Decrease of Fisher information and the information geometry of evolution equations for quantum mechanical probability amplitudes

Carlo Cafaro[1] and Paul M. Alsing[2]
[1]*SUNY Polytechnic Institute, 12203 Albany, New York, USA*
[2]*Air Force Research Laboratory, Information Directorate, 13441 Rome, New York, USA*

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The relevance of the concept of Fisher information is increasing in both statistical physics and quantum computing. From a statistical mechanical standpoint, the application of Fisher information in the kinetic theory of gases is characterized by its decrease along the solutions of the Boltzmann equation for Maxwellian molecules in the two-dimensional case. From a quantum mechanical standpoint, the output state in Grover's quantum search algorithm follows a geodesic path obtained from the Fubini-Study metric on the manifold of Hilbert-space rays. Additionally, Grover's algorithm is specified by constant Fisher information. In this paper, we present an information geometric characterization of the oscillatory or monotonic behavior of statistically parametrized squared probability amplitudes originating from special functional forms of the Fisher information function: constant, exponential decay, and power-law decay. Furthermore, for each case, we compute both the computational speed and the availability loss of the corresponding physical processes by exploiting a convenient Riemannian geometrization of useful thermodynamical concepts. Finally, we briefly comment on the possibility of using the proposed methods of information geometry to help identify a suitable trade-off between speed and thermodynamic efficiency in quantum search algorithms.

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I. INTRODUCTION

The importance of the concept of Fisher information is increasing in both classical and quantum settings, ranging from foundational aspects of theoretical physics, including statistical physics, to quantum computing. In Ref. [1] the Fisher information was regarded as a measure of the degree of disorder of an isolated statistical system. In particular, it was shown that by minimizing the Fisher information subject to suitable physical constraints, the resulting equilibrium probability density function satisfied the correct differential equations for the system (including, among others, Schrödinger's wave equation, the Klein-Gordon equation, and the Maxwell-Boltzmann law). Interestingly, in Ref. [1] it was suggested that the Fisher information specifies an arrow of time that points in the direction of decreasing accuracy for the determination of the mean value of the statistical parameter that specifies the system. Connections between the decrease of Fisher information and the second law of thermodynamics were, to some extent, explored in Refs. [2,3]. In Ref. [4] the concept of Fisher information was employed to present a systematic approach to deriving Lagrangians of relevance in physics. Of particular interest are the applications of the notion of Fisher information in quantum theory. For example, in Ref. [5] the principle of minimum Fisher information is used to derive the many-particle time-dependent Schrödinger equation. In Ref. [6] it was proposed that the classical Fisher information of a quantum observable is a measure of the robustness of the observable with respect to noise. Indeed, it was shown that Fisher information is proportional to the rate of entropy increase of the observable when the quantum system is subjected to a Gaussian diffusive process. In Ref. [7], in an effort to advance the information approach to physics by linking the classical Lagrangian approach to mechanics and the concept of Fisher information, a general notion of kinetic energy with respect to a parameter was introduced and its consequences were discussed. For an extended presentation of the role of Fisher information in physics, we refer to Ref. [8]. In addition to covering foundational aspects of physics, the use of Fisher information has also been extended to problems in statistical physics from a more applied perspective. The application of Fisher information to the kinetic theory of gases started with the investigation carried out by McKean in Ref. [9]. In that work, the monotonic decreasing behavior of the Fisher information was observed while studying a one-dimensional toy model of a Maxwellian gas. Following this line of investigation, the decrease of Fisher information along the solutions of the linear Fokker-Planck equation was reported by Toscani in Ref. [10]. In Ref. [11] it was shown that Fisher information also decreases along the Boltzmann equation for Maxwellian molecules in two dimensions. For a generalization of this finding extended to higher dimensions, we refer to Ref. [12]. Finally, studying the spatially homogeneous Landau equation for Maxwellian molecules, the nonincreasing behavior of the Fisher information was reported in Refs. [13–15].

From a quantum computing viewpoint, quantum Fisher information can be physically interpreted by observing that its square root is proportional to the statistical speed, that is, the instantaneous rate of change of the absolute statistical distance between two pure states in the Hilbert space (or, more generally, in the space of density operators for general mixtures) along the path parametrized by a given statistical parameter. The absolute statistical distance, in turn, is the maximum number of distinguishable states along the parametrized path, optimized over all quantum measurements. The role played by Fisher information in quantum information science is also becoming increasingly important. First, we recall that variational principle driven Riemannian geometrizations of Grover's original quantum search algorithms appear in both nonadiabatically [16] and adiabatically [17,18] constrained dynamical settings. In the latter framework, the link between the Bures metric of two density matrices [19] and the Riemann metric tensor underlying the adiabatic evolution is of particular significance. Second, we observe that it is known that there are quantum speed limits for either isolated quantum systems evolving (both nonadiabatically [20] and adiabatically [21]) according to a unitary dynamics or open quantum systems coupled to an environment [22–24]. In the latter case, as mentioned earlier, the Fisher information plays a key role in the geometric interpretation of quantum speed limits of dynamical evolutions in quantum computing based on the notion of statistical distance between quantum states, either pure or mixed [25,26]. In particular, when taking into consideration open-system dynamics where dissipative effects may occur, the temporal behavior of the Fisher information plays a key role in the determination of a bound to the speed of evolution of the quantum system [23]. Third, we point out that dissipation may have a constructive role in certain tasks of interest for quantum information processing [27]. For example, it is known that dissipation can be used in a constructive manner in quantum search problems [28–31]. For instance, in Ref. [31] it was shown that introducing dissipation into Grover's original quantum search algorithm has positive effects because it leads to a more robust search where the oscillations between target and nontarget items can be damped out.

The lack of a unifying theoretical framework for all the fundamental issues outlined in the first, second, and third points motivate us to pursue here an information geometric analysis wherein Riemannian geometry, probability calculus, and the statistical thermodynamical nature of Fisher information all simultaneously play a crucial role. An important finding of great utility in our proposed information geometric investigation is that the output state in Grover's quantum search algorithm follows a geodesic path obtained from the Fubini-Study metric on the manifold of Hilbert-space rays and, additionally, Grover's algorithm is specified by constant Fisher information [32–36].

In this paper, we use methods of information geometry to characterize the oscillatory or monotonic behavior of statistically parametrized squared probability amplitudes that correspond to suitably chosen functional forms of the Fisher information function: constant, exponential decay, and power-law decay. Moreover, for each case, we find both the computational speed and the availability loss of the corresponding physical processes by making use of a convenient Riemannian geometrization of useful thermodynamical concepts. Finally, we propose the use of methods of information geometry to help identify a suitable trade-off between speed and thermodynamic efficiency in quantum search algorithms.

The layout of the remainder of this paper is as follows. In Sec. II we introduce the concept of Fisher information in both classical and quantum information theory. In Sec. III we use the notion of Fisher information in order to quantify the concept of quantum distinguishability for both pure and mixed quantum states. In Sec. IV focusing on pure states and using variational calculus techniques, we present an explicit derivation of the information geometric evolution equations of quantum mechanical probability amplitudes for arbitrary forms of the Fisher information function. In Sec. V we apply the main results obtained in the previous section to three special scenarios: constant Fisher information, exponential decay and power-law decay. In particular, the oscillatory or monotonic behaviors of the statistically parameterized squared probability amplitudes are reported. In Sec. VI we discuss the link among physical systems, Fisher information functions, and geodesic paths on Riemannian manifolds. In Sec. VII we first review some basic material on a Riemannian geometric characterization of thermodynamic concepts. Special attention is devoted to the concepts of thermodynamic length and dissipated availability (or availability loss [37]) and their link with the notion of Fisher information. Then, for each of the three illustrative examples considered in Sec. V, we compute both the availability loss and the computational speed of the quantum process that corresponds to each selected functional form of the Fisher information. Finally, our concluding remarks appear in Sec. VIII.

II. FISHER INFORMATION

In this section, we briefly introduce the concept of Fisher information in both classical and quantum information-theoretic settings.

A. Classical framework

In the framework of classical information theory, the Fisher information $\mathcal{F}(\theta)$ quantifies the amount of information that an observable random variable X carries about an unknown parameter θ upon which the probability distribution $p(x|\theta) = p_\theta(x)$ depends. For a continuous random variable X, the classical Fisher information $\mathcal{F}(\theta)$ is defined as

$$\mathcal{F}_{\text{classical}}(\theta) \overset{\text{def}}{=} \left\langle \left[\frac{\partial \log p(x|\theta)}{\partial \theta} \right]^2 \right\rangle = \int p(x|\theta) \left[\frac{\partial \log p(x|\theta)}{\partial \theta} \right]^2 dx. \tag{1}$$

In this paper, log denotes the natural logarithmic function. We note that, by means of simple algebra, $\mathcal{F}(\theta)$ in Eq. (1) can be rewritten in terms of the probability amplitude $\sqrt{p(x|\theta)}$, a fundamental quantity in quantum theory:

$$\mathcal{F}(\theta) = 4 \int \left[\frac{\partial \sqrt{p(x|\theta)}}{\partial \theta} \right]^2 dx. \tag{2}$$

The quantity $\partial_\theta[\log p(x|\theta)]$ with $\partial_\theta \overset{\text{def}}{=} \frac{\partial}{\partial \theta}$ in Eq. (1) is known as the score while the probability distribution $p(x|\theta)$ is known as the likelihood function. Observe that, exploiting the normalization condition for $p(x|\theta)$, the expectation value of the score is zero:

$$\langle \partial_\theta \log p(x|\theta) \rangle = 0. \tag{3}$$

Therefore, from Eqs. (1) and (3), we conclude that the Fisher information $\mathcal{F}(\theta)$ can be regarded as the variance of the score function. For the sake of completeness, we note that for a discrete random variable X, the classical Fisher information $\mathcal{F}(\theta)$ is defined as

$$\mathcal{F}(\theta) \overset{\text{def}}{=} \sum_{i=1}^{n} p_i \left(\frac{\partial \log p_i}{\partial \theta}\right)^2 = \sum_{i=1}^{n} \frac{\dot{p}_i^2}{p_i}, \tag{4}$$

where $p_i = p_i(x|\theta)$ and $\dot{p}_i \overset{\text{def}}{=} \frac{\partial p_i}{\partial \theta}$. In anticipation of the formal comparison with the definition of the quantum Fisher information to be considered in the next subsection, observe that the score function $\partial_\theta[\log p_i(x|\theta)]$ in Eq. (4) satisfies the following relation:

$$\frac{1}{2}\left(p_i \frac{\partial \log p_i}{\partial \theta} + \frac{\partial \log p_i}{\partial \theta} p_i\right) = \frac{\partial p_i}{\partial \theta}. \tag{5}$$

For a detailed discussion of the intimate link between the Fisher information and the Shannon entropy, we refer to Ref. [38]. Finally, for an intriguing statistical mechanical interpretation of the Fisher information, we refer to Ref. [39].

B. Quantum framework

In quantum information theory, the concept of Fisher information can be introduced in the context of a single parameter estimation problem. This problem concerns the inference of the value of a coupling constant θ in the Hamiltonian $\mathcal{H}_\theta$,

$$\mathcal{H}_\theta \overset{\text{def}}{=} \hbar h_0 \theta, \tag{6}$$

of a probe system by observing the evolution of the probe due to $\mathcal{H}_\theta$. In Eq. (6), $\hbar$ is the reduced Planck constant, θ is assumed to have units of frequency, and h_0 is a dimensionless coupling Hamiltonian. The quantum Fisher information $\mathcal{F}_{\text{quantum}}(\theta)$ is defined as [40]

$$\mathcal{F}_{\text{quantum}}(\theta) \overset{\text{def}}{=} \max_{\{\mathcal{X}(x)\}} [\mathcal{F}(\theta)], \tag{7}$$

with $\mathcal{F}(\theta)$ given by

$$\mathcal{F}(\theta) \overset{\text{def}}{=} \int p(x|\theta)\left[\frac{\partial \log p(x|\theta)}{\partial \theta}\right]^2 dx. \tag{8}$$

The quantity $\{\mathcal{X}(x)\}$ in Eq. (7) denotes a generalized measurement where $\mathcal{X}(x)$ are non-negative, Hermitian operators that satisfy the completeness relation,

$$\int \mathcal{X}(x)\, dx = \mathbf{1}, \tag{9}$$

with $\mathbf{1}$ denoting the unit operator. Furthermore, the probability distribution $p(x|\theta)$ in Eq. (8) is defined as

$$p(x|\theta) \overset{\text{def}}{=} \text{tr}[\mathcal{X}(x)\rho(\theta)], \tag{10}$$

where x labels the outcomes of the measurement and it need not be a single continuous real variable. It can also be discrete or multivariate, for instance. The symbol "tr" in Eq. (10) denotes the usual trace operation. The quantity $\rho(\theta)$ in Eq. (10) denotes a curve on the space of density operators parametrized by the parameter θ. Observe that while the classical distinguishability metric satisfies the relation

$$ds_{\text{PD}}^2 = \mathcal{F}_{\text{classical}}(\theta)\, d\theta^2, \tag{11}$$

the quantum distinguishability metric fulfills the condition

$$ds_{\text{DO}}^2 = \mathcal{F}_{\text{quantum}}(\theta)\, d\theta^2. \tag{12}$$

Note that PD in Eq. (11) and DO in Eq. (12) denote probability distributions and density operators, respectively. Braunstein and Caves showed that $\mathcal{F}_{\text{quantum}}(\theta)$ can be written as [40]

$$\mathcal{F}_{\text{quantum}}(\theta) = \langle L^2(\theta)\rangle \overset{\text{def}}{=} \text{tr}[\rho(\theta)L^2(\theta)], \tag{13}$$

where L is the so-called symmetric logarithmic derivative operator. This operator is defined implicitly in terms of the following relation:

$$\frac{1}{2}(\rho L + L\rho) = \frac{\partial \rho}{\partial \theta}, \tag{14}$$

with

$$\frac{\partial \rho}{\partial \theta} = -i[T(\theta), \rho(\theta)], \tag{15}$$

where i is the imaginary unit. By replacing both the trace with the integral (or summation) and the density operator with the probability density function, we observe the formal analogies between Eqs. (4) and (13) and Eqs. (5) and (14), respectively. The quantity $T(\theta) = T_\theta$ in Eq. (15) is the Hermitian generator of displacements in the parameter θ defined as

$$T_\theta(t) \overset{\text{def}}{=} i\frac{\partial U_\theta(t)}{\partial \theta} U_\theta^\dagger(t). \tag{16}$$

