

Role of short periodic orbits in quantum maps with continuous openingsCarlos A. Prado,^{1,2} Gabriel G. Carlo,^{3,*} R. M. Benito,⁴ and F. Borondo⁵¹*Comisión Nacional de Energía Atómica, Departamento de Física, Av. del Libertador 8250, 1429 Buenos Aires, Argentina*²*Departamento de Física, FCEyN, Universidad de Buenos Aires, Ciudad Universitaria, 1428 Buenos Aires, Argentina*³*Comisión Nacional de Energía Atómica, CONICET, Departamento de Física, Av. del Libertador 8250, 1429 Buenos Aires, Argentina*⁴*Grupo de Sistemas Complejos and Departamento de Física, Escuela Técnica Superior de Ingenieros Agrónomos, Universidad Politécnica de Madrid, 28040 Madrid, Spain*⁵*Departamento de Química, and Instituto de Ciencias Matemáticas (ICMAT), Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain*

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We apply a recently developed semiclassical theory of short periodic orbits to the continuously open quantum tribaker map. In this paradigmatic system the trajectories are partially bounced back according to continuous reflectivity functions. This is relevant in many situations that include optical microresonators and more complicated boundary conditions. In a perturbative regime, the shortest periodic orbits belonging to the classical repeller of the open map—a cantor set given by a region of exactly zero reflectivity—prove to be extremely robust in supporting a set of long-lived resonances of the continuously open quantum maps. Moreover, for steplike functions a significant reduction in the number needed is obtained, similarly to the completely open situation. This happens despite a strong change in the spectral properties when compared to the discontinuous reflectivity case. In order to give a more realistic interpretation of these results we compare with a Fresnel-type reflectivity function.

DOI: [10.1103/PhysRevE.97.042211](https://doi.org/10.1103/PhysRevE.97.042211)**I. INTRODUCTION**

In many experimental situations like in the case of optical cavities [1–3], the knowledge of properties of an open system becomes crucial. This is also a very interesting theoretical problem, even from a pure mathematical point of view [4]. At the classical level, these situations are usually modeled by eliminating all the trajectories that arrive at a given region of phase space (the opening) giving rise to a fractal invariant set, the repeller. The quantum analogs of these systems are characterized by a set of resonances and the number of long-lived states scales with the Planck constant as $\hbar^{-d/2}$, where $d + 1$ is the fractal dimension of the classical repeller. This is the so-called fractal Weyl law [5–8]. However, the reflection mechanisms at the boundaries are usually more complicated than this complete opening [9]. The first step in order to understand these mechanisms is considering a constant reflectivity R , meaning that the classical trajectories arriving to the opening are partially reflected. In these cases we have a multifractal behavior [10], and the usual fractal Weyl law needs to be nontrivially modified. This has been done for the case of maps [11], which are very suitable models for more complicated systems.

There is another point of view to study this problem which is based on the semiclassical theory of short periodic orbits (POs) for open quantum maps [12]. The shortest POs contained in the classical repeller are used to construct a basis set of scar functions which expands the quantum repeller and is suitable to express the quantum nonunitary operators [13–15]. Recently,

we have extended the short POs theory to partially open quantum maps where a fraction of the quantum probability is reflected [16].

The next step for a better understanding of the properties derived from more realistic reflection mechanisms at the boundary [1] is to consider a reflectivity function. In this paper we apply our semiclassical theory to a continuously open tribaker map. Although we will drop the word “partially” from here on in order to simplify the notation, this is obviously a particular case of a partially open map. We take into account a step function of the Fermi-Dirac kind in order to smooth the boundaries of the opening, and a sinusoidal function which provides with a more generic profile. We have found strong changes in the spectral behavior with respect to the discontinuous openings. Despite this, the shortest POs belonging to the classical repeller (corresponding to the fully open scenario) explain the main properties of these maps, in a perturbative regime (i.e., for low values of the reflectivity). Moreover, in the step function case there is still a significant reduction in the number of POs needed for the semiclassical calculations. A physical interpretation in terms of the index of refraction of microresonators is also given.

