Fractional superstatistics from a kinetic approach

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Through a kinetic approach, in which temperature fluctuations are taken into account, we obtain generalized fractional statistics interpolating between Fermi-Dirac and Bose-Einstein statistics. The latter correspond to the superstatistical analogues of the Polychronakos and Haldane-Wu statistics. The virial coefficients corresponding to these statistics are worked out and compared to those of an ideal two-dimensional anyon gas. It is shown that the obtained statistics reproduce correctly the second and third virial coefficients of an anyon gas. On this basis, a link is established between the statistical parameter and the strength of fluctuations. A further generalization is suggested by allowing the statistical parameter to fluctuate. As a by-product, superstatistics of ewkons, introduced recently to deal with dark energy [Phys. Rev. E **94**, 062115 (2016)], are also obtained within the same method.

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I. INTRODUCTION

The statistics of all observed particles are covered by the two well-known realizations of quantum statistics: the Bose-Einstein (BE) statistics and the Fermi-Dirac (FD) statistics. Bosons, obeying BE statistics, are characterized by a wave function that does not change under the exchange of any two particles, while the many-body wave function of a system of fermions, obeying FD statistics, changes sign under the same process; that is, $|\psi_{2,1}\rangle = \pm|\psi_{1,2}\rangle$, where + (−) refers to bosons (fermions). While this is true for all elementary particles, for some time already there has been an ongoing interest in the physics of quasiparticles obeying fractional statistics. Aside from being interesting in their own right in the context of mathematical physics, current interest in fractional statistics is motivated by its relevance in a number of physical processes, such as the fractional quantum Hall effect [1,2], high-temperature superconductivity [3–5], low-dimensional interacting systems [6], cold atomic gases [7], nuclear matter [8], and models of dark matter [9].

Fractional statistics arises when the many-body wave function of a system of indistinguishable particles is allowed to acquire an arbitrary phase under the exchange of two particles. Leinaas and Myrheim [10] showed that the wave function in a two-dimensional system can pick up an arbitrary phase $e^{i\pi\alpha}$, where α is a real number, $\alpha \in [0,1]$ (mod. 2), when two particles are swapped; that is, $|\psi_{2,1}\rangle = e^{i\pi\alpha}|\psi_{1,2}\rangle$. Later Wilczek [11] coined the name *anyons* for particles obeying these peculiar statistics.[1]

Numerous extensions of the conventional BE and FD statistics have been suggested, either from quantum mechanical considerations or more rooted in statistical mechanics. In the formalism of fractional exclusion statistics, Haldane [13] defined a generalized Pauli exclusion principle, allowing an interpolation expression between the bosonic and fermionic limits. Later, Wu [14] obtained a distribution function for particles obeying the Haldane principle. Polychronakos [15] defined a different form of fractional statistics, with the advantage of being generalizable to interacting particles.

Nevertheless, the relation between fractional statistics and anyons is quite elusive. So far, the expression for the occupation numbers in an ideal gas of anyons has remained unknown, and statistical mechanical models of anyons are still incomplete. A promising idea to circumvent this issue is to consider a more general form of fractional statistics by adding one or a few parameters into the occupation numbers. In the seminal work of Rovenchak [16,17], fractional statistics are generalized in the sense of Tsallis statistics [18] or incomplete statistics [19]. Such generalized fractional statistics allow establishing an approximate correspondence within an anyon gas, because of the extra-parameter that underpins the occupation numbers. In such an approach, the generalization is introduced *phenomenologically*. It can also be justified by maximizing a more general form of entropy [20,21].

In this paper we take a different path. The key idea is to consider that the temperature has small fluctuations around some mean value and to use basic probability rules to establish the occupation numbers. The statistics are then given by a mixture of statistics, characterized by the mean temperature and a parameter that defines the strength of fluctuations. Such an approach has two advantages : first, the generalization of the statistics is based upon an empirical fact, namely, fluctuations, which opens the door for an experimental validation. Second, fractional statistics *à la* Tsallis represent only a particular case of the present approach, as a consequence of a specific type of fluctuations. More general statistics can be generated following different forms of fluctuations [22,23]. Note that experimental

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[1]Anyons are generally classified as Abelian and non-Abelian. Excitations corresponding to Abelian anyons have been detected experimentally; they play an important role in the fractional quantum Hall effect [12]. Non-Abelian anyons have not been definitively detected, although actively studied.

evidence for statistics close to Tsallis statistics but with tiny corrections can be found in the literature [24].

