Self-avoiding walk on a square lattice with correlated vacancies

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The self-avoiding walk on the square site-diluted correlated percolation lattice is considered. The Ising model is employed to realize the spatial correlations of the metric space. As a well-accepted result, the (generalized) Flory's mean-field relation is tested to measure the effect of correlation. After exploring a perturbative Fokker-Planck-like equation, we apply an enriched Rosenbluth Monte Carlo method to study the problem. To be more precise, the winding angle analysis is also performed from which the diffusivity parameter of Schramm-Loewner evolution theory (κ) is extracted. We find that at the critical Ising (host) system, the exponents are in agreement with Flory's approximation. For the off-critical Ising system, we find also a behavior for the fractal dimension of the walker trace in terms of the correlation length of the Ising system $\xi(T)$, i.e., $D_F^{\text{SAW}}(T) - D_F^{\text{SAW}}(T_c) \sim \frac{1}{\sqrt{\xi(T)}}$.

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I. INTRODUCTION

The effect of environmental disorder on critical behaviors is a long-standing problem in condensed-matter systems. The self-avoiding walk (SAW) as a realization of many physical systems (such as polymers) in such media has attracted much attention in the literature. The problem is more interesting when the disorder is itself critical or generally self-affine, since two diverging lengths (one for the SAW and another for the host media) compete, which may lead to nontrivial effects [1]. Since only the end parts of the SAW can lie on dead ends, the asymptotic behavior of its gyration radius is expected to be dominated by a backbone structure, rather than by that of a full fractal lattice. In percolation clusters, the fact that the backbone fractal dimension is different from the spectral dimension of the system reveals that random walks and SAWs probe different properties of the fractal lattice [2].

The problem of a SAW in dilute systems was first studied by Chakraberti *et al.* [3]. They showed, based on renormalization-group ideas, that disorder does not change the properties of the model. This was challenged by Kremer [4] for a dilute diamond lattice, and by Aharony *et al.* for other fractal lattices [5], depending on their backbone fractal dimension. Kremer's findings [4] apparently violated the Harris criterion, according to which for $\alpha > 0$ ($\alpha \equiv$ the exponent of the heat capacity) systems (such as SAW), disorder should be relevant. The results showed that for $p > p_c$ ($p_c \equiv$ the percolation threshold), the ν exponent ($\equiv$ the exponent of the end-to-end distance; see the following sections) is identical to the pure SAW on a regular lattice, and for $p = p_c$ it is in accordance with the Aharony generalization of Flory's mean-field formula [6,7] $\nu = 3/(2+\bar{d})$ ($\approx \frac{2}{3}$ for the diamond lattice), in which $\bar{d} = d - \beta/\nu_{\text{perc}}$, d is the spatial dimension of the system, and β and ν_{perc} are the exponents of the percolation problem. The β exponent is defined for the percolation probability in the vicinity of the critical point [$P(p) \propto |p - p_c|^{\beta}$], whereas the ν_{perc} exponent is defined for the correlation length $\xi(p) \propto |p - p_c|^{-\nu_{\text{perc}}}$. The two-dimensional result $\nu_{\text{perc}} = 4/3$ and $\beta = 5/36$ is exact, whereas for the three-dimensional case $\nu_{\text{perc}} = 0.88$ and $\beta = 0.41$, which have been obtained by numerical simulations [8]. It was also suggested that a more appropriate generalization of Flory's result is $\nu = \frac{1}{\tilde{d}}(\frac{3\tilde{d}}{2+\tilde{d}})$, where $\tilde{d}$ is the spectral dimension of the fractal space. The dependence of ν on many other fractal parameters has been investigated [2,9–11]. The exponent has also been derived within many other approximations [12–16]. For a good review, see [1]. Nakayashi *et al.* argued that the large change of ν for $p = p_c$ should be the result of large errors, and the exponent does not change even at $p = p_c$ [17–20], the result which was argued by other authors [1].

The most conclusive field-theoretical results were found by Meir and Harris [21] and then extended [22]. By starting from a Landau-Ginsberg-Wilson Hamiltonian, it was found that for noncritical disorder $p > p_c$ the RG flow is toward the pure SAW fixed point, and for the critical disorder ($p = p_c$), $\nu_p = \frac{1}{2}(1 + \epsilon/8 + 15\epsilon^2/256)$, in which $\epsilon \equiv 4 - d$. In this respect, one expects that for $p > p_c$ and the large spatial scales $r \gg \zeta_p$ (in which ζ_p is the percolation correlation length), the properties of the pure SAW are retrieved, whereas for $r \ll \zeta_p$ the behaviors of the model at $p = p_c$ are seen [23]. Despite a huge literature, the effect of spatial correlation in metric space in not yet known. In addition, the conformal invariance of the SAW on critical fractal lattices is another important question that has not been addressed in the literature. In the present paper, we dilute the host system by removing some sites and letting the random walker pass only through the remaining (active) area. The pattern of this diluteness is considered to be correlated, which is realized by means of the Ising model. We call the resulting lattice *the site-diluted Ising-correlated lattice*. The importance of these correlations is also argued in detail in terms of a Fokker-Planck-like equation. The Monte Carlo method as well as the winding angle method are used

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to extract the various exponents of the self-avoiding random walkers, or equivalently the polymers.

The paper is organized as follows: In the following section we discuss the motivation for this study, and we introduce and describe the model. The numerical methods and details are explored in Sec. III. The results are presented in Sec. IV, which has two subsections: Sec. IV A contains the critical results, and Sec. IV B presents the power-law behaviors in the off-critical temperatures. The final section contains our conclusions.

II. THE CONSTRUCTION OF THE PROBLEM

The notion of critical phenomena on fractal lattices originated mainly in the work of Gefen *et al.* [24]. The concept can be generalized to dilute systems that become fractal in some limits. Examples include magnetic material in porous media [25–32], fluid dynamics in porous media [33,34], and self-organized criticality on percolation lattices [35,36]. Among the statistical models, there is a great deal of interest in simple random walks (RWs) and self-avoiding walks (SAWs) in dilute systems. One example is the self-non-intersecting chains of monomers in disordered systems, which have strong connections to experiments [1]. In addition, a correlated SAW serves as an important realization for protein folding in lattices [37], in which the correlations are mostly realized by Ising interactions between monomers [38].