This paper is organized as follows: In Sec. II we make a brief description of our semiclassical approach and define the system used, i.e., the classical and quantum continuously open tribaker map. In Sec. III we apply the short POs theory and discuss the results. Our conclusions are presented in Sec. IV.

II. CONTINUOUSLY OPEN MAPS

Classical and quantum chaos have benefited from the study of maps, which capture all the essential properties of

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more complicated dynamical systems [17–19]. Open maps are transformations of the 2-torus where trajectories disappear when they reach an open region in the bidimensional phase space. The intersection of the forward and backwards trapped sets (trajectories that do not escape either in the past or in the future) form the repeller, an invariant of fractal dimension. Partially open maps are those in which the opening does not absorb all the trajectories that arrive at it but reflects back a certain amount. Previously [16], we have investigated a constant function given by a reflectivity $R \in [0 : 1]$. Here we consider two different functions of the phase space, $F_R(q, p)$, where now R is a parameter that performs the transition between the minimum amount of reflection $R = 0$ (which is not the completely open case) and the closed system ($R = 1$). This is intended to capture the general properties of optical microcavities and more complicated boundary conditions.

Multifractality manifests itself through a measure that now is not uniformly distributed on the repeller. In each phase space region X_i , this measure depends on the average intensity I_t when $t \rightarrow \infty$ of a number N_{ic} of random initial conditions taken inside X_i . The initial intensity is $I_0 = 1$ for each trajectory and changes to $I_{t+1} = F_R(q, p)I_t$ each time it hits the opening [20]. The finite time measure for X_i is given by $\mu_{t,i}^b = \langle I_{t,i} \rangle / \sum_i \langle I_{t,i} \rangle$ where the average is over the initial conditions in the given phase space region. If we consider this to be the analog of the backwards trapped set of open maps, we can evolve backwards and obtain $\mu_{t,i}^f$ the analog of the forward trapped set. Their intersection gives what we call the *continuous repeller* $\mu_{t,i}$.

To quantize a map we impose boundary conditions for both the position and momentum representations by taking $\langle q + 1 | \psi \rangle = e^{i2\pi\chi_q} \langle q | \psi \rangle$, and $\langle p + 1 | \psi \rangle = e^{i2\pi\chi_p} \langle p | \psi \rangle$, with $\chi_q, \chi_p \in [0, 1)$. In a Hilbert space of finite dimension $N = (2\pi\hbar)^{-1}$, the semiclassical limit corresponds to $N \rightarrow \infty$, and the propagator is given by a $N \times N$ matrix. Position and momentum eigenstates are given by $|q_j\rangle = |(j + \chi_q)/N\rangle$ and $|p_j\rangle = |(j + \chi_p)/N\rangle$ with $j \in \{0, \dots, N-1\}$. A discrete Fourier transform gives $\langle p_k | q_j \rangle = \frac{1}{\sqrt{N}} e^{-2i\pi(j+\chi_q)(k+\chi_p)/N} \equiv (G_N^{\chi_q, \chi_p})$. The opening (we take a strip parallel to the p axis) is quantized as a projection operator P on its complement, so the open map is of the general form $\tilde{U} = PUP$, where U is the propagator for the closed one. Here we take an opening function so the projector becomes $\sqrt{F_R} \times \mathbb{1}$, where the identity has the dimension of the escape region. This map has N right eigenvectors $|\Psi_j^R\rangle$ and N left ones $\langle\Psi_j^L|$, which are mutually orthogonal $\langle\Psi_j^L|\Psi_k^R\rangle = \delta_{jk}$, and that are associated to resonances z_j . Our normalization is such that $\langle\Psi_j^R|\Psi_j^R\rangle = \langle\Psi_j^L|\Psi_j^L\rangle$.