The unitary evolution operator $U_\theta(t)$ is generated by the Hamiltonian $\mathcal{H}_\theta(t)$

$$\mathcal{H}_\theta(t)U_\theta(t) = i\hbar\frac{\partial U_\theta(t)}{\partial t}, \tag{17}$$

where

$$\rho_\theta(0) \to \rho_\theta(t) \overset{\text{def}}{=} U_\theta(t)\rho_\theta(0)U_\theta^\dagger(t). \tag{18}$$

The dagger symbol "†" in Eq. (16) denotes the usual Hermitian conjugate operation. Observe that if $\mathcal{H}_\theta(t) = \hbar h_0 \theta$ is a constant quantity, using Eq. (16), one finds that $T_\theta(t) = h_0 t$. Then, for pure states $\rho_\theta^2 = \rho_\theta$, it can be shown that [41]

$$\mathcal{F}_{\text{quantum}}(\theta) = 4\sigma_{T_\theta(t)}^2 = 4\left[\left\langle T_\theta^2(t)\right\rangle - \langle T_\theta(t)\rangle^2\right]. \tag{19}$$

For the sake of completeness, we remark that in the case of mixed states, the variance provides an upper bound on the quantum Fisher information [42]. Furthermore, in the case of time estimation, we have

$$\theta \mapsto t, \quad T_\theta(t) \mapsto \mathcal{H}(t), \quad \mathcal{F}_{\text{quantum}}(\theta) \mapsto \mathcal{F}_{\text{quantum}}(t), \tag{20}$$

and Eq. (19) becomes

$$\mathcal{F}_{\text{quantum}}(t) = \frac{4}{\hbar^2}\sigma_{\mathcal{H}(t)}^2 = \frac{4}{\hbar^2}[\langle \mathcal{H}^2(t)\rangle - \langle \mathcal{H}(t)\rangle^2]. \tag{21}$$

The quantum Fisher information $\mathcal{F}_{\text{quantum}}(\theta)$ can be interpreted in an efficient manner as the square of a statistical speed $v_{\mathcal{F}}$ [22,43]:

$$\mathcal{F}_{\text{quantum}}(\theta) = v_{\mathcal{F}}^2 \overset{\text{def}}{=} \left[\frac{dl(\theta)}{d\theta}\right]^2. \tag{22}$$

The quantity $v_{\mathcal{F}}$ in Eq. (22) denotes the rate of change with respect to the parameter θ of the absolute statistical distance $l(\theta)$ between two pure states (or, in general, density operators for general mixtures) in the Hilbert space. The absolute statistical distance $l(\theta)$ equals the maximum number of distinguishable states along the path $\rho(\theta) = \rho_\theta$ parametrized by θ, optimized over all possible generalized quantum measurements. For further details on the quantum Fisher information, we refer to Refs. [42,44–46].

III. INFORMATION GEOMETRY AND QUANTUM DISTINGUISHABILITY

In this section, we briefly present suitable information geometric measures of quantum distinguishability for both pure and mixed states.

A. Pure states

Classical probability distributions can be distinguished by means of the so-called classical Fisher-Rao information metric tensor $g_{ij}^{(\mathrm{FR})}(\theta)$ given by [47]

$$g_{ij}^{(\mathrm{FR})}(\theta) \overset{\mathrm{def}}{=} \int p(x|\theta)\frac{\partial \log [p(x|\theta)]}{\partial\theta^i}\frac{\partial \log [p(x|\theta)]}{\partial\theta^j}\,dx = 4\int \frac{\partial\sqrt{p(x|\theta)}}{\partial\theta^i}\frac{\partial\sqrt{p(x|\theta)}}{\partial\theta^j}\,dx. \tag{23}$$

One possible way of transitioning from the classical to the quantum settings is to replace the integral and the probability density function $p(x|\theta) = p_\theta(x) = p_\theta$ in Eq. (23) with the trace operation and the density operator ρ_θ, respectively. Then, the quantum version of $g_{ij}^{(\mathrm{FR})}(\theta)$ in Eq. (23) becomes the so-called Wigner-Yanase metric $g_{ij}^{(\mathrm{WY})}(\theta)$ [44,48],

$$g_{ij}^{(\mathrm{WY})}(\theta) = 4\mathrm{tr}\,[(\partial_i\sqrt{\rho_\theta})(\partial_j\sqrt{\rho_\theta})] = 4\mathrm{tr}[(\partial_i\rho_\theta)(\partial_j\rho_\theta)], \tag{24}$$

since $\rho_\theta = \rho_\theta^2$ with ρ_θ being a pure state. As pointed out in Ref. [44], quantum generalizations of the Fisher information are not unique. Observe that $\partial_i\rho_\theta$ in Eq. (24) can be written as

$$\partial_i\rho_\theta = \partial_i(|\psi_\theta\rangle\langle\psi_\theta|) = |\partial_i\psi_\theta\rangle\langle\psi_\theta| + |\psi_\theta\rangle\langle\partial_i\psi_\theta|. \tag{25}$$

Therefore, after some straightforward algebra, we find

$$(\partial_i\rho_\theta)(\partial_j\rho_\theta) = \langle\psi_\theta|\partial_j\psi_\theta\rangle|\partial_i\psi_\theta\rangle\langle\psi_\theta| + |\partial_i\psi_\theta\rangle\langle\partial_j\psi_\theta| + \langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle|\psi_\theta\rangle\langle\psi_\theta| + \langle\partial_i\psi_\theta|\psi_\theta\rangle|\psi_\theta\rangle\langle\partial_j\psi_\theta|. \tag{26}$$

Using Eq. (26), $\mathrm{tr}[(\partial_i\rho_\theta)(\partial_j\rho_\theta)]$ in Eq. (24) can be recast as

$$\mathrm{tr}[(\partial_i\rho_\theta)(\partial_j\rho_\theta)] = \langle\psi_\theta|(\partial_i\rho_\theta)(\partial_j\rho_\theta)|\psi_\theta\rangle = \langle\psi_\theta|\partial_j\psi_\theta\rangle\langle\psi_\theta|\partial_i\psi_\theta\rangle + \langle\psi_\theta|\partial_i\psi_\theta\rangle\langle\partial_j\psi_\theta|\psi_\theta\rangle + \langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle + \langle\partial_i\psi_\theta|\psi_\theta\rangle\langle\partial_j\psi_\theta|\psi_\theta\rangle. \tag{27}$$

Using the normalization condition $\langle\psi_\theta|\psi_\theta\rangle = 1$, we have $\langle\partial_j\psi_\theta|\psi_\theta\rangle = -\langle\psi_\theta|\partial_j\psi_\theta\rangle$. Therefore, $\mathrm{tr}[(\partial_i\rho_\theta)(\partial_j\rho_\theta)]$ in Eq. (27) becomes

$$\mathrm{tr}[(\partial_i\rho_\theta)(\partial_j\rho_\theta)] = \langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle + \langle\partial_i\psi_\theta|\psi_\theta\rangle\langle\partial_j\psi_\theta|\psi_\theta\rangle. \tag{28}$$

Following the line of reasoning presented in Ref. [49], we observe that we can write the inner product $\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle$ as

$$\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle = \gamma_{ij} + i\sigma_{ij}, \tag{29}$$

where

$$\gamma_{ij} \overset{\mathrm{def}}{=} \mathrm{Re}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle] \quad \text{and} \quad \sigma_{ij} \overset{\mathrm{def}}{=} \mathrm{Im}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle], \tag{30}$$

respectively. Note that Re (z) and Im (z) denote the real and the imaginary part of a complex quantity z, respectively. Observe that γ_{ij} and σ_{ij} are symmetric and antisymmetric quantities, respectively. Indeed,

$$\gamma_{ji} = \mathrm{Re}[\langle\partial_j\psi_\theta|\partial_i\psi_\theta\rangle] = \mathrm{Re}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle^*] = \mathrm{Re}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle] = \gamma_{ij} \tag{31}$$

and

$$\sigma_{ji} = \mathrm{Im}[\langle\partial_j\psi_\theta|\partial_i\psi_\theta\rangle] = \mathrm{Im}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle^*] = -\mathrm{Im}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle] = -\sigma_{ij}. \tag{32}$$

Since $\sigma_{ij} = -\sigma_{ji}$, $\sigma_{ij}d\theta^i d\theta^j = 0$. Finally, by using Eqs. (28), (29), and (30), $g_{ij}^{(\mathrm{WY})}(\theta)$ in Eq. (24) becomes

$$g_{ij}^{(\mathrm{WY})}(\theta) = 4\{\mathrm{Re}[\langle\partial_i\psi_\theta|\partial_j\psi_\theta\rangle] + \langle\partial_i\psi_\theta|\psi_\theta\rangle\langle\partial_j\psi_\theta|\psi_\theta\rangle\}. \tag{33}$$

For the sake of completeness, we recall that

$$g_{ij}^{(\mathrm{FS})}(\theta) = \tfrac{1}{4}g_{ij}^{(\mathrm{WY})}(\theta), \tag{34}$$

where $g_{ij}^{(\mathrm{FS})}(\theta)$ denotes the Fubini-Study metric. The infinitesimal line element ds_{FS}^2 corresponding to the Fubini-Study metric tensor $g_{ij}^{(\mathrm{FS})}(\theta)$ is given by

$$ds_{\mathrm{FS}}^2 = g_{ij}^{(\mathrm{FS})}(\theta)\,d\theta^i\,d\theta^j. \tag{35}$$

The metric tensor components $g_{ij}^{(\mathrm{FS})}(\theta)$ must be such that [49] (1) they transform properly under a change of the coordinates $\theta \to \theta' = \theta'(\theta)$, (2) they are invariant under gauge transformations, $\psi(\theta) \to \psi'(\theta) = e^{i\alpha(\theta)}\psi(\theta)$, and (3) they define a positive definite metric tensor. Imposing these conditions, it can be shown that ds_{FS}^2 can be defined as

$$ds_{\mathrm{FS}}^2 \overset{\mathrm{def}}{=} \|d\psi\|^2 - |\langle\psi|d\psi\rangle|^2 = \langle d\psi|d\psi\rangle - \langle d\psi|\psi\rangle\langle\psi|d\psi\rangle = \langle d\psi_\perp|d\psi_\perp\rangle = 1 - |\langle\psi'|\psi\rangle|^2, \tag{36}$$

where $|d\psi\rangle$ and $|d\psi_\perp\rangle$ are given by

$$|d\psi\rangle \overset{\mathrm{def}}{=} |\psi'\rangle - |\psi\rangle, \quad \text{and} \quad |d\psi_\perp\rangle \overset{\mathrm{def}}{=} |d\psi\rangle - |\psi\rangle\langle\psi|d\psi\rangle, \tag{37}$$

respectively. For the sake of clarity, we remark that $|\psi\rangle$ and $|\psi'\rangle$ are two neighboring normalized pure states, $|d\psi\rangle$ is the difference between them, and $|d\psi_\perp\rangle$ is the projection of $|d\psi\rangle$ orthogonal to $|\psi\rangle$. Expanding $|\psi\rangle$ and $|\psi'\rangle$ with respect to an orthonormal basis $\{|m\rangle\}$ with $m \in \{1,\ldots, N\}$, we obtain

$$|\psi\rangle \overset{\mathrm{def}}{=} \sum_{m=1}^{N}\sqrt{p_m(\theta)}e^{i\phi_m(\theta)}|m\rangle \quad \text{and} \quad |\psi'\rangle \overset{\mathrm{def}}{=} \sum_{m=1}^{N}\sqrt{p_m + dp_m}e^{i(\phi_m + d\phi_m)}|m\rangle, \tag{38}$$

respectively. Substituting Eq. (38) into Eq. (36) and recalling Eq. (34), after some tedious but straightforward algebra [35], the infinitesimal Wigner-Yanase line element $ds^2_{\text{WY}} = 4ds^2_{\text{FS}}$ becomes

$$ds^2_{\text{WY}} = \left\{\sum_{m=1}^{N} \frac{\dot{p}_m^2}{p_m} + 4\left[\sum_{m=1}^{N} p_m \dot{\phi}_m^2 - \left(\sum_{m=1}^{N} p_m \dot{\phi}_m\right)^2\right]\right\} d\theta^2, \tag{39}$$

where

$$\dot{p}_m \overset{\text{def}}{=} \frac{dp_m}{d\theta} \quad \text{and} \quad \dot{\phi}_m \overset{\text{def}}{=} \frac{d\phi_m}{d\theta}. \tag{40}$$

In the next subsection, we move our discussion from pure states to density operators.

