Two order parameters for the Kuramoto model on complex networks

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We investigate the behavior of two different order parameters for the Kuramoto model in the desynchronized phase. Since the primary role of the order parameter is to distinguish different phases, we focus on the ability to discern the desynchronized phase from the synchronized one on complex networks with the size N. From the exact derivation of the difference between two order parameters, Δ, on a star network, we find that these order parameters disagree in the desynchronized phase. We also show that the hub plays an important role and provide an analytic conjecture on the condition that the two order parameters agree with each other as $N \to \infty$. The conjecture is numerically confirmed.

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I. INTRODUCTION

Phase transitions have been one of the traditional subjects in statistical physics [1–3]. The first step to the quantitative description of the phase transition is to identify the order parameter (OP). The most fundamental role of the OP is the identification of distinctive phases. Quantitatively, the average vanishes on one side of the transition and moves away from zero on the other side in general. Thus, the onset of the nonzero value of the OP specifies the transition threshold. Therefore, defining a good OP is one of the most important steps to understand various properties of phase transitions. Examples of good OPs are the magnetization in ferromagnetic transitions [4], the difference of density between liquid and gas in liquid-gas transitions [5], and the electron pair amplitude in superconducting transitions [6].

Sometimes, however, a misbehaving OP has bothered physicists in diverse topics. For example, the number of corners or the corner density is claimed to be not a good OP for the low-dimensional self-avoiding work model of protein folding. The difference between the densities of vertical and horizontal bonds is suggested as a better OP [7]. In the homogeneous plaquette Ising model, the OP involving the product of two nearest-neighbor spins on the same row of the square lattice was suggested as a good OP instead of the usual magnetization [8].

Recently, coupled oscillatory systems have attracted interest from researchers for their theoretical importance and potential applications (see for example Ref. [9]). The Kuramoto model is one of the well-known models for coupled oscillatory systems [10]. The original Kuramoto model is defined on the complete graph in which all oscillators are coupled to all the others. The low-dimensional Kuramoto model on regular lattices has been also investigated [11–14]. With the recent development of the complex network theory, the Kuramoto model on complex networks also has attracted many researchers [15–20]. More recently, some variants of the Kuramoto model showed very intriguing transition natures such as the discontinuous or explosive transition [21–26].

Like many other systems studied in statistical physics, the simplest analytic approaches to the Kuramoto model generally rely on the mean-field theory. The critical coupling strength of the Kuramoto model on the complete graph is generally known as the Kuramoto value. The mean-field analysis gives that the critical coupling strength on a complex network is determined by the Kuramoto value, rescaled by the first and second moments of the degree distribution [15–17,20]. Thus, the mean-field analysis predicts that the critical coupling vanishes in the thermodynamic limit on a graph with a high degree heterogeneity. However, the evidence obtained from extensive numerical analyses shows that the critical coupling strength converges to a nonzero value, in striking contrast to the prediction of mean-field analysis [17–19]. Therefore, the question left is what causes the disagreement between the mean-field result and the numerical expectations. As one of the possible origins of such discrepancy, the use of different OPs has been conjectured [17,27], but the problem still remains open.

To study the phase transitions in the Kuramoto model, two slightly different OPs are widely used depending on the underlying topology. As already addressed, choosing a better OP is the important first step to investigate various synchronization transitions observed in Kuramoto-type models as in many other systems showing phase transitions. Furthermore, the use of different OPs has been conjectured [17,27] to cause the disagreement between the mean-field result and numerical predictions. However, how the choice of OP affects the synchronization transition in the Kuramoto model is not fully understood. Therefore, finding the difference between the two OPs for the Kuramoto model is very important. In this paper, we focus on the capability of the two OPs to separate the desynchronized phase from the synchronized one depending on the underlying topology. In the desynchronized (or disordered) phase, both OPs should vanish in the thermodynamic limit if they are good OPs. We, however, analytically and numerically show that the two OPs do not coincide with each other when the connectivity of the underlying network is highly heterogeneous. This suggests that the OP should be carefully chosen when the underlying topology is highly heterogeneous.

