto \mu\delta_{ab}u$. From the boundary condition (where we ignore surface tension for simplicity), there is necessarily a pressure appearing to compensate for this, $P \simeq \mu u$. Hence, on the fiber section, the longitudinal force per unit area is $\overline{\mathcal{F}} = \tau_{\underline{33}} = 2\mu\partial_s v - \mu u = 3\mu\partial_s v$, and we recover the standard Trouton enhancement ratio [36] 3/2. Indeed, in the end we get a factor 3 instead of the factor 2 that would have been found if we had forgotten the pressure contribution. To summarize, in order to satisfy the boundary constraint, a pressure must appear when we have a gradient in the longitudinal velocity, and it transforms a 2 into a 3 in the longitudinal viscous forces because this pressure also acts on the sections.

2. Hagen-Poiseuille profile

If we now consider a gradient in the gradient of the longitudinal velocity ($\partial_s^2 v \neq 0$), it will induce a gradient of radial infall ($\partial_s u = -\partial_s^2 v$). This gradient of radial infall induces viscous forces per unit area applied on fiber sections, of the form $\mathcal{F}^a = \tau^{\underline{3}a} \simeq \mu(\partial_s u/2)y^a$. We must realize that the component $\tau^{\underline{3}a}$ gives the sectional components of the force per unit area applied onto the fiber sections, but $\tau_{\underline{3}a}n^a$ gives also the longitudinal forces applied on the boundary. A way to have a longitudinal component of viscous forces on the fiber side is to have a HP profile, that is, a parabolic profile. Indeed, a velocity profile $\overline{V} = v^{(1)}r^2$ creates $\tau_{\underline{3}a} \simeq \mu 2 v^{(1)} y_a$ and if $v^{(1)} = -\partial_s u/4 = \partial_s^2 v/4$, then the boundary constraint is satisfied. If the fiber radius is not constant along the fiber ($\mathcal{H} \neq 0$), this reasoning is slightly altered, thus explaining the corresponding contribution in Eq. (7.12). To summarize, when we have a gradient of stretching, then we have a gradient of infall velocity which creates a component $\tau_{\underline{3}a}$ on the fiber side, which in turn needs to be canceled by a HP profile so as to satisfy the boundary conditions, and eventually when everything is taken into account, there is no sectional component for the forces per unit area on sections ($\mathcal{F}_a \simeq 0$).

3. Dipolar longitudinal velocity

Let us consider a simple case in which there is no longitudinal velocity $v = v^{(0)}_\varnothing$, but we allow for a gradient in the sectional components of the FCL velocity ($\partial_s U_a \neq 0$). This induces a force per unit area applied on sections of the form $\mathcal{F}^a = \tau^{\underline{3}a} \simeq \mu\partial_s U_a$. Following the same reasoning as in the previous section, we realize that the flow must adapt in a way which creates an additional contribution for $\tau^{\underline{3}a}$ in order to satisfy the boundary constraint. If we consider a dipolar modulation of the longitudinal velocity $\overline{V} = v^{(0)}_a y^a$, then it creates a stress $\tau^{\underline{3}a} = \mu v^{(0)}_a$. If $v^{(0)}_a = -\partial_s U_a$, then the boundary constraint is satisfied, and given that we have considered $v = v^{(0)}_\varnothing = 0$, this is equivalent to the constraint (7.12b) once we use (2.28). Physically, if neighbor sections slide along each other (that is, if they have a relative velocity which is sectional), the viscous forces force them to rotate so that they do not slide along each other. To conclude, we might think that a gradient in the sectional velocity would create a sectional force per unit area on sections, but in fact the boundary condition would ensure that this is not the case, just like in the previous section. Only when considering higher order constraints will there be a sectional viscous force per unit area on sections, and this is why the lowest order model is a string model where viscous forces per area are necessarily longitudinal.

4. Dipolar pressure constraint

If we have a dipole modulation of the radial infall ($u^{(0)}_a \neq 0$), then following the reasoning of Sec. VII F 1, it will induce a dipole in the pressure so as to satisfy boundary constraints. Indeed, the sectional velocity contains $V^a = y^a/2(u^{(0)}_b y^b)$ and, thus, $\tau_{ab} \supset \mu\delta_{ab}(u^{(0)}_c y^c)$. In order to compensate for this component, we need a pressure gradient $P^{(0)}_a = \mu u^{(0)}_a$, which once all other constraints are used gives (7.17b). To be fully correct, we should also mention that $u^{(0)}_a$ also contributes as $\tau_{ab} \supset \mu y_{(a} u^{(0)}_{b)}$, but there is also a parabolic sectional component $V_a = \widehat{V}^{(1)}_a r^2$ which contributes as $\tau_{ab} \supset 4\mu y_{(a}\widehat{V}^{(1)}_{b)}$, and from the orthoradial constraint they must cancel. Again, the bottom line is that at lowest order the system adapts so that $\tau_{\underline{3}a} \simeq 0$, but this property also implies that there is no sectional component for the viscous forces per unit area on sections.

G. Comparison with rod models

The methods based on rod models follow a slightly different logical route in order to obtain a one-dimensional reduction. Indeed, in our method we solve for the constraints and then use them to obtain the volumic forces. In rod models, we solve instead only *some* of the boundary constraints, those needed to get the pressure moments, and then we use them to compute the forces per unit area applied on fiber sections. These are then integrated to get total forces and total torques applied on sections, allowing to establish a momentum balance equation and an angular momentum balance equation on a slice of fluid contained between two infinitesimally close sections. The information of the boundary constraints which was not used explicitly is then used implicitly because we use that no force is applied on the sides of the infinitesimal slice, so the balance equations involve only the forces and torques on the fiber sections. If surface tension effects are considered, then they are of course added to the side of the infinitesimal slice. In this section, we collect the expressions for forces and torques that we find with our formalism, so as to facilitate comparisons with existing literature using rod models.

1. Viscous forces

The total force on sections is

$$\boldsymbol{F} \simeq \int_{r=0}^{r=R} \boldsymbol{\mathcal{F}} r\, dr\, d\theta, \tag{7.50}$$

where we recall the definition (5.4) for the forces per unit area on sections. This relation is approximate because we have neglected the effect of noncircular sections. If we were to take into account correctly the effect of a noncircular shape, then we would need to perform a change of variables as in Appendix B. At lowest order the total longitudinal force is given by

$$\overline{F} \simeq \pi R^2 \overline{\mathcal{F}}^{(0)}_{\varnothing} = \pi R^2 \left(-\frac{\nu}{R} + 3\mu \partial_s v\right). \tag{7.51}$$

Note that in order to write a momentum balance equation on an infinitesimal slice, we should also consider (i) long distance forces $\boldsymbol{g}$ in the bulk of the slice, and (ii) the effect of surface tension on the side of the slice [e.g., Eq. (49) of Ribe *et al.* [23]]. This can also be computed by integrating all the volumic forces on the infinitesimal slice, as the total lineic force is obtained from

$$\frac{d\boldsymbol{F}^{\rm tot}}{ds} \equiv \int_{r=0}^{r=R} \boldsymbol{f} h r\, dr\, d\theta, \tag{7.52}$$

where we recall that the total volumic forces are given by (5.9). The factor h, which mathematically is $\sqrt{g_{ij}}$ with the metric (2.35), takes into account the fact that if the fiber is curved, then for an infinitesimal slice there is more fluid in the exterior of curvature and less in the interior of curvature. At lowest order we get simply

$$\frac{d\boldsymbol{F}^{\rm tot}}{ds} = \pi R^2 \boldsymbol{g} + 2\partial_s(\pi \nu R \boldsymbol{T}) + \partial_s(\overline{F}\boldsymbol{T}). \tag{7.53}$$

The first term is the effect of gravity, the second is the effect of surface tension on the side of the infinitesimal slice, and the last is the net effect of viscous forces on sections. Using (7.51), the total lineic force is simply

$$\frac{d\boldsymbol{F}^{\rm tot}}{ds} = \pi R^2 \boldsymbol{g} + \partial_s(\pi \nu R \boldsymbol{T}) + 3\mu \partial_s(\pi R^2 \partial_s v \boldsymbol{T}), \tag{7.54}$$

in agreement with the right-hand side of (7.22). If we had ignored the effect due to the induced pressure on the sections [given by the first term of (7.51)], we would have overestimated the total lineic force by a factor 2 as seen when comparing (7.54) with (7.53). So, we can state that if for viscous forces the effect of the constrained pressured is to enhance by a factor $\frac{3}{2}$ the total forces, then for surface tension the effect of the constrained pressure is a reduction by a factor 2.

2. Viscous torques

Still neglecting the effect of noncircular sections, the total torque applied on a fiber section is given by

$$\boldsymbol{\Gamma} \simeq \int_{r=0}^{r=R} (y^a \boldsymbol{d}_a) \times \boldsymbol{\mathcal{F}}\, d\theta\, r dr \tag{7.55}$$

or in components (that is, using $\boldsymbol{\Gamma} = \Gamma^a \boldsymbol{d}_a + \overline{\Gamma}\boldsymbol{T}$)

$$\Gamma^a \simeq -\int_{r=0}^{r=R} \widetilde{y}^a \overline{\mathcal{F}}\, d\theta\, r\, dr, \tag{7.56}$$

$$\overline{\Gamma} \simeq \int_{r=0}^{r=R} \widetilde{y}_b \mathcal{F}^b d\theta\, r\, dr. \tag{7.57}$$

In order to obtain the lowest order expressions for the torque, we only need to keep contributions which are linear in y^a in the components of $\boldsymbol{\mathcal{F}}$, and we find

$$\Gamma_a \simeq \frac{\pi R^4}{4} \varepsilon_a^{\ b} \overline{\mathcal{F}}^{(0)}_b = -\frac{\pi R^4}{4} \widetilde{\overline{\mathcal{F}}}^{(0)}_a, \quad \overline{\Gamma} \simeq \frac{\pi R^4}{2} \mathring{\mathcal{F}}^{(0)}. \tag{7.58}$$

Note that $\widehat{V}^{(0)}_{ab}$ does not contribute to the torque as it corresponds to a shear flow inside the section.

The expression of torques takes a simple form when expressed in terms of fluid vorticity. At lowest order, the vorticity components obtained from (4.37) with the lowest order constraints replaced are given by

$$\varpi^{\underline{3}} = \dot{\phi} + O(\epsilon_R^2), \tag{7.59a}$$

$$\varpi^a = \omega^a + \kappa^a v + O(\epsilon_R^2). \tag{7.59b}$$

The longitudinal component of the torque is then found from

$$\begin{aligned}\mathring{\mathcal{F}}^{(0)} &\simeq \mu\left[\partial_s\dot{\phi} - \tfrac{1}{2}\kappa^a\left(\widetilde{\omega}_a + v_a^{(0)}\right)\right] = \mu\left(\partial_s\dot{\phi} + \widetilde{\kappa}_a\omega^a\right)\\ &= \mu(\partial_s\varpi)^{\underline{3}},\end{aligned} \tag{7.60}$$

where in the second equality we have used the lowest order constraint (7.12b) and in the third we used the property (2.24b). This component of the torque is induced by twisting (longitudinal difference of vorticity) and its physical origin is thus obvious.

The sectional torque is found from

$$\begin{aligned}-\widetilde{\overline{\mathcal{F}}}^{(0)}_a &\simeq -\frac{\nu}{R}\kappa_a + 3\mu\left[\partial_s(\omega_a + \kappa_a v) - \frac{3}{2}\kappa_a\partial_s v - \dot{\phi}\widetilde{\kappa}_a\right]\\ &= -\frac{\nu}{R}\kappa_a + 3\mu(\partial_s\varpi)_a - \frac{9}{2}\mu\kappa_a\partial_s v.\end{aligned} \tag{7.61}$$

The physical origin of the second term is simple. The sectional component of vorticity corresponds to a rotation around a sectional axis. If we consider two neighbor sections which have different sectional vorticities, as would happen if the fiber is bent, then the fluid located inside will be squeezed on one side and stretched on the other side, that is, there will appear a dipole of stretching. Then, from the viscous forces induced, this creates a sectional torque. With this naive view we would get a factor 2 and not a factor 3, but as in Sec. VII F 1, there is also a dipolar pressure which is induced to satisfy the boundary conditions, and it implies again the appearance of the Trouton factor enhancement $\frac{3}{2}$.