B. Density operators

In the case of density operators, one needs to consider the quantum analog $\mathcal{M}_{\vec{\rho}}$ of the probability simplex [50,51],

$$\mathcal{M}_{\vec{\rho}} \overset{\text{def}}{=} \left\{\vec{\rho} \in \mathcal{L}(\mathcal{H}) : \vec{\rho} \overset{\text{def}}{=} \sum_{i,j=1}^{N} \rho^{ij} \vec{e}_{ij}, \vec{\rho} = \vec{\rho}^{\dagger}, \text{tr}(\vec{\rho}) = 1, \vec{\rho} \geqslant 0\right\}, \tag{41}$$

where $\mathcal{L}(\mathcal{H})$ denotes the linear space of all linear operators on a N-dimensional Hilbert space $\mathcal{H}$ with density operators $\vec{\rho}$ written as vectors in $\mathcal{L}(\mathcal{H})$. The space $\mathcal{M}_{\vec{\rho}}$ in Eq. (41) is an (N^2-1)-dimensional *real* manifold with nontrivial boundary. An arbitrary linear operator vector $\vec{V}$ on $\mathcal{H}$ can be decomposed with respect to an operator vector basis $\vec{e}_{ij} \overset{\text{def}}{=} |i\rangle\langle j|$ with i, $j = 1, \ldots, N$ as

$$\vec{V} = \sum_{i,j=1}^{N} \langle i|\vec{V}|j\rangle \vec{e}_{ij} = \sum_{i,j=1}^{N} V^{ij} \vec{e}_{ij}. \tag{42}$$

The tangent space at $\vec{\rho}$ is characterized by an (N^2-1)-dimensional *real* vector space of traceless Hermitian operators $\vec{T}$,

$$\vec{T} = \sum_{i,j=1}^{N} T^{ij} \vec{e}_{ij}, \tag{43}$$

with $\text{tr}(\vec{T}) = 0$. The action of 1-forms $\tilde{F}$, expanded in terms of the dual basis $\tilde{\omega}^{ji} \overset{\text{def}}{=} |i\rangle\langle j|$,

$$\tilde{F} \overset{\text{def}}{=} \sum_{i,j=1}^{N} F_{ij} \tilde{\omega}^{ji}, \tag{44}$$

on density operators $\vec{\rho}$ is defined as

$$\begin{aligned}\tilde{F}(\vec{\rho}) &\overset{\text{def}}{=} \langle \tilde{F}, \vec{\rho}\rangle = \sum_{i,j,l,k=1}^{N} F_{ij} \rho^{lk} \langle \tilde{\omega}^{ji}, \vec{e}_{lk}\rangle \\ &= \sum_{i,j,l,k=1}^{N} F_{ij} \rho^{lk} \delta_l^j \delta_k^i = \sum_{i,j=1}^{N} F_{ij} \rho^{ji} = \text{tr}(\tilde{F}\vec{\rho}) \overset{\text{def}}{=} \langle \tilde{F}\rangle.\end{aligned} \tag{45}$$

Therefore, from Eq. (45), a Hermitian 1-form $\tilde{F} = \tilde{F}^{\dagger}$ is an ordinary quantum observable with $\langle \tilde{F}, \vec{\rho}\rangle = \langle \tilde{F}\rangle$. A metric structure $\mathbf{g}_{\vec{\rho}}(\cdot,\cdot)$ on the manifold $\mathcal{M}_{\vec{\rho}}$ can be introduced by specifying the action of the metric on a pair of 1-forms $\tilde{A}$ and $\tilde{B}$ as

$$\begin{aligned}\mathbf{g}_{\vec{\rho}}(\tilde{A}, \tilde{B}) &\overset{\text{def}}{=} \left\langle \frac{\tilde{A}\tilde{B} + \tilde{B}\tilde{A}}{2}\right\rangle = \text{tr}\left[\left(\frac{\tilde{A}\tilde{B} + \tilde{B}\tilde{A}}{2}\right)\vec{\rho}\right] \\ &= \text{tr}\left[\frac{\tilde{A}}{2}(\vec{\rho}\tilde{B} + \tilde{B}\vec{\rho})\right] = \langle \tilde{A}, \mathcal{R}_{\vec{\rho}}(\tilde{B})\rangle,\end{aligned} \tag{46}$$

where $\mathcal{R}_{\vec{\rho}}(\tilde{B})$ is the raising operator that maps 1-forms (lower covariant components) to vectors (upper contravariant components) [50]:

$$\mathcal{R}_{\vec{\rho}}(\tilde{B}) \overset{\text{def}}{=} \frac{\vec{\rho}\tilde{B} + \tilde{B}\vec{\rho}}{2}. \tag{47}$$

The metric $\mathbf{g}_{\vec{\rho}}(\tilde{A}, \tilde{B})$ in Eq. (46) is constructed in terms of statistical correlations of quantum observables. By means of the lowering operator $\mathcal{L}_{\vec{\rho}}(\vec{A})$ that maps vectors to 1-forms [50],

$$\mathcal{L}_{\vec{\rho}}(\vec{A}) \overset{\text{def}}{=} \mathcal{R}_{\vec{\rho}}^{-1}(\vec{A}), \tag{48}$$

the action of the metric tensor $g_{\vec{\rho}}(\cdot,\cdot)$ on a pair of vectors $\vec{A}$ and $\vec{B}$ can be defined as

$$\mathbf{g}_{\vec{\rho}}(\vec{A}, \vec{B}) \overset{\text{def}}{=} \langle \mathcal{L}_{\vec{\rho}}(\vec{A}), \vec{B}\rangle = \text{tr}[\vec{B}\mathcal{L}_{\vec{\rho}}(\vec{A})]. \tag{49}$$

Finally, the quantum line element for density operators is given by

$$ds^2_{\text{DO}} \overset{\text{def}}{=} \mathbf{g}_{\vec{\rho}}(d\vec{\rho}, d\vec{\rho}), \tag{50}$$

where $d\vec{\rho} \overset{\text{def}}{=} (\vec{\rho} + d\vec{\rho}) - \vec{\rho}$ with

$$\vec{\rho} \overset{\text{def}}{=} \sum_{j=1}^{N} p^j |j\rangle\langle j| \quad \text{and} \quad \vec{\rho} + d\vec{\rho} \overset{\text{def}}{=} \sum_{j=1}^{N} (p^j + dp^j)|j'\rangle\langle j'|, \tag{51}$$

and $|j'\rangle \overset{\text{def}}{=} e^{id\theta h}|j\rangle$. The quantity $e^{id\theta h}$ denotes an infinitesimal unitary transformation on the orthonormal basis that diagonalizes $\vec{\rho}$, while h is the Hermitian operator that generates the infinitesimal unitary basis transformations. After some algebra, $d\vec{\rho}$ can be rewritten as

$$d\vec{\rho} \overset{\text{def}}{=} \sum_{j=1}^{N} dp^j|j\rangle\langle j| + id\theta \sum_{j,\,k=1}^{N}(p^j - p^k)h_{kj}|k\rangle\langle j|, \tag{52}$$

where $h_{kj} \overset{\text{def}}{=} \langle k|h|j\rangle$. Finally, by substituting Eq. (52) into Eq. (50), ds^2_{DO} becomes [50]

$$ds^2_{\text{DO}} = \mathbf{g}_{\vec{\rho}}(d\vec{\rho}, d\vec{\rho}) \overset{\text{def}}{=} \text{tr}[d\vec{\rho}\mathcal{L}_{\vec{\rho}}(d\vec{\rho})] = \left[\sum_{k=1}^{N}\frac{1}{p^k}\left(\frac{dp^k}{d\theta}\right)^2 + 2\sum_{j\neq k}\frac{(p^j - p^k)^2}{(p^j + p^k)}|h_{jk}|^2\right]d\theta^2. \tag{53}$$

Notice that the quantum line element in Eq. (53) is identical to the distinguishability metric for density operators obtained by Braunstein and Caves in Ref. [40] by optimizing over all generalized quantum measurements for distinguishing among neighboring quantum states. We also point out that for pure states, the line element in Eq. (53) becomes the usual Fubini-Study metric, a gauge-invariant metric on the *complex* projective Hilbert space [49]. For the sake of completeness, we also point out that ds^2_{DO} in Eq. (53) was originally regarded as the infinitesimal form of a distance between density operators in Ref. [52] and interpreted as a generalization of transition probabilities to mixed states in Ref. [53]. Indeed, the Bures distance $d_{\text{Bures}}(\rho_1, \rho_2)$ between two mixed density operators ρ_1 and ρ_2 is given by [52]

$$d_{\text{Bures}}(\rho_1, \rho_2) \overset{\text{def}}{=} \sqrt{2}[1 - \mathcal{F}(\rho_1, \rho_2)]^{\frac{1}{2}}, \tag{54}$$

where $\mathcal{F}(\rho_1, \rho_2)$ is the so-called Uhlmann fidelity defined as [53]

$$\mathcal{F}(\rho_1, \rho_2) \overset{\text{def}}{=} \text{tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}. \tag{55}$$

The infinitesimal Bures line element ds^2_{Bures} can be expressed in terms of the Bures distance between two infinitesimally close density matrices as [19]

$$ds^2_{\text{Bures}} \overset{\text{def}}{=} d^2_{\text{Bures}}(\rho, \rho + d\rho) = \frac{1}{2}\sum_{i,j}\frac{|\langle i|d\rho|j\rangle|^2}{p^i + p^j}, \tag{56}$$

where ρ and $d\rho$ in Eq. (56) are defined as

$$\rho \overset{\text{def}}{=} \sum_{i=1}^{N} p^i|i\rangle\langle i| \quad \text{and}$$

$$d\rho \overset{\text{def}}{=} \sum_{i=1}^{N} dp^i|i\rangle\langle i| + \sum_{i=1}^{N} p^i|di\rangle\langle i| + \sum_{i=1}^{N} p^i|i\rangle\langle di|, \tag{57}$$

respectively. By substituting the expression for $d\rho$ in Eq. (57) into Eq. (56), ds^2_{Bures} becomes

$$ds^2_{\text{Bures}} = \frac{1}{4}\left[\sum_{i=1}^{N}\frac{(dp^i)^2}{p^i} + 2\sum_{i\neq j}\frac{(p^i - p^j)^2}{p^i + p^j}|\langle i|dj\rangle|^2\right]. \tag{58}$$

Observe that, modulo an irrelevant constant factor, ds^2_{Bures} in Eq. (58) and ds^2_{DO} in Eq. (53) are identical. Similarly, note that when $[\rho, d\rho] \overset{\text{def}}{=} \rho d\rho - d\rho\rho = 0$, the Bures metric essentially becomes the Fisher-Rao information metric. Furthermore, for pure states, $\rho \overset{\text{def}}{=} |\psi\rangle\langle\psi|$ with $d\rho \overset{\text{def}}{=} |d\psi\rangle\langle\psi| + |\psi\rangle\langle d\psi|$, the Bures metric becomes the Fubini-Study metric,

$$\begin{aligned} ds^2_{\text{Bures}} &= \sum_{i\in\ker(\rho)}|\langle d\psi|i\rangle|^2 = \langle d\psi|(1 - |\psi\rangle\langle\psi|)|d\psi\rangle \\ &= \langle d\psi|d\psi\rangle - \langle d\psi|\psi\rangle\langle\psi|d\psi\rangle = ds^2_{\text{FS}}. \end{aligned} \tag{59}$$

For a more detailed presentation of this material, we refer to Refs. [50,54]. As a final remark, we point out that the distinguishability of mixed density operators can be quantified in terms of several metrics within the information geometric framework. For further details on this specific issue, we refer to Refs. [47,55,56].

IV. GEODESIC PATHS IN THE PROJECTIVE SPACE

In this section, after pointing out our working assumptions, we use methods of variational calculus to extremize the action functional expressed in terms of the infinitesimal Fubini-Study line element. The extremization procedure leads to determination of the geodesic paths followed by the quantum-mechanical probability amplitudes of pure quantum states.

A. Variance of the phase changes

Recall that given two normalized pure states $|\psi\rangle$ and $|\psi'\rangle$ as defined in Eq. (38), the Fubini-Study metric becomes

$$ds^2_{\text{FS}} = \frac{1}{4}\left\{\sum_{m=1}^{N}\frac{\dot{p}_m^2}{p_m} + 4\left[\sum_{m=1}^{N}p_m\dot{\phi}_m^2 - \left(\sum_{m=1}^{N}p_m\dot{\phi}_m\right)^2\right]\right\}d\theta^2. \tag{60}$$

In terms of the variance of phase changes $\sigma^2_{\dot{\phi}}$,

$$\sigma^2_{\dot{\phi}} \overset{\text{def}}{=} \sum_{m=1}^{N}p_m\dot{\phi}_m^2 - \left(\sum_{m=1}^{N}p_m\dot{\phi}_m\right)^2, \tag{61}$$

with $0 \leqslant \sigma^2_{\dot{\phi}}$, the metric in Eq. (60) becomes

$$ds^2_{\text{FS}} = \frac{1}{4}\left[\sum_{m=1}^{N}\frac{\dot{p}_m^2}{p_m} + 4\sigma^2_{\dot{\phi}}\right]d\theta^2. \tag{62}$$

Following a remark made in Ref. [40], we point out that a suitable choice of an orthonormal basis $\{|k\rangle\}$ makes $\sigma^2_{\dot{\phi}}$ equal to zero. Specifically, the condition that must be satisfied by $\{|k\rangle\}$ is that for any $1 \leqslant k \leqslant N$,

$$p_k\left(d\phi_k - \sum_{j=1}^{N} p_j d\phi_j\right) = 0. \tag{63}$$

Indeed, after some simple algebraic manipulations, Eq. (63) can be written as

$$\sum_{k=1}^{N} p_k (d\phi_k)^2 - \left(\sum_{k=1}^{N} p_k d\phi_k\right)^2 = 0, \tag{64}$$

that is, using Eq. (61), $\sigma_{\dot{\phi}}^2 = 0$. In particular, any basis $\{|k\rangle\}$ that satisfies Eq. (63) is such that its basis vectors also satisfy the relation

$$\mathrm{Im}\,(\langle k|\psi\rangle\langle k|d\psi_{\perp}\rangle) = 0, \tag{65}$$

with $|\psi\rangle$ and $|d\psi_{\perp}\rangle$ given in Eqs. (38) and (37), respectively. To verify the relation in Eq. (65), recall that $|d\psi\rangle \overset{\text{def}}{=} |\psi'\rangle - |\psi\rangle$, $|d\psi_{\perp}\rangle \overset{\text{def}}{=} |d\psi\rangle - |\psi\rangle\langle\psi|d\psi\rangle$, where $|\psi\rangle$ and $|\psi'\rangle$ can be written as

$$|\psi\rangle \overset{\text{def}}{=} \sum_{j=1}^{N} \sqrt{p_j} e^{-i\phi_j} |j\rangle \tag{66}$$

and

$$|\psi'\rangle \overset{\text{def}}{=} \sum_{j=1}^{N} \sqrt{p_j + dp_j} e^{-i(\phi_j + d\phi_j)} |j\rangle, \tag{67}$$

respectively. Using these previous four relations, $\langle\psi|k\rangle\langle k|d\psi_{\perp}\rangle$ becomes

$$\begin{aligned}\langle\psi|k\rangle\langle k|d\psi_{\perp}\rangle &= \sqrt{p_k} e^{-i\phi_k}[\langle k|\psi'\rangle - \langle k|\psi\rangle\langle\psi|\psi'\rangle] \\ &= \sqrt{p_k} e^{-i\phi_k}\langle k|\psi'\rangle - \sqrt{p_k} e^{-i\phi_k}\langle k|\psi\rangle\langle\psi|\psi'\rangle.\end{aligned} \tag{68}$$