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II. THE KURAMOTO MODEL AND TWO ORDER PARAMETERS

We consider an unweighted and undirected network of N coupled phase oscillators. The phase of oscillator i, denoted by $\theta_i(t)$ $(i = 1,\ldots,N)$, evolves in time as

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i). \tag{1}$$

Here, ω_i is the natural frequency of oscillator i distributed with a given probability density $g(\omega)$ and λ stands for the coupling strength. The connections among oscillators are represented by the adjacency matrix $\mathbf{A}$, i.e., $A_{ij} = 1$ if two oscillators i and j are connected, while $A_{ij} = 0$ otherwise. On the complete graph, λ in Eq. (1) should be changed into λ/N.

The most widely used OP is defined as (for example, see Ref. [21])

$$Re^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{2}$$

Here R and Ψ are the magnitude and phase of the complex OP, respectively. Equation (2) is an especially convenient choice for the Kuramoto model on the complete graph or homogeneous network to apply the ordinary mean-field theory. For example, on the complete graph in which every oscillator is connected to all others, Eq. (1) is simplified as

$$\dot{\theta}_i = \omega_i + \lambda R \sin(\Psi - \theta_i), \tag{3}$$

using Eq. (2). In Eq. (3) the interaction between any given oscillator i is effectively described as an interaction with a single oscillator with phase Ψ. Since Ψ also evolves in time, $\Psi(t)$ can be expressed as $\Psi(t) = \Omega t + \Psi_0$ in the steady synchronous regime. Here Ω is a constant group angular velocity and Ψ_0 is a constant initial phase. By introducing phase deviations, $\Delta\theta_i \equiv \Psi - \theta_i$, Eq. (3) is rewritten as

$$\Delta\dot{\theta}_i = \Delta\omega_i - \lambda R \sin \Delta\theta_i, \tag{4}$$

where $\Delta\omega_i = \Omega - \omega_i$. Thus, the coupled N equations effectively become N decoupled equations, which makes the analytical treatment much easier.

On the other hand, Ichinomya [20] introduced another OP for the mean-field analysis of the synchronization transition of the Kuramoto model on a network with heterogeneous degree distribution [20]. This OP is a weighted sum,

$$re^{i\Psi'} = \frac{1}{N\langle k\rangle}\sum_{j=1}^{N} k_j e^{i\theta_j}, \tag{5}$$

on discrete and finite networks [15–20,24–26]. Using mean-field theory and Eq. (5), Eq. (1) is rewritten as (see Appendix A)

$$\dot{\theta}_i = \omega_i + \lambda k_i r \sin(\Psi' - \theta_i). \tag{6}$$

Here k_i $(\equiv \sum_{j=1}^{N} A_{ij})$ is the degree of the node i. As shown in Eq. (6), the use of r has great advantages for the Kuramoto model on the heterogeneous networks because the N coupled equations become N decoupled equations as in Eq. (4), which makes the mean-field approach simple. As one can easily find, the usefulness of each OP defined in Eqs. (2) and (5) strongly depends on the underlying topology. More precisely, if the structure of the underlying network is rather homogeneous, then choosing Eq. (2) is better to apply the mean-field theory, while Eq. (5) can be better for heterogeneously connected oscillators. Therefore, it is natural to ask a question whether R and r predict the same phase or not, regardless of the underlying topology.