In the above-mentioned literature on the SAW in disordered systems, the site-diluteness of the host media is commonly realized by the percolation theory, in which no correlation between active sites (through which the random walker can pass) is considered, and there has been little attention given to the correlation effects in the host system. Recently, it was shown that involving (Ising-type) correlations in the diluteness pattern of the media dramatically changes the properties of the sandpile model with respect to that on the uncorrelated percolation lattice [36]. More interesting effects are for the critical host system in which two diverging lengths (one for the dynamical model and another for the host system) compete. This motivates one to consider the SAW on the correlated site-diluted lattices, which is the aim of the present paper. The motivations of the present work are twofold:

(i) What is the effect of correlation in the diluteness pattern of the lattice (which is Ising-type in this paper)? More precisely, does Flory's generalized relation work for the SAW on the correlated dilute system?

(ii) What are the behaviors of the SAW in the off-critical host system? More importantly, how does the fractal dimension of the SAW traces change in the vicinity of the critical point?

To incorporate the correlations in the diluteness pattern of the host media, we have used the Ising model. Let us explain the construction of the lattice in more detail. Consider a two-dimensional square $L_0 \times L_0$ lattice comprised of some random active and inactive sites. The random walkers are restricted to taking their steps only on the active sites. The *active area* is defined as the area that is formed from the set of active sites. We represent the status of sites by the field s, which is 1 (0) for an active (inactive) site. Therefore, the overall status of the system is known if the configuration of this field is given, i.e., $\{s_i\}_{i=1}^{L}$. We use the Ising model to model the pattern of s_i's, which is defined on the regular square $L_0 \times L_0$ lattice with spins $\sigma = 2(s - 1/2) = \pm 1$. Therefore, the pattern of the active area of the original system is obtained by the Ising model for which the spins play the role of the field of activity-inactivity of the media, and the correlations are simply controlled by the artificial temperature T (which has nothing to do with the real temperature). By "temperature," we mean the control parameter that tunes the correlations of the host system. It is notable that the activity configuration of the media is quenched, i.e., when an Ising configuration is obtained, SAW samples are generated in the resulting dilute lattice.

The Ising Hamiltonian (H) is

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j - h \sum_i \sigma_i, \quad \sigma_i = \pm 1 \tag{1}$$

in which J is the coupling constant, h is the magnetic field (which is supposed to be zero in this paper), and σ_i and σ_j are the spins at sites i and j, respectively, having the values ∓ 1 (as introduced above). $\langle i,j \rangle$ shows that sites i and j are nearest neighbors. $J > 0$ corresponds to a ferromagnetic system (positively correlated host lattice), whereas $J < 0$ is for an antiferromagnetic system (negatively correlated host lattice). We emphasize that in this paper we use the Ising model as the metric space. Our model is not a magnetic one, instead the spins show the activity state of the sites. The artificial temperature T controls the correlations and the population of the active sites to the total number of sites, and it also controls the heterogeneity. The population of the active site can be directly controlled by h, which determines the preferred direction of the spins in the Ising model. For $h = 0$, the model is well known to exhibit nonzero magnetization per site $M = \langle \sigma_i \rangle$ at temperatures below the critical temperature T_c. Although we set $h = 0$ throughout this paper, we would like to mention some points concerning this parameter here. In the Ising model, the magnetization has a discontinuity at $h = 0$ along the $T < T_c$ line, i.e., for $h = 0^+$ and $T < T_c$ we have $M > 0$, whereas for the case $h = 0^-$ and $T < T_c$ we have $M < 0$. We can have a percolation description of the Ising model that is controlled by T and h as follows: In each T and h the system is composed of some spin clusters. Let us consider only up-spin clusters, bearing in mind that the system has the symmetry $h \to -h$ and $\sigma_i \to -\sigma_i$. We define $h_{\text{th}}(T)$ as the magnetic-field threshold below which there is no spanning cluster of parallel spins and above which some spanning clusters appear. Apparently for $T = 0$ all spins align in the same direction and $H_{\text{th}}(T = 0) = 0^+$. Also for $T = \infty$ the spins are uncorrelated and take the up direction with the probability $\frac{1}{2}e^h/\cosh h$. There are two transitions in the Ising model: the magnetic (paramagnetic to ferromagnetic) transition and the percolation transition (in which the connected geometrical spin clusters percolate). For the two-dimensional (2D) regular Ising model at $h = 0$ these two transitions occur simultaneously [39], although it is not the case for all versions of the Ising model, e.g., for the site-diluted Ising model [32].

Fokker-Planck-like approximation

We define the Ising model on the $L_0 \times L_0$ square lattice. Then by solving Eq. (1) for $h = 0$, an Ising sample at a temperature $T \leqslant T_c$ is made, and some self-avoiding random walkers start the motion (from the boundary or bulk, depending on the statistical observables) on the largest spanning cluster of the sample. This is defined as the connected spin cluster, which contains the same spin sites and also connects opposite boundaries of the system. Note that the walks are only possible on the active ($\sigma = +1$) sites in the spanning cluster. Let us call this host area the *active space*. The active-space coordination number is defined as $z_j \equiv \sum_{i\in\delta_j} \delta_{\sigma_i,1}$, where δ_j is the set of neighbors of the ith site, and δ is the Kronecker delta function. Let us denote the trace of a SAW up to time t by γ_t, the tip of the trace by $\vec{r}_t$, and the number of accessible neighbors at t by Z_t. Therefore, when a random walker reaches the site $\vec{r}$ at time t, it has ways Z_t to move in the next time, which is

$$\begin{aligned} Z_t(\vec{r}_t = \vec{r}) &\equiv z_{\vec{r}} - \sum_{t'\leqslant t}\sum_{\vec{r}'\in\delta\vec{r}} \delta(\vec{r}_{t'},\vec{r}')s_{\vec{r}'}\theta(\vec{r},\vec{r}') \\ &= \sum_{t'\leqslant t}\sum_{\vec{r}'\in\delta\vec{r}}[\delta_{t,t'} - \delta(\vec{r}_{t'},\vec{r})]s_{\vec{r}'}\theta(\vec{r},\vec{r}'). \end{aligned} \tag{2}$$

In this relation, $\delta\vec{r}$ is the set of active neighbors of $\vec{r}$ and $\delta(\vec{r}_t,\vec{r}) = 1$ if the random walker is in site $\vec{r}$ at time t, and zero otherwise, $s_{\vec{r}} = 1$ if the site $\vec{r}$ is active and zero otherwise, and $\theta(\vec{r}',\vec{r}) \equiv 1 - \delta(\vec{r}',\vec{r})$, which is apparently unity in this expression, but it becomes important in the following perturbation expansions. Note that $s_{\vec{r}} = \frac{1}{2}[\sigma(\vec{r}) + 1]$. When $\sum_{t'\leqslant t}\sum_{\vec{r}'\in\delta\vec{r}} \delta(\vec{r}_{t'},\vec{r}')s_{\vec{r}'} = z_{\vec{r}}$, there will no longer be a way to move further, and a new process should start. Otherwise each of the $Z_t(\vec{r}_t)$ sites is chosen with the same probability, i.e., $1/Z_t(\vec{r}_t)$.