A. Semiclassical theory

We have recently developed a semiclassical theory [16] that can be directly applied to obtain the resonances of continuously open maps by means of their shortest POs. We now give a brief description of the main details. Let γ be a PO of fundamental period L that belongs to a continuously open map. We can define coherent states $|q_j, p_j\rangle$ associated to each point of the orbit. We then construct L linear combinations of them $|\phi_\gamma^m\rangle$ ($m \in \{0, \dots, L-1\}$). Finally, the right $|\psi_{\gamma,m}^R\rangle$

and left $\langle\psi_{\gamma,m}^L|$ scar functions for the PO are defined through the propagation of these linear combinations under the continuously open map \tilde{U} , up to approximately the system's Ehrenfest time τ . Normalization ($\mathcal{N}_\gamma^{R,L}$) is chosen in such a way that $\langle\psi_{\gamma,m}^R|\psi_{\gamma,m}^R\rangle = \langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^L\rangle$ and $\langle\psi_{\gamma,m}^L|\psi_{\gamma,m}^R\rangle = 1$.

We then select a number of POs, N^{POs} , from the whole set up to a period L , in order to cover the continuous repeller. In this work we use all POs up to period L that are inside the repeller (of the fully open case). We also consider a few of them that are outside, N_{\max}^{outPOs} , having the greatest values of μ , and optimized to provide with the most uniform covering possible of the continuous repeller. We solve a generalized eigenvalue problem in this basis set to obtain the semiclassical eigenstates (for details see Ref. [16]).

B. The tribaker map

All calculations are performed on the tribaker map

$$\mathcal{B}(q, p) = \begin{cases} (3q, p/3) & \text{if } 0 \leq q < 1/3 \\ (3q - 1, [p + 1]/3) & \text{if } 1/3 \leq q < 2/3. \\ (3q - 2, [p + 2]/3) & \text{if } 2/3 \leq q < 1 \end{cases} \quad (1)$$

This is an area-preserving, uniformly hyperbolic, piecewise-linear, and invertible map with Lyapunov exponent $\lambda = \ln 3$. The opening region is the domain $1/3 < q < 2/3$ of the reflectivity function F_R . We use two kinds of functions; in the first place we consider

$$F_R(q, p) = \begin{cases} (1 - R)/[1 + \exp[-A(q - B)]] + R & \text{if } q > 1/2 \\ (1 - R)/(1 + \exp[-A((1 - q) - B)]) + R, & \text{if } q < 1/2 \end{cases} \quad (2)$$

which is a step function of the Fermi-Dirac kind. We take $A = 120$ and $B = 0.63$, which corresponds to approximately a value 1 at $q = 1/3$ and $q = 2/3$ and a bottom at R in the middle of the opening region. This function aims to smoothing the transition between the closed and the open regions of the phase space, but retaining at the same time some of the properties of the discontinuous case, like the flat bottom and the quick drop in reflectivity. The other function is given by

$$F_R(q, p) = [(1 - R) \cos(6\pi q) + (1 + R)]/2, \quad (3)$$

which is essentially a sinusoid that matches a value 1 at the boundaries of the opening and a minimum given by R . This function is intended to capture the main properties of optical setups like in microlasers experiments.

The quantum version of the tribaker map, U^B uses the discrete Fourier transform $G_N^{1/2, 1/2}$ with antiperiodic boundary conditions ($\chi_q = \chi_p = 1/2$) to preserve time reversal and parity. For the details of the position representation the tribaker map see Refs. [21,22]. The continuously open quantum tribaker map is then given by means of the operator

$$P = \begin{pmatrix} \mathbb{1}_{N/3} & 0 & 0 \\ 0 & \sqrt{F_R} \mathbb{1}_{N/3} & 0 \\ 0 & 0 & \mathbb{1}_{N/3} \end{pmatrix}, \quad (4)$$

applied in such a way to preserve the original symmetries.