III. DIFFERENCE BETWEEN r AND R

Due to the factor $k_j/\langle k\rangle$ in Eq. (5), hubs or oscillators with large degrees contribute more to r, which causes overestimation of r. Especially, even though each oscillator has randomly assigned phase $\theta_i(t)$ in which $R = 0$, r still remains at some nonzero value in the limit $N \to \infty$. In order to find how such overestimation occurs, we focus on the difference between the two OPs,

$$\Delta = r - R = \frac{1}{N}\sum_{j}^{N}\left[\frac{k_j}{\langle k\rangle} - e^{-i(\Psi-\Psi')}\right] e^{i(\theta_j - \Psi')}. \tag{7}$$

For an exact analysis of the overestimation of r, we first consider the star network as an underlying topology. Since there is only one hub in the star network, the degree of the hub, k_h, and the average degree $\langle k\rangle$ are exactly expressed by N, i.e, $k_h = N - 1$ and $\langle k\rangle = 2(N-1)/N \simeq 2$ for $N \gg 1$. The other oscillators have only one connection to the hub; thus $k_i = 1$ if i is not the hub. Thus, Eq. (5) becomes

$$re^{i\Psi'} \simeq \frac{1}{2N}\sum_{j\neq h}^{N} e^{i\theta_j} + \frac{1}{2}e^{i\theta_h}, \tag{8}$$

where the subscript h stands for the hub. The first term comes from the contribution of oscillators with $k_i = 1$ and the second term is that of the hub to r. Similarly, we rewrite Eq. (2) as

$$Re^{i\Psi} = \frac{1}{N}\sum_{j\neq h}^{N} e^{i\theta_j} + \frac{1}{N}e^{i\theta_h}. \tag{9}$$

From Eqs. (8) and (9), Eq. (7) becomes

$$\Delta(N) = \frac{1}{2N}(e^{-i\Psi'} - 2e^{-i\Psi})\sum_{j\neq h}^{N} e^{i\theta_j} + \frac{1}{2}e^{-i\Psi'}e^{i\theta_h} - \frac{1}{N}e^{-i\Psi}e^{i\theta_h}. \tag{10}$$

Here, $\theta_h(t)$ is the phase of the oscillator at the hub. In the limit $N \to \infty$, the last term in Eq. (10) can be ignored. In the case of $\lambda > \lambda_c$ in which the macroscopic portion of the oscillators is mutually synchronized, the summation in the first term in Eq. (10) should scale as

$$\sum_{j\neq h}^{N} e^{i\theta_j} \sim N. \tag{11}$$

Thus, Δ remains constant ($\Delta > 0$), which might not cause any fundamental change in the behavior of OP.

However, one should be cautious to choose OP when $\lambda < \lambda_c$ in which θ_i is randomly distributed. By the definition of R, $R = 0$ for randomly distributed θ_i. In contrast to the value of R, if the contribution of the hub to r becomes significantly

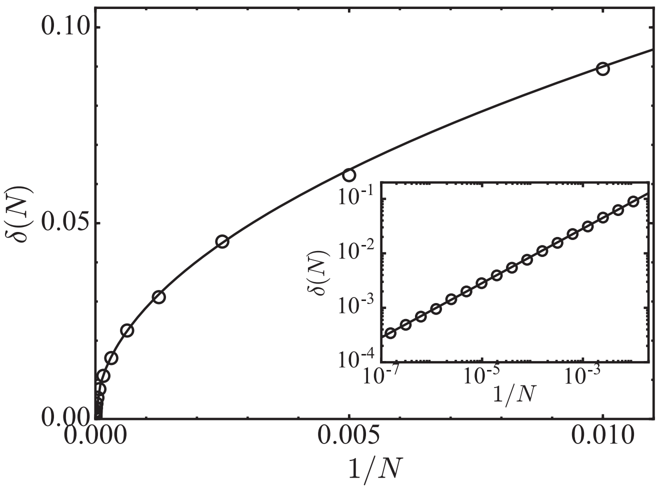

FIG. 1. Plot of $\delta(N)$ against $1/N$ on the star network when $\lambda < \lambda_c$. The solid line represents the analytic expectation $\delta(N)$ ($\equiv [1/2 - \Delta(N)]) \simeq N^{-1/2}$. Inset: The same plot in log-log scale. The slope of the solid line in the inset is 1/2.