Observe that $\langle\psi|k\rangle = \sqrt{p_k} e^{-i\phi_k}$, $\langle k|\psi\rangle = \langle\psi|k\rangle^* = \sqrt{p_k} e^{i\phi_k}$ and, limiting our attention to the second order expansion of $|\psi'\rangle$ with respect to $d\phi_k$ and dp_k, the expressions for $\langle\psi|\psi'\rangle$ and $\langle k|\psi'\rangle$ in Eq. (68) become

$$\begin{aligned}\langle\psi|\psi'\rangle = 1 &- \frac{1}{8}\sum_{j=1}^{N} \frac{(dp_j)^2}{p_j} + i\sum_{j=1}^{N} p_j d\phi_j \\ &+ \frac{i}{2}\sum_{j=1}^{N} dp_j\, d\phi_j - \frac{1}{2}\sum_{j=1}^{N} p_j (d\phi_j)^2\end{aligned} \tag{69}$$

and

$$\begin{aligned}\langle k|\psi'\rangle = \sqrt{p_k} e^{i\phi_k}\bigg[1 + i(d\phi_k) - \frac{1}{2}(d\phi_k)^2 + \frac{1}{2}\frac{dp_k}{p_k} \\ + \frac{i}{2}\frac{dp_k\, d\phi_k}{p_k} - \frac{1}{8}\frac{(dp_k)^2}{p_k^2}\bigg],\end{aligned} \tag{70}$$

respectively. Substituting Eqs. (69) and (70) into Eq. (68), after some algebra, we obtain Eq. (65). In conclusion, it is always possible to assume $\sigma_{\dot{\phi}}^2 = 0$ given an appropriate choice of basis $\{|k\rangle\}$. In what follows, we assume to be working under such a condition.

B. Extremizing the action functional

For the sake of convenience, recall that the relation between the Wigner-Yanase and the Fubini-Study infinitesimal line elements is $ds_{\mathrm{WY}}^2 = 4 ds_{\mathrm{FS}}^2$, where ds_{FS}^2 is given by

$$ds_{\mathrm{FS}}^2 \overset{\text{def}}{=} \frac{1}{4}\left\{\sum_{k=1}^{N} \frac{\dot{p}_k^2}{p_k} + 4\left[\sum_{k=1}^{N} \dot{\phi}_k^2 p_k - \left(\sum_{k=1}^{N} \dot{\phi}_k p_k\right)^2\right]\right\} d\theta^2, \tag{71}$$

with $\dot{p}_k$ and $\dot{\phi}_k$ defined in Eq. (40). Observe that

$$\sum_{k=1}^{N} \dot{\phi}_k^2 p_k - \left(\sum_{k=1}^{N} \dot{\phi}_k p_k\right)^2 = \langle\dot{\phi}^2\rangle - \langle\dot{\phi}\rangle^2 = \sigma_{\dot{\phi}}^2, \tag{72}$$

where $\langle\cdot\rangle$ denotes the averaging operation and $\phi \overset{\text{def}}{=} (\phi_1, \ldots, \phi_N)$. Therefore, using Eq. (72), Eq. (71) can be rewritten as

$$ds_{\mathrm{FS}}^2 = \tfrac{1}{4}\{\mathcal{F}(\theta) + 4\sigma_{\dot{\phi}}^2\} d\theta^2, \tag{73}$$

where $\mathcal{F}(\theta)$ denotes the Fisher information function defined as

$$\mathcal{F}(\theta) \overset{\text{def}}{=} \sum_{k=1}^{N} \frac{\dot{p}_k^2}{p_k}. \tag{74}$$

As pointed out earlier, we assume $\sigma_{\dot{\phi}}^2 = 0$. Then, the action functional to consider is given by

$$\mathcal{S} \overset{\text{def}}{=} \int ds_{\mathrm{FS}} = \int \sqrt{ds_{\mathrm{FS}}^2} = \frac{1}{2}\int \mathcal{F}^{\frac{1}{2}}(\theta) d\theta, \tag{75}$$

that is, more formally,

$$\mathcal{S}[p] \overset{\text{def}}{=} \int \mathcal{L}(\dot{p}, p, \theta) d\theta, \tag{76}$$

where $p \overset{\text{def}}{=} (p_1, \ldots, p_N)$ and $\mathcal{L}(\dot{p}, p, \theta)$ denotes the Lagrangian-like function defined as

$$\mathcal{L}(\dot{p}, p, \theta) \overset{\text{def}}{=} \tfrac{1}{2}\mathcal{F}^{\frac{1}{2}}(\theta), \tag{77}$$

with $\mathcal{F}(\theta)$ defined in Eq. (74). We wish to determine the probability paths $p \overset{\text{def}}{=} (p_1, \ldots, p_N)$ with $p_k = p_k(\theta)$ for any $1 \leqslant k \leqslant N$ that make the action functional $\mathcal{S}[p]$ in Eq. (76) stationary subject to the conservation of the probability condition

$$\sum_{k=1}^{N} p_k = 1. \tag{78}$$

Generally speaking, an action functional $\mathcal{S}[p] \overset{\text{def}}{=} \int \mathcal{L}(\dot{p}, p, \theta) d\theta$ has a stationary value if $\delta\mathcal{S} = 0$. It happens that for any $\bar{k}$ with $1 \leqslant \bar{k} \leqslant N$, we have

$$\begin{aligned}\delta\mathcal{S} &= \int \left(\frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\delta\dot{p}_{\bar{k}} + \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}}\delta p_{\bar{k}}\right) d\theta \\ &= \int \left[\frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\frac{d(\delta p_{\bar{k}})}{d\theta} + \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}}\delta p_{\bar{k}}\right] d\theta \\ &= \int \frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\frac{d(\delta p_{\bar{k}})}{d\theta} d\theta + \int \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}}\delta p_{\bar{k}}\, d\theta.\end{aligned} \tag{79}$$

Integrating by parts the first term in the last line of Eq. (79), we find

$$\delta\mathcal{S} = \frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\delta p_{\bar{k}} - \int \frac{d}{d\theta}\left(\frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\right)\delta p_{\bar{k}}\,d\theta + \int \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}}\delta p_{\bar{k}}\,d\theta. \tag{80}$$

We point out that, in the variational calculus scheme being considered here, only the probability paths $p_{\bar{k}}(\theta)$ are being varied while the endpoints are being kept fix, that is, $\delta p_{\bar{k}}(\theta_{\mathrm{i}}) = \delta p_{\bar{k}}(\theta_{\mathrm{f}}) = 0$. Therefore, the condition $\delta\mathcal{S} = 0$ becomes

$$\int\left[\frac{d}{d\theta}\left(\frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\right) - \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}}\right]\delta p_{\bar{k}}\,d\theta = 0. \tag{81}$$

Since Eq. (81) must be satisfied for any small change $\delta p_{\bar{k}}$, the condition $\delta\mathcal{S} = 0$ leads to the so-called Euler-Lagrange differential equations:

$$\frac{d}{d\theta}\left(\frac{\partial\mathcal{L}}{\partial\dot{p}_{\bar{k}}}\right) - \frac{\partial\mathcal{L}}{\partial p_{\bar{k}}} = 0. \tag{82}$$

Returning to our specific problem, we wish to find the stationary value of the action functional

$$\mathcal{S}[p] = \int\left[\mathcal{L}(\dot{p},p,\theta) - \lambda\left(\sum_{k=1}^{N} p_k - 1\right)\right]d\theta, \tag{83}$$

where λ in Eq. (83) is the Lagrange multiplier coefficient and $\mathcal{L}(\dot{p},p,\theta)$ is the Lagrangian-like function given in Eq. (77). Consider the following change of variables [57]:

$$p_k(\theta) \to q_k(\theta), \quad \text{with} \quad p_k(\theta) \overset{\text{def}}{=} q_k^2(\theta). \tag{84}$$

In terms of the probability amplitude variables $q_k(\theta)$, Eqs. (74) and (78) become

$$\mathcal{F}(\theta) = 4\sum_{k=1}^{N}\dot{q}_k^2 \tag{85}$$

and

$$\sum_{k=1}^{N} q_k^2 = 1, \tag{86}$$

respectively. Using Eqs. (85) and (86), the action functional in Eq. (83) becomes

$$\mathcal{S}_{\text{new}}[q] = \int \mathcal{L}_{\text{new}}(\dot{q},q,\theta)d\theta, \tag{87}$$

where $q \overset{\text{def}}{=} (q_1,\ldots,q_N)$, and

$$\mathcal{L}_{\text{new}}(\dot{q},q,\theta) \overset{\text{def}}{=} \left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}} - \lambda\left(\sum_{k=1}^{N} q_k^2 - 1\right). \tag{88}$$

Following the line of reasoning outlined before, we find that

$$\delta\mathcal{S}_{\text{new}} = \frac{\delta\mathcal{S}_{\text{new}}}{\delta q_{\bar{k}}}\delta q_{\bar{k}} = 0, \quad \forall\ 1 \leqslant \bar{k} \leqslant N \tag{89}$$

leads to the following Euler-Lagrange differential equations:

$$\frac{d}{d\theta}\left(\frac{\partial\mathcal{L}_{\text{new}}}{\partial\dot{q}_{\bar{k}}}\right) - \frac{\partial\mathcal{L}_{\text{new}}}{\partial q_{\bar{k}}} = 0. \tag{90}$$

A straightforward computation yields the following three equalities:

$$\frac{\partial\mathcal{L}_{\text{new}}}{\partial\dot{q}_{\bar{k}}} = \frac{\dot{q}_{\bar{k}}}{\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}}},$$
$$\frac{d}{d\theta}\left(\frac{\partial\mathcal{L}_{\text{new}}}{\partial\dot{q}_{\bar{k}}}\right) = \frac{\ddot{q}_{\bar{k}}}{\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}}} - \frac{\dot{q}_{\bar{k}}^2\ddot{q}_{\bar{k}}}{\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{3}{2}}},$$
$$\frac{\partial\mathcal{L}_{\text{new}}}{\partial q_{\bar{k}}} = -2\lambda q_{\bar{k}}. \tag{91}$$

Employing the three relations in Eq. (91), the Euler-Lagrange equations in Eq. (90) become

$$\frac{\ddot{q}_{\bar{k}}}{\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}}} - \frac{\dot{q}_{\bar{k}}^2\ddot{q}_{\bar{k}}}{\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{3}{2}}} + 2\lambda q_{\bar{k}} = 0, \tag{92}$$

that is,

$$\ddot{q}_{\bar{k}} - \frac{\dot{q}_{\bar{k}}^2\ddot{q}_{\bar{k}}}{\sum_{k=1}^{N}\dot{q}_k^2} + 2\lambda\left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}} q_{\bar{k}} = 0. \tag{93}$$

Observe that in terms of the Lagrangian-like function $\mathcal{L}$ defined in Eq. (77) expressed in terms of the probability amplitudes q_k, we find that $\mathcal{L}(\theta)$ and $\dot{\mathcal{L}}(\theta)/\mathcal{L}(\theta)$ are given by

$$\mathcal{L}(\theta) = \frac{1}{2}\mathcal{F}^{\frac{1}{2}}(\theta) = \left(\sum_{k=1}^{N}\dot{q}_k^2\right)^{\frac{1}{2}} \quad \text{and} \quad \frac{\dot{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)} = \frac{\dot{q}_{\bar{k}}\ddot{q}_{\bar{k}}}{\sum_{k=1}^{N}\dot{q}_k^2}, \tag{94}$$

respectively. Then, using the equalities in Eq. (94), the Euler-Lagrange equations expressed in Eq. (93) become

$$\ddot{q}_{\bar{k}} - \frac{\dot{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)}\dot{q}_{\bar{k}} + 2\lambda_{\text{FS}}\mathcal{L}(\theta)q_{\bar{k}} = 0, \tag{95}$$

where λ_{FS} is the Lagrange multiplier coefficient obtained within the framework of the Fubini-Study metric. For the sake of completeness, we remark that if we had used the Wigner-Yanase metric, Eq. (95) would have been written as

$$\ddot{q}_{\bar{k}} - \frac{\dot{\mathcal{L}}(\theta)}{\mathcal{L}(\theta)}\dot{q}_{\bar{k}} + \frac{\lambda_{\text{WY}}}{2}\mathcal{L}(\theta)q_{\bar{k}} = 0. \tag{96}$$