The last term of (7.61) is more subtle and it has been ignored in Ribe [22] and Ribe *et al.* [23]. Indeed, *stretching* implies the viscous force (7.51), *twisting* implies the longitudinal torque (7.60), and *bending* induces the second term of the sectional torque (7.61), and these geometries have been considered separately in these references even though they can have mixed effects. However, the coupling of stretching with curvature has been ignored, and it happens to have an effect on the torque which is contained in the last term in Eq. (7.61). We stress that this effect arises at the same order, and is not an order ϵ_R^2 correction.

In a simple case, the physical origin of this last term can also be understood. Let us ignore surface tension and consider a stationary regime with no axial rotation ($\omega_a = \dot{\phi} = 0$) and constant curvature ($\partial_s \kappa^a = 0$). The expression of the sectional force (7.61) is simply

$$-\overline{\widetilde{\mathcal{F}}}_a^{(0)} \simeq -\tfrac{3}{2}\mu\kappa_a\partial_s v. \quad (7.62)$$

As the fiber is stretched ($\partial_s v > 0$), there is a radial infall since $u = -\partial_s v$. Since the fiber is curved, the particles in the exterior of curvature ($\widetilde{\kappa}_a y^a > 0$) are compressed while they move closer to the FCL, and conversely the particles in the interior of curvature ($\widetilde{\kappa}_a y^a < 0$) are stretched while they move toward the FCL. As a result of viscous forces, this creates a contribution to the sectional torque. And, as usual, we get a factor 3/2 enhancement, the Trouton ratio, exactly like for the other contribution to the bending torque.

The first term of (7.61) comes from surface tension effects. If the FCL is curved, then $\widetilde{\kappa}^a$ points in the exterior of curvature (or $-\widetilde{\kappa}_a$ points toward the center of curvature of the FCL). This means that extrinsic curvature is increased in the exterior (for points such that $\widetilde{\kappa}_a y^a > 0$) and decreased in the interior (for points such that $\widetilde{\kappa}_a y^a < 0$). From Young-Laplace law, this induces a dipole in pressure, with more pressure in the outside than in the inside as can be seen on the constraint (7.17b). This dipolar pressure distribution creates in turn a sectional torque.

However, if we perform an angular momentum balance equation on an infinitesimal slice, one should also add (i) the torque of long distance forces $\boldsymbol{g}$ in the bulk of the slice and (ii) the contribution of surface tension on the side of the slice which also creates a torque [e.g., Eq. (50) of Ribe *et al.* [23]]. Just as for the momentum balance equation, the angular momentum balance equation is best computed by integrating the torques of volumic forces on the infinitesimal slice, as the total lineic torque is obtained from

$$\frac{d\boldsymbol{\Gamma}^{\rm tot}}{ds} \equiv \int_{r=0}^{r=R} (y^a \boldsymbol{d}_a) \times \boldsymbol{f}\, h\, d\theta\, r\, dr. \quad (7.63)$$

We find that it can be expressed as

$$\frac{d\boldsymbol{\Gamma}^{\rm tot}}{ds} = \boldsymbol{T} \times \boldsymbol{F} + \frac{d\boldsymbol{\gamma}}{ds}, \quad (7.64)$$

where the first term involves the sectional part of the total viscous forces defined in Eq. (7.50), and the second term is given at lowest order by

$$\frac{d\boldsymbol{\gamma}}{ds} \simeq \partial_s \boldsymbol{\Gamma} + \frac{\pi R^4}{4}\left(\widetilde{\boldsymbol{\kappa}} \times \boldsymbol{g} + 4\frac{\nu}{R}\mathcal{H}\boldsymbol{\kappa}\right). \quad (7.65)$$

It appears clearly that the last term on the right-hand side comes from surface tension effects on the side of the infinitesimal slice, whereas the first term is the net effect of torques applied on the sections of the infinitesimal slice. The middle term is the torque induced by gravity, which comes from the fact that when the fiber is curved, then the center of mass of an infinitesimal slice is not exactly on the FCL, but is instead offset by $R^2\widetilde{\boldsymbol{\kappa}}/4$. In components (7.65) reads as simply

$$\left[\frac{d\boldsymbol{\gamma}}{ds}\right]^a \simeq \partial_s \Gamma^a - \widetilde{\kappa}^a \overline{\Gamma} + \nu\pi R^3 \mathcal{H}\kappa^a, \quad (7.66a)$$

$$\left[\frac{d\boldsymbol{\gamma}}{ds}\right]^3 \simeq \partial_s \overline{\Gamma} + \widetilde{\kappa}^a \Gamma_a, \quad (7.66b)$$

where the expressions of the torques applied on fiber sections are given by (7.58) with (7.60) and (7.61). For completeness, we report the explicit result which is

$$\begin{aligned}\left[\frac{d\boldsymbol{\gamma}}{ds}\right]^a \simeq \frac{\pi R^4}{4}\Big(&\overline{g}\kappa_a + \frac{\nu\mathcal{H}\kappa_a}{R} - 12\mu\mathcal{H}\widetilde{\kappa}_a\dot{\phi} - 2\mu\kappa_b\kappa^b\omega_a \\ &+ 2\mu\kappa_a\kappa^b\omega_b - 6\mu\mathcal{H}\kappa_a\partial_s v - \frac{\nu\partial_s\kappa_a}{R} + 12\mu\mathcal{H}v\partial_s\kappa_a \\ &+ \frac{3}{2}\mu\partial_s v\partial_s\kappa_a - 3\mu\dot{\phi}\partial_s\widetilde{\kappa}_a - 5\mu\widetilde{\kappa}_a\partial_s\dot{\phi} + 12\mu\mathcal{H}\partial_s\omega_a \\ &- \frac{3}{2}\mu\kappa_a\partial_s^2 v + 3\mu v\partial_s^2\kappa_a + 3\mu\partial_s^2\omega_a\Big), \end{aligned} \quad (7.67)$$

$$\begin{aligned}\left[\frac{d\boldsymbol{\gamma}}{ds}\right]^3 \simeq \frac{\pi R^4}{4}\big(&-g^a\kappa_a - 3\mu\kappa_a\kappa^a\dot{\phi} + 8\mu\mathcal{H}\widetilde{\kappa}^a\omega_a \\ &+ 3\mu v\widetilde{\kappa}^a\partial_s\kappa_a + 2\mu\omega^a\partial_s\widetilde{\kappa}_a + 8\mu\mathcal{H}\partial_s\dot{\phi} \\ &+ 5\mu\widetilde{\kappa}^a\partial_s\omega_a + 2\mu\partial_s^2\dot{\phi}\big). \end{aligned} \quad (7.68)$$

3. Sectional forces and dipolar HP profile

The component of the longitudinal velocity $v_b^{(1)} r^2 y^b$ can be considered as a dipolar HP profile, as it is a parabolic profile with a dipolar modulation. This velocity component induces a force per area on sections $\mathcal{F}^a = \tau^{3a} \simeq \mu 2 y^a (v_b^{(1)} y^b) + \mu v_a^{(1)} r^2$ and it remains undetermined by the boundary constraints. When averaged over directions, it contributes to the forces per unit area as $\mathcal{F}^a \simeq 2\mu v_a^{(1)} r^2$, and after integration over the whole section, it contributes to the sectional part of the total viscous force. Indeed, the sectional components of the total viscous force applied on sections [defined in Eq. (7.50)] are

$$\begin{aligned} F_a \simeq \pi R^4\big(&-2\mu v_a^{(1)} + 3\mu\mathcal{H}\kappa_a\dot{\phi} + \tfrac{1}{4}\mu\kappa^b\widetilde{\kappa}_a\omega_b \\ &- \tfrac{15}{4}\mu\mathcal{H}\widetilde{\kappa}_a\partial_s v + \tfrac{1}{4}\mu\dot{\phi}\partial_s\kappa_a + 3\mu\mathcal{H}v\partial_s\widetilde{\kappa}_a \\ &+ \tfrac{1}{8}\mu\partial_s v\partial_s\widetilde{\kappa}_a + \tfrac{1}{2}\mu\kappa_a\partial_s\dot{\phi} + 3\mu\mathcal{H}\partial_s\widetilde{\omega}_a \\ &- \tfrac{3}{8}\mu\widetilde{\kappa}_a\partial_s^2 v + \tfrac{1}{4}\mu v\partial_s^2\widetilde{\kappa}_a - \tfrac{1}{4}\mu\kappa_b\kappa^b\widetilde{\omega}_a + \tfrac{1}{4}\mu\partial_s^2\widetilde{\omega}_a\big). \end{aligned} \quad (7.69)$$

As the rotation of sections is constrained by (7.12b), then an angular momentum balance equation would in fact determine the value of $v_a^{(1)}$ because part of the torque balance equation (7.64) comes from $\boldsymbol{T} \times \boldsymbol{F}$. In our method, it is determined from $\overline{\mathcal{D}}_a^{(0)}$ (see Table V), that is, from the dipole of the longitudinal part of the Navier-Stokes equation. Indeed, it determines the rate of change of the local dipolar longitudinal velocity $v_a^{(0)}$ which is related to the vorticity and the local rotation rate of the fluid on the FCL [see the discussion which follows (7.12)], and it thus contains the same information as the angular momentum balance equation used in rod models. The expression obtained for $v_a^{(1)}$ is reported in Appendix F, and once replaced in Eq. (7.69), the sectional components of the total viscous force applied on sections read as

$$\begin{aligned}F_a \simeq \pi R^4\Big(&-\frac{\nu}{2R}\partial_s\widetilde{\kappa}_a + 3\mu\mathcal{H}\kappa_a\dot{\phi} + \frac{3}{4}\mu\kappa^b\widetilde{\kappa}_a\omega_b\\&-\frac{9}{2}\mu\mathcal{H}\widetilde{\kappa}_a\partial_s v + \frac{3}{4}\mu\dot{\phi}\partial_s\kappa_a + 3\mu\mathcal{H}v\partial_s\widetilde{\kappa}_a\\&-\frac{3}{8}\mu\partial_s v\partial_s\widetilde{\kappa}_a + \frac{5}{4}\mu\kappa_a\partial_s\dot{\phi} + 3\mu\mathcal{H}\partial_s\widetilde{\omega}_a - \frac{1}{2}v\dot{\phi}\kappa_a\\&-\frac{21}{8}\mu\widetilde{\kappa}_a\partial_s^2 v + \frac{3}{4}\mu v\partial_s^2\widetilde{\kappa}_a - \frac{3}{4}\mu\kappa_b\kappa^b\widetilde{\omega}_a + \frac{3}{4}\mu\partial_s^2\widetilde{\omega}_a\\&-\frac{1}{2}\dot{\phi}\omega_a - \frac{1}{4}\mu\widetilde{\kappa}_b\omega^b\kappa_a + \frac{3}{4}\widetilde{\kappa}_a v\partial_s v + \frac{3}{4}\widetilde{\omega}_a\partial_s v\Big).\end{aligned} \tag{7.70}$$

4. Rod models and their validity

As explained in the previous sections, the constitutive relations of rod models are based on the determination of forces per unit area on fiber sections, and boundary constraints are only used explicitly to determine the pressure profile. The boundary constraints are then used implicitly in balance equations.

At lowest order, only the momentum balance equation is used, and we find (7.23) which is also exactly what is found in the viscous string model. This equation determines the motion of the FCL, and thus the rotation rate ω^a can be inferred from it. Given that the fiber vorticity is constrained to match the rotation rate of the fluid located on the FCL (see Sec. VII B 4), the angular momentum balance equation is in fact used to determine the sectional components of the viscous forces per unit area $\mathcal{F}^a$, as explained in the previous section, and it corresponds to the addition of an order ϵ_R^2 correction.

To summarize, the rod model amounts to using Eq. (7.23), formally exactly like the viscous string model, but it differs from it in the expression of the force used on the right-hand side which has sectional components. Hence, the rod model corresponds to

$$\mathcal{D}_t\boldsymbol{\mathcal{V}}_{\rm Cen} \simeq \boldsymbol{g} + \frac{1}{\pi R^2}\partial_s\boldsymbol{F}, \tag{7.71}$$

with the force given by

$$\boldsymbol{F} \equiv (\pi R^2 3\partial_s v + \nu\pi R)\boldsymbol{T} + F^a\boldsymbol{d}_a, \tag{7.72}$$

and where we recall the definition of the convective derivative $\mathcal{D}_t \equiv \partial_t + v\partial_s$. The components of the left-hand side of Eq. (7.71) are obtained from Eqs. (7.21) exactly like for the string model. The components of the first term on the right-hand side are obtained from Eqs. (7.22) as in the string model. However, the rod model differs from the string model thanks to the sectional components of the total viscous force which are reported in Eq. (7.70). Note that from the relations of Sec. II D we must use

$$\partial_s(F^a\boldsymbol{d}_a) = \partial_s F^a\boldsymbol{d}_a + \boldsymbol{T}\tilde{\kappa}_a F^a \tag{7.73}$$

so as to evaluate the last term in Eq. (7.71). Furthermore, the dynamics of $\dot{\phi}$ is given by the lowest order dynamical equation (7.18c) as in our model, and it is required since $\dot{\phi}$ appears in Eq. (7.70). In the rod model of Ref. [23], this is equivalently found from the longitudinal part of the momentum balance equation, even though it does not appear so explicitly as the physical case studied is stationary.