large, or more precisely, if there is a node i whose degree scales as $k_i \sim N$, then Eq. (7) predicts that $r \neq 0$ is possible in the limit $N \to \infty$, even though θ_i is randomly distributed. For example, on a star network in a desynchronized phase, $\sum_{j\neq h}^{N} \exp(i\theta_j)$ in Eq. (10) scales as $|\sum_{j\neq h}^{N} \exp(i\theta_j)| \sim N^{1/2}$, since θ_i is randomly distributed. The last term in Eq. (10) can be ignored when $N \gg 1$. Thus, Eq. (10) becomes

$$\Delta(N) \sim 1/2 - N^{-1/2}, \tag{12}$$

when $N \gg 1$. Therefore, on a star network in which $k_h \simeq N$, we expect $\Delta(N) \to 1/2$ from Eq. (10) in the limit $N \to \infty$. In order to analyze the behavior of $\Delta(N)$ on a star network, we also define $\delta(N)$ as

$$\delta(N) \equiv [1/2 - \Delta(N)] \sim N^{-1/2}. \tag{13}$$

IV. NUMERICAL RESULTS

In the numerical analyses, we use networks with $N = 10^2 \sim 10^7$. In order to investigate the capability of two OPs to separate the desynchronized phase from the synchronized one, we consider the behavior of $\Delta(N)$ or $\delta(N)$ for $\lambda = 0$ to guarantee $R = 0$ for any distribution of ω_i in the limit $N \to \infty$. Thus, we randomly assign θ_i ($\in [0,2\pi]$) (see Appendix B) to each node i and measure R and r. Both OPs are averaged over 1000 network realizations. In Fig. 1 we display the behavior of $\delta(N)$ in the desynchronized phase obtained from star networks with various N. The data in Fig. 1 clearly show that $\delta(N) \to 0$ as $1/N \to 0$. This perfectly agrees with the analytic expectation, $\Delta(N) \to 1/2$ in the limit $N \to \infty$. From the best fit of the data to $\delta(N) \sim N^{-\alpha}$ we obtain $\alpha \simeq 1/2$ (see the inset of Fig. 1).

Equations (5)–(7) provide an important clue to understand the behavior of Δ when the system is in the desynchronized phase, i.e., when θ_i is distributed randomly. Now, let us consider a scale-free network (SFN) as an underlying topology connecting each oscillator in the system. SFNs are generally characterized by their degree distribution, $P(k) \sim k^{-\gamma}$. On a SFN, the degree of the largest hub scales as $k_{\max} \sim N^{\frac{1}{\gamma-1}}$. In Eq. (7), the contribution of the second term in the summation would be negligible in the desynchronized phase as $N \to \infty$. Therefore, if the degree of the largest hub grows faster than $O(N)$, then we expect that Δ approaches a nonzero value in the limit $N \to \infty$. From this condition, we expect that $\Delta(N) > 0$ in the limit $N \to \infty$ for $\gamma \leqslant 2$.