On the other hand, by defining the probability of a site being occupied at t, $p(\vec{r},t) \equiv \langle\delta(\vec{r}_t,\vec{r})\rangle$ (in which $\langle\,\rangle$ is the ensemble average *for a fixed disorder configuration*; for the averaging over both random walks and disorder, we use the notation $\langle\langle\,\rangle\rangle$), one can easily show that

$$\begin{aligned} p(\vec{r},t) &= s_{\vec{r}}\left\langle \Theta(\vec{r},t-\tau)\sum_{\vec{r}'\in\delta\vec{r}} \frac{\delta(\vec{r}_{t-\tau},\vec{r}')}{Z_{t-1}(\vec{r}_{t-\tau} = \vec{r}')} s_{\vec{r}'}\right\rangle \\ &= \sum_{n=0}^{\infty}\sum_{\vec{r}'\in\delta\vec{r}} s_{\vec{r}}s_{\vec{r}'}\left\langle \frac{\Theta(\vec{r},t-\tau)\delta(\vec{r}_{t-\tau},\vec{r}')}{z_{\vec{r}'}}\epsilon(t,\vec{r}')^n\right\rangle, \end{aligned} \tag{3}$$

where $\epsilon(t,\vec{r}') \equiv \frac{\sum_{t'\leqslant t-\tau}\sum_{\vec{r}''\in\delta\vec{r}'} \delta(\vec{r}_{t'},\vec{r}'')\theta(\vec{r}'',\vec{r})s_{\vec{r}''}}{z_{\vec{r}'}} < 1$, and $\Theta(\vec{r},t) \equiv 1 - \sum_{t'\leqslant t} \delta(\vec{r}_{t'},\vec{r})$ is a nonlocal detector operator that is unity if the point $\vec{r}$ has not been visited up to time t and zero otherwise [i.e., $w(\vec{r},t) \equiv 1 - P(\vec{r},t) \equiv \langle\Theta(\vec{r},t)\rangle = 1 - \sum_{t'\leqslant t} p(\vec{r},t)$]. In obtaining this equation, we have used the fact that for a random walker to be in site $\vec{r}$ at time t, it should have been in its neighbors at time $t-\tau$. The factor $s_{\vec{r}} \times \Theta(\vec{r},t-\tau)$ on the right-hand side of the first line of Eq. (3) ensures that site $\vec{r}$ is active and has not been visited before. It is notable that this equation is true for a quenched percolation configuration of the metric system, i.e., $\{s_i\}_{i=1}^N$, and it has not been averaged over s_i's.

To obtain a differential equation, we separate the $n = 0$ term from the others and take this term to the left. By subtracting $p(\vec{r},t-\tau)$ from both sides, and ignoring the $n \geqslant 3$ terms, we obtain the relation

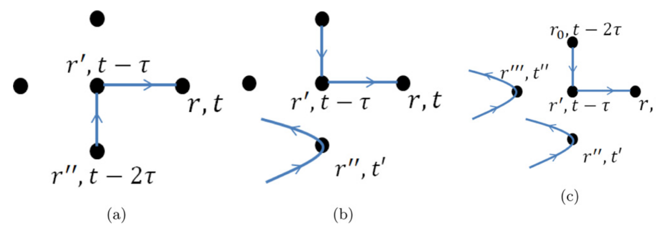


FIG. 1. Three situations of SAWs reaching the point $\vec{r}$ at time t.

$$\begin{aligned} &\partial_t p(\vec{r},t) - Dw(\vec{r},t)\nabla^\alpha p(\vec{r},t) \\ &= \sum_{\vec{r}'\in\delta\vec{r}}\sum_{\vec{r}''\in\delta\vec{r}'} \frac{s_{\vec{r}}s_{\vec{r}'}s_{\vec{r}''}}{z(\vec{r}')^2} I_1(\vec{r},\vec{r}',\vec{r}'',t-\tau) \\ &\quad + \sum_{\substack{\vec{r}'\in\delta\vec{r} \\ \vec{r}'',\vec{r}'''\in\delta\vec{r}'}} \frac{s_{\vec{r}}s_{\vec{r}'}s_{\vec{r}''}s_{\vec{r}'''}}{z(\vec{r}')^3} I_2(\vec{r},\vec{r}',\vec{r}'',\vec{r}''',t-\tau) + O(\epsilon^3), \end{aligned} \tag{4}$$

where $\partial_t p(\vec{r},t) \equiv \frac{1}{\tau}[p(\vec{r},t) - p(\vec{r},t-\tau)]$, $\nabla^\alpha p(\vec{r},t) \equiv a^{-\alpha}(\sum_{\vec{r}'\in\delta\vec{r}} s_{\vec{r}}s_{\vec{r}'}\frac{p(\vec{r}',t-\tau)}{z_{\vec{r}'}/4} - 4p(\vec{r},t-\tau))$, $D \equiv \frac{a^\alpha}{4\tau}$, a is the lattice constant, and α is the order of fractional derivative, which is not *a priori* known and should be determined by the fractal dimension of the host. It is evident that $\alpha = 2$ for a regular lattice. It is notable that $\delta(\vec{r}_t,\vec{r})\Theta(\vec{r},t) \equiv \delta(\vec{r}_t,\vec{r})$. In the above equation, we have defined

$$\begin{aligned} I_1(\vec{r},\vec{r}',\vec{r}'',t) &\equiv \theta(\vec{r}'',\vec{r})\frac{1}{\tau}\int_0^t \langle\Theta(\vec{r},t)\delta(\vec{r}',t)\delta(\vec{r}'',t')\rangle dt', \\ I_2(\vec{r},\vec{r}',\vec{r}'',\vec{r}''',t) &\equiv \theta(\vec{r}'',\vec{r})\theta(\vec{r}''',\vec{r}) \\ &\quad \times\frac{1}{\tau}\iint_0^t \langle\Theta(\vec{r},t)\delta(\vec{r}',t)\delta(\vec{r}'',t')\delta(\vec{r}''',t'')\rangle \\ &\quad \times dt'dt''. \end{aligned} \tag{5}$$