Finally, we already mentioned that for straight fibers the model is improved by considering the full expression of the boundary curvature [7] given by Eq. (6.11) instead of the lowest order $1/R$. We can use a similar ansatz for curved fibers by noting that in Eq. (7.72), the term $\nu\pi R$ is in fact $\nu\pi\mathcal{K}_\emptyset R^2$ at lowest order in ϵ_R. Hence, from Eq. (7.34a) an improved rod model is obtained by the replacement

$$\nu\pi R \to \nu\pi R\left[1 - R^2\left(\tfrac{3}{2}\mathcal{H}^2 + \tfrac{1}{2}\kappa_a\kappa^a + \partial_s\mathcal{H}\right)\right] \tag{7.74}$$

in Eq. (7.72). This improved rod model is necessary to compute in Ref. [28] the Rayleigh-Plateau instability of a viscous fiber.

The validity of rod models is limited by the fact that we are considering one correction while discarding other sources of corrections, whereas in our approach we systematically consider all corrections of order ϵ_R^2. Among the effects ignored there are the following:

(i) the difference between the velocity of the central line and the velocity of the fluid on the central line (see Sec. IV I);

(ii) the HP profile induced by the constraint (7.12a) which mixes the fluid particles belonging to neighboring sections;

(iii) the shape moments which are sourced and invalidate the assumption that the sections remain circular.

Furthermore, even though it is computationally involved, our approach allows to find the corrections up to any order when rod models would fail because of the impossibility to deal with the mixing sections. In principle, we have a clear recursive algorithm made of constraints replacements in fundamental dynamical equations.

However, there are cases in which rod models capture the essential corrections. First, shape moments appear in all intermediary expressions but do not appear in the final dynamical equations (G3) and (G4), hence, if we are not interested in the fiber sections shape but only on the central line, they can be ignored. Furthermore, if we are considering the steady motion in a rotating frame, and if the Rosby number is very small, that is, for high rotation rates, then it does not matter if we have ignored most of the corrective effects. Indeed, the boundary constraints do not involve the frame rotation, and frame rotation enters essentially only in the determination of $v_a^{(1)}$ from $\overline{\mathcal{D}}_a^{(0)}$, or equivalently in the determination of the total sectional forces since it is related through (7.69). In this regime, the fast rotation induces a sectional force and its expression should be captured correctly by an angular momentum balance equation thanks to (7.64), provided the last term in (7.61) is correctly included. In the end, rod models take only some corrective terms, but if the system is considered in a fast-rotating frame, these retained corrective terms should also be the most important ones, and rod models should lead to a reliable extension of the viscous string model.

VIII. CONCLUSION

We have developed all the theoretical tools which are required to obtain a general one-dimensional description of curved fibers. A concrete application for toroidal viscous fibers is presented separately in Pitrou [28]. From a theoretical point of view, our $2+1$ splitting, our use of fiber adapted coordinates, and more importantly our parametrization of the velocity field in terms of STF tensors allow for a clear discussion about constraints and dynamical equations. It avoids the cluttered component by component expressions which are usual in such context [18], and it bears a more transparent geometrical meaning since all quantities are analyzed in terms of their monopole, dipole, quadrupole, and higher order multipoles. We find that it is the natural language which allows to overcome the complexity of equations for curved viscous fibers. Indeed, the use of an adapted formalism is the key to understand in depth apparently complex problems. From a practical or computational point of view, this STF based approach is very powerful as it is possible to handle tensors with appropriate abstract tensor packages, and to this end we used xAct [35]. The corresponding notebooks are available upon request from the author.

The main results of this article are the following:

(i) We have recovered the standard results of axisymmetric fibers at lowest order in Eqs. (6.3) and (6.10), including also axial rotation.

(ii) The first corrections for this model are collected in Eq. (6.23).

(iii) We extended these results to include a second set of corrections and these can be found in Appendix E.

(iv) The main purpose of this article was to develop a formalism for curved fibers and we first rederived the viscous string model whose central equations are (7.18), (7.25), and (7.20). Its covariant formulation is summarized in Eqs. (7.28) and (7.29).

(v) We found the first corrections for curved fibers in full generality in Eqs. (G3), (G4), and (7.41), and these are relevant when ϵ_R is not so small since they are of order ϵ_R^2.

(vi) Elliptic shape perturbations are sourced at that order and their dynamics is governed at that order by (7.42) with (F7) replaced.

(vii) In particular, when restricting to straight fibers, the dynamical equation for the evolution of elliptic shape perturbations takes the simple form (7.44).

(viii) Finally, when comparing with rod models methods, we have exhibited a missing term in the expression (7.61) for the torque applied on fiber sections.

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APPENDIX A: STF FORMALISM

1. Extraction of STF tensors

Integrals on directions are simply

$$\int \frac{d\theta}{2\pi} n^{a_1}\dots n^{a_{2n}} = \frac{(2n-1)!!}{(2n)!!}\delta^{(a_1 a_2}\dots\delta^{a_{2n-1}a_{2n})},$$
$$\int \frac{d\theta}{2\pi} n^{a_1}\dots n^{a_{2n+1}} = 0. \tag{A1}$$

Here, $\delta^{(a_1 a_2}\dots\delta^{a_{2n-1}a_{2n})}$ means that the indices need to be fully symmetrized among the $(2n-1)!!$ possible permutations. The lowest nonvanishing integrals are

$$\int \frac{d\theta}{2\pi} n^a n^b = \frac{1}{2}\delta^{ab}, \tag{A2}$$

$$\int \frac{d\theta}{2\pi} n^a n^b n^c n^d = \frac{1}{8}(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}). \tag{A3}$$

We recall that in general for STF tensors, we can use the multi-index notation $K \equiv a_1\dots a_k$ or $L \equiv b_1\dots b_\ell$. However, in order to avoid confusion on multi-indices in this section and the next one, we use the weaker multi-index notation $a_K \equiv a_1\dots a_k$ or $b_L \equiv b_1\dots b_\ell$. Let us define

$$I^{a_L}_{b_L} \equiv \delta^{a_1}_{\langle b_1}\dots\delta^{a_\ell}_{b_\ell\rangle}, \quad J^{a_L}_{b_L} \equiv \epsilon^{a_1}{}_{\langle b_1}\delta^{a_2}_{b_2}\dots\delta^{a_\ell}_{b_\ell\rangle}, \tag{A4}$$

where we recall that when indices are enclosed in $\langle\dots\rangle$ we must take the symmetric trace-free part. Clearly, these quantities are just related by

$$\epsilon^{a_1}_{c}\, I^{ca_{L-1}}_{b_L} = J^{a_L}_{b_L}, \quad \epsilon^{a_1}_{c}\, J^{ca_{L-1}}_{b_L} = -I^{a_L}_{b_L}. \tag{A5}$$

They have the interesting properties

$$I^{ca_{L-1}}_{cb_{L-1}} = I^{a_{L-1}}_{b_{L-1}}, \quad I^a_a = 2, \quad I^a_b\epsilon^b_a = 0, \tag{A6a}$$

$$J^{ca_{L-1}}_{cb_{L-1}} = J^{a_{L-1}}_{b_{L-1}}, \quad J^a_a = 0, \quad J^a_b\epsilon^b_a = 2, \tag{A6b}$$

$$I^{a_L}_{b_L}I^{b_L}_{a_L} = 2, \quad J^{a_L}_{b_L}J^{b_L}_{a_L} = 2, \quad I^{a_L}_{b_L}J^{b_L}_{a_L} = 0. \tag{A7}$$

The tensors (A4) are used when computing the following integrals on direction vectors:

$$2^\ell \int \frac{d\theta}{2\pi} n^{a_L} n_{\langle b_K\rangle} = \delta^\ell_k\, I^{a_L}_{b_L}, \tag{A8a}$$

$$2^\ell \int \frac{d\theta}{2\pi} \epsilon^a_c n^c n^{a_{L-1}} n_{\langle b_K\rangle} = \delta^\ell_k\, J^{a_L}_{b_L}. \tag{A8b}$$

It is then immediate to show that for a scalar function expanded in STF tensors as

$$S = \sum_\ell S_L n^{\langle L\rangle}, \tag{A9}$$

then the STF moments can be extracted through

$$S_L = 2^{-\ell}\int \frac{d\theta}{2\pi} n_{\langle L\rangle} S. \tag{A10}$$

This type of integral is very well suited for a tensor computer algebra system such as xAct [35] since we need only to implement the rules (A1).

2. Products of STF tensor

As explained in Sec. III B, STF tensors in two dimensions are irreducible representations of SO(2). If the tensor has ℓ indices, then it is in the representation D_ℓ. When we have a product of two STF tensors of rank ℓ and ℓ', it means we have the tensor product of the representations $D_\ell \otimes D_{\ell'}$. This is not irreducible, but it can be decomposed in irreducible representations. Given that the dimension of D_ℓ is 2 (except for D_0 for which the dimension is 1), then this tensor product is of dimension $2 \times 2 = 4$. When decomposed in irreducible representations, it is either of the form $D_{|\ell-\ell'|} \oplus D_{\ell+\ell'}$ if $\ell \neq \ell'$, or $D_{\ell+\ell'} \oplus D_0 \oplus D_0$ if $\ell = \ell'$. When counting the dimensions, it is the statement that $2 \times 2 = 2 + 2$ in the former case, and $2 \times 2 = 2 + 1 + 1$ in the latter case.

In order to see in practice how this decomposition is performed, let us consider two STF tensors A_K and B_L. If we assume first that $k < \ell$, then

$$A_{a_K} B_{b_L} = A_{\langle a_K} B_{b_L \rangle} + A^{c_K} B_{c_K \langle b_{L-K}} I^{a_K}_{b_K \rangle}. \tag{A11}$$

Under this form, we have indeed decomposed the product into two irreducible parts $D_{|k-\ell|}$ and $D_{|k+\ell|}$ (that is, $2 \times 2 = 2 + 2$), which are respectively the STF tensors $A^{c_K} B_{c_K \langle b_{L-K} \rangle}$ and $A_{\langle a_K} B_{b_L \rangle}$.

However, if the tensors are of equal rank ($k = \ell$), this gets slightly different since

$$A_{a_L} B_{b_L} = A_{\langle a_L} B_{b_L \rangle} + \tfrac{1}{2} A^{c_L} B_{c_L} I^{a_L}_{b_L} + \tfrac{1}{2} \epsilon^c{}_d A_{c c_{L-1}} B^{d c_{L-1}} J^{a_L}_{b_L}. \tag{A12}$$

In that case, we have decomposed the product as a sum of two scalar functions (both corresponding to the representation D_0) and an element of $D_{2\ell}$ (that is, $2 \times 2 = 2 + 1 + 1$) which are, respectively, $A^{c_L} B_{c_L}$, $\epsilon^c_d A_{c c_{L-1}} B^{d c_{L-1}}$ and the STF tensor $A_{\langle a_L} B_{b_L \rangle}$.

In both cases we get (with $k \leqslant \ell$)

$$A_K B_L y^K y^L = A_{\langle K} B_{L \rangle} y^K y^L + \frac{r^{2k}}{2^k} A^K B_{K c_{L-K}} y^{c_{L-K}}. \tag{A13}$$

If we consider the expansion (3.2), we can first remove the traces in the L indices to recast the expansion as

$$V_a(y^i,t) = \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} V^{(n)}_{a \langle L \rangle}(s,t) y^L r^{2n}. \tag{A14}$$

Each $V^{(n)}_{a\langle L \rangle}$ can be handled exactly as a product of tensors A_a and B_L. If $\ell > 1$, we can use the decomposition (A11), but if $\ell = 1$, we must use (A12). Combining all the terms in the sum (3.2), we finally conclude that the decomposition of a 2-vector field in terms of irreps is necessarily of the form (3.10).