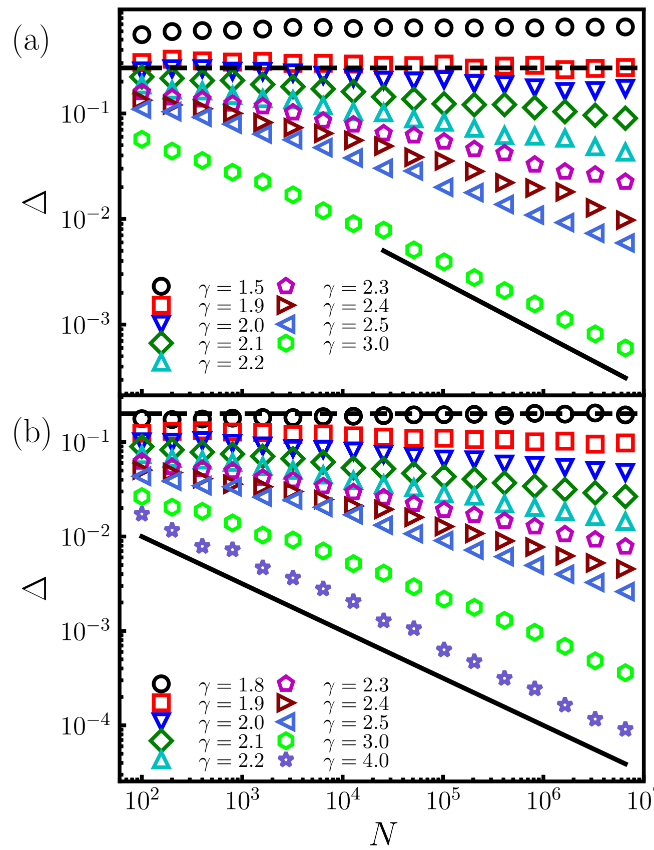

FIG. 2. Plot of $\Delta(N)$ against N on SFNs constructed from (a) the configuration model and (b) the static model with $\langle k \rangle = 4$. The solid lines represent the relation $\Delta(N) \simeq N^{-0.5}$. The dashed lines denote $\Delta(N) = \text{const}$.

In order to verify our conjecture for the condition of vanishing Δ on SFNs, we numerically measure $\Delta(N)$ on two different types of SFNs, the configuration model [28] and the static model [29] when $\lambda = 0$ ($< \lambda_c$). The results are shown in Fig. 2. In the configuration model, the assigned degree of each node i, k_i, is drawn from the power-law distribution $P(k) = (\gamma - 1)k^{-\gamma}$ and $k_i \in [1,\infty]$ for $\gamma > 1$. Therefore, the average degree, $\langle k \rangle$, depends on γ; i.e., $\langle k \rangle$ changes as γ changes. On the other hand, $\langle k \rangle$ can be fixed to a constant value in the static model, regardless of γ. In Fig. 2(b), we set $\langle k \rangle = 4$. As shown in Fig. 2(a), $\Delta(N)$ approaches zero as $\Delta(N) \sim N^{-\alpha'}$ for $\gamma > 2$. The value of α' depends on γ. When γ is large enough (or $\gamma > 3$), we find that $\alpha' \to 1/2$ and α' continuously decreases as γ decreases. When $\gamma = 2$, the behavior of $\Delta(N)$ seems to be marginal, and $\Delta(N)$ remains constant (or $\alpha' = 0$) for $\gamma < 2.0$ as shown in Fig. 2(a).

On SFNs generated by the static model, the similar behavior is observed as shown in Fig. 2(b). As for the case on SFNs generated by the configuration model, we find $\alpha' \to 1/2$ for $\gamma > 3$. The value of α' decreases as γ decreases. In contrast to the SFNs generated by the configuration model, $\gamma = 2$ does not seem to be marginal for the SFNs generated by the static model. When $\gamma = 2$, $\Delta(N)$ tends to decrease as N increases, even though the value of α' is quite small. Thus, we expect that $\Delta(N) \to 0$ in the limit $N \to \infty$. From the data in Fig. 2(b) we expect that the marginal value of γ lies in the range $1.8 < \gamma < 1.9$ for the SFNs generated by the static model.

V. SUMMARY AND DISCUSSION

In summary, we investigate the difference between two OPs for the Kuramoto model. Since the primary role of the OP is to distinguish the desynchronized phase (or disordered phase) from the synchronized phase (or ordered phase), we mainly focus on the difference between two OPs in the desynchronized phase. Based on the analytic derivation for the difference between two OPs on the star network, we find that the growth of the hub's degree plays an important role. Since the hub's degree in the star network grows as N, the behaviors of two OPs on the star network are different in the desynchronized phase, i.e., $\Delta(N \to \infty) = 1/2$ as in Eq. (12). The conjecture is verified through the numerical analysis. This result also provides a clue to understand the different behavior of two OPs on heterogeneous networks. Since the degree of the largest hub in a SFN grows as $k_{\max} \sim N^{\frac{1}{\gamma-1}}$, we expect that $\gamma = 2$ is marginal. From the numerical analysis on SFNs, we find that $\gamma = 2$ is marginal for the configuration model. On the other hand, the marginal value of γ on SFNs generated by the static model slightly decreases and we expect that the marginal value is in the range $1.8 < \gamma < 1.9$ for the static model.