To find I_1, we write it as the form

$$\begin{aligned} I_1 &= \frac{1}{\tau}\int_0^{t-\tau} \langle\Theta(\vec{r},t-\tau)\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t')\rangle\theta(\vec{r}'',\vec{r})dt' \\ &\quad \times[1 - \delta(t',t-2\tau) + \delta(t',t-2\tau)] \\ &= \frac{1}{\tau}\langle\Theta(\vec{r},t-\tau)\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t-2\tau)\rangle\theta(\vec{r}'',\vec{r}) \\ &\quad +\frac{1}{\tau}\int_0^{t-\tau} \langle\Theta(\vec{r},t-\tau)\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t')\rangle \\ &\quad \times[1 - \delta(t',t-2\tau)]\theta(\vec{r}'',\vec{r})dt'. \end{aligned} \tag{6}$$

For the first term we have the situation that has been schematically shown in Fig. 1(a), and the second term is equivalent to Fig. 1(b). We have separated these two terms since their behaviors are expected to be different. The quantities $\delta(\vec{r}',t-\tau)$ and $\delta(\vec{r}'',t-2\tau)$ in $\langle\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t-2\tau)\rangle$ are maximally correlated, and their multiplication forms a new field $\delta^{(2)}(\vec{r},\vec{r}',\vec{r}'',t) \equiv \frac{1}{\tau}\Theta(\vec{r},t)\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t-2\tau)$ and $p^{(2)}(\vec{r}',\vec{r}'',t) \equiv \langle\delta^{(2)}(\vec{r}',\vec{r}'',t)\rangle$. For the second term, however, the fields are expected to be nearly independent, due to their temporal distance, i.e., $\int_0^{t-\tau} \langle\Theta(\vec{r},t)\delta(\vec{r}',t-\tau)\delta(\vec{r}'',t')\rangle[1 - \delta(t',t-2\tau)]dt' \approx w(\vec{r},t-\tau)p(\vec{r}',t-\tau)P(\vec{r}'',t-2\tau)$.

Therefore,

$$I_1 \simeq \theta(\vec{r}'',\vec{r})\bigg[p^{(2)}(\vec{r}',\vec{r}'',t) + \frac{1}{\tau}w(\vec{r},t-\tau)p(\vec{r}',t-\tau)P(\vec{r}'',t-2\tau)\bigg]. \quad (7)$$

Now let us consider the second integral, for which we do the same procedure. By multiplying the expression (which is equal to unity)

$$1 = \{[1-\delta(t',t-2\tau)][1-\delta(t'',t-2\tau)] + \delta(t',t-2\tau) + \delta(t'',t-2\tau) - \delta(t',t-2\tau)\delta(t'',t-2\tau)\} \quad (8)$$

by I_2, we can do the same procedure. These quantities determine the configuration of the SAWs, entering the points $\vec{r}$, $\vec{r}'$, and $\vec{r}''$ at times t, t', and t''. The contribution of the first term of the second line has been shown in Fig. 1(c), and the other terms only improve the contributions of Figs. 1(b) and 1(a). If we pick up only the first term, we obtain

$$\begin{aligned}\tau I_2&(\text{first term})\\ &= \theta(\vec{r}'',\vec{r})\theta(\vec{r}''',\vec{r})\iint_0^{t-\tau}\langle\Theta(\vec{r},t)\delta(\vec{r}',t-\tau)\\ &\quad\times\delta(\vec{r}'',t')\delta(\vec{r}''',t'')\delta(\vec{r}_0,t-2\tau)\rangle\\ &\quad\times[1-\delta(t',t-2\tau)][1-\delta(t'',t-2\tau)]dt'dt''\\ &\approx \tau\theta(\vec{r}'',\vec{r})\theta(\vec{r}''',\vec{r})p^{(2)}(\vec{r}',\vec{r}_0,t)P(\vec{r}'',t-\tau)P(\vec{r}''',t-\tau). \quad (9)\end{aligned}$$

In this equation, we have inserted a trivial term $\delta(\vec{r}_0,t-2\tau)$ in the expression in which $\vec{r}_0$ has been shown in Fig. 1(c). This insertion changes nothing, since the random walker has apparently been in $\vec{r}_0$ at time $t-2\tau$. Finally, we obtain

$$\begin{aligned}\partial_t p(\vec{r},t) &- Dw(\vec{r},t)\nabla^\alpha p(\vec{r},t)\\ &= \sum_{\vec{r}'\in\delta\vec{r}}\sum_{\vec{r}''\in\delta\vec{r}'}\frac{s_{\vec{r}}s_{\vec{r}'}s_{\vec{r}''}}{z(\vec{r}')^2}\theta(\vec{r}'',\vec{r})\bigg[p^{(2)}(\vec{r}',\vec{r}'',t)\\ &\quad+\frac{1}{\tau}w(\vec{r},t-\tau)p(\vec{r}',t-\tau)P(\vec{r}'',t-2\tau)\bigg]\\ &\quad+\sum_{\substack{\vec{r}'\in\delta\vec{r}\\ \vec{r}'',\vec{r}'''\in\delta\vec{r}'}}\frac{s_{\vec{r}}s_{\vec{r}'}s_{\vec{r}''}s_{\vec{r}'''}}{z(\vec{r}')^3}\theta(\vec{r}'',\vec{r})\theta(\vec{r}''',\vec{r})\\ &\quad\times p^{(2)}(\vec{r}',\vec{r}_0,t)P(\vec{r}'',t-\tau)P(\vec{r}''',t-\tau)+O(\epsilon^3). \quad (10)\end{aligned}$$