APPENDIX B: ALTERNATE SHAPE REPRESENTATION

We can consider the following STF moments:

$$\mathcal{M}_L \equiv 2^\ell \int y_{\langle L \rangle} \bar{\rho}(y^b) d^2 y^b, \tag{B1}$$

which are integrals on the fiber section which should capture its shape. $\bar{\rho}$ is a step function which is unity if there is a fluid particle and vanishes otherwise. These moments are built just like the material moments of constant density extended objects, or like the electric moments of uniformly charged extended objects [37]. These moments can be related to the moments of radial dimensions $\mathcal{R}_L$ defined in Eq. (4.12), but the relation is nonlinear. To see this, let us change variables and define rescaled coordinates

$$z^a \equiv \frac{y^a}{1 + \widehat{\mathcal{R}}_L n^L}, \quad y^a = z^a (1 + \widehat{\mathcal{R}}_L n^L). \tag{B2}$$

The Jacobian of the transformation is

$$d^2 y^b = J d^2 z^b, \quad J = (1 + \widehat{\mathcal{R}}_L n^L)^2 \tag{B3}$$

and the integrals (B1) are recast as

$$\mathcal{M}_L = 2^\ell \frac{2\pi R^{\ell+2}}{\ell+2} \int (1 + \widehat{\mathcal{R}}_M n^M)^{\ell+2} n_{\langle L \rangle} \frac{d\theta}{2\pi}. \tag{B4}$$

For the monopole we obtain

$$\mathcal{M}_\varnothing = \pi R^2 \left(1 + \sum_{\ell=1}^{\infty} 2^{-\ell} \widehat{\mathcal{R}}_L \widehat{\mathcal{R}}^L \right) \simeq \pi R^2, \tag{B5}$$

which is just the area of the section. If $\ell > 0$, then the other geometric multipoles are simply approximated by (keeping only linear terms)

$$\mathcal{M}_L \simeq 2\pi R^{\ell+2} \widehat{\mathcal{R}}_L = 2\pi R^{2(\ell+1)} \mathcal{R}_L, \tag{B6}$$

where (A8) was used to compute the integral (B4).

The kinematic equation, giving the evolution in time of these moments, is found from the conservation equation of the density function $\bar{\rho}$:

$$\partial_t \bar{\rho} + \mathcal{V}^i_R \partial_i \bar{\rho} = \partial_t \bar{\rho} + \mathcal{V}^3_R \partial_s \bar{\rho} + \mathcal{V}^a_R \partial_a \bar{\rho} = 0. \tag{B7}$$

Indeed, integrating over directions as in Eq. (B1), and after integrations by parts, we get simply

$$\partial_t \mathcal{M}^L = -\int y^{\langle L \rangle} \left(\mathcal{V}^3_R \partial_s \bar{\rho} - \bar{\rho} \partial_a \mathcal{V}^a_R \right) d^2 y + \ell \int y^{\langle a_{L-1}} \mathcal{V}^{a_\ell \rangle}_R \bar{\rho} \, d^2 y. \tag{B8}$$

$\mathcal{V}^a_R$ needs to be expressed in terms of its multipoles. To this end, we first express it in terms of V^a from the relation (4.10b), and then use the expansion (3.10) for V^a. As for $\mathcal{V}^3_R$, we should first use that $\mathcal{V}^3_R = \mathcal{V}^3_R / h$, expand $1/h = 1/(1 + \widetilde{\kappa}_a y^a)$ which brings increasing powers of $\widetilde{\kappa}_a y^a$, and then we should relate it to $\overline{V}$ from (4.10b) so as to use the expansion (3.9) for $\overline{V}$. The angular integrals can then be performed with (A8). The resulting dynamical equations for these shape multipoles $\mathcal{M}^L$ are rather complicated because of the high powers in the curvature vector κ^a, but they are linear in both the velocity multipoles and the shape multipoles $\mathcal{M}^L$. Instead, the dynamical equation (4.25) was still linear in velocity multipoles, but extremely nonlinear in the shape multipoles $\mathcal{R}_L$. Hence, it is not surprising that the relation between the two types of multipoles (B4) is very nonlinear. Since the normal vector and thus the extrinsic curvature are more easily expressed with the multipoles $\mathcal{R}_L$ as seen on (4.21), we chose to work with these multipoles so as to be able to include surface tension effects.

APPENDIX C: VELOCITY OF THE COINCIDENT POINT

Let us define the space-time Cartesian coordinates $X^{\hat{\mu}} = (t,x^{\mu})$ with $\hat{\mu} = 0,1,2,3$ and the space-time fiber adapted coordinates $Y^{\hat{\imath}} = (t,y^{\hat{\imath}})$ with $\hat{\imath} = 0,1,2,3$. Then, each coordinate system is a function of the other one, that is, we have the functions $X^{\hat{\mu}}(Y^{\hat{\imath}})$ and $Y^{\hat{\imath}}(X^{\hat{\mu}})$ which are related by

$$\frac{\partial X^{\hat{\mu}}}{\partial Y^{\hat{\imath}}}\frac{\partial Y^{\hat{\imath}}}{\partial X^{\hat{\nu}}} = \delta^{\hat{\mu}}_{\hat{\nu}}, \quad \frac{\partial Y^{\hat{\imath}}}{\partial X^{\hat{\mu}}}\frac{\partial X^{\hat{\mu}}}{\partial Y^{\hat{\jmath}}} = \delta^{\hat{\imath}}_{\hat{\jmath}}. \tag{C1}$$

In particular, using the first relation for $\hat{\mu} = \mu$ and $\hat{\nu} = 0$, and the second relation with $\hat{\imath} = i$ and $\hat{\jmath} = 0$ we get

$$0 = \left.\frac{\partial x^{\mu}}{\partial t}\right|_y + \left.\frac{\partial x^{\mu}}{\partial y}\frac{\partial y^i}{\partial t}\right|_x = \left.\frac{\partial x^{\mu}}{\partial t}\right|_y + d_i^{\mu}\left.\frac{\partial y^i}{\partial t}\right|_x, \tag{C2a}$$

$$0 = \left.\frac{\partial y^i}{\partial t}\right|_x + \left.\frac{\partial y^i}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial t}\right|_y = \left.\frac{\partial y^i}{\partial t}\right|_x + d^i_{\mu}\left.\frac{\partial x^{\mu}}{\partial t}\right|_y \tag{C2b}$$

and we recover (4.7).

APPENDIX D: CARTAN STRUCTURE RELATION

In this Appendix, we build on the four-dimensional perspective of the previous section. Let us define the space-time tetrad

$$d_{\underline{\hat{\imath}}} = (d_0, d_{\underline{i}}). \tag{D1}$$

It is made from the spatial orthonormal basis on which we have added a time directed vector

$$d_0^{\hat{\mu}} = \delta_0^{\hat{\mu}} \Rightarrow \partial_s d_0 = \partial_t d_0 = 0. \tag{D2}$$

Let us define the infinitesimal rotation matrices

$$[J_i]_{jk} \equiv \eta_{ijk} \Rightarrow [J_i, J_j] = -\eta_{ijk} J_k, \tag{D3}$$

where η_{ijk} is the permutation symbol with $\eta_{123} = 1$ and where the sum on the index k is implied. We can define an operator valued (rotation valued) one-form in the four-dimensional classical space-time by

$$\mathbf{\Omega} \equiv \kappa^{\underline{i}} J_i \, ds + \omega^{\underline{i}} J_i \, dt. \tag{D4}$$

The components of this form in the Cartesian canonical basis are

$$\mathbf{\Omega} = \Omega_{\hat{\mu}} dx^{\hat{\mu}}, \quad \Omega_{\hat{\mu}} \equiv \kappa^{\underline{i}} J_i \delta^3_{\hat{\mu}} + \omega^{\underline{i}} J_i \delta^0_{\hat{\mu}}. \tag{D5}$$

This form is clearly the connection form of the Cartan formalism since (still with the sum on k implied)

$$\partial_s d_{\underline{j}} = [\Omega_3]_{jk} d_{\underline{k}} \Leftrightarrow \partial_s d_{\underline{j}} = \kappa^{\underline{i}}[J_i]_{jk} d_{\underline{k}} \Leftrightarrow \partial_s d_{\underline{j}} = \kappa \times d_{\underline{j}},$$

$$\partial_t d_{\underline{j}} = [\Omega_0]_{jk} d_{\underline{k}} \Leftrightarrow \partial_t d_{\underline{j}} = \omega^{\underline{i}}[J_i]_{jk} d_{\underline{k}} \Leftrightarrow \partial_t d_{\underline{j}} = \omega \times d_{\underline{j}},$$

which in a four-dimensional perspective reads exactly as the first Cartan structure equation

$$\partial_{\hat{\mu}} d_{\underline{\hat{\jmath}}} = [\Omega_{\hat{\mu}}]_{\hat{\jmath}\hat{k}} d_{\underline{\hat{k}}}. \tag{D6}$$

Then, since the classical space-time is flat, the second Cartan structure equation reads as [38]

$$d\mathbf{\Omega} + \mathbf{\Omega} \wedge \mathbf{\Omega} = 0. \tag{D7}$$

Expressed explicitly in a basis of two-forms $dy^{\hat{\imath}} \wedge dy^{\hat{\jmath}} = dy^{\hat{\imath}} \otimes dy^{\hat{\jmath}} - dy^{\hat{\jmath}} \otimes dy^{\hat{\imath}}$, the terms of this equation read as

$$d\mathbf{\Omega} = \tfrac{1}{2}[\partial_t \kappa^{\underline{i}} - \partial_s \omega^{\underline{i}}] J_i \, dt \wedge ds,$$

$$\mathbf{\Omega} \wedge \mathbf{\Omega} = \tfrac{1}{2}\omega^{\underline{i}}\kappa^{\underline{j}}[J_i, J_j] dt \wedge ds = \tfrac{1}{2}[\kappa \times \omega]^{\underline{i}} J_i \, dt \wedge ds$$

and therefore we recover the structure relation (2.18).

APPENDIX E: SECOND SET OF CORRECTIONS FOR AXISYMMETRIC VISCOUS FIBERS

As discussed in Sec. VI A, the dynamical equation (6.10b) is part of the first set of corrections, implying that (6.23b) is in fact part of the second set of corrections, so we only report the second corrections for $\partial_t v$ and $\partial_t \ln R$. The second corrections to the dynamical equation for v are

$$\begin{aligned}
\partial_t v \supset R^4 \big[& -\tfrac{1}{2}\mathcal{H}^3\dot{\phi}^2 - \tfrac{1}{4}\mathcal{H}\dot{\phi}^2\partial_s\mathcal{H} + 48\mathcal{H}^5\partial_s v + \tfrac{543}{4}\mathcal{H}^3\partial_s\mathcal{H}\partial_s v + \tfrac{453}{8}\mathcal{H}(\partial_s\mathcal{H})^2\partial_s v \\
& + \tfrac{117}{8}\mathcal{H}^3(\partial_s v)^2 + \tfrac{147}{16}\mathcal{H}\partial_s\mathcal{H}(\partial_s v)^2 + \tfrac{1}{2}\mathcal{H}(\partial_s v)^3 - \tfrac{11}{4}\mathcal{H}^2\dot{\phi}\partial_s\dot{\phi} - \tfrac{1}{6}\dot{\phi}\partial_s\mathcal{H}\partial_s\dot{\phi} \\
& - \tfrac{7}{6}\mathcal{H}(\partial_s\dot{\phi})^2 + \tfrac{315}{8}\mathcal{H}^2\partial_s v\partial_s^2\mathcal{H} + 12\partial_s\mathcal{H}\partial_s v\partial_s^2\mathcal{H} + \tfrac{5}{16}(\partial_s v)^2\partial_s^2\mathcal{H} + \tfrac{321}{4}\mathcal{H}^4\partial_s^2 v \\
& + \tfrac{1}{24}\dot{\phi}^2\partial_s^2 v + \tfrac{951}{8}\mathcal{H}^2\partial_s\mathcal{H}\partial_s^2 v + \tfrac{33}{2}(\partial_s\mathcal{H})^2\partial_s^2 v + \tfrac{327}{16}\mathcal{H}^2\partial_s v\partial_s^2 v + \tfrac{45}{16}\partial_s\mathcal{H}\partial_s v\partial_s^2 v \\
& + \tfrac{17}{32}(\partial_s v)^2\partial_s^2 v + \tfrac{339}{16}\mathcal{H}\partial_s^2\mathcal{H}\partial_s^2 v + \tfrac{67}{16}\mathcal{H}(\partial_s^2 v)^2 - \tfrac{7}{6}\mathcal{H}\dot{\phi}\partial_s^2\dot{\phi} - \tfrac{19}{48}\partial_s\dot{\phi}\partial_s^2\dot{\phi} \\
& + \tfrac{93}{16}\mathcal{H}\partial_s v\partial_s^3\mathcal{H} + \tfrac{15}{8}\partial_s^2 v\partial_s^3\mathcal{H} + \tfrac{135}{4}\mathcal{H}^3\partial_s^3 v + \tfrac{207}{8}\mathcal{H}\partial_s\mathcal{H}\partial_s^3 v + \tfrac{7}{2}\mathcal{H}\partial_s v\partial_s^3 v \\
& + \tfrac{47}{16}\partial_s^2\mathcal{H}\partial_s^3 v + \tfrac{33}{64}\partial_s^2 v\partial_s^3 v - \tfrac{5}{48}\dot{\phi}\partial_s^3\dot{\phi} + \tfrac{15}{32}\partial_s v\partial_s^4\mathcal{H} + \tfrac{63}{16}\mathcal{H}^2\partial_s^4 v \\
& + \tfrac{15}{8}\partial_s\mathcal{H}\partial_s^4 v - \tfrac{3}{64}\partial_s v\partial_s^4 v + \tfrac{1}{48}\partial_s^6 v \\
& + \nu\big(\tfrac{41}{8}\mathcal{H}^3\partial_s^2\mathcal{K} + \tfrac{53}{16}\mathcal{H}\partial_s\mathcal{H}\partial_s^2\mathcal{K} + \tfrac{3}{8}\mathcal{H}\partial_s v\partial_s^2\mathcal{K} + \tfrac{1}{4}\partial_s^2\mathcal{H}\partial_s^2\mathcal{K} + \tfrac{1}{8}\partial_s^2 v\partial_s^2\mathcal{K} \\
& + \tfrac{65}{16}\mathcal{H}^2\partial_s^3\mathcal{K} - \tfrac{1}{16}\mathcal{H}v\partial_s^3\mathcal{K} + \tfrac{3}{4}\partial_s\mathcal{H}\partial_s^3\mathcal{K} + \tfrac{11}{96}\partial_s v\partial_s^3\mathcal{K} - \tfrac{1}{16}\mathcal{H}\partial_t\partial_s^2\mathcal{K} \\
& + \tfrac{29}{32}\mathcal{H}\partial_s^4\mathcal{K} - \tfrac{1}{48}v\partial_s^4\mathcal{K} - \tfrac{1}{48}\partial_t\partial_s^3\mathcal{K} + \tfrac{3}{64}\partial_s^5\mathcal{K}\big)\big].
\end{aligned} \tag{E1}$$