The rescaling of the Lagrange multiplier coefficient in transitioning from the Fubini-Study to the Wigner-Yanase cases occurs in order to satisfy the conservation of probability condition in both scenarios. Finally, recalling that

$$\mathcal{L}_{\text{FS}} = \tfrac{1}{2}\mathcal{F}^{\frac{1}{2}} \quad \text{and} \quad \mathcal{L}_{\text{WY}} = \mathcal{F}^{\frac{1}{2}}, \tag{97}$$

that is, $\mathcal{F} = 4\mathcal{L}_{\text{FS}}^2 = \mathcal{L}_{\text{WY}}^2$, in terms of the Fisher information function, Eqs. (95) and (96) become

$$\ddot{q}_{\bar{k}} - \frac{1}{2}\frac{\dot{\mathcal{F}}(\theta)}{\mathcal{F}(\theta)}\dot{q}_{\bar{k}} + \lambda_{\text{FS}}\mathcal{F}^{\frac{1}{2}}(\theta)q_{\bar{k}} = 0 \tag{98}$$

and

$$\ddot{q}_{\bar{k}} - \frac{1}{2}\frac{\dot{\mathcal{F}}(\theta)}{\mathcal{F}(\theta)}\dot{q}_{\bar{k}} + \frac{\lambda_{\text{WY}}}{2}\mathcal{F}^{\frac{1}{2}}(\theta)q_{\bar{k}} = 0, \tag{99}$$

respectively. In general, for each $\bar{k}$ with $1 \leqslant \bar{k} \leqslant N$, the integration of the previous second-order N ordinary differential equations (ODEs) leads to a formal expression of $q_{\bar{k}}(\theta)$. Specifically, each $q_{\bar{k}}(\theta)$ is the superposition of two linearly independent solutions of the ODEs expressed in terms of two real constants of integration $c_{\bar{k}}^{(1)}$ and $c_{\bar{k}}^{(2)}$. In particular, the formal expressions of such independent solutions appear in terms of the Lagrange multiplier λ_{FS} and depend on the particular functional dependence of the Fisher information function on the parameter θ. Therefore, in principle, the exact expression of these independent solutions requires one to express the Lagrange multiplier in terms of the characteristic parameters that specify the Fisher information function by imposing that $\frac{1}{4}F(\theta)$ equals the sum of $\dot{q}_{\bar{k}}^2(\theta)$ with $1 \leqslant \bar{k} \leqslant N$. Finally, since $q_{\bar{k}}^2(\theta)$ are probabilities, the $2N$ integration constants $c_{\bar{k}}^{(1)}$ and $c_{\bar{k}}^{(2)}$ have to be chosen in such a manner that $q_{\bar{k}}^2(\theta)$ add up to unity and $0 \leqslant q_{\bar{k}}^2(\theta) \leqslant 1$ for any $1 \leqslant \bar{k} \leqslant N$. In what follows, we shall take into consideration the integration of Eq. (98) for various functional forms of the Fisher information function.

V. ILLUSTRATIVE EXAMPLES

In this section we determine the geodesic paths followed by the quantum-mechanical probability amplitudes of pure quantum states for three distinct functional forms of the Fisher information function: constant Fisher information, exponential decay, and power-law decay.

A. Example one: Constant Fisher information

In the working assumption that $\mathcal{F}(\theta)$ takes a constant value $\mathcal{F}_0$, Eq. (98) describes a simple harmonic oscillator:

$$\ddot{q}_{\bar{k}} + \lambda_{\text{FS}}\mathcal{F}_0^{\frac{1}{2}} q_{\bar{k}} = 0. \tag{100}$$

Integration of Eq. (100) leads to the following general solution for the geodesic path of quantum-mechanical probability amplitudes $q_{\bar{k}}(\theta)$,

$$q_{\bar{k}}(\theta) = c_{\bar{k}}^{(1)} \cos\left(\mathcal{F}_0^{\frac{1}{4}}\sqrt{\lambda_{\text{FS}}}\theta\right) + c_{\bar{k}}^{(2)} \sin\left(\mathcal{F}_0^{\frac{1}{4}}\sqrt{\lambda_{\text{FS}}}\theta\right), \tag{101}$$

where $c_{\bar{k}}^{(1)}$ and $c_{\bar{k}}^{(2)}$ are two integration constants. Therefore, assuming for the sake of clarity that $\bar{k} = 1, 2$, probabilities $p_1(\theta)$ and $p_2(\theta) \overset{\text{def}}{=} 1 - p_1(\theta)$ can be written as

$$p_1(\theta) = \cos^2\left(\mathcal{F}_0^{\frac{1}{4}}\sqrt{\lambda_{\text{FS}}}\theta\right), \quad \text{and}$$
$$p_2(\theta) = \sin^2\left(\mathcal{F}_0^{\frac{1}{4}}\sqrt{\lambda_{\text{FS}}}\theta\right), \tag{102}$$

respectively. The value of the Lagrange multiplier coefficient λ_{FS} in Eq. (102) can be explicitly obtained by requiring that

$$\frac{\dot{p}_1^2(\theta)}{p_1(\theta)} + \frac{\dot{p}_2^2(\theta)}{p_2(\theta)} = \mathcal{F}_0. \tag{103}$$

By substituting Eq. (102) into Eq. (103), we obtain

$$\lambda_{\text{FS}} = \tfrac{1}{4}\mathcal{F}_0^{\frac{1}{2}}. \tag{104}$$

Observe that in the case of the analog counterpart of Grover's quantum search algorithm $\mathcal{F}_0 = 4$ and, thus, $\lambda_{\text{FS}} = 1/2$ and $\lambda_{\text{WY}} = 1$. Therefore, the failure and success probabilities are given by $p_1(\theta) = \cos^2(\theta)$ and $p_2(\theta) = \sin^2(\theta)$, respectively. In Fig. 1 we observe an oscillatory behavior of the success (dotted) and failure (solid) probabilities in the case of constant Fisher information.

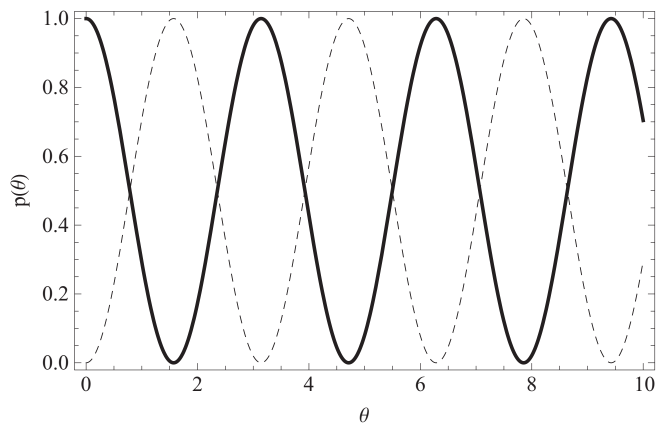

FIG. 1. Oscillatory behavior of the success (dotted) and failure (solid) probabilities in the case of constant Fisher information.

B. Example two: Exponential decay

In this subsection, we assume that the Fisher information function is a monotonically decreasing function that exhibits exponentially decaying behavior,

$$\mathcal{F}(\theta) \overset{\text{def}}{=} \mathcal{F}_0 e^{-\xi\theta}, \tag{105}$$

with $\mathcal{F}_0$ and ξ being positive real constant coefficients. In this working assumption, Eq. (98) describes the equation of an aging spring in the presence of damping,

$$\ddot{q}_{\bar{k}} + \frac{\xi}{2}\dot{q}_{\bar{k}} + \lambda_{\text{FS}}\mathcal{F}_0^{\frac{1}{2}} e^{-\frac{\xi}{2}\theta} q_{\bar{k}} = 0, \tag{106}$$

where $\dot{q}_{\bar{k}} \overset{\text{def}}{=} dq_{\bar{k}}/d\theta$. Equation (106) can be analytically integrated, and a closed form solution can be found.

In what follows, we consider the classical second order ordinary differential equation that describes an aging spring with damping,

$$m\ddot{x} + b\dot{x} + ke^{-\eta t}x = 0, \tag{107}$$

that is,

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}e^{-\eta t}x = 0. \tag{108}$$

In Eq. (108), $m > 0$ is the mass, $k > 0$ is the value of the spring constant at $t = 0$, $b > 0$ is the constant damping coefficient, $\eta \overset{\text{def}}{=} -\frac{1}{k(t)}\frac{d[k(t)]}{dt} \in \mathbb{R}_+\backslash\{0\}$, and $\dot{x} \overset{\text{def}}{=} dx/dt$. Equations (106) and (108) are essentially identical once we impose that

$$\theta = t, \quad \xi = \frac{2b}{m} = 2\eta \quad \text{and} \quad \lambda_{\text{FS}}\mathcal{F}_0^{\frac{1}{2}} = \frac{k}{m}. \tag{109}$$

To integrate Eq. (107), we employ two convenient mathematical tricks. First, we make a change of variables,

$$x(t) \to y(t) : x(t) \overset{\text{def}}{=} y(t)e^{-\frac{b}{2m}t}. \tag{110}$$

From Eq. (110), we get

$$\dot{x} = \dot{y}e^{-\frac{b}{2m}t} - \frac{b}{2m}ye^{-\frac{b}{2m}t} \quad \text{and}$$
$$\ddot{x} = \ddot{y}e^{-\frac{b}{2m}t} - \frac{b}{m}\dot{y}e^{-\frac{b}{2m}t} + \frac{b^2}{4m^2}ye^{-\frac{b}{2m}t}. \tag{111}$$

Using the two relations in Eqs. (111), Eq. (107) becomes

$$m\left(\ddot{y} - \frac{b}{m}\dot{y} + \frac{b^2}{4m^2}y\right) + b\left(\dot{y} - \frac{b}{2m}y\right) + ke^{-\eta t}y = 0, \tag{112}$$

that is,

$$m\ddot{y} + \left(ke^{-\eta t} - \frac{b^2}{4m}\right)y = 0. \tag{113}$$

At this point, let us consider the change of the independent variable

$$t \to s(t) : s(t) \overset{\text{def}}{=} \alpha e^{\beta t}, \tag{114}$$

that is,

$$t = \frac{1}{\beta}\log\left(\frac{s}{\alpha}\right), \tag{115}$$

with α and β being real coefficients. From Eq. (114), we obtain after some algebra,

$$\frac{d}{dt} \overset{\text{def}}{=} \beta s\frac{d}{ds}, \quad \text{and} \quad \frac{d^2}{dt^2} \overset{\text{def}}{=} \beta^2 s\left(\frac{d}{ds} + s\frac{d^2}{ds^2}\right). \tag{116}$$

Using the relations in Eq. (116), Eq. (113) becomes

$$s^2y'' + sy' + \frac{1}{m\beta^2}\left[k\left(\frac{s}{\alpha}\right)^{-\frac{\eta}{\beta}} - \frac{b^2}{4m}\right]y = 0, \tag{117}$$

where $y' \overset{\text{def}}{=} dy/ds$. At this point, we impose that

$$-\frac{\eta}{\beta} \overset{\text{def}}{=} 2, \quad \text{and} \quad \frac{k}{m\beta^2\alpha^{-\frac{\eta}{\beta}}} \overset{\text{def}}{=} 1, \tag{118}$$

that is,

$$\alpha \overset{\text{def}}{=} \frac{2}{\eta}\sqrt{\frac{k}{m}} \quad \text{and} \quad \beta \overset{\text{def}}{=} -\frac{1}{2}\eta, \tag{119}$$

and, consequently,

$$s(t) = \frac{2}{\eta}\sqrt{\frac{k}{m}}e^{-\frac{1}{2}\eta t}. \tag{120}$$

Using the relations in Eq. (119), the linear second-order differential equation in Eq. (117) becomes

$$s^2y'' + sy' + \left[1 - \left(\frac{b}{m\eta}\right)^2\right]y = 0. \tag{121}$$

Equation (121) is Bessel's equation of order $\frac{b}{m\eta} \geqslant 0$, and its integration leads to the following general solution [58]:

$$y(s) = c_1\mathcal{J}_{+\frac{b}{m\eta}}(s) + c_2\mathcal{J}_{-\frac{b}{m\eta}}(s), \tag{122}$$

that is, using Eqs. (110), (114), and (119),

$$x(t) = c_1e^{-\frac{b}{2m}t}\mathcal{J}_{+\frac{b}{m\eta}}\left(\frac{2}{\eta}\sqrt{\frac{k}{m}}e^{-\frac{\eta}{2}t}\right)$$
$$+ c_2e^{-\frac{b}{2m}t}\mathcal{J}_{-\frac{b}{m\eta}}\left(\frac{2}{\eta}\sqrt{\frac{k}{m}}e^{-\frac{\eta}{2}t}\right), \tag{123}$$

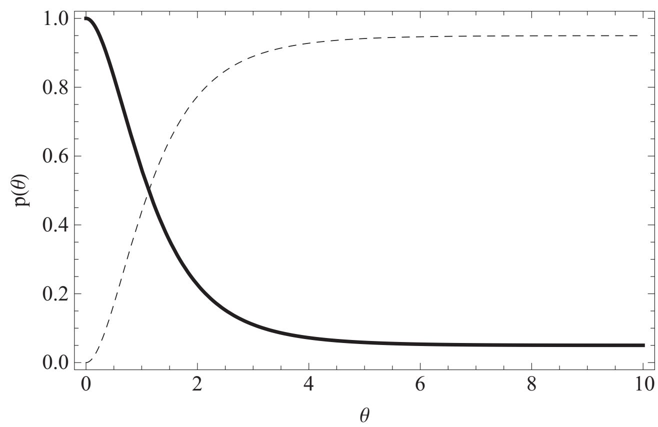

FIG. 2. Monotonic behavior of the success (dotted) and failure (solid) probabilities in the case of exponential decay of the Fisher information.

where c_1 and c_2 are two real integration constants, and $\mathcal{J}_\nu(x)$ denotes the Bessel function of the first kind of order $\nu \geqslant 0$ [58]. Finally, using Eqs. (109) and (123), the geodesic path of the quantum-mechanical probability amplitudes $q_{\bar{k}}(\theta)$ becomes

$$q_{\bar{k}}(\theta) = c_{\bar{k}}^{(1)}e^{-\frac{\xi}{4}\theta}\mathcal{J}_{+1}\left(\frac{4}{\xi}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}e^{-\frac{\xi}{4}\theta}\right)$$
$$+ c_{\bar{k}}^{(2)}e^{-\frac{\xi}{4}\theta}\mathcal{J}_{-1}\left(\frac{4}{\xi}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}e^{-\frac{\xi}{4}\theta}\right), \tag{124}$$

where $c_{\bar{k}}^{(1)}$ and $c_{\bar{k}}^{(2)}$ are two real integration constants. In Fig. 2, setting $\mathcal{F}_0 = 1, \xi = 2$ and preserving the normalization constraint, we observe a monotonic behavior of the success (dotted) and failure (solid) probabilities in the case of exponential decay of the Fisher information.