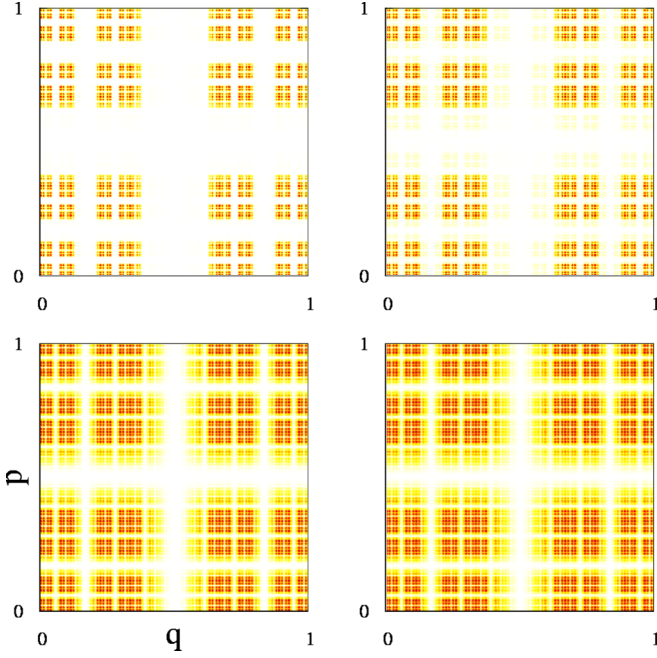


FIG. 1. Classical measure $\mu_{t,i}$ on the 2-torus for the continuously open tribaker map. In the upper panels we show the step opening, in the lower ones the sinusoidal opening. In the left column $R = 0.01$, and in the right one $R = 0.1$.

Figure 1 shows the finite time continuous repeller $\mu_{t,i}$ at time $t = 10$. In the upper panels the step reflectivity is represented, while in the lower ones we find the sinusoidal case. In the left column $R = 0.01$, and in the right one $R = 0.1$.

III. RESULTS

In order to characterize the spectral behavior of continuously open maps we have evaluated the local dimension $d_{\text{loc}} = \{\ln[M(N)] - \ln[M(N/3)]\} / \ln(3)$; $M(N)$ is the number of resonances satisfying $|z_j| > \nu_c$, and we have chosen $N = 3^5$ and $N = 3^9$. This is a convenient quantity that looks into the details of the spectral scaling behavior [11]. We show the results in Fig. 2 for $R = 0$, $R = 0.001$, $R = 0.01$, and $R = 0.1$ (see caption for details). It is clear that the marked oscillatory behavior typical of the discontinuous opening [11] is almost completely absent in our cases. This seems to be valid even for the lowest R values and in the large N limit. This strong change in the spectral features suggests that the resonances follow a different kind of Weyl law (a different regime, at least). Nevertheless, there is a small portion of them above a given ν_c that approximately follows a scaling ruled by the dimension of the repeller. For $N = 3^5$, we take $\nu_c = 0.81$ for the step reflectivity case and $\nu_c = 0.91$ for the sinusoidal case.

Next, we apply the semiclassical theory to construct an approximation to the continuously open quantum tribaker map for $N = 3^5$, several values of R , and considering POs up to period $L = 7$. A convenient quantum phase space representation can be obtained by means of the symmetrical operator \hat{h}_j [12] associated to the right $|\Psi_j^R\rangle$ and left $\langle\Psi_j^L|$ long-lived semiclassical eigenstates, which are related to the