As for the radius evolution, it should be corrected at that order by

$$\begin{aligned}\partial_t \ln R \supset R^4\big[&-\tfrac{9}{4}\mathcal{H}^4\partial_s v-\tfrac{63}{16}\mathcal{H}^2\partial_s\mathcal{H}\partial_s v-\tfrac{15}{32}(\partial_s\mathcal{H})^2\partial_s v-\tfrac{3}{8}\mathcal{H}^2(\partial_s v)^2-\tfrac{1}{16}\partial_s\mathcal{H}(\partial_s v)^2\\&+\tfrac{1}{8}\mathcal{H}\dot\phi\partial_s\dot\phi+\tfrac{1}{48}(\partial_s\dot\phi)^2-\tfrac{3}{4}\mathcal{H}\partial_s v\partial_s^2\mathcal{H}-\tfrac{51}{16}\mathcal{H}^3\partial_s^2 v-\tfrac{63}{32}\mathcal{H}\partial_s\mathcal{H}\partial_s^2 v\\&-\tfrac{13}{32}\mathcal{H}\partial_s v\partial_s^2 v-\tfrac{9}{64}\partial_s^2\mathcal{H}\partial_s^2 v-\tfrac{3}{64}(\partial_s^2 v)^2+\tfrac{1}{48}\dot\phi\partial_s^2\dot\phi-\tfrac{3}{64}\partial_s v\partial_s^3\mathcal{H}\\&-\tfrac{27}{32}\mathcal{H}^2\partial_s^3 v-\tfrac{5}{32}\partial_s\mathcal{H}\partial_s^3 v-\tfrac{3}{64}\partial_s v\partial_s^3 v-\tfrac{1}{64}\mathcal{H}\partial_s^4 v+\tfrac{1}{128}\partial_s^5 v\\&-\nu\big(\tfrac{3}{16}\mathcal{H}^2\partial_s^2\mathcal{K}+\tfrac{1}{32}\partial_s\mathcal{H}\partial_s^2\mathcal{K}+\tfrac{3}{32}\mathcal{H}\partial_s^3\mathcal{K}+\tfrac{1}{96}\partial_s^4\mathcal{K}\big)\big].\end{aligned}\tag{E2}$$

APPENDIX F: HIGHER ORDER CONSTRAINTS FOR CURVED FIBERS

From the boundary constraint (5.5) we get the additional contributions

$$\begin{aligned}\widehat{V}_a^{(1)} \supset R^2\big(&-\tfrac{12}{5}\widehat{V}_a^{(2)}-3\mathcal{H}v_a^{(1)}-\tfrac{21}{80}\varepsilon_a{}^c\widehat{V}_{bc}^{(1)}\kappa^b+\tfrac{7}{2}\mathcal{H}^2\kappa_a\dot\phi+\tfrac{3}{128}\kappa_a\kappa_b\kappa^b\dot\phi\\&+\tfrac{63}{40}\mathcal{H}\kappa^b\widetilde{\kappa}_a\omega_b-\tfrac{63}{40}\mathcal{H}\kappa_b\kappa^b\widetilde{\omega}_a+\tfrac{5}{8}\kappa_a\dot\phi\partial_s\mathcal{H}-\tfrac{19}{4}\mathcal{H}^2\widetilde{\kappa}_a\partial_s v-\tfrac{3}{256}\kappa_b\kappa^b\widetilde{\kappa}_a\partial_s v\\&-\tfrac{31}{32}\widetilde{\kappa}_a\partial_s\mathcal{H}\partial_s v-\tfrac{3}{5}\partial_s v_a^{(1)}+\mathcal{H}\dot\phi\partial_s\kappa_a+\tfrac{7}{2}\mathcal{H}^2 v\partial_s\widetilde{\kappa}_a+\tfrac{3}{128}v\kappa_b\kappa^b\partial_s\widetilde{\kappa}_a\\&+\tfrac{5}{8}v\partial_s\mathcal{H}\partial_s\widetilde{\kappa}_a+\tfrac{1}{8}\mathcal{H}\partial_s v\partial_s\widetilde{\kappa}_a+\tfrac{103}{40}\mathcal{H}\kappa_a\partial_s\dot\phi+\tfrac{7}{2}\mathcal{H}^2\partial_s\widetilde{\omega}_a+\tfrac{3}{128}\kappa_b\kappa^b\partial_s\widetilde{\omega}_a\\&+\tfrac{5}{8}\partial_s\mathcal{H}\partial_s\widetilde{\omega}_a-\tfrac{1}{16}\partial_s\widetilde{\kappa}_a\partial_s^2 v+\mathcal{H}v\partial_s^2\widetilde{\kappa}_a+\mathcal{H}\partial_s^2\widetilde{\omega}_a-\tfrac{53}{32}\mathcal{H}\widetilde{\kappa}_a\partial_s^2 v-\tfrac{7}{64}\widetilde{\kappa}_a\partial_s^3 v\big),\end{aligned}\tag{F1}$$

$$\begin{aligned}P_\varnothing^{(0)} \supset -\nu R\big(\tfrac{3}{2}\mathcal{H}^2+\tfrac{1}{2}\kappa_a\kappa^a+\partial_s\mathcal{H}\big)+R^2\big(&-P_\varnothing^{(1)}-3\mathcal{H}^2\partial_s v-\tfrac{1}{4}\kappa_a\kappa^a\partial_s v-\tfrac{9}{4}\partial_s\mathcal{H}\partial_s v\\&+\tfrac{1}{2}v\kappa^a\partial_s\kappa_a+\tfrac{1}{2}\kappa^a\partial_s\omega_a-\tfrac{9}{4}\mathcal{H}\partial_s^2 v-\tfrac{3}{8}\partial_s^3 v\big),\end{aligned}\tag{F2}$$

$$\begin{aligned}P_a^{(0)} \supset \nu R\big(&-\tfrac{3}{2}\varepsilon_a^c\mathcal{R}_{bc}\kappa^b+\tfrac{7}{2}\mathcal{H}^2\widetilde{\kappa}_a+\tfrac{3}{4}\kappa_b\kappa^b\widetilde{\kappa}_a+2\widetilde{\kappa}_a\partial_s\mathcal{H}+\mathcal{H}\partial_s\widetilde{\kappa}_a\big)\\&+R^2\big(-P_a^{(1)}-\tfrac{8}{5}\widehat{V}_a^{(2)}+4\mathcal{H}v_a^{(1)}+\tfrac{9}{20}\varepsilon_a{}^c\widehat{V}_{bc}^{(1)}\kappa^b-5\mathcal{H}^2\kappa_a\dot\phi-\tfrac{15}{32}\kappa_a\kappa_b\kappa^b\dot\phi\\&+\tfrac{13}{10}\mathcal{H}\kappa^b\widetilde{\kappa}_a\omega_b-\tfrac{13}{10}\mathcal{H}\kappa_b\kappa^b\widetilde{\omega}_a-\tfrac{5}{2}\kappa_a\dot\phi\partial_s\mathcal{H}+\tfrac{17}{2}\mathcal{H}^2\widetilde{\kappa}_a\partial_s v+\tfrac{31}{64}\kappa_b\kappa^b\widetilde{\kappa}_a\partial_s v\\&+\tfrac{43}{8}\widetilde{\kappa}_a\partial_s\mathcal{H}\partial_s v+\tfrac{8}{5}\partial_s v_a^{(1)}-2\mathcal{H}\dot\phi\partial_s\kappa_a-\tfrac{1}{2}v\kappa^b\widetilde{\kappa}_a\partial_s\kappa_b-5\mathcal{H}^2 v\partial_s\widetilde{\kappa}_a\\&-\tfrac{15}{32}v\kappa_b\kappa^b\partial_s\widetilde{\kappa}_a-\tfrac{5}{2}v\partial_s\mathcal{H}\partial_s\widetilde{\kappa}_a+\tfrac{1}{2}\mathcal{H}\partial_s v\partial_s\widetilde{\kappa}_a-\tfrac{7}{10}\mathcal{H}\kappa_a\partial_s\dot\phi-\tfrac{1}{2}\kappa^b\widetilde{\kappa}_a\partial_s\omega_b\\&-5\mathcal{H}^2\partial_s\widetilde{\omega}_a-\tfrac{15}{32}\kappa_b\kappa^b\partial_s\widetilde{\omega}_a-\tfrac{5}{2}\partial_s\mathcal{H}\partial_s\widetilde{\omega}_a+\tfrac{41}{8}\mathcal{H}\widetilde{\kappa}_a\partial_s^2 v+\tfrac{1}{4}\partial_s\widetilde{\kappa}_a\partial_s^2 v\\&-2\mathcal{H}v\partial_s^2\widetilde{\kappa}_a-2\mathcal{H}\partial_s^2\widetilde{\omega}_a+\tfrac{11}{16}\widetilde{\kappa}_a\partial_s^3 v\big).\end{aligned}\tag{F3}$$