C. Example three: Power-law decay

In this subsection, we assume that the Fisher information function is a monotonically decreasing function that exhibits power-law decay behavior,

$$\mathcal{F}(\theta) \overset{\text{def}}{=} \frac{\mathcal{F}_0}{(1 + \Omega\theta)^n}, \tag{125}$$

where $\mathcal{F}_0$, Ω, and $n \geqslant 0$ are real constant coefficients. Using Eq. (125), Eq. (98) becomes

$$\ddot{q}_{\bar{k}} + \frac{n\Omega}{2}\frac{1}{1 + \Omega\theta}\dot{q}_{\bar{k}} + \frac{\lambda_{\text{FS}}\mathcal{F}_0^{\frac{1}{2}}}{(1 + \Omega\theta)^{\frac{n}{2}}}q_{\bar{k}} = 0, \tag{126}$$

where $\dot{q}_{\bar{k}} \overset{\text{def}}{=} dq_{\bar{k}}/d\theta$. Equation (126) is a linear second-order ordinary differential equation with varying coefficients. Its analytical integration is nontrivial for arbitrary values of $n \geqslant 0$. However, in what follows, we use a mathematical trick that allows to deduce a closed form solution for Eq. (126) for a specific choice of the constants Ω and n in Eq. (125). We proceed as follows.

Consider the second order linear differential equation with time-dependent coefficients,

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0, \tag{127}$$

where $\dot{x} \overset{\text{def}}{=} dx/dt$. Next, consider the following change of independent variable:

$$t \to s : s \overset{\text{def}}{=} f(t). \tag{128}$$

After some algebra, we obtain

$$\frac{dx}{dt} = \frac{ds}{dt}\frac{dx}{ds} \tag{129}$$

and

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{ds}{dt}\frac{dx}{ds}\right) = \frac{ds}{dt}\frac{d}{ds}\left(\frac{ds}{dt}\frac{dx}{ds}\right)$$

$$= \frac{ds}{dt}\frac{d}{ds}\left(\frac{ds}{dt}\right)\frac{dx}{ds} + \frac{ds}{dt}\frac{ds}{dt}\frac{d}{ds}\left(\frac{dx}{ds}\right)$$

$$= \frac{ds}{dt}\frac{d}{dt}\left(\frac{ds}{dt}\right)\frac{dt}{ds}\frac{dx}{ds} + \frac{d^2x}{ds^2}\left(\frac{ds}{dt}\right)^2, \tag{130}$$

that is,

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2x}{ds^2}. \tag{131}$$

Substituting Eqs. (129) and (131) into Eq. (127), we obtain

$$\left(\frac{ds}{dt}\right)^2\frac{d^2x}{ds^2} + \frac{d^2s}{dt^2}\frac{dx}{ds} + p(t)\frac{ds}{dt}\frac{dx}{ds} + q(t)x = 0, \tag{132}$$

that is, after some algebraic manipulations,

$$\frac{d^2x}{ds^2} + \frac{\frac{d^2s}{dt^2} + p(t)\frac{ds}{dt}}{\left(\frac{ds}{dt}\right)^2}\frac{dx}{ds} + \frac{q(t)}{\left(\frac{ds}{dt}\right)^2}x = 0, \tag{133}$$

where $x = x(s)$. Let us define the quantities A and B as

$$A \overset{\text{def}}{=} \frac{q(t)}{\left(\frac{ds}{dt}\right)^2}, \quad \text{and } B \overset{\text{def}}{=} \frac{\frac{d^2s}{dt^2} + p(t)\frac{ds}{dt}}{\left(\frac{ds}{dt}\right)^2}, \tag{134}$$

respectively. If we are able to select a suitable change of independent variables $t \to s \overset{\text{def}}{=} f(t)$ such that both A and B are constant quantities, integration of Eq. (127) reduces to integration of the following second-order linear differential equation with constant coefficients:

$$\frac{d^2x}{ds^2} + B\frac{dx}{ds} + Ax = 0. \tag{135}$$

We recall that for $B^2 < 4A$, the system that evolves according to Eq. (135) exhibits an underdamped oscillatory motion. Instead, when $B^2 > 4A$, the system manifests over-damped motion. Finally, when $B^2 = 4A$, the system is characterized by a critically damped motion. We can now return to our problem of integrating Eq. (126) and exploit the above mentioned mathematical reasoning. From Eqs. (98) and (127), replacing the independent variable t with θ, we have

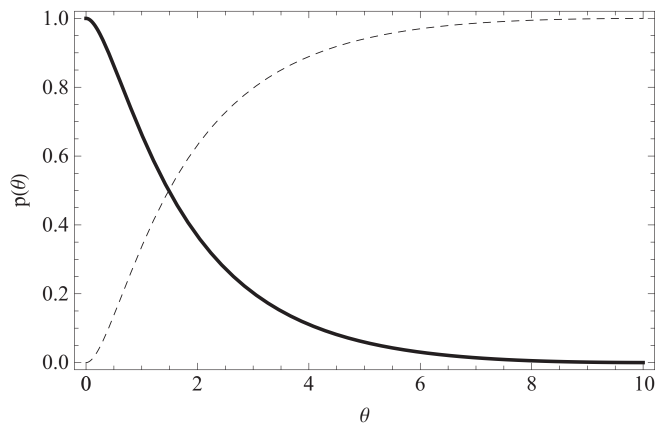

FIG. 3. Monotonic behavior of the success (dotted) and failure (solid) probabilities in the case of power-law decay of the Fisher information.

$$p(\theta) \overset{\text{def}}{=} -\frac{1}{2}\frac{\dot{\mathcal{F}}}{\mathcal{F}}, \quad \text{and } q(\theta) \overset{\text{def}}{=} \lambda_{\text{FS}}\mathcal{F}^{\frac{1}{2}}. \tag{136}$$

Substituting Eq. (136) into Eq. (134) and imposing that A and B are constant coefficients, we obtain, after some algebra, the following suitable change of independent variables:

$$\theta \to s : s(\theta) \overset{\text{def}}{=} \frac{1}{B}\log\left[1 + \frac{B}{\sqrt{A}}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}\theta\right], \tag{137}$$

together with the following two-parameter functional form for the Fisher information function $\mathcal{F}(\theta; A, B)$:

$$\mathcal{F}(\theta; A, B) \overset{\text{def}}{=} \frac{\mathcal{F}_0}{\left[1 + \frac{B}{\sqrt{A}}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}\theta\right]^4}. \tag{138}$$

In summary, we have shown that if $n = 4$ and Ω in Eq. (125) is defined as

$$\Omega = \Omega(A, B) \overset{\text{def}}{=} \frac{B}{\sqrt{A}}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}, \tag{139}$$

there is a suitable change of independent variables defined in Eq. (137) that makes Eq. (126) a linear second-order differential equation with constant coefficients. Thus, it can now be integrated in a straightforward manner. For instance, in the case of critical damping where $B^2 = 4A$ in Eq. (135), the general solution for the geodesic path of quantum-mechanical probability amplitudes $q_{\bar{k}}(\theta)$ becomes

$$q_{\bar{k}}(\theta) \overset{\text{def}}{=} \frac{c_{\bar{k}}^{(1)} + c_{\bar{k}}^{(2)}\frac{1}{B}\log\left(1 + \frac{B}{\sqrt{A}}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}\theta\right)}{\left[1 + \frac{B}{\sqrt{A}}\sqrt{\lambda_{\text{FS}}}\mathcal{F}_0^{\frac{1}{4}}\theta\right]^{\frac{1}{2}}}, \tag{140}$$

where $c_{\bar{k}}^{(1)}$ and $c_{\bar{k}}^{(2)}$ are two real integration constants. In Fig. 3, setting $A = \frac{1}{4}$, $B = 1$, $\mathcal{F}_0 = 1$, and preserving the normalization constraint, we observe a monotonic behavior of the success (dotted) and failure (solid) probabilities in the case of power-law decay of the Fisher information.

VI. ON PHYSICAL SYSTEMS, FISHER INFORMATION, AND GEODESIC PATHS

In this section, we present some clarifying remarks on the link among physical systems, Fisher information functions, and geodesic paths on Riemannian manifolds.

A. General remarks

We point out that classical Fisher information can be computed by considering parametric probability distributions $p_\theta(x) \overset{\text{def}}{=} |\psi_\theta(x)|^2$ that emerge from the absolute square of parametric quantum mechanical wave functions $\psi_\theta(x)$. Similarly, quantum Fisher information can be defined by means of parametric rank-one projections that can be regarded as density operators $\rho_\theta(x)$ constructed from the above mentioned wave functions $\psi_\theta(x)$. The functional form of such parametric quantum mechanical wave functions $\psi_\theta(x)$ depends on the particular choice of the unitary evolution operator $U_\theta(t)$. The operator $U_\theta(t)$ is generated by the parameter-dependent Hamiltonian $\mathcal{H}_\theta(t)$ that specifies the physical system under consideration. Furthermore, $\mathcal{H}_\theta(t)$ acts as the Hermitian generator of temporal displacements and satisfies the relation $\mathcal{H}_\theta(t)U_\theta(t) = i\hbar\partial_t U_\theta(t)$ with $\partial_t \overset{\text{def}}{=} \partial/\partial_t$. The value of the parameter of interest θ that specifies the Hamiltonian $\mathcal{H}_\theta(t)$ is inferred by observing the evolution of the probe system due to $\mathcal{H}_\theta(t)$. More specifically, the observation of the probe system requires finding measurements that are capable of optimally resolving parameter-dependent neighboring quantum states. Such an optimal resolution is achieved by employing statistical distinguishability in order to define a Riemannian metric on the manifold of quantum mechanical density operators. Then, the Fisher information appears in the infinitesimal line element ds^2_{FS} on such a manifold, namely, $ds^2_{\text{FS}} \overset{\text{def}}{=} (1/4)\{\mathcal{F}(\theta) + 4\sigma^2_\phi\}d\theta^2$. Finally, by integrating the geodesic equations on this Riemannian manifold, one can obtain the geodesic paths for the probability amplitude variables $q_\theta(x)$ with $p_\theta(x) \overset{\text{def}}{=} q^2_\theta(x)$.

For pure states, the Fisher information $\mathcal{F}_\theta(t)$ reduces to a multiple of the variance $\sigma^2_{T_\theta}$ of the Hermitian generator T_θ of displacements in θ. Specifically, $\mathcal{F}_\theta(t) = 4\sigma^2_{T_\theta} \overset{\text{def}}{=} 4\langle(T_\theta - \langle T_\theta\rangle)^2\rangle$ with $T_\theta(t) \overset{\text{def}}{=} i[\partial_\theta U_\theta(t)]U^\dagger_\theta(t)$, $\partial_\theta \overset{\text{def}}{=} \partial/\partial_\theta$, and $\langle T_\theta\rangle \overset{\text{def}}{=} \text{tr}(\rho_\theta T_\theta)$. To simplify the discussion throughout, we refer to a single parameter of interest θ. However, our analysis can be generalized in principle to multiple parameters of interest. In general, the selected parameters of interest are experimentally controllable quantities. For instance, external magnetic field intensity, phase difference, temperature, spin-spin coupling constant, volume per particle, reciprocal temperature, and computing time are all suitable examples of experimentally controllable parameters of interest. In particular, for probe systems such as Bose-Einstein condensates and nanomagnetic bits, the external magnetic field is usually used as a parameter of interest. For a quantum oscillator in the presence of dephasing noise, the phase difference plays the role of the parameter of interest. Moreover, temperature, spin-spin coupling constant, and external magnetic field intensity are three convenient parameters of interest for Ising spin models. For both a classical ideal gas and a van der Waals gas, the volume per particle and the reciprocal temperature $\beta \overset{\text{def}}{=} (k_B T)^{-1}$ with k_B denoting the Boltzmann constant are convenient control parameters. Finally, for probe systems described by quantum search Hamiltonians, the computing time can play the role of the parameter of interest.

B. From thermodynamics to quantum metrology

The Fisher information can assume a variety of functional forms with respect to the parameter of interest. For instance, within the Fisher information approach to thermodynamics via the Schrödinger equation [59–61], it can be shown that the Fisher information $\mathcal{F}_{\text{HO}}(\theta)$ that emerges from the thermal description of the one-dimensional quantum mechanical harmonic oscillator is proportional to the harmonic oscillator's specific heat C_V [61]:

$$\mathcal{F}_{\text{HO}}(\theta) = C_V \frac{e^{-\hbar\omega\theta}}{\theta^2}. \tag{141}$$

In Eq. (141) the parameter θ denotes the reciprocal temperature β while ω is the frequency of the oscillator.