In this equation, $P(\vec{r},t)$ requires the full information about the status of the walker in the past times, i.e., it is a field that carries the information on the history of the random walk, whereas $p^{(2)}$ is a local field. Now let us average over the disorder, i.e., take a configurational average from Eq. (10) over the s_i configuration. The correlation of the noise $\{s_i\}_{i=1}^N$ surely affects the resultant equation. The equation involves the moments of s (up to the fourth moment in the above equation). When s_i's are uncorrelated, we obtain that $\langle s_{\vec{r}_1},s_{\vec{r}_2},\dots,s_{\vec{r}_n}\rangle$ are equal to s^n, where $s=\langle s_{\vec{r}}\rangle$. Also, $\langle s_{\vec{r}_1},s_{\vec{r}_2},\dots,s_{\vec{r}_n}G\rangle$ (where G is some function that depends on the s configuration) can safely be approximated by $s^n\langle G\rangle$. Apparently this is not true for correlated noises in which $\langle s_{\vec{r}_1}s_{\vec{r}_2}\rangle \neq s^2$. In the above equation, $\Gamma^{(2)}(|\vec{r}-\vec{r}'|)\equiv\langle s_{\vec{r}}s_{\vec{r}'}\rangle$, $\Gamma^{(3)}(\vec{r}_1,\vec{r}_2,\vec{r}_3)\equiv\langle s_{\vec{r}_1}s_{\vec{r}_2}s_{\vec{r}_3}\rangle$, and $\Gamma^{(4)}(\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4)\equiv\langle s_{\vec{r}_1}s_{\vec{r}_2}s_{\vec{r}_3}s_{\vec{r}_4}\rangle$ have appeared, which are temperature-dependent and are calculated using the Ising autocorrelation functions.

This analysis has been presented to highlight the important effect of correlations in the metric space of the SAW. The equation governing the SAW on the site-diluted system is therefore a fractional nonlocal Fokker-Planck equation (as is evidently true for all SAWs), and also the $(n+1)$-point Ising correlation function appears in the nth term of the perturbative expansion. The fact that the obtained equation is perturbative and involves nonlinear-nonlocal functions make it less efficient, therefore numerical studies are crucial to understanding its properties.

III. NUMERICAL METHODS, MONTE CARLO APPROACH, AND SLE THEORY

In this paper, we have used the enriched Rosenbluth method. To describe the method, let us consider a growing polymer chain (or a self-avoiding random walker) at the nth step, for which the $(n+1)$th monomer should be added to the chain in an active neighboring site. In the Rosenbluth-Rosenbluth (RR) method, we give weight $W(N)\equiv(\prod_{t=1}^N Z_t)^{-1}$ to each sample configuration, in which Z_t has been defined in (2). The configurational average is then defined by

$$\langle R^2\rangle \equiv \frac{\sum_i W_i(N)R_i^2}{\sum_i W_i(N)}, \quad (11)$$

in which i runs over distinct configurations, W_i is its weight, and R_i is the end-to-end distance, which is the Euclidean distance between the start point and the end point of the curve. The end-to-end critical exponent (ν) is defined by

$$\langle R^2\rangle \sim N^{2\nu}, \quad (12)$$

which is the inverse of the fractal dimension of the random-walk trace, i.e., $\nu=\frac{1}{D_f}$. To obtain the fractal dimension, one can use the box-counting. In the box-counting scheme, the fractal dimension is defined by the relation $N(L)\sim L^{D_f}$, in which $N(L)$ is the length of the stochastic curve (SAW) inside a box of linear size L. It is notable that all polymers have the same length N in this averaging. For the enrichment follow the Grassberger method [40], which is as follows: If $W(N)$ is above a certain threshold, we add a new walker and give the new and old walker half the original weight. If $W(N)$ is below a certain threshold, then we eliminate it with the probability $p=1/2$ and double the weights of the remaining half.

By means of this method, we calculate the ν exponent (of the end-to-end distance, to be defined later) as well as the fractal dimension of the SAW using the box-counting method. To be more precise, we have also used the winding-angle statistics to extract the diffusivity parameter of Schramm-Loewner evolution (SLE).

As a well-known fact, the critical 2D models have special algebraic and geometrical properties. The algebraic properties of these models are described within conformal field theories. However, these theories are unable to uncover the geometrical features of these models since it concerns the local fields defined in these models. SLE theory aims to describe the interfaces of 2D critical models via growth processes. Thanks to this theory, a deep connection between the local properties and the global (geometrical) features of the 2D critical models has been discovered. These nonintersecting interfaces are assumed to have two essential properties, namely conformal invariance and the domain Markov property [41].

In the SLE theory, one replaces the critical curve by a dynamical one. We consider the model on the upper half-plane, i.e., $H = \{z \in \mathbb{C}, \text{Im}z \geqslant 0\}$. Let us denote the curve up to time t as γ_t and the hull K_t as the set of points that are located exactly on the γ_t trace, or are disconnected from infinity by γ_t. The complement of K_t is $H_t := H \backslash K_t$, which is simply connected. According to the Riemann mapping theorem, there is always a conformal mapping $g_t(z)$ (in two dimensions) that maps $H_t \rightarrow H$. The map $g_t(z)$ (commonly referred to as a uniformizing map, meaning that it uniformizes the γ_t trace to the real axis) is the unique conformal map with $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{z^2})$ as $z \rightarrow \infty$ known as hydrodynamical normalization. Loewner showed that this mapping satisfies the following equation [41–44]:

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \tag{13}$$

with the initial condition $g_t(z) = z$ and for which the tip of the curve (up to time t) is mapped to the point ξ_t on the real axis. For fixed z, $g_t(z)$ is well-defined up to time τ_z for which $g_t(z) = \xi_t$. The more formal definition of the hull is therefore $K_t = \{z \in H : \tau_z \leqslant t\}$. For more information, see [41,43]. For the critical models, it has been shown [42] that ξ_t (referred to as the driving function) is a real-valued function proportional to the one-dimensional Brownian motion $\xi_t = \sqrt{\kappa} B_t$ in which κ is known as the diffusivity parameter. SLE aims to analyze these critical curves by classifying them to the one-parameter classes represented by κ. The relation between the fractal dimension of the curves $D_f \equiv \frac{1}{\nu}$ and the diffusivity parameter (κ) is $D_f = 1 + \frac{\kappa}{8}$.

The important tests of SLE are left passage probability [45,46], direct SLE mapping [47,48], and the winding angle statistics [49]. The latter is defined by the relation

$$\langle \theta^2 \rangle = \kappa \log R, \tag{14}$$

where R is the end-to-end distance and θ is the total winding angle of the movement up to the end point with respect to a global direction. Noting that $R \sim N^{\nu}$, one finds that $\langle \theta^2 \rangle = 8\nu(D_F - 1) \log N$. This slope is exactly 2 for the SAW on the regular lattice, i.e., $T = 0$.