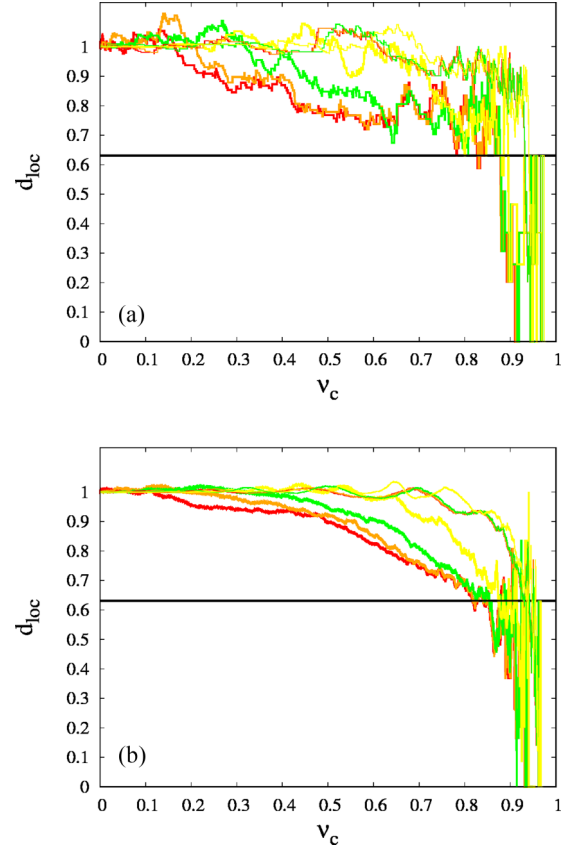


FIG. 2. Local dimension d_{loc} behavior as a function of ν_c . (a) Results for $N = 3^5$; (b) corresponding to $N = 3^9$. Thinner lines represent the sinusoidal opening, and the thicker ones the step opening. By using darker to lighter shades of gray (red, orange, green, and yellow) we display the cases with $R = 0$, $R = 0.001$, $R = 0.01$, and $R = 0.1$. The black horizontal line stands for the fractal dimension of the repeller [i.e., $\ln(2)/\ln(3)$].

eigenvalue z_j :

$$\hat{h}_j = \frac{|\Psi_j^R\rangle\langle\Psi_j^L|}{\langle\Psi_j^L|\Psi_j^R\rangle}. \quad (5)$$

We sum the first j of these projectors [15] (corresponding to the eigenvalues with the greatest moduli, $|z_j| \geq |z_{j'}|$ with $j \leq j'$) up to ν_c :

$$\hat{Q}_j \equiv \sum_{j'=1}^j \hat{h}_{j'}. \quad (6)$$

Their phase space representation by means of coherent states $|q, p\rangle$ is given by

$$h_j(q, p) = |\langle q, p | \hat{h}_j | q, p \rangle|, \quad (7)$$

$$Q_j(q, p) = |\langle q, p | \hat{Q}_j | q, p \rangle|. \quad (8)$$

This is the semiclassical quantum continuous repeller $Q_{\nu_c}^{\text{sc}}$. In Fig. 3 we show the $Q_{\nu_c}^{\text{sc}}$ obtained by using just the POs belonging to the repeller. The upper panels correspond to the step reflectivity function and the lower ones to the sinusoidal case. In the left column we have taken $R = 0.01$, while in

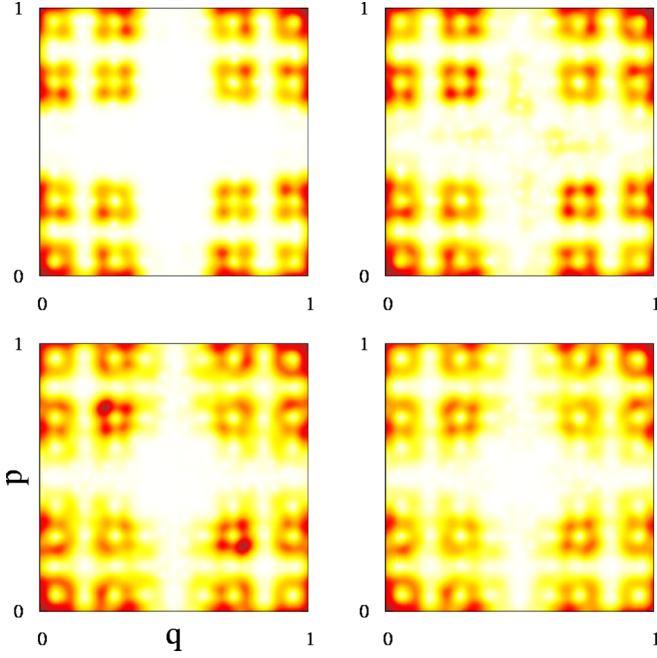


FIG. 3. Semiclassical quantum continuous repeller $Q_{v_c}^{sc}$ over the semiclassical long-lived resonances with eigenvalue moduli greater than v_c . Upper panels correspond to the step opening ($v_c = 0.81$) and lower ones to the sinusoidal opening ($v_c = 0.91$). In the left column $R = 0.01$, and in the right one $R = 0.1$.

the right one $R = 0.1$. The overlaps between these normalized distributions with the ones obtained by using the exact eigenstates, calculated as $O = \iint Q_{v_c}(q, p) Q_{v_c}^{sc}(q, p) dq dp$, are greater than $O = 0.99$, in all cases.