From the higher order of the Navier-Stokes equation that we consider as constraints, we get

$$\begin{aligned}v^{(2)} = \frac{\nu}{R}\Big(&-\frac{1}{16}\mathcal{H}^3-\frac{5}{64}\mathcal{H}\kappa_a\kappa^a-\frac{7}{64}\kappa^a\partial_s\kappa_a+\frac{1}{32}\partial_s^2\mathcal{H}\Big)\\&-\frac{3}{16}\mathcal{H}(\partial_s v)^2-\frac{1}{64}\overline{g}\kappa_a\kappa^a+\frac{3}{32}\mathcal{H}(v)^2\kappa_a\kappa^a+\frac{3}{16}\mathcal{H}v\kappa^a\Omega_a+\frac{3}{32}\mathcal{H}\Omega_a\Omega^a\\&+\frac{1}{32}U^a\kappa^b\widetilde{\kappa}_a\Omega_b+\frac{1}{64}\kappa_a\kappa^a\overline{I}+\frac{3}{32}\kappa^a\dot\phi\widetilde{\Omega}_a-\frac{1}{64}\kappa^a\overline{\Omega}\widetilde{\Omega}_a+\frac{1}{32}U^a\kappa_a\kappa^b\widetilde{\Omega}_b\\&+\frac{3}{16}\mathcal{H}v\kappa^a\omega_a-\frac{1}{32}\widetilde{\kappa}^a\dot\phi\omega_a+\frac{3}{32}\widetilde{\kappa}^a\overline{\Omega}\omega_a+\frac{3}{16}\mathcal{H}\Omega^a\omega_a+\frac{3}{32}\mathcal{H}\omega_a\omega^a+\frac{1}{64}\widetilde{\kappa}^a\omega_a\bar\omega\\&+\frac{1}{64}U^a\kappa_b\kappa^b\widetilde{\omega}_a-\frac{9}{64}\mathcal{H}\kappa_a\kappa^a\partial_s v+\frac{7}{64}v\kappa_a\kappa^a\partial_s v+\frac{7}{64}\kappa^a\Omega_a\partial_s v+\frac{7}{64}\kappa^a\omega_a\partial_s v\\&+\frac{9}{16}\mathcal{H}\partial_s\mathcal{H}\partial_s v+\frac{5}{64}(v)^2\kappa^a\partial_s\kappa_a+\frac{3}{32}\widetilde{\kappa}^a\dot\phi\partial_s\kappa_a+\frac{3}{32}v\Omega^a\partial_s\kappa_a+\frac{3}{32}v\omega^a\partial_s\kappa_a\\&-\frac{3}{64}\kappa^a\partial_s v\partial_s\kappa_a+\frac{3}{64}v\partial_s\kappa_a\partial_s\kappa^a+\frac{1}{16}\dot\phi\partial_s\dot\phi+\frac{1}{16}\overline{\Omega}\partial_s\dot\phi+\frac{1}{16}v\kappa^a\partial_s\omega_a\\&+\frac{3}{32}\Omega^a\partial_s\omega_a+\frac{3}{32}\omega^a\partial_s\omega_a+\frac{3}{64}\partial_s\kappa^a\partial_s\omega_a+\frac{1}{64}\kappa_a\kappa^a\partial_t\bar U-\frac{1}{64}\kappa^a\partial_t\omega_a\\&+\frac{9}{16}\mathcal{H}^2\partial_s^2 v-\frac{33}{128}\kappa_a\kappa^a\partial_s^2 v-\frac{9}{64}\partial_s v\partial_s^2 v+\frac{9}{64}v\kappa^a\partial_s^2\kappa_a+\frac{9}{64}\kappa^a\partial_s^2\omega_a+\frac{9}{32}\mathcal{H}\partial_s^3 v+\frac{3}{64}\partial_s^4 v,\end{aligned}\tag{F4}$$

$$
\begin{aligned}
v_a^{(1)} =& \frac{\nu}{4R}\partial_s\widetilde{\kappa}_a + \frac{1}{4}\upsilon\kappa_a\dot{\phi} + \frac{1}{4}\upsilon\kappa_a\overline{\Omega} + \frac{1}{4}\dot{\phi}\Omega_a + \frac{1}{4}\overline{\Omega}\Omega_a + \frac{1}{4}\dot{\phi}\omega_a + \frac{1}{4}\overline{\Omega}\omega_a - \frac{1}{4}\kappa^b\widetilde{\kappa}_a\omega_b \\
&+ \frac{1}{8}\kappa_a\widetilde{\kappa}^b\omega_b + \frac{1}{4}\kappa_b\kappa^b\widetilde{\omega}_a + \frac{3}{8}\mathcal{H}\widetilde{\kappa}_a\partial_s\upsilon - \frac{3}{8}\upsilon\widetilde{\kappa}_a\partial_s\upsilon - \frac{3}{8}\widetilde{\Omega}_a\partial_s\upsilon - \frac{3}{8}\widetilde{\omega}_a\partial_s\upsilon - \frac{1}{4}\dot{\phi}\partial_s\kappa_a \\
&+ \frac{1}{4}\partial_s\upsilon\partial_s\widetilde{\kappa}_a - \frac{3}{8}\kappa_a\partial_s\dot{\phi} + \frac{9}{8}\widetilde{\kappa}_a\partial_s^2\upsilon - \frac{1}{4}\upsilon\partial_s^2\widetilde{\kappa}_a - \frac{1}{4}\partial_s^2\widetilde{\omega}_a,
\end{aligned} \tag{F5}
$$

$$
\begin{aligned}
\widehat{V}_a^{(2)} =& \frac{\nu}{R}\left(-\frac{5}{32}\varepsilon_a{}^c\mathcal{R}_{bc}\kappa^b + \frac{5}{128}\mathcal{H}^2\widetilde{\kappa}_a + \frac{5}{128}\kappa_b\kappa^b\widetilde{\kappa}_a - \frac{5}{128}\widetilde{\kappa}_a\partial_s\mathcal{H} - \frac{1}{64}\mathcal{H}\partial_s\widetilde{\kappa}_a + \frac{1}{64}\partial_s^2\widetilde{\kappa}_a\right) \\
&- \frac{1}{32}\varepsilon_a{}^c\widehat{V}_{bc}^{(1)}\kappa^b - \frac{5}{192}\Omega^2\widetilde{\kappa}_a - \frac{5}{64}(\dot{\phi})^2\widetilde{\kappa}_a - \frac{1}{256}(\partial_s\upsilon)^2\widetilde{\kappa}_a - \frac{5}{192}(\upsilon)^2\kappa_b\kappa^b\widetilde{\kappa}_a \\
&+ \frac{25}{256}\kappa_a\kappa_b\kappa^b\dot{\phi} - \frac{1}{16}\widetilde{\kappa}_a\dot{\phi}\overline{\Omega} + \frac{1}{24}\widetilde{\kappa}_a\overline{\Omega}^2 - \frac{3}{32}\upsilon\kappa^b\widetilde{\kappa}_a\Omega_b - \frac{1}{24}\widetilde{\kappa}_a\Omega_b\Omega^b + \frac{1}{24}\upsilon\kappa_b\kappa^b\widetilde{\Omega}_a \\
&+ \frac{1}{24}\kappa^b\Omega_b\widetilde{\Omega}_a + \frac{3}{16}\mathcal{H}\kappa^b\widetilde{\kappa}_a\omega_b - \frac{5}{48}\upsilon\kappa^b\widetilde{\kappa}_a\omega_b - \frac{7}{48}\widetilde{\kappa}_a\Omega^b\omega_b + \frac{1}{24}\kappa^b\widetilde{\Omega}_a\omega_b \\
&- \frac{5}{64}\widetilde{\kappa}_a\omega_b\omega^b - \frac{3}{16}\mathcal{H}\kappa_b\kappa^b\widetilde{\omega}_a + \frac{5}{96}\upsilon\kappa_b\kappa^b\widetilde{\omega}_a + \frac{5}{96}\kappa^b\Omega_b\widetilde{\omega}_a + \frac{5}{96}\kappa^b\omega_b\widetilde{\omega}_a \\
&- \frac{5}{32}\mathcal{H}\upsilon\widetilde{\kappa}_a\partial_s\upsilon + \frac{235}{1536}\kappa_b\kappa^b\widetilde{\kappa}_a\partial_s\upsilon - \frac{1}{16}\kappa_a\dot{\phi}\partial_s\upsilon + \frac{5}{64}\kappa_a\overline{\Omega}\partial_s\upsilon - \frac{5}{32}\mathcal{H}\widetilde{\Omega}_a\partial_s\upsilon \\
&- \frac{5}{32}\mathcal{H}\widetilde{\omega}_a\partial_s\upsilon + \frac{9}{64}\widetilde{\kappa}_a\partial_s\mathcal{H}\partial_s\upsilon + \frac{3}{32}\upsilon\dot{\phi}\partial_s\kappa_a + \frac{3}{32}\upsilon\overline{\Omega}\partial_s\kappa_a - \frac{35}{384}\upsilon\kappa^b\widetilde{\kappa}_a\partial_s\kappa_b \\
&- \frac{7}{96}\widetilde{\kappa}_a\omega^b\partial_s\kappa_b + \frac{19}{96}\kappa^b\widetilde{\omega}_a\partial_s\kappa_b + \frac{25}{256}\upsilon\kappa_b\kappa^b\partial_s\widetilde{\kappa}_a - \frac{1}{8}\kappa^b\omega_b\partial_s\widetilde{\kappa}_a + \frac{7}{32}\mathcal{H}\partial_s\upsilon\partial_s\widetilde{\kappa}_a \\
&- \frac{9}{64}\upsilon\partial_s\upsilon\partial_s\widetilde{\kappa}_a + \frac{3}{16}\mathcal{H}\kappa_a\partial_s\dot{\phi} - \frac{1}{96}\upsilon\kappa_a\partial_s\dot{\phi} - \frac{1}{96}\Omega_a\partial_s\dot{\phi} - \frac{1}{96}\omega_a\partial_s\dot{\phi} - \frac{5}{48}\partial_s\kappa_a\partial_s\dot{\phi} \\
&+ \frac{3}{32}\dot{\phi}\partial_s\omega_a + \frac{3}{32}\overline{\Omega}\partial_s\omega_a - \frac{83}{384}\kappa^b\widetilde{\kappa}_a\partial_s\omega_b + \frac{57}{256}\kappa_b\kappa^b\partial_s\widetilde{\omega}_a - \frac{9}{64}\partial_s\upsilon\partial_s\widetilde{\omega}_a \\
&+ \frac{9}{64}\mathcal{H}\widetilde{\kappa}_a\partial_s^2\upsilon - \frac{1}{16}\upsilon\widetilde{\kappa}_a\partial_s^2\upsilon - \frac{1}{16}\widetilde{\Omega}_a\partial_s^2\upsilon - \frac{1}{16}\widetilde{\omega}_a\partial_s^2\upsilon + \frac{13}{64}\partial_s\widetilde{\kappa}_a\partial_s^2\upsilon - \frac{1}{64}\dot{\phi}\partial_s^2\kappa_a \\
&+ \frac{5}{128}\partial_s\upsilon\partial_s^2\widetilde{\kappa}_a - \frac{9}{64}\kappa_a\partial_s^2\dot{\phi} + \frac{9}{128}\widetilde{\kappa}_a\partial_s^3\upsilon - \frac{1}{64}\upsilon\partial_s^3\widetilde{\kappa}_a - \frac{1}{64}\partial_s^3\widetilde{\omega}_a,
\end{aligned} \tag{F6}
$$

$$
\begin{aligned}
\widehat{V}_{ab}^{(1)} =& \frac{\nu}{R}\left(\mathcal{R}_{ab} + \frac{1}{3}\kappa_{\langle a}\kappa_{b\rangle}\right) + \kappa_{\langle a}\kappa_{b\rangle}\left[\frac{1}{6}(\upsilon)^2 + \frac{5}{8}\partial_s\upsilon\right] + \frac{1}{4}\kappa_{\langle a}\widetilde{\kappa}_{b\rangle}\dot{\phi} - \frac{1}{4}\upsilon\kappa_{\langle a}\partial_s\kappa_{b\rangle} \\
&- \frac{1}{4}\kappa_{\langle a}\partial_s\omega_{b\rangle} + \frac{1}{6}(\omega_{\langle a} + \Omega_{\langle a})(\omega_{b\rangle} + \Omega_{b\rangle}) + \frac{1}{3}\upsilon\kappa_{\langle a}(\omega_{b\rangle} + \Omega_{b\rangle}).
\end{aligned} \tag{F7}
$$