Numerical details

At $T = T_c$, for which the Ising model becomes critical, some power-law behaviors emerge. The method to simulate the system in the vicinity of this point is important, due to the problem of critical slowing down. To avoid this problem, we have used the Wolff Monte Carlo method to generate Ising samples. Our ensemble averaging contains both random-walks averaging as well as Ising-percolation lattice averaging. For the latter case we have generated 2×10^3 Ising uncorrelated samples for each temperature on the lattice size $L_0 = 2048$. To make the Ising samples uncorrelated, between each successive sampling, we have implied $L^2/3$ random spin flips and let the sample equilibrate by $500L^2$ Monte Carlo steps. The main lattice has been chosen to be square, for which the Ising critical temperature is $T_c \approx 2.269$. Only the samples with temperatures $T \leqslant T_c$ have been generated, since the spanning

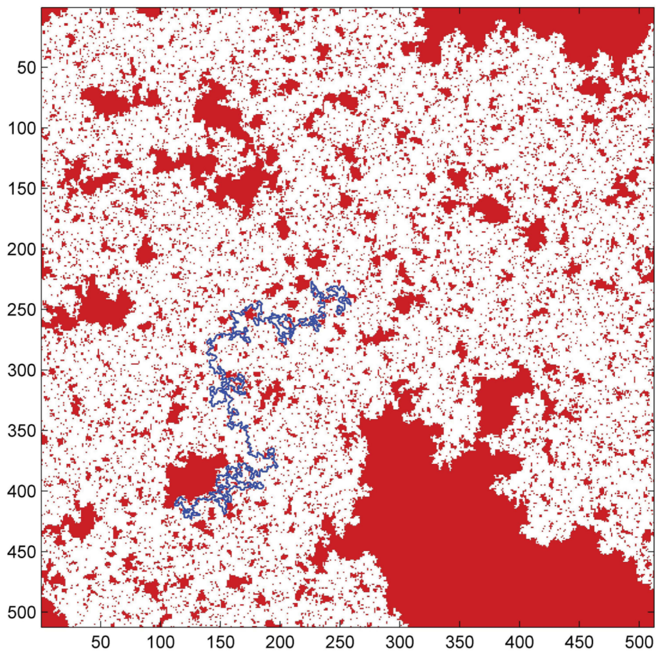


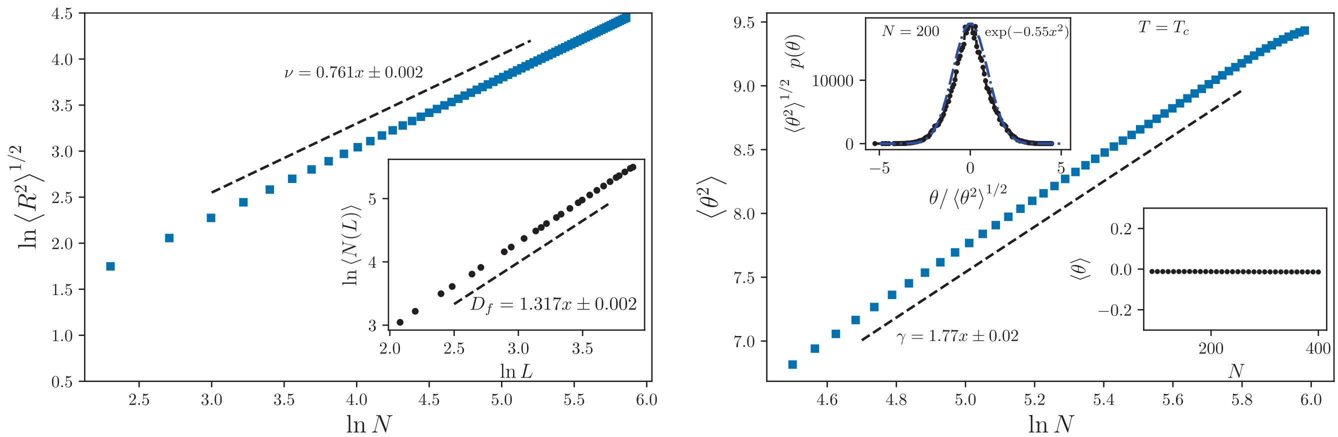
FIG. 2. $N = 2000$ bulk SAW sample in an Ising sample media in a 512×512 lattice at $T = T_c$ (blue lines). The red sites represent the forbidden (inactive) sites, and the white sites are representative of the active ones.

clusters (active space) are present only for this case. As stated in the previous section, the random walkers move only on the active space, which is defined as the set of connected sites having the same (majority) spin, which connects two opposite boundaries. The temperatures considered in this paper are $T = T_c - \delta t_1 \times i$ ($i = 1,2,\ldots,5$ and $\delta t_1 = 0.01$) to obtain the statistics in the close vicinity of the critical temperature $T_c \simeq 2.269$ (note that the model shows nontrivial power-law behaviors in the vicinity of the critical temperature) and $T = T_c - \delta t_2 \times i$ ($i = 1,2,\ldots,10$ and $\delta t_2 = 0.05$) for the more distant temperatures. To equilibrate the Ising sample and obtain the desired samples, we have started from the high temperatures ($T > T_c$). For each temperature, 2×10^6 SAWs were generated for 2×10^3 Ising samples (for each Ising sample, 10^3 avalanche samples were generated and each Ising sample had its own particle dynamics to reach a steady state). We have used the Hoshen-Kopelman [50] algorithm for identifying the clusters in the lattice.

Once a spanning Ising percolation cluster is obtained, the SAW simulations begin. Figure 2 is a 512×512 sample at $T = T_c$ in which the red (white) sites show the inactive (active) sites and an $N = 2000$ length SAW (which has started from the bulk and moves only on the white sites) has been shown in blue. The geometrical properties of these walks are investigated in this paper.

IV. RESULTS

Two cases have been considered separately: The critical fractal (self-affine) host space ($T = T_c$) and the supercritical one ($T < T_c$). In the latter case, the trend of the exponents to the critical case is obtained. It is expected that the critical behaviors of the ordinary SAW on the regular lattice are retrieved in the limit $T \rightarrow 0$. For all temperatures in the range $T < T_c$, power-



(a) (b)

FIG. 3. (a) $\log_{10}\langle R^2\rangle^{\frac{1}{2}}$ in terms of $\log_{10} N$ for $T = T_c$. Inset: $\log_{10}\langle N(L)\rangle$ in terms of $\log_{10} L$. (b) $\langle\theta^2\rangle$ in terms of $\ln N$ with the slope $\gamma = 1.77 \pm 0.02$ for $T = T_c$. Upper inset: the distribution function of θ for $N = 200$. Lower inset: $\langle\theta\rangle$ in terms of N, which is zero.

law behaviors have been observed. We argue that there are two fixed points in the problem, namely $T = 0$ (the IR fixed point) and $T = T_c$ (the UV fixed point). The critical exponents are nearly constant for most of the phase space and show deviations in the vicinity of the critical temperature.