Finally, and in order to examine the details of these quantum continuous repellers, we calculate the performance P [15], defined as the fraction of long-lived eigenvalues semiclassically reproduced within an error given by $\epsilon = \sqrt{[\text{Re}(z_i^{ex}) - \text{Re}(z_i^{sc})]^2 + [\text{Im}(z_i^{ex}) - \text{Im}(z_i^{sc})]^2}$. In the latter expression z_i^{ex} and z_i^{sc} are the exact eigenvalues and those given by the semiclassical theory, respectively. We consider only the eigenvalues with modulus greater than v_c . We calculate the number of scar functions N_{SF} as a fraction of N that are needed in order to obtain as many semiclassical eigenvalues inside the $\epsilon = 0.001$ vicinity of the corresponding exact ones in order to reach $P \geq 0.8$. The fraction N_{SF}/N is a measure of the morphology of the quantum continuous repeller. In fact, the larger this number the more interconnected the POs belonging to the open repeller are. In this sense, it quantifies the departure from the completely open case.

In Fig. 4 we show the fraction N_{SF}/N needed to reach $P \geq 0.8$ as a function of $R \in [0 : 0.1]$. The upper and thinner lines correspond to the sinusoidal opening, while the mainly lower and thicker ones to the step opening. The blue (black) lines with squares correspond to the case in which we only take POs that belong to the repeller. The green (gray) lines with circles display the results when taking into account a small maximum number of POs outside of the repeller, $N_{\max}^{\text{outPO}} = 5$. There is no significant improvement in the calculations when considering these POs, underlining the fact that the main role is played by the repeller. It is also clear from Fig. 4 that the step reflectivity function is able to keep the reduction in the

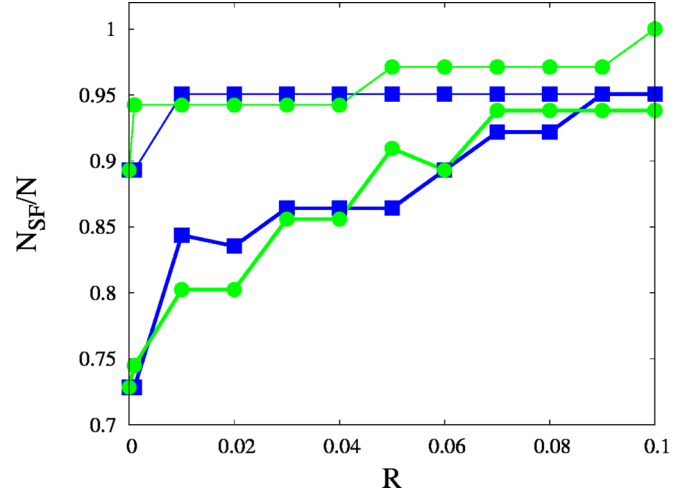


FIG. 4. Fraction of scar functions N_{SF}/N needed to reach $P = 0.8$ as a function of the parameter R . The blue (black) lines with squares correspond to the cases considering only POs inside the repeller. Green (gray) lines with circles correspond to considering $N_{\max}^{\text{outPO}} = 5$ outside of it. Thinner lines represent the sinusoidal opening, while the thicker ones the step case.

number of POs needed that is typical in the discontinuous openings [16]. This is not the case for the sinusoidal openings, though the spectral behavior seems to be qualitatively similar in the sense that only small oscillations coming from the multifractal sampling are present [11]. It is interesting to notice that there is no important change that can be appreciated either in the classical and quantum continuous repellers or in the d_{loc} behavior, between $R = 0.01$ and $R = 0.1$ in the sinusoidal opening; on the other hand the step opening shows clear differences.