From the incompressibility constraint (5.15) we also obtain

$$
\begin{aligned}
P_\varnothing^{(1)} =& \frac{\nu}{4R}(-\mathcal{H}^2 - \kappa_a\kappa^a + \partial_s\mathcal{H}) + \frac{1}{2}\Omega^2 + \frac{1}{2}(\dot{\phi})^2 - \frac{3}{8}(\partial_s\upsilon)^2 + \frac{1}{2}(\upsilon)^2\kappa_a\kappa^a + \dot{\phi}\overline{\Omega} + \upsilon\kappa^a\Omega_a + \upsilon\kappa^a\omega_a \\
&+ \Omega^a\omega_a + \frac{1}{2}\omega_a\omega^a - \frac{1}{8}\kappa_a\kappa^a\partial_s\upsilon + \frac{1}{4}\upsilon\kappa^a\partial_s\kappa_a + \frac{1}{4}\kappa^a\partial_s\omega_a + \frac{1}{4}\partial_s^3\upsilon,
\end{aligned} \tag{F8}
$$

$$
\begin{aligned}
P_a^{(1)} =& \frac{\nu}{R}\left(-\frac{3}{4}\varepsilon_a{}^c\mathcal{R}_{bc}\kappa^b + \frac{3}{16}\mathcal{H}^2\widetilde{\kappa}_a + \frac{3}{16}\kappa_b\kappa^b\widetilde{\kappa}_a - \frac{3}{16}\widetilde{\kappa}_a\partial_s\mathcal{H} + \frac{1}{8}\mathcal{H}\partial_s\widetilde{\kappa}_a - \frac{1}{8}\partial_s^2\widetilde{\kappa}_a\right) \\
&- \frac{1}{8}\Omega^2\widetilde{\kappa}_a - \frac{3}{8}(\dot{\phi})^2\widetilde{\kappa}_a + \frac{9}{32}(\partial_s\upsilon)^2\widetilde{\kappa}_a - \frac{1}{8}(\upsilon)^2\kappa_b\kappa^b\widetilde{\kappa}_a - \frac{3}{16}\kappa_a\kappa_b\kappa^b\dot{\phi} - \frac{1}{2}\widetilde{\kappa}_a\dot{\phi}\overline{\Omega} \\
&- \frac{1}{4}\upsilon\kappa^b\widetilde{\kappa}_a\Omega_b - \frac{1}{2}\upsilon\kappa^b\widetilde{\kappa}_a\omega_b - \frac{1}{2}\widetilde{\kappa}_a\Omega^c\omega_c - \frac{3}{8}\widetilde{\kappa}_a\omega_c\omega^c + \frac{1}{4}\upsilon\kappa_b\kappa^b\widetilde{\omega}_a + \frac{1}{4}\kappa^b\Omega_b\widetilde{\omega}_a \\
&+ \frac{1}{4}\kappa^b\omega_b\widetilde{\omega}_a + \frac{3}{4}\mathcal{H}\upsilon\widetilde{\kappa}_a\partial_s\upsilon + \frac{1}{8}\kappa_b\kappa^b\widetilde{\kappa}_a\partial_s\upsilon - \frac{1}{2}\kappa_a\dot{\phi}\partial_s\upsilon - \frac{1}{8}\kappa_a\overline{\Omega}\partial_s\upsilon + \frac{3}{4}\mathcal{H}\widetilde{\Omega}_a\partial_s\upsilon \\
&+ \frac{3}{4}\mathcal{H}\widetilde{\omega}_a\partial_s\upsilon + \frac{1}{4}\upsilon\dot{\phi}\partial_s\kappa_a + \frac{1}{4}\upsilon\overline{\Omega}\partial_s\kappa_a - \frac{1}{16}\upsilon\kappa^b\widetilde{\kappa}_a\partial_s\kappa_b - \frac{3}{16}\upsilon\kappa_b\kappa^b\partial_s\widetilde{\kappa}_a - \frac{3}{8}\upsilon\partial_s\upsilon\partial_s\widetilde{\kappa}_a \\
&- \frac{1}{4}\upsilon\kappa_a\partial_s\dot{\phi} - \frac{1}{4}\Omega_a\partial_s\dot{\phi} - \frac{1}{4}\omega_a\partial_s\dot{\phi} + \frac{1}{4}\partial_s\kappa_a\partial_s\dot{\phi} + \frac{1}{4}\dot{\phi}\partial_s\omega_a + \frac{1}{4}\overline{\Omega}\partial_s\omega_a
\end{aligned}
$$

$$
-\frac{1}{16}\kappa^b\widetilde{\kappa}_a\partial_s\omega_b-\frac{3}{16}\kappa_b\kappa^b\partial_s\widetilde{\omega}_a-\frac{3}{8}\partial_s v\partial_s\widetilde{\omega}_a+\frac{1}{4}v\widetilde{\kappa}_a\partial_s^2 v+\frac{1}{4}\widetilde{\Omega}_a\partial_s^2 v+\frac{1}{4}\widetilde{\omega}_a\partial_s^2 v
-\frac{1}{8}\partial_s\widetilde{\kappa}_a\partial_s^2 v+\frac{1}{8}\dot{\phi}\partial_s^2\kappa_a+\frac{3}{16}\partial_s v\partial_s^2\widetilde{\kappa}_a+\frac{1}{8}\kappa_a\partial_s^2\dot{\phi}-\frac{3}{8}\widetilde{\kappa}_a\partial_s^3 v+\frac{1}{8}v\partial_s^3\widetilde{\kappa}_a+\frac{1}{8}\partial_s^3\widetilde{\omega}_a. \tag{F9}
$$

APPENDIX G: HIGHER ORDER CORRECTIONS FOR CURVED FIBERS

Once the boundary constraints of Sec. VII C 6 and Appendix F are replaced, the dynamical evolution for the longitudinal velocity and the FCL velocity are given by

$$
\begin{aligned}
\partial_t v ={}& \overline{g}+\frac{v\mathcal{H}}{R}-\overline{I}-2U^a\widetilde{\Omega}_a-U^a\widetilde{\omega}_a+6\mathcal{H}\partial_s v-v\partial_s v-\partial_t\bar{U}+3\partial_s^2 v\\
&+vR\left(\frac{5}{4}\mathcal{H}^3+\frac{1}{4}\mathcal{H}\kappa_a\kappa^a+\frac{15}{4}\mathcal{H}\partial_s\mathcal{H}+\frac{1}{2}\kappa^a\partial_s\kappa_a+\frac{5}{4}\partial_s^2\mathcal{H}\right)\\
&+R^2\Big(\Omega^2\mathcal{H}-8v_{\varnothing}^{(2)}+\mathcal{H}v^2\kappa_a\kappa^a-5v^{(1)a}\widetilde{\kappa}_a+\mathcal{H}\dot{\phi}^2+2\mathcal{H}\dot{\phi}\overline{\Omega}+2\mathcal{H}v\kappa^a\Omega_a+\frac{1}{2}\kappa^a\dot{\phi}\widetilde{\Omega}_a\\
&+2\mathcal{H}v\kappa^a\omega_a+\frac{1}{2}\widetilde{\kappa}^a\dot{\phi}\omega_a+\widetilde{\kappa}^a\overline{\Omega}\omega_a+2\mathcal{H}\Omega^a\omega_a+\mathcal{H}\omega_a\omega^a+12\mathcal{H}^3\partial_s v\\
&-\frac{9}{4}\mathcal{H}\kappa_a\kappa^a\partial_s v+\frac{5}{4}v\kappa_a\kappa^a\partial_s v+\frac{5}{4}\kappa^a\Omega_a\partial_s v+\frac{5}{4}\kappa^a\omega_a\partial_s v+21\mathcal{H}\partial_s\mathcal{H}\partial_s v\\
&-\frac{3}{4}\mathcal{H}(\partial_s v)^2+\frac{3}{2}\mathcal{H}v\kappa^a\partial_s\kappa_a+\frac{1}{2}v^2\kappa^a\partial_s\kappa_a+\frac{1}{4}\widetilde{\kappa}^a\dot{\phi}\partial_s\kappa_a+\frac{1}{2}v\Omega^a\partial_s\kappa_a+\frac{1}{2}v\omega^a\partial_s\kappa_a\\
&+\frac{1}{8}\kappa^a\partial_s v\partial_s\kappa_a+\dot{\phi}\partial_s\dot{\phi}+\overline{\Omega}\partial_s\dot{\phi}+\frac{3}{2}\mathcal{H}\kappa^a\partial_s\omega_a+\frac{1}{2}v\kappa^a\partial_s\omega_a+\frac{1}{2}\Omega^a\partial_s\omega_a\\
&+\frac{1}{2}\omega^a\partial_s\omega_a+3\partial_s v\partial_s^2\mathcal{H}+18\mathcal{H}^2\partial_s^2 v-\frac{3}{8}\kappa_a\kappa^a\partial_s^2 v+6\partial_s\mathcal{H}\partial_s^2 v-\frac{3}{4}\partial_s v\partial_s^2 v\\
&+\frac{1}{4}v\kappa^a\partial_s^2\kappa_a+\frac{1}{4}\kappa^a\partial_s^2\omega_a+6\mathcal{H}\partial_s^3 v+\frac{3}{4}\partial_s^4 v\Big)+O\left(\epsilon_R^3\right),
\end{aligned}\tag{G1}
$$

$$
\begin{aligned}
(\partial_t U)_a ={}& g_a-\frac{v\widetilde{\kappa}_a}{R}+(v)^2\widetilde{\kappa}_a-2\tilde{U}_a\overline{\Omega}-I_a+2\bar{U}\widetilde{\Omega}_a+2v\widetilde{\Omega}_a+2v\widetilde{\omega}_a-3\widetilde{\kappa}_a\partial_s v\\
&+vR\left(\frac{3}{4}\varepsilon_a^c\mathcal{R}_{bc}\kappa^b-\frac{53}{16}\mathcal{H}^2\widetilde{\kappa}_a-\frac{9}{16}\kappa_b\kappa^b\widetilde{\kappa}_a-\frac{35}{16}\widetilde{\kappa}_a\partial_s\mathcal{H}-\frac{7}{8}\mathcal{H}\partial_s\widetilde{\kappa}_a-\frac{1}{8}\partial_s^2\widetilde{\kappa}_a\right)\\
&+R^2\Big(-\frac{24}{5}\widehat{V}_a^{(2)}-8\mathcal{H}v_a^{(1)}-\frac{33}{20}\varepsilon_a^c\widehat{V}_{bc}^{(1)}\kappa^b-\frac{1}{8}\Omega^2\widetilde{\kappa}_a-\frac{1}{8}v^2\kappa_b\kappa^b\widetilde{\kappa}_a+12\mathcal{H}^2\kappa_a\dot{\phi}\\
&+\frac{9}{32}\kappa_a\kappa_b\kappa^b\dot{\phi}-\frac{5}{8}\widetilde{\kappa}_a\dot{\phi}^2-\widetilde{\kappa}_a\dot{\phi}\overline{\Omega}-\frac{1}{4}v\kappa^b\widetilde{\kappa}_a\Omega_b-\frac{1}{4}v\kappa^b\widetilde{\kappa}_a\omega_b+\frac{29}{10}\mathcal{H}\kappa_a\widetilde{\kappa}^b\omega_b\\
&-\frac{1}{2}\widetilde{\kappa}_a\Omega^b\omega_b-\frac{3}{8}\widetilde{\kappa}_a\omega_b\omega^b+\frac{1}{4}\kappa^b\Omega_b\widetilde{\omega}_a+\frac{1}{4}\kappa^b\omega_b\widetilde{\omega}_a+3\kappa_a\dot{\phi}\partial_s\mathcal{H}-18\mathcal{H}^2\widetilde{\kappa}_a\partial_s v\\
&+\frac{3}{4}\mathcal{H}v\widetilde{\kappa}_a\partial_s v-\frac{19}{64}\kappa_b\kappa^b\widetilde{\kappa}_a\partial_s v-\frac{1}{4}\kappa_a\dot{\phi}\partial_s v-\frac{3}{8}\kappa_a\overline{\Omega}\partial_s v+\frac{3}{4}\mathcal{H}\widetilde{\Omega}_a\partial_s v\\
&+\frac{3}{4}\mathcal{H}\widetilde{\omega}_a\partial_s v-\frac{51}{8}\widetilde{\kappa}_a\partial_s\mathcal{H}\partial_s v+\frac{3}{32}\widetilde{\kappa}_a(\partial_s v)^2-\frac{6}{5}\partial_s v_a^{(1)}+4\mathcal{H}\dot{\phi}\partial_s\kappa_a+\frac{1}{4}v\dot{\phi}\partial_s\kappa_a\\
&+\frac{3}{4}v\overline{\Omega}\partial_s\kappa_a+\frac{19}{32}v\kappa^b\widetilde{\kappa}_a\partial_s\kappa_b-\frac{9}{32}v\kappa_a\widetilde{\kappa}^b\partial_s\kappa_b+12\mathcal{H}^2v\partial_s\widetilde{\kappa}_a+3v\partial_s\mathcal{H}\partial_s\widetilde{\kappa}_a\\
&+\frac{1}{2}\mathcal{H}\partial_s v\partial_s\widetilde{\kappa}_a-\frac{1}{8}v\partial_s v\partial_s\widetilde{\kappa}_a+\frac{69}{10}\mathcal{H}\kappa_a\partial_s\dot{\phi}-\frac{1}{2}v\kappa_a\partial_s\dot{\phi}-\frac{1}{4}\Omega_a\partial_s\dot{\phi}-\frac{1}{4}\omega_a\partial_s\dot{\phi}\\
&+\frac{3}{4}\partial_s\kappa_a\partial_s\dot{\phi}+\frac{1}{4}\dot{\phi}\partial_s\omega_a+\frac{3}{4}\overline{\Omega}\partial_s\omega_a+\frac{1}{4}\bar{\omega}\partial_s\omega_a+\frac{19}{32}\kappa^b\widetilde{\kappa}_a\partial_s\omega_b-\frac{9}{32}\kappa_a\widetilde{\kappa}^b\partial_s\omega_b\\
&+12\mathcal{H}^2\partial_s\widetilde{\omega}_a+3\partial_s\mathcal{H}\partial_s\widetilde{\omega}_a+\frac{1}{8}\partial_s v\partial_s\widetilde{\omega}_a-\frac{1}{4}\partial_s\widetilde{\kappa}_a\partial_t v-\frac{1}{4}\kappa_a\partial_t\dot{\phi}-\frac{63}{8}\mathcal{H}\widetilde{\kappa}_a\partial_s^2 v\\
&+\frac{3}{8}v\widetilde{\kappa}_a\partial_s^2 v+\frac{1}{4}\widetilde{\Omega}_a\partial_s^2 v+\frac{1}{4}\widetilde{\omega}_a\partial_s^2 v-\frac{3}{8}\partial_s\widetilde{\kappa}_a\partial_s^2 v+\frac{3}{8}\dot{\phi}\partial_s^2\kappa_a+4\mathcal{H}v\partial_s^2\widetilde{\kappa}_a
\end{aligned}
$$