A. Critical temperature

The characterization of the fixed points in any perturbed statistical model is very important, since it yields information about its large-scale behaviors. Some critical systems on the uncorrelated percolation lattice show a fixed point at $p = p_c$ (which is unstable toward the stable $p = 1$ fixed point) [33,35]. The Ising metric space when seen as a percolation lattice has a chance to show a similar phenomenon, i.e., it has a fixed point at $T = T_c$. In this section, we concentrate on the critical temperature case $T = T_c$. The run times in this case are large due to a critical slowing down.

Figure 3(a) shows $\log\sqrt{\langle R^2\rangle}$ in terms of $\log N$ (the length of a polymer), which is linear with the well-defined slope $\nu = 0.761 \pm 0.002$ [for a definition of the exponents, see Eq. (12)]. This corresponds to $D_F = 1.314 \pm 0.003$. This is confirmed by the inset graph in which $\log N(L)$ has been sketched in terms of $\log L$ with the exponent $D_F^{\text{box counting}} = 1.317 \pm 0.002$. Therefore, Flory's relation predicts that the effective dimension of the critical Ising model is $\bar{d}^{\text{Flory}} = 1.94$. This should be compared with the obtained fractal dimension of the critical Ising model, which is $\bar{d} = \frac{187}{96} \simeq 1.948$ [8], for which Flory's approximation yields $\nu_{T_c}^{\text{Flory}} = \frac{288}{379} \simeq 0.760$ [51]. It is also notable that the exponents of the SAW on the critical uncorrelated percolation clusters are $\bar{d}_{p_c}^{(\text{2D percolation})} = \frac{91}{49} \simeq 1.857$, $\nu_{p_c}^{(\text{SAW on 2D percolation})} = \frac{147}{189} \simeq 0.77$, and $D_F^{(\text{SAW on 2D percolation})} \simeq 1.286$. Despite these results, one can hardly convince oneself that the exponents for the critical correlated and uncorrelated percolation lattices are meaningfully different. The excellent agreement between our Monte Carlo calculations and Flory's approximation encourages one to extend this theory to all temperatures.

The SLE diffusivity parameter is consistent with Flory's approximation. To study this, the winding angle test has been calculated [Fig. 3(b)]. It is seen that $\langle\theta^2\rangle$ behaves logarithmically with respect to N, which confirms the prediction of the SLE theory. The lower inset reveals that $\langle\theta\rangle = 0$ and the upper inset shows the Gaussian form of $p(\theta)$. The slope of the semilog plot is 1.76 ± 0.04, which is equivalent to the diffusivity parameter $\kappa = 2.26 \pm 0.07$ [see Eq. (14) for the definition]. Therefore, the universality class of $\text{SAW}_{T=T_c}$ is distinct from the one for the SAW on the regular lattice for which $\kappa = \frac{8}{3} \simeq 2.67$, i.e., $\delta\kappa \equiv \kappa_{T=0}^{\text{SAW}} - \kappa_{T=T_c}^{\text{SAW}} = 0.41 \pm 0.07$. The other exponents corresponding to a winding angle test are $\nu^{(\kappa)} = 0.778 \pm 0.005$, $D_F^{\kappa} = 1.284 \pm 0.008$, which are more or less in agreement with the above results. The results have been gathered in Table I.

B. Off-critical temperatures

The behavior of the SAW for all temperatures shows its overall structure. For all temperatures in range, the SAWs show power-law behaviors with some well-defined exponents. Note that $T = 0$ is the regular system. Our observations show that the exponents are rapidly saturated and become nearly constant with small fluctuations for low temperatures toward the $T = 0$ results. For example, Fig. 4(a) shows this behavior (see the upper inset) for the fractal dimension (possibly a finite-size effective exponent) obtained by the box-counting

TABLE I. The exponents of $\text{SAW}_{T=T_c}$ obtained by two methods. In the first row we have reported the results calculating ν and D_F by using the end-to-end distance (ETED) analysis, whereas the second row has been obtained by calculating κ using the winding angle (WA) distribution. The central charge has been calculated using the relation $c = 1 - \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$, and the t parameter has been defined by the relation $c = 1 - \frac{6}{t(t+1)}$.

	κ	ν	D_F	$\bar{d}^{\text{Flory}}$	c_κ	t
ETED	2.51(3)	0.761(2)	1.314(3)	1.94(1)	−0.32(1)	1.68(1)
WA	2.27(6)	0.778(5)	1.284(8)	1.85(2)	−0.96(4)	1.32