Even though the mean-field analysis of the Kuramoto model on SFNs with relatively homogeneous natural frequency distribution such as a Gaussian or Lorentzian distribution is known to undergo a transition only for $\gamma > 3$ [16], the choice of a suitable OP can be crucial, especially in numerical analysis, due to the behavior of $\Delta(N)$. On SFNs, $\Delta(N)$ decreases as $\Delta(N) \simeq N^{-0.5}$ for $\gamma > 3$. Thus, there is still a high possibility that Eq. (5) overestimates the degree of coherence in the desynchronized phase due to the strong finite-size effect even for $\gamma > 3$ as in Fig. 2. More importantly, the mean-field analysis and the numerical expectations on SFNs with $\gamma \leqslant 3$ do not coincide [15–20]. Our results provide quantitative evidence that the disagreement comes from the use of different OPs as suggested in Refs. [17,27]. This noncoincidence should come from the fact that $\alpha' < 1/2$ for $\gamma \leqslant 3$. Furthermore, some real networks have been reported to have $\gamma < 2$ [30,31] including ecological networks. In such systems, our results clearly show that the use of Eq. (5) is not suitable to study the synchronization transition. Therefore, careful consideration should be given to the analysis of the synchronization transition when Eq. (5) is used as an OP.

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APPENDIX A: DERIVATION OF EQ. (6)

In the mean-field theory, Eq. (1) on a complex network can be rewritten as

$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{k_j=1}^{k_{\max}} \sum_{\theta_j} \frac{k_i k_j P(k_i)}{\langle k \rangle} \frac{N(k_j;\theta_j)}{N(k_j)} \sin(\theta_j - \theta_i), \tag{A1}$$

where $N(k_j;\theta_j)$ is the number of oscillators with phase θ_j for a given degree k_j, and $N(k_j)$ is the total number of oscillators of degree k_j. $P(k)$ for a finite network becomes $P(k_j) = N(k_j)/N$. Then, Eq. (A1) becomes

$$\dot{\theta}_i(t) = \omega_i + \lambda \sum_{j=1}^{N} \frac{k_i k_j}{\langle k \rangle N} \sin(\theta_j - \theta_i). \tag{A2}$$

The same equation can be obtained from the annealed network approach [25]. By substituting Eq. (5) into Eq. (A2) we finally obtain Eq. (6).

APPENDIX B: DISTRIBUTION OF θ

In order to guarantee that the system is in the desynchronized phase we use a random uniform distribution for $\theta_i \in [0,2\pi]$. In the desynchronized phase, $P(\theta(t=\infty))$ reduces to the random uniform distribution using any initial distribution of $P(\theta)$ if ω_i's are not identical, as shown in Fig. 3. In Fig. 3, we have shown the following: if $P(\omega)$, and $P(\theta(t=0))$ are the Gaussian distributions with zero means and standard deviations $\sigma_{\theta(t=0)} = 0.01$ and $\sigma_\omega = 1.0$, then $P(\theta)$ reduces to a random uniform distribution in the steady state. For $\sigma_{\theta(t=0)} = 0$, we obtain the same result if $\sigma_\omega > 0$.