$$
- \frac{1}{4} v^2 \partial_s^2 \widetilde{\kappa}_a + \frac{9}{16} \partial_s v \partial_s^2 \widetilde{\kappa}_a + \frac{3}{8} \kappa_a \partial_s^2 \dot{\phi} + 4\mathcal{H} \partial_s^2 \widetilde{\omega}_a - \frac{1}{2} v \partial_s^2 \widetilde{\omega}_a + \frac{1}{8} \widetilde{\kappa}_a \partial_s \partial_t v
- \frac{1}{4} \partial_s \partial_t \widetilde{\omega}_a - \frac{21}{16} \widetilde{\kappa}_a \partial_s^3 v + \frac{3}{8} v \partial_s^3 \widetilde{\kappa}_a + \frac{3}{8} \partial_s^3 \widetilde{\omega}_a \Big) + O\left(\epsilon_R^3\right). \tag{G2}
$$

If the higher moments of Navier-Stokes equation and the incompressibility constraint are then used to replace the unknown variables as summarized in Table V, and also if time derivatives in corrective terms are replaced using the lower order dynamical equations, we obtain the closed and final results with order ϵ_R^2 corrections included:

$$
\begin{aligned}
\partial_t v = {} & \overline{g} + \frac{\nu \mathcal{H}}{R} - \overline{I} - 2U^a \widetilde{\Omega}_a - U^a \widetilde{\omega}_a + 6\mathcal{H} \partial_s v - v \partial_s v - \partial_t \bar{U} + 3\partial_s^2 v \\
& + \nu R \left(\frac{7}{4} \mathcal{H}^3 + \frac{3}{4} \mathcal{H} \kappa_a \kappa^a + \frac{15}{4} \mathcal{H} \partial_s \mathcal{H} + \frac{1}{4} \kappa^a \partial_s \kappa_a + \partial_s^2 \mathcal{H} \right) \\
& + R^2 \Big(\Omega^2 \mathcal{H} + \frac{1}{4} \mathcal{H} v^2 \kappa_a \kappa^a + \mathcal{H} \dot{\phi}^2 + 2\mathcal{H} \dot{\phi} \overline{\Omega} + \frac{1}{2} \mathcal{H} v \kappa^a \Omega_a - \frac{3}{4} \mathcal{H} \Omega_a \Omega^a + \kappa^a \dot{\phi} \widetilde{\Omega}_a + \frac{5}{4} \kappa^a \overline{\Omega} \widetilde{\Omega}_a \\
& + \frac{1}{2} \mathcal{H} v \kappa^a \omega_a - \frac{1}{2} \widetilde{\kappa}^a \dot{\phi} \omega_a - \frac{3}{4} \widetilde{\kappa}^a \overline{\Omega} \omega_a + \frac{1}{2} \mathcal{H} \Omega^a \omega_a + \frac{1}{4} \mathcal{H} \omega_a \omega^a + 12\mathcal{H}^3 \partial_s v \\
& - 3\mathcal{H} \kappa_a \kappa^a \partial_s v + 2v \kappa_a \kappa^a \partial_s v + 2\kappa^a \Omega_a \partial_s v + 2\kappa^a \omega_a \partial_s v + \frac{33}{2} \mathcal{H} \partial_s \mathcal{H} \partial_s v + \frac{3}{4} \mathcal{H} (\partial_s v)^2 \\
& + \frac{3}{2} \mathcal{H} v \kappa^a \partial_s \kappa_a - \frac{1}{4} v^2 \kappa^a \partial_s \kappa_a + \frac{3}{4} \widetilde{\kappa}^a \dot{\phi} \partial_s \kappa_a - \frac{1}{4} v \Omega^a \partial_s \kappa_a - \frac{1}{4} v \omega^a \partial_s \kappa_a \\
& - \frac{3}{8} \kappa^a \partial_s v \partial_s \kappa_a - \frac{3}{8} v \partial_s \kappa_a \partial_s \kappa^a + \frac{1}{2} \dot{\phi} \partial_s \dot{\phi} + \frac{1}{2} \overline{\Omega} \partial_s \dot{\phi} + \frac{3}{2} \mathcal{H} \kappa^a \partial_s \omega_a - \frac{1}{4} v \kappa^a \partial_s \omega_a \\
& - \frac{1}{4} \Omega^a \partial_s \omega_a - \frac{1}{4} \omega^a \partial_s \omega_a - \frac{3}{8} \partial_s \kappa^a \partial_s \omega_a + 3\partial_s v \partial_s^2 \mathcal{H} + \frac{27}{2} \mathcal{H}^2 \partial_s^2 v - \frac{57}{16} \kappa_a \kappa^a \partial_s^2 v \\
& + 6\partial_s \mathcal{H} \partial_s^2 v + \frac{3}{8} \partial_s v \partial_s^2 v + \frac{3}{8} v \kappa^a \partial_s^2 \kappa_a + \frac{3}{8} \kappa^a \partial_s^2 \omega_a + \frac{15}{4} \mathcal{H} \partial_s^3 v + \frac{3}{8} \partial_s^4 v \Big) + O\left(\epsilon_R^3\right),
\end{aligned} \tag{G3}
$$

$$
\begin{aligned}
(\partial_t U)_a = {} & g_a - \frac{\nu \widetilde{\kappa}_a}{R} + (v)^2 \widetilde{\kappa}_a - 2\tilde{U}_a \overline{\Omega} - I_a + 2\bar{U} \widetilde{\Omega}_a + 2v \widetilde{\Omega}_a + 2v \widetilde{\omega}_a - 3\widetilde{\kappa}_a \partial_s v \\
& + \nu R \left(-\frac{31}{8} \mathcal{H}^2 \widetilde{\kappa}_a - \frac{7}{8} \kappa_b \kappa^b \widetilde{\kappa}_a - \frac{13}{8} \widetilde{\kappa}_a \partial_s \mathcal{H} - \frac{9}{4} \mathcal{H} \partial_s \widetilde{\kappa}_a - \frac{3}{4} \partial_s^2 \widetilde{\kappa}_a \right) \\
& + R^2 \Big(\frac{1}{2} v^2 \kappa_b \kappa^b \widetilde{\kappa}_a + 12\mathcal{H}^2 \kappa_a \dot{\phi} - 2\mathcal{H} v \kappa_a \dot{\phi} - \frac{1}{4} \widetilde{\kappa}_a \dot{\phi}^2 - 2\mathcal{H} v \kappa_a \overline{\Omega} - \widetilde{\kappa}_a \dot{\phi} \overline{\Omega} \\
& - \frac{3}{4} \widetilde{\kappa}_a \overline{\Omega}^2 - 2\mathcal{H} \dot{\phi} \Omega_a - 2\mathcal{H} \overline{\Omega} \Omega_a + v \kappa^b \widetilde{\kappa}_a \Omega_b + \widetilde{\kappa}_a \Omega_b \Omega^b - \frac{1}{2} \kappa^b \Omega_b \widetilde{\Omega}_a \\
& - \frac{1}{2} v \kappa_a \kappa^b \widetilde{\Omega}_b - 2\mathcal{H} \dot{\phi} \omega_a - 2\mathcal{H} \overline{\Omega} \omega_a + v \kappa^b \widetilde{\kappa}_a \omega_b + 2\mathcal{H} \kappa_a \widetilde{\kappa}^b \omega_b + \frac{3}{2} \widetilde{\kappa}_a \Omega^b \omega_b \\
& - \frac{1}{2} \kappa^b \widetilde{\Omega}_a \omega_b + \frac{1}{2} \widetilde{\kappa}_a \omega_b \omega^b + 3\kappa_a \dot{\phi} \partial_s \mathcal{H} - 21\mathcal{H}^2 \widetilde{\kappa}_a \partial_s v + \frac{9}{2} \mathcal{H} v \widetilde{\kappa}_a \partial_s v \\
& - \frac{27}{16} \kappa_b \kappa^b \widetilde{\kappa}_a \partial_s v - \frac{1}{2} \kappa_a \dot{\phi} \partial_s v - \frac{11}{4} \kappa_a \overline{\Omega} \partial_s v + \frac{9}{2} \mathcal{H} \widetilde{\Omega}_a \partial_s v + \frac{9}{2} \mathcal{H} \widetilde{\omega}_a \partial_s v \\
& - \frac{27}{4} \widetilde{\kappa}_a \partial_s \mathcal{H} \partial_s v + \frac{15}{16} \widetilde{\kappa}_a (\partial_s v)^2 + 6\mathcal{H} \dot{\phi} \partial_s \kappa_a - \frac{1}{2} v \dot{\phi} \partial_s \kappa_a - \frac{1}{2} v \overline{\Omega} \partial_s \kappa_a \\
& + \frac{3}{8} v \kappa^b \widetilde{\kappa}_a \partial_s \kappa_b + \frac{1}{2} \widetilde{\kappa}_a \omega^b \partial_s \kappa_b - \kappa^b \widetilde{\omega}_a \partial_s \kappa_b + 12\mathcal{H}^2 v \partial_s \widetilde{\kappa}_a + \frac{1}{2} \kappa^b \omega_b \partial_s \widetilde{\kappa}_a \\
& + 3v \partial_s \mathcal{H} \partial_s \widetilde{\kappa}_a - \frac{9}{2} \mathcal{H} \partial_s v \partial_s \widetilde{\kappa}_a + \frac{9}{4} v \partial_s v \partial_s \widetilde{\kappa}_a + 8\mathcal{H} \kappa_a \partial_s \dot{\phi} - \frac{1}{2} v \kappa_a \partial_s \dot{\phi} \\
& - \frac{1}{2} \Omega_a \partial_s \dot{\phi} - \frac{1}{2} \omega_a \partial_s \dot{\phi} + 2\partial_s \kappa_a \partial_s \dot{\phi} - \frac{1}{2} \dot{\phi} \partial_s \omega_a - \frac{1}{2} \overline{\Omega} \partial_s \omega_a + \frac{3}{8} \kappa^b \widetilde{\kappa}_a \partial_s \omega_b \\
& + \frac{1}{2} \kappa_a \widetilde{\kappa}^b \partial_s \omega_b + 12\mathcal{H}^2 \partial_s \widetilde{\omega}_a + 3\partial_s \mathcal{H} \partial_s \widetilde{\omega}_a + \frac{9}{4} \partial_s v \partial_s \widetilde{\omega}_a - \frac{69}{4} \mathcal{H} \widetilde{\kappa}_a \partial_s^2 v
\end{aligned}
$$

$$+\frac{3}{2}v\widetilde{\kappa}_a\partial_s^2 v+\frac{3}{2}\widetilde{\Omega}_a\partial_s^2 v+\frac{3}{2}\widetilde{\omega}_a\partial_s^2 v-\frac{21}{4}\partial_s\widetilde{\kappa}_a\partial_s^2 v+\frac{3}{4}\dot{\phi}\partial_s^2\kappa_a+6\mathcal{H}v\partial_s^2\widetilde{\kappa}_a$$
$$-\frac{3}{8}\partial_s v\partial_s^2\widetilde{\kappa}_a+\frac{5}{4}\kappa_a\partial_s^2\dot{\phi}+6\mathcal{H}\partial_s^2\widetilde{\omega}_a-\frac{27}{8}\widetilde{\kappa}_a\partial_s^3 v+\frac{3}{4}v\partial_s^3\widetilde{\kappa}_a+\frac{3}{4}\partial_s^3\widetilde{\omega}_a\Big)+O\left(\epsilon_R^3\right). \tag{G4}$$

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