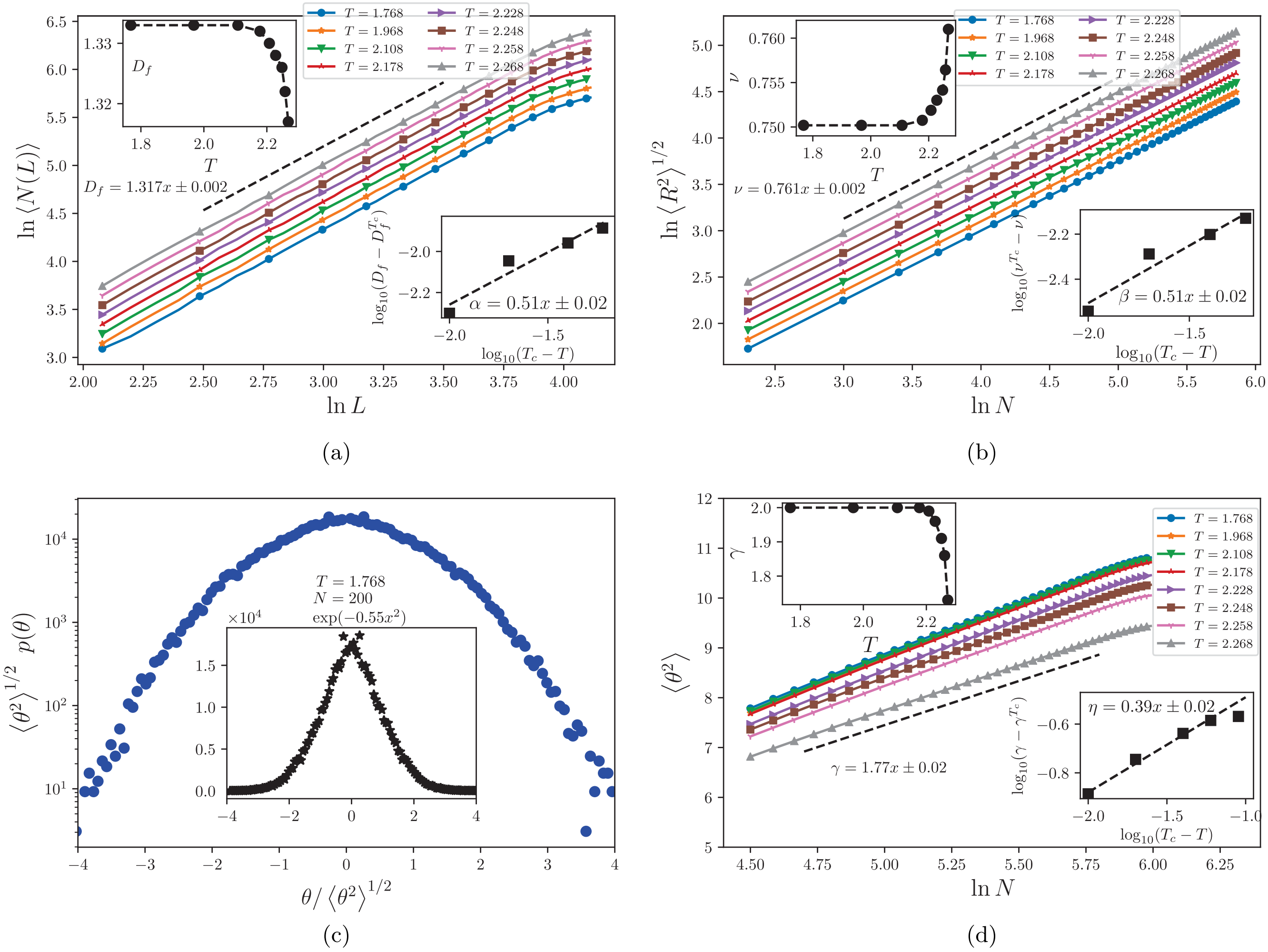


FIG. 4. (a) $\ln\langle N(L)\rangle$ in terms of $\ln L$ for various rates of temperature. Upper inset: D_F in terms of T; lower inset: power-law behavior of the fractal dimension. (b) $\ln\langle R^2\rangle^{\frac{1}{2}}$ in terms of $\ln N$ for various rates of temperature. Upper inset: the ν exponent in terms of T; lower inset: power-law behavior of ν. (c) The distribution of the winding angle θ for $N = 200$ and $T = 1.768$. (d) $\langle\theta^2\rangle$ in terms of $\ln N$ with the slope γ which is T-dependent. Upper inset: γ in terms of T; lower inset: power-law behavior of γ.

method $[D_F(T)]$. Interestingly, we have observed that in the vicinity of the critical temperature, some power-law behaviors arise in terms of $|T - T_c|$. The lower inset of this figure reveals that

$$\log_{10}|D_F(T) - D_F(T_c)| = \alpha \log_{10}\epsilon + \text{const}, \quad (15)$$

in which $\epsilon \equiv \frac{|T-T_c|}{T_c}$ and α is a new exponent whose value has been obtained using the least-squares estimator (LSE) method. The same feature is seen for the $\nu(T)$ exponent from the analysis of $\sqrt{\langle R^2\rangle}$ [Fig. 4(b)], in which the corresponding exponent β is the same as $\alpha = 0.51 \pm 0.02$, which is understood by the relation $D_F(T) = \frac{1}{\nu(T)}$.

Now let us consider the winding angle statistics. This quantity shows a Gaussian distribution with variance proportional to the logarithm of the length of chain. As an example, we have shown $\langle\theta^2\rangle^{\frac{1}{2}} p(\theta)$ in terms of $\theta/\langle\theta^2\rangle^{\frac{1}{2}}$ for $N = 200$ in Fig. 4(c), which is apparently Gaussian. We have calculated the slope of $\langle\theta^2\rangle$ in terms of $\ln N$ [the γ exponent in Fig. 4(d)], which is 2 for low temperatures. In the vicinity of T_c one can easily show that

$$\gamma(T) - \gamma(T_c) = A\epsilon^{\alpha} + B\epsilon^{2\alpha}. \quad (16)$$

For temperatures very close to T_c, the first term on the right-hand side is dominant, which leads to a power-law behavior [lower inset of Fig. 4(d)]. Therefore, the exponent for temperatures close to T_c should be more or less equal to α, as is seen. Note that the discrepancy comes from the nonlinear behavior for lower temperatures.

The main finding of this section is therefore an exponent α whose closest fractional value is $\frac{1}{2}$. Noting that the correlation length of the 2D Ising model ξ scales with temperature as $\xi \sim |T - T_c|^{-1}$, one finds the following scaling relation:

$$D_F^{\text{SAW}}(T) - D_F^{\text{SAW}}(T_c) \sim \frac{1}{\sqrt{\xi}}. \quad (17)$$

V. DISCUSSION AND CONCLUSION

Many features of random walks on random fractal lattices are known. Since these host systems have commonly been considered uncorrelated, introducing correlation in the host system is important and interesting, which leads to some nontrivial effects on the statistics of self-avoiding random walkers. In this paper, we have considered the SAW on the Ising-correlated site-diluted percolation lattice whose correlations are controlled by the temperature T. The importance of the correlations has been argued in terms of a stochastic differential equation. The enriched Rosenbluth method as well as winding angle statistics have been employed to obtain the critical exponents of the system in both the $T = T_c$ and $T < T_c$ cases. We found that the exponents at $T = T_c$ are in agreement with Flory's approximation. The winding angle analysis more or less showed the same features. This suggests that the ν exponent depends only on the effective fractal dimension of the host system. For temperatures in the range $T < T_c$, some other interesting power-law behaviors have arisen. The calculated exponents reveal that $D_F(T) - D_F(T_c) \sim \frac{1}{\sqrt{\xi(T)}}$, in which $\xi(T)$ is the correlation length of the off-critical Ising system. The winding angle statistics also confirms this result. If the findings of Kremer (that the disorder is irrelevant) are applicable to all disordered systems, then one finds that this relation for the fractal dimension is a finite-size effect, which should be investigated further in the literature.

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