In order to be more precise about the different behavior of these two reflectivity functions in terms of the number of scar functions needed to obtain the semiclassical resonances, we have studied the performance of our theory for different values of the parameter A of the step function. We have compared this with the performance of the sinusoidal reflectivity; in both cases we have taken $R = 0$ and considered only POs inside the repeller. From Fig. 5, the effect of varying A from 60 to 120 becomes clear (we have also adjusted B in order to keep the opening approximately between 1/3 and 2/3). The step function goes from a shape that is similar to the sinusoidal opening to one that resembles the discontinuous one [see Figs. 5(a) and 5(b), respectively]. In this transition the fraction of scar functions needed decreases from approximately the one corresponding to the sinusoidal opening to a much lower one. In conclusion, the reason for the difference seen in Fig. 4 can now be understood: the system is much more closed for the sinusoidal opening than for the step function case with the original value for A . As a result more of the chaotic saddle survives in the long-lived states, and more information is needed to reconstruct them from a semiclassical basis. But by just varying A the step function can be turned into a reflectivity shape that needs the same additional information.

Though our theoretical model is very useful to isolate the effect of just smoothing out the borders of the opening regions and then compare it with a more generic sinusoidal reflectivity

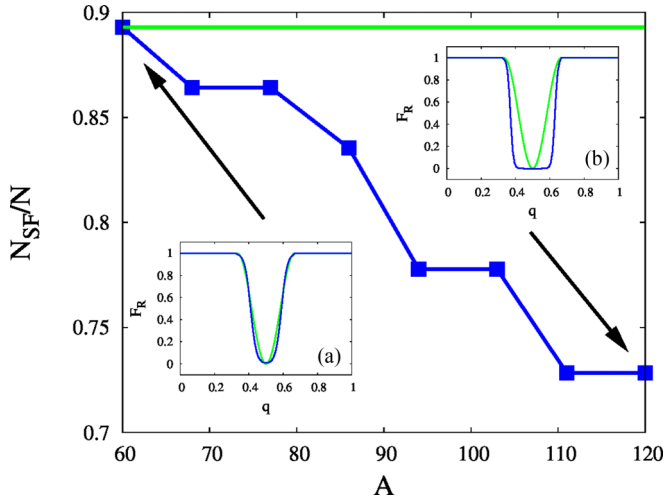


FIG. 5. Fraction of scar functions N_{SF}/N needed to reach $P = 0.8$ as a function of the parameter A of the step reflectivity function. The blue (black) lines with squares correspond to the cases considering only POs inside the repeller and $R = 0$. The green (gray) horizontal line represents the performance for the sinusoidal opening. We compare the shape of the step functions for $A = 60$ and $A = 120$ with the sinusoidal F_R in (a) and (b), respectively.

function, a clear link with their physical meaning is in order. For that purpose we have calculated the behavior of a Fresnel type reflectivity function, proposed in Ref. [23] as a way to take into account the partial escape at dielectric interfaces in microcavities. For TM polarization our reflectivity function is given by

$$F_n(q, p) = \left(\frac{\sqrt{1 - \{n \sin[y(q)]\}^2} - n \cos[y(q)]}{\sqrt{1 - \{n \sin[y(q)]\}^2} + n \cos[y(q)]} \right)^2, \quad (9)$$