In the framework of quantum metrology for a general Hamiltonian parameter $\mathcal{H}_\theta$ [62,63], it happens that the maximum quantum Fisher information is given by

$$\mathcal{F}_{\max}(\theta) \overset{\text{def}}{=} [\lambda_{\max}(h_\theta) - \lambda_{\min}(h_\theta)]^2. \tag{142}$$

The quantities $\lambda_{\max}(h_\theta)$ and $\lambda_{\min}(h_\theta)$ denote the maximal and the minimal eigenvalues of the generator h_θ of parameter translation with respect to θ,

$$h_\theta \overset{\text{def}}{=} i(\partial_\theta U_\theta)U^\dagger_\theta, \tag{143}$$

with $U_\theta \overset{\text{def}}{=} e^{-i\mathcal{H}_\theta t}$ in Eq. (143). For instance, for a spin-1/2 particle in an external magnetic field $\vec{B} \overset{\text{def}}{=} B\hat{n}_\theta$ with $\hat{n}_\theta \overset{\text{def}}{=} (\cos(\theta), 0, \sin(\theta))$, the interaction Hamiltonian $\mathcal{H}_\theta$ can be written as

$$\mathcal{H}_\theta \overset{\text{def}}{=} B[\cos(\theta)\sigma_x + \sin(\theta)\sigma_z]. \tag{144}$$

The Hamiltonian in Eq. (144) was written by setting the electric charge e, the mass m, and the speed of light c equal to one. Furthermore, σ_x and σ_z are Pauli operators. The parameter θ is the angle between the z axis and the magnetic field $\vec{B}$. In this case, it can be shown that $\mathcal{F}_{\max}(\theta)$ in Eq. (142) is constant in θ and equals

$$\mathcal{F}_{\max} = 4B^2 \sin^2(Bt). \tag{145}$$

Therefore, $\mathcal{F}_{\max}$ oscillates with respect to time t and exhibits a period $T \overset{\text{def}}{=} \pi/B$.

For the sake of completeness, we point out that the definition of the Fisher information is not limited to Hamiltonian systems. For instance, in the context of an information geometric approach to complex systems in the presence of limited information [64,65], the Fisher information of Gaussian statistical models is such that $\mathcal{F}_{\text{Gaussian}}(\theta) \propto 1/\theta^2$ with θ denoting the standard deviation of the zero mean one-dimensional Gaussian random variable that specifies the statistical model being considered.

C. Analog quantum search

The information geometric analysis performed in this paper can be especially relevant to the quantum search problem [66]. We recall that Grover's original quantum search algorithm can be viewed as a definite discrete-time sequence of elementary unitary transformations acting on qubits from a digital quantum computing perspective. In particular, given an initial input state, the output of the algorithm becomes the input state after the action of the sequence of unitary transformations. Furthermore, the length of the algorithm is equal to the number of unitary transformations that characterize the quantum computational software. Finally, the failure probability after k iterations of Grover's original search algorithm periodically oscillates as k increases. In Ref. [67], an analog counterpart of Grover's algorithm was proposed. The search problem was recast in terms of finding the normalized target eigenstate $|w\rangle$ corresponding to the only nonzero eigenvalue E of an Hamiltonian $\mathcal{H}_w \overset{\text{def}}{=} E|w\rangle\langle w|$ acting on a *complex* N-dimensional Hilbert space. The search ends when the system evolves from the initial state $|s\rangle$ into the state $|w\rangle$ with quantum overlap $x = \cos(\theta) \overset{\text{def}}{=} \langle s|w\rangle$. Such evolution is the continuous-time quantum mechanical Schrödinger evolution under the time-independent Hamiltonian [67]:

$$\mathcal{H}_{\text{Farhi-Gutmann}} \overset{\text{def}}{=} E|w\rangle\langle w| + E|s\rangle\langle s|. \tag{146}$$

From a physics standpoint, the Hamiltonian formulation of Grover's search Hamiltonian can be understood in terms of Rabi oscillations between the source and the target states [68]. We emphasize that it is possible to consider a generalized version of $\mathcal{H}_{\text{Farhi-Gutmann}}$ in Eq. (146) in terms of a more general time-independent quantum search Hamiltonian $\mathcal{H}_{\text{oscillation}}$ that describes an oscillation between the two states $|s\rangle$ and $|w\rangle$ [69],

$$\mathcal{H}_{\text{oscillation}} \overset{\text{def}}{=} E[\alpha|w\rangle\langle w| + \beta|w\rangle\langle s| + \gamma|s\rangle\langle w| + \delta|s\rangle\langle s|], \tag{147}$$

where α, β, γ, δ are *complex* expansion coefficients. We also remark that once the digital-to-analog transition is completed, information geometry can be employed to view Grover's iterative procedure as a geodesic path on the manifold of parametric density operators of pure quantum states built from the continuous approximation of the parametric quantum output state in Grover's algorithm. In particular, the Fisher information is computed from the probability distribution vector with oscillating components that characterize the Groverian geodesic paths and happens to be constant.

An alternative to Grover's original quantum search algorithm is Grover's fixed-point search algorithm [70]. In particular, the failure probability after k recursive steps of such algorithm decreases monotonically and converges to zero as k increases. An analog counterpart of a fixed-point search algorithm can be recovered by considering time-dependent Hamiltonians for both fixed-point nonadiabatic [71] and adiabatic [72] quantum search. These time-dependent Hamiltonians can be recast as

$$\mathcal{H}_{\text{fixed-point}}(t) \overset{\text{def}}{=} f_1(t)[I - |s\rangle\langle s|] + f_2(t)[I - |w\rangle\langle w|], \tag{148}$$

where $|s\rangle$ is the initial state of the quantum system, $|w\rangle$ is the target state and I denotes the identity operator. In the framework of adiabatic quantum search, $f_1(t) \overset{\text{def}}{=} 1 - s(t)$ and $f_2(t) \overset{\text{def}}{=} s(t)$ with $s(t)$ being the so-called schedule of the search algorithm. It was shown in Ref. [72] that for a suitable choice of parameters that parametrize the schedule $s(t)$, the Hamiltonian $\mathcal{H}(t)$ can exhibit both a Grover-like scaling and the fixed-point property. In particular, such Hamiltonian can drive the system toward a fixed point. Once the digital-to-analog transition is performed, information geometry could be exploited to regard Grover's fixed-point algorithm recursive procedure as a geodesic path on the manifold of parametric density operators of pure quantum states built from the continuous approximation of the parametric quantum output state in Grover's fixed point algorithm. In particular, the Fisher information is computed from the probability distribution vector with nonoscillating components that characterize the fixed-point Groverian geodesic paths and happens to be monotonically decreasing with respect to the parameter of interest chosen to parametrize the geodesic paths on the underlying manifold. For a recent preliminary investigation of these ideas, we refer to Ref. [36].

In view of these considerations, we have considered in this paper functional forms of the Fisher information that could be of relevance in the framework of analog quantum search with search Hamiltonians given in Eqs. (146), (147), and (148). More specifically, the quantum mechanical evolution under the Grover-like search Hamiltonians (GSH) in Eqs. (146) and (147) generate wave functions that lead to periodically oscillating probability distributions with constant Fisher information. Instead, the quantum mechanical evolution under the fixed-point search Hamiltonian (FPSH) in Eqs. (148) can generate wavefunctions that lead to monotonically convergent probability distributions with decreasing Fisher information. Clearly, a deeper understanding of the connection between the Fisher information and the schedule of the quantum algorithm remains to be uncovered in order to provide a rigorous mapping between our geometric analysis and the Hamiltonian formulation of the problem. In particular, it remains to be understood how to exactly quantify the speed at which the Hamiltonian can drive the system toward the target state (that is, the soft or strong nature of monotonic convergence toward the target state) is related to both the functional forms of the schedule and the Fisher information.

Despite these unresolved issues, we believe that our work represents a nontrivial step forward towards the accomplishment of such challenging goals. We also remark that our information geometric analysis can be extended in a number of ways. For instance, we limited our analysis to a single parameter of interest and, in addition, we considered only special monotonically decreasing Fisher information functions. However, the extension of our work to arbitrary functional forms of the Fisher information, depending or not on more than one parameter of interest, seems to be outside the reach of analytical treatment. In this regard, it may be helpful to familiarize with recent numerical strategies to find optimal protocols as geodesics on a Riemannian manifold [73]. In particular, an oscillating Fisher information $\mathcal{F}(\theta)$ would require the integration of a differential equation that describes a damped harmonic oscillator with θ-dependent damping coefficient given in terms of $\dot{\mathcal{F}}/\mathcal{F}$ with $\dot{\mathcal{F}} \stackrel{\text{def}}{=} d\mathcal{F}/d\theta$. In this respect, it may be useful to better understand the very recent asymptotic stability property for such a type of differential equation [74].

For the time being, we remark that in the case of constant Fisher information, one deals with geodesic paths that satisfy a differential equation that formally resembles that of a simple harmonic oscillator and obtains oscillatory output probabilities. In the case of exponential decay of the Fisher information, one observes geodesic paths that satisfy a differential equation that formally resembles that of an aging spring in the presence of damping together with monotonic output probabilities. Finally, when the Fisher information exhibits a power-law decay, geodesic paths satisfying an ordinary differential equation that resembles that of a critically damped harmonic oscillator and leads to monotonic output probabilities. The presence of damping effects seems to lead to the characteristic monotonic behavior of the quantum mechanical probability amplitude squared. Therefore, it is reasonable to further investigate this plausible connection between Fisher information and dissipative effects in an effort to render any such connection more rigorous. This latter point shall be addressed in the next section by exploiting a Riemannian geometric characterization of thermodynamic concepts.

VII. RIEMANNIAN GEOMETRIC VIEWPOINT OF THERMODYNAMIC CONCEPTS

An efficient thermodynamic process occurs by minimum dissipation or maximum power. In particular, dissipation can be quantified in terms of the amount of work lost in the process [37]. Availability loss (that is, dissipated availability or irreversibility) and entropy production are the two most common measures of dissipation in thermodynamics [75]. In a Riemannian approach to thermodynamics, both availability loss and entropy production are related to the concept of thermodynamic length. However, while in the former case one deals with the so-called energy version of the thermodynamic length, in the latter case the so-called entropy version of the thermodynamic length is taken into consideration [76]. Specifically, optimum paths that minimize entropy production are commonly referred to as *optimum cooling paths* (that is, maximum reversibility paths) and characterize a thermodynamic process that occurs at constant thermodynamic speed [77–80]. The notions of thermodynamic length and dissipated availability will be discussed in the next subsection.

A. Preliminaries

Consider a physical system, small in mass and extent, surrounded by an (infinite) environment with temperature T_0 and pressure p_0 which are unaffected by any process experienced by the system. An arbitrary process can be viewed as an interaction between the system and the environment, once one includes in the system as much material or machinery that is affected by the process itself. Under these working assumptions, Gibbs introduced a quantity Φ (that is, the Gibbs free energy [81]) defined as

$$\Phi \stackrel{\text{def}}{=} E + p_0 V - T_0 S, \tag{149}$$

where E is the energy of the system, V is its volume, S denotes its entropy, and Gibbs showed that (for further details, see Ref. [82])

$$\Delta\Phi \leqslant 0, \tag{150}$$

where $\Delta\Phi$ is the increase in the quantity Φ. The availability (or available energy) Λ of the system and the environment is defined as [83]

$$\Lambda \stackrel{\text{def}}{=} \Phi - \Phi_{\text{min}}, \tag{151}$$

where Φ_{min} is the minimum possible value of Φ attained when the system is in a state from which no spontaneous changes can happen. Such a state of the system is the state of stable equilibrium (or, more generally, maximum stability) and is characterized by a pressure p_0 and a temperature T_0. The availability Λ in Eq. (151) represents the maximum value of the useful work, that is to say, work in excess of that done against the environment that could be obtained from the system and the environment via any arbitrary process, without intervention of other bodies:

$$\Lambda = W_{\text{excess}}. \tag{152}$$

We point out that for the most stable state of the system, $\Lambda = 0$. Furthermore, for any state of any system immersed in a stable environment, $\Lambda \geqslant 0$.

Transitioning from a conventional to a geometrical setting, the so-called thermodynamic length $\mathcal{L}_{\text{th.}}$ of a curve $\theta^\mu = \theta^\mu(t)$ parametrized by t with $0 \leqslant t \leqslant \tau$ is defined as [37]

$$\mathcal{L}_{\text{th.}} \stackrel{\text{def}}{=} \int_0^\tau \sqrt{g_{\mu\nu}(\theta)\frac{d\theta^\mu}{dt}\frac{d\theta^\nu}{dt}}\,dt, \tag{153}$$

where $g_{\mu\nu}(\theta)$ in Eq. (153) denotes the so-called thermodynamic metric tensor given by [84,85]

$$\begin{aligned} g_{\mu\nu}(\theta) &\stackrel{\text{def}}{=} \frac{\partial^2\psi}{\partial\theta^\mu\partial\theta^\nu} = -\frac{\partial\langle X_\mu\rangle}{\partial\theta^\nu} \\ &= \langle (X_\mu - \langle X_\mu\rangle)(X_\nu - \langle X_\nu\rangle)\rangle. \end{aligned} \tag{154}$$

The quantity ψ in Eq. (154) denotes the free entropy,

$$\psi \stackrel{\text{def}}{=} \log(\mathcal{Z}) = -\beta\Phi = \mathcal{S} - \theta^\mu\langle X_\mu\rangle, \tag{155}$$

where $\mathcal{Z}, \Phi, \mathcal{S}, \beta \stackrel{\text{def}}{=} \frac{1}{k_B T}$, and k_B are the partition function, free energy, entropy, reciprocal temperature (T), and Boltzmann constant, respectively. The variables $\{X_\mu(x)\}$ are thermodynamic variables that specify the Hamiltonian of the system (for instance, internal energy and volume) while x belongs to the configuration space. Furthermore, the time-dependent θ's are experimentally controllable parameters of the system that specify the accessible thermodynamic state space of the system. Expectation values in Eqs. (154) and (155) are defined with respect to the probability distribution $p(x|\theta)$ (Gibbs ensemble),

$$p(x|\theta) \stackrel{\text{def}}{=} \frac{1}{\mathcal{Z}}e^{-\beta\mathcal{H}(x,\theta)} = \frac{1}{\mathcal{Z}}e^{-\theta^\mu(t)X_\mu(x)}, \tag{156}$$

where, adopting the Einstein convention, repeated lower and upper indices are summed over. We point out that, using Eqs. (154), (155), and (156), the thermodynamic metric tensor can be shown to be equal to the Fisher-Rao information metric tensor:

$$g_{\mu\nu}(\theta) \overset{\text{def}}{=} \frac{\partial^2 \psi}{\partial\theta^\mu \partial\theta^\nu} = \int p(x|\theta) \frac{\partial \log p(x|\theta)}{\partial\theta^\mu} \frac{\partial \log p(x|\theta)}{\partial\theta^\nu} dx. \tag{157}$$