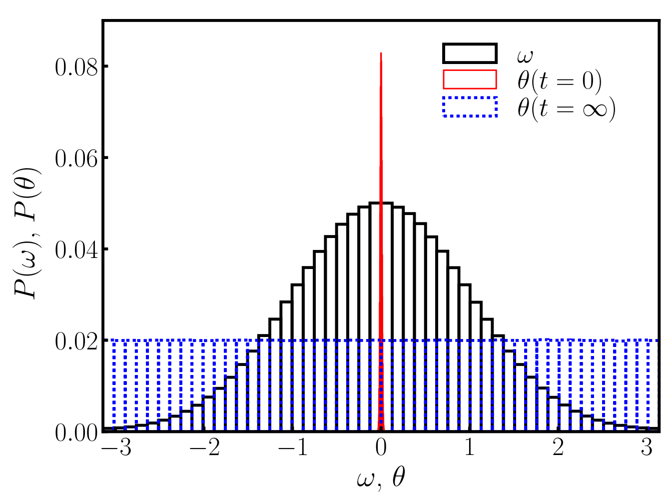

FIG. 3. Plot of $P(\omega)$ (gray solid boxes), $P(\theta(t=0))$ (black solid boxes), and $P(\theta(t=\infty))$ (dashed boxes) on SFNs with $N = 10^7$. $P(\omega)$ and $P(\theta(t=0))$ are given by the Gaussian distribution with standard deviations $\sigma_{\theta(t=0)} = 0.01$ and $\sigma_\omega = 1.0$, respectively.

[1] S.-K. Ma, *Modern Theory of Critical Phenomena* (Addison-Wesley, Redwood City, 1982).

[2] J. J. Binney, N. J. Dowrick, A. J. Fisher, and M. E. J. Newman, *The Theory of Critical Phenomena: An Introduction to the Renormalization Group* (Oxford University Press, Oxford, 2002).

[3] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971).

[4] L. P. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palciauskas, M. Rayl, J. Swift, D. Aspnes, and J. Kane, Rev. Mod. Phys. **39**, 395 (1967).

[5] J. M. H. Levelt Sengers, Physica **73**, 73 (1974).

[6] P. Heller, Rep. Prog. Phys. **30**, 731 (1967).

[7] D. P. Foster and C. Pinettes, Phys. Rev. E **77**, 021115 (2008).

[8] S. Davatolhagh, D. Dariush, and L. Separdar, Phys. Rev. E **81**, 031501 (2010).

[9] G. V. Osipov, J. Kurths, and C. Zhou, *Synchronization in Oscillatory Networks* (Springer, Berlin, 2007).

[10] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer, Berlin, 1984).

[11] H. Daido, Phys. Rev. Lett. **61**, 231 (1988).

[12] H. Hong, H. Park, and M. Y. Choi, Phys. Rev. E **70**, 045204(R) (2004).

[13] H. Hong, H. Park, and M. Y. Choi, Phys. Rev. E **72**, 036217 (2005).

[14] H. Hong, H. Chaté, H. Park, and L.-H. Tang, Phys. Rev. Lett. **99**, 184101 (2007).

[15] J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005).

[16] D.-S. Lee, Phys. Rev. E **72**, 026208 (2005).

[17] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).

[18] Y. Moreno and A. F. Pacheco, Europhys. Lett. **68**, 603 (2004).

[19] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **98**, 034101 (2007).

[20] T. Ichinomiya, Phys. Rev. E **70**, 026116 (2004).

[21] J. Gómez-Gardeñes, S. Gómez, A. Arenas, and Y. Moreno, Phys. Rev. Lett. **106**, 128701 (2011).

[22] X. Hu, S. Bocacaletti, W. Huang, X. Zhang, Z. Liu, S. Guan, and C.-H. Lai, Sci. Rep. **4**, 7262 (2014).

[23] X. Zhang, Y. Zou, S. Boccaletti, and Z. Liu, Sci. Rep. **4**, 5200 (2014).

[24] T. K. Dal'Maso Peron and F. A. Rodrigues, Phys. Rev. E **86**, 056108 (2012).

[25] B. C. Coutinho, A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **87**, 032106 (2013).

[26] P. Ji, T. K. D. M. Peron, F. A. Rodrigues, and J. Kurths, Sci. Rep. **4**, 4783 (2014).

[27] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).

[28] M. Molloy and B. Reed, Random Struct. Alg. **6**, 161 (1995).

[29] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).

[30] S. Valverde and R. V. Solé, Europhys. Lett. **72**, 858 (2005).

[31] J. M. Montoya, S. L. Pimm, and R. V. Solé, Nature (London) **442**, 259 (2006).