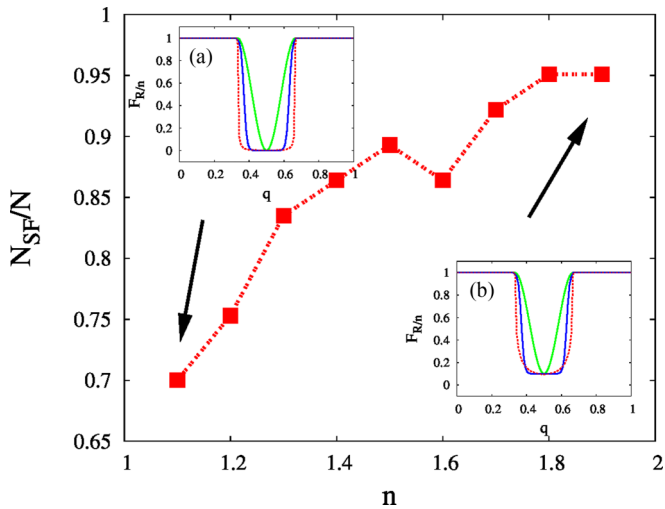


FIG. 6. Fraction of scar functions N_{SF}/N needed to reach $P = 0.8$ as a function of the index of refraction n of the microresonator. The red (dark gray) dotted line with squares correspond to considering only POs inside the repeller. In inset (a) we compare the shape of the Fresnel reflectivity for $n = 1.1$ [red (dark gray) dotted line] with the step [blue (black) line] and sinusoidal [green (gray) line] functions for $R = 0$. In inset (b) the same is done for $n = 1.9$ and $R = 0.1$.

where $y(q) = 3 \arcsin(1/n)(2q - 1)$, and n is the index of refraction of the microresonator. We notice that the opening region is the same as in the previous cases, i.e., the middle third of the tribaker map, and not the one given by the critical angle of total internal reflection. This is because the tribaker map is a good model to generate a chaotic saddle but if opened in one of its thirds; when the opening is larger than this the remaining repeller vanishes quickly. We have considered $n \in [1.1, 1.9]$ and calculated the fraction of scar functions needed in our semiclassical theory, results which are displayed in Fig. 6.

In this range, the main parameter of the step and sinusoidal reflectivity functions, i.e., R , can be directly associated to the index of refraction n (A and B are just to control the degree of smoothing of the sharp borders of the opening for the step function). In fact, the behavior of the Fresnel reflectivity is very similar to that of the step function seen in Fig. 4. This study also reveals that not only the width of the opening is relevant, but also its depth. As a matter of fact, if we look at Figs. 6(a) and 6(b) it becomes clear that the bottom of the three reflectivities almost coincides.

IV. CONCLUSIONS

We have applied the recently developed short POs theory for partially open quantum maps to the case where the reflectivity function is continuous on the phase space. This situation is relevant for actual experiments with microresonators like those used to produce microlasers [3]. In particular, recent developments in this area show that boundary conditions can be highly non trivial [1], and this of course impacts on many properties.

By considering a Fermi-Dirac-type step function we have tried to capture the essential features of continuity versus discontinuity, smoothing the sharp borders at the opening. On the other hand, we have also used a sinusoidal function that models more generic situations. A parameter R allows us to control the degree of reflectivity by determining the bottom of these functions. In both cases the spectral properties change with respect to the discontinuous opening. The typical strong oscillations in the scaling of the number of resonances as a function of the eigenvalue moduli above a given threshold ν_c are almost completely lost, even for the lowest values of the parameter R . This suggest that a different Weyl law (or regime) should be investigated for these cases.

Nevertheless and quite surprisingly, our semiclassical theory revealed that the role played by the shortest POs that belong to the completely open repeller is very robust, at least for $R < 0.1$ and $N = 3^5$, where we have made our calculations. These orbits give the fundamental structure on which the quantum continuous repeller is constructed. This is underlined by the fact that including a few POs outside of it does not significantly change the picture. Moreover, in the case of the step opening we are able to use a relatively small amount of POs inside the perturbative region. We notice that the step reflectivity function corresponds to a greater opening than that of the cosine one. In fact, if we change the value of the parameter A the amount of scar functions needed grows and can reach the one needed for the sinusoidal reflectivity. Finally, the parameter R that controls the bottom of both reflectivity functions has a very direct correspondence with the index of

refraction of microresonators. If we consider a Fresnel type of reflectivity we obtain a very similar behavior to that of the step function in the perturbative region. This region corresponds to a low to medium index microstadium according to Ref. [23]. Our studies have also revealed that not only the size of the opening but also its depth are relevant for the semiclassical behavior.

These findings could lead to new experiments in order to detect this structure, and also to optimize the cavity design. In the future we plan to use this insight on the morphology of the

eigenfunctions to find the scaling of the spectra and the reasons behind it.

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