The quantity $g_{\mu\nu}(\theta)$ in Eq. (157) is a Riemannian metric on the manifold of thermodynamic states. The thermodynamic length in Eq. (153) has dimensions of speed and its physical interpretation is related to the concept of availability loss (or, dissipated availability) $\Lambda_{\text{dissipated}}$ in a thermodynamic process [37,75]:

$$\Lambda_{\text{dissipated}} \overset{\text{def}}{=} \int_0^\tau g_{\mu\nu}(\theta) \frac{d\theta^\mu}{dt} \frac{d\theta^\nu}{dt} dt. \tag{158}$$

The quantity $\Lambda_{\text{dissipated}}$ can be expressed in terms of the so-called thermodynamic divergence of the path $\mathcal{D}$ [86,87],

$$\mathcal{D} \overset{\text{def}}{=} \tau \cdot \Lambda_{\text{dissipated}}, \tag{159}$$

where, in the context of Riemannian geometry, $\mathcal{D}/2\tau$ is also known as the energy of the path parametrized with t where $0 \leqslant t \leqslant \tau$. Indeed, considering Eqs. (153) and (158), the application of the Cauchy-Schwarz inequality leads to the following inequality:

$$\Lambda_{\text{dissipated}} \geqslant \frac{\mathcal{L}^2_{\text{th.}}}{\tau}, \tag{160}$$

that is, $\mathcal{D} \geqslant \mathcal{L}^2_{\text{th.}}$ (the divergence-length inequality expresses the fact that the minimum divergence of the path is the square of the thermodynamic length). The equality in Eq. (160) is obtained only for the most favorable time parametrization, which occurs when

$$\|\dot{\theta}(t)\| \overset{\text{def}}{=} \left(g_{\mu\nu}(\theta) \frac{d\theta^\mu}{dt} \frac{d\theta^\nu}{dt} \right)^{\frac{1}{2}} = \text{const}, \tag{161}$$

with the constant equal to $\mathcal{L}_{\text{th.}}/\tau$. Therefore, a thermodynamic process dissipates minimum availability when it proceeds at constant speed.

B. Illustrative examples

Given the equivalence between the Fisher-Rao information metric and the thermodynamic metric tensor, we can apply the concepts of thermodynamic length and availability loss to our selected illustrative examples discussed in Sec. V. We recall that our output probability paths $p_{\tilde{k}}(\theta)$ are parametrized by a single statistical parameter θ that denotes the computational time of a quantum process.

In general, the geodesic equations satisfied by the statistical parameters $\theta^\mu = \theta^\mu(t)$ with $1 \leqslant \mu \leqslant |\Theta|$, where $|\Theta|$ denotes the cardinality of the set Θ of statistical parameters, are given by

$$\frac{d^2\theta^\mu}{dt^2} + \Gamma^\mu_{\nu\rho} \frac{d\theta^\nu}{dt} \frac{d\theta^\rho}{dt} = 0. \tag{162}$$

The quantities $\Gamma^\mu_{\nu\rho}$ in Eq. (162) are the connection coefficients defined as

$$\Gamma^\mu_{\nu\rho} \overset{\text{def}}{=} \tfrac{1}{2} g^{\mu\alpha}(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}), \tag{163}$$

where $\partial_\nu \overset{\text{def}}{=} \frac{\partial}{\partial\theta^\nu}$. In our analysis, we have

$$ds^2_{\text{FS}} = g_{\theta\theta}(\theta) d\theta^2, \quad \text{with } g_{\theta\theta}(\theta) \overset{\text{def}}{=} \tfrac{1}{4}\mathcal{F}(\theta). \tag{164}$$

Using Eqs. (164) and (163), the geodesic equation in Eq. (162) becomes

$$\frac{d^2\theta}{dt^2} + \frac{1}{2\mathcal{F}} \frac{d\mathcal{F}}{d\theta} \left(\frac{d\theta}{dt}\right)^2 = 0. \tag{165}$$

From the integration of Eq. (165), we can also consider the so-called computational speed defined as

$$v(t) \overset{\text{def}}{=} \left[g_{\theta\theta}(\theta) \left(\frac{d\theta}{dt}\right)^2 \right]^{\frac{1}{2}} = \frac{1}{2}\sqrt{\mathcal{F}[\theta(t)]} \frac{d\theta}{dt}. \tag{166}$$

In what follows, we compute the availability loss $\Lambda_{\text{dissipated}}$ in Eq. (158) and the computational speed v in Eq. (166) after integrating the nonlinear ordinary differential equation in (165) whose structure clearly depends on the functional form of the Fisher information function $\mathcal{F}$. Below, we consider the three cases considered in Sec. V.

1. Example one: Constant Fisher information

In this case, since $\mathcal{F}(\theta) \overset{\text{def}}{=} \mathcal{F}_0$, Eq. (165) becomes

$$\frac{d^2\theta}{dt^2} = 0. \tag{167}$$

Assuming as initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \dot{\theta}_0$, we obtain

$$\theta(t) = \theta_0 + \dot{\theta}_0(t - t_0). \tag{168}$$

Furthermore, the availability loss $\Lambda_{\text{dissipated}}$ in Eq. (158) becomes

$$\Lambda_{\text{dissipated}}(\tau) = \frac{\mathcal{F}_0}{4} \dot{\theta}_0^2 \tau. \tag{169}$$

Finally, the computational speed v in Eq. (166) is given by

$$v = \frac{1}{2}\sqrt{\mathcal{F}_0}\dot{\theta}_0. \tag{170}$$

We notice that the quantum process proceeds at constant speed and, thus, dissipates minimum availability. Moreover, the dissipated availability grows linearly with τ (that is, the length of the parametrization interval).

2. Example two: Exponential decay

In this case, since $\mathcal{F}(\theta) \overset{\text{def}}{=} \mathcal{F}_0 e^{-\xi\theta}$, Eq. (165) becomes

$$\frac{d^2\theta}{dt^2} - \frac{\xi}{2} \left(\frac{d\theta}{dt}\right)^2 = 0. \tag{171}$$

Assuming initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \dot{\theta}_0$, integration of Eq. (171) yields

$$\theta(t) = \theta_0 - \frac{2}{\xi} \log\left[1 - \xi\frac{\dot{\theta}_0}{2}(t - t_0)\right]. \tag{172}$$

TABLE I. Behavior of availability losses, computational speeds, and geodesic paths for different physical scenarios that can arise from different functional forms of the Fisher information. GSH and FPSH denote Grover-like search Hamiltonians and fixed-point-like search Hamiltonians, respectively.

Fisher information	Geodesic paths	Physical system	Probability	Availability loss	Speed
Constant	Simple harmonic oscillator	GSH	Oscillatory	Higher	Higher
Exponential decay	Aging spring with damping	Strong convergence FPSH	Monotonic	Lower	Lower
Power law decay	Critically damped harmonic oscillator	Soft convergence FPSH	Monotonic	Lower	Lower

Furthermore, the availability loss $\Lambda_{\text{dissipated}}$ in Eq. (158) becomes

$$\Lambda_{\text{dissipated}}(\tau) = \frac{\mathcal{F}_0}{4}\dot{\theta}_0^2 e^{-\xi\theta_0}\tau. \tag{173}$$

Finally, the computational speed v in Eq. (166) is given by

$$v = \frac{1}{2}\sqrt{\mathcal{F}_0} e^{-\frac{\xi}{2}\theta_0}\dot{\theta}_0. \tag{174}$$

In analogy to the first example, the quantum process proceeds at constant speed and, thus, dissipates minimum availability. Moreover, the dissipated availability grows linearly with τ. However, comparing Eqs. (169) and (170) with Eqs. (173) and (174), we observe that while the computational speed of the process is smaller in this second case, the availability loss is also smaller.

3. *Example three: Power-law decay*

In this case, since $\mathcal{F}(\theta) \overset{\text{def}}{=} \frac{\mathcal{F}_0}{(1+\Omega\theta)^4}$, Eq. (165) becomes

$$\frac{d^2\theta}{dt^2} - \frac{2\Omega}{1+\Omega\theta}\left(\frac{d\theta}{dt}\right)^2 = 0. \tag{175}$$

Assuming initial conditions $\theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \dot{\theta}_0$, integrating Eq. (175), we obtain

$$\theta(t) = \frac{(1+\Omega\theta_0)^2 + \Omega\dot{\theta}_0\left[(t-t_0) - \frac{1+\Omega\theta_0}{\Omega\dot{\theta}_0}\right]}{\Omega^2\dot{\theta}_0\left[\frac{1+\Omega\theta_0}{\Omega\dot{\theta}_0} - (t-t_0)\right]}. \tag{176}$$

Furthermore, the availability loss $\Lambda_{\text{dissipated}}$ in Eq. (158) becomes

$$\Lambda_{\text{dissipated}}(\tau) = \frac{\mathcal{F}_0}{4}\frac{\dot{\theta}_0^2}{(1+\Omega\theta_0)^2}\tau. \tag{177}$$

Finally, the computational speed v in Eq. (166) is given by

$$v = \frac{1}{2}\sqrt{\mathcal{F}_0}\frac{1}{1+\Omega\theta_0}\dot{\theta}_0. \tag{178}$$

In analogy to the first and second examples, the quantum process proceeds at constant speed and, thus, dissipates minimum availability. In addition, the dissipated availability grows linearly with τ. However, comparing Eqs. (169) and (170) with Eqs. (177) and (178), we observe that while the computational speed of the process is smaller in this third case, the availability loss is also smaller. In Table I we report the observed behavior of availability losses, computational speeds, and geodesic paths for different physical scenarios that can arise from different functional forms of the Fisher information.

VIII. CONCLUDING REMARKS

In this paper, we presented an information geometric characterization of the oscillatory or monotonic behavior of statistically parametrized squared probability amplitudes originating from special functional forms of the Fisher information function: constant, exponential decay, and power-law decay. Furthermore, for each case, we computed both the computational speed and the availability loss of the corresponding physical processes by employing a convenient Riemannian geometrization of thermodynamical concepts. In what follows, we outline our main findings in a more detailed fashion:

(1) We provided a dynamical information geometric characterization of the Fisher information function via an explicit derivation of the Euler-Lagrange equations satisfied by the quantum-mechanical probability amplitudes of pure states using variational calculus techniques applied to an action functional defined in terms of either the Fubini-Study [see Eq. (98)] or the Wigner-Yanase [see Eq. (99)] metric tensors.

(2) We analyzed the parametric behavior of the squared probability amplitudes arising from three different classes of Fisher information functions: constant Fisher information, exponential decay, and power-law decay. In the first case, we observed oscillatory behavior of the output probabilities (Fig. 1) that arises from the integration of a differential equation describing a simple harmonic oscillator [see Eq. (100)]. In the second case, we reported monotonic behavior of the output probabilities (Fig. 2) that originates from the integration of a differential equation characterizing an aging spring in the presence of damping [see Eq. (106)]. Finally, in the third case, we observed monotonic behavior of the output probabilities (Fig. 3). In particular, upon a suitable change of variables, the reported behavior of the output probabilities can be explained as emerging from the integration of a differential equation describing a critically damped harmonic oscillator [see Eqs. (126), (135), and (137)]. The overall picture emerging from the analysis of these three cases inspired us to further investigate the connection between the Fisher information and dissipative effects.

(3) We used the Riemannian geometrization of thermodynamical concepts, including thermodynamic speed and dissipated availability, to study the behavior of both the availability loss [see Eq. (158)] and the computational speed [see Eq. (166)] of the quantum processes specified in terms of the previously mentioned output probability paths. Specifically, after finding the optimal parametrization of the statistical variable θ that specifies our output probabilities $p_{\tilde{k}}(\theta)$, we evaluated both the availability loss and the computational speed along the optimal geodesic paths corresponding to the above mentioned three scenarios [see Table I together with Eqs. (169), (170), (173),

(174), (177), and (178)]. Our main finding here is that a greater computational speed comes necessarily at the expense of a greater availability loss.

As a final remark, we recall that from a quantum mechanical standpoint, the output state in Grover's quantum search algorithm follows a geodesic path obtained from the Fubini-Study metric on the manifold of Hilbert-space rays. In addition, Grover's algorithm is specified by a constant Fisher information. A topic of great interest in quantum computing is the investigation of constructive uses of dissipation. For instance, in Ref. [31] it was shown that it is possible to modify Grover's algorithm by introducing a suitable amount of dissipation in such a manner that the newly obtained algorithm, while preserving the typical number of queries $O(\sqrt{N/M})$ (where N is the number of items and M is the number of target items), gains robustness by damping out the oscillations between the target and nontarget states. Furthermore, the problem of designing quantum algorithms that are both fast and thermodynamically efficient is a very challenging and relevant problem [88]. To the best of our knowledge, there does not exist any conclusive investigation that concerns this type of issue. In Ref. [89] however, it was shown there that the faster one seeks to implement a shortcut, the higher is the thermodynamic cost of realizing the associated quantum process.

Despite the limits of our investigation, we are confident that our information geometric analysis of the evolution of quantum systems combined with thermodynamical considerations can be especially relevant to information physicists and, more specifically, quantum information theorists with particular interest in thermodynamical aspects of quantum information. We also strongly believe that the significance of our work runs far deeper than what is presently understood. However, significant further exploration is needed to make a precise formal connection among parameter-dependent probe Hamiltonians, Fisher information, and optimal cooling paths on the underlying parameter manifold. In conclusion, based also on our recent findings in quantum computing [36], statistical mechanics [90,91], and information geometry [92,93], we have reason to believe that our information geometric analysis presented in this paper will pave the way to further quantitative investigations on the role played by the Fisher information function in the trade-off between speed and thermodynamic efficiency in quantum search algorithms.

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