A picture of this paradigm can be given using the adiabatic ansatz: Let ζ be some fluctuating intensive quantity. The state space X covered by the system during its evolution can be partitioned into small cells characterized by a sharp value of ζ. Within each cell, the system is described by the conditional distribution $p(A \mid \zeta)$, to be found in a state $A \in X$. As ζ varies adiabatically from cell to cell, the joint distribution of finding the system with a sharp value of ζ in a specific state A is given by a mixture of probabilities: $p(A,\zeta) = p(A \mid \zeta)p(\zeta)$ (Bayes theorem). The resulting statistics $p(A)$ for finding the system in the state A is obtained through marginalization:

$$p(A) = \int p(A \mid \zeta)p(\zeta)\,d\zeta. \tag{1}$$

Such an approach is usually referred to as *superstatistics* since it consists of a superposition of statistics. The formalism has been introduced in Ref. [22], but such a pattern has a long tradition in statistical mechanics, and some features have been anticipated earlier [25–28].

The paper is organized as follows. In Sec. II we derive fractional superstatistics following mostly the approach introduced in Refs. [21,29], but allowing for temperature fluctuations. We establish the superstatistics corresponding to Polychronakos and Haldane-Wu statistics. We also obtain a generalization of "ewkons" statistics, introduced recently as a candidate to model dark energy [30,31]. In Sec. III we derive the virial coefficients corresponding to those statistics and compare them to those of a two-dimensional ideal anyon gas. In Sec. IV a further generalization is proposed by allowing the statistical parameter to fluctuate within a distribution function assigned to it. We comment and summarize in Sec. V.

II. FRACTIONAL SUPERSTATISTICS FROM A KINETIC APPROACH

Consider a system *at thermal equilibrium* at a fixed temperature T, $\beta = 1/T$ (hereafter the Boltzmann constant is set to unity), and focus on the change of the level mean occupation $n_i(t) \equiv n(t,E_i)$ due to the transition *to* and *from* the level E_i. In the decorrelation approximation, valid in the thermodynamic limit, the change of $n_i(t)$ is governed by the following master equation:

$$\begin{aligned}\frac{dn_i(t)}{dt} &= \pi(t,E_{i+1} \to E_i) + \pi(t,E_{i-1} \to E_i)\\ &\quad - \pi(t,E_i \to E_{i+1}) - \pi(t,E_i \to E_{i-1}),\end{aligned} \tag{2}$$

where $\pi(t,E_i \to E_{i+1})$ defines the transition probability from the state E_i to the state E_{i+1}. The latter is related to the transition rate $r(t,E_i,\Delta E)$ from the state E_i to the state E_{i+1} as [21]

$$\pi(t,E_{i+1} \to E_i) = r(t,E_i,\Delta E)\phi(n_i)\psi(n_{i+1}), \tag{3}$$

where $\phi_i \equiv \phi(n_i)$ and $\psi_{i+1} \equiv \psi(n_{i+1})$ are some functions depending, respectively, on the occupational distribution of the initial state E_i and the arrival state E_{i+1}. The functions ϕ_i and ψ_i can inhibit or enhance the transition probability from a site to another and hence define the underlying *exclusion principle* governing the particle kinetics. ϕ_i is proportional to the probability of finding the occupation number n_i in the state E_i, and ψ_i is proportional to the probability of introducing an extra particle into a state with occupational number n_i. In this sense, they can be thought of as the semiclassical analogues of the quantum creation and annihilation operators matrix elements in second quantization:

$$\begin{aligned}\phi(n_i) &\propto |\langle n_{i-1}|\widehat{a}_{n_i}|n_i\rangle|^2,\\ \psi(n_i) &\propto |\langle n_{i+1}|\widehat{a}^{\dagger}_{n_i}|n_i\rangle|^2.\end{aligned} \tag{4}$$

Notice that the transition probability from an empty state must be zero, hence $\phi(0) = 0$. Also, if the arrival state is empty, there is no manifestation of the exclusion principle, and the probability transition is not affected, that is, $\psi(0) = 1$. Rewriting Eq. (2) as a continuity equation and considering Brownian particles, one can relate the functions ϕ_i and ψ_i in the following fashion [21]:

$$\frac{\phi_i}{\psi_i} = e^{-\beta\epsilon_i}, \tag{5}$$

where $\epsilon_i \equiv E_i - \mu$ is the single particle energy defined up to the chemical potential μ. The left-hand side of Eq. (5) contains the classical or quantum behavior of the system and expresses the underlying exclusion principle, and the right-hand side is just the probability of the state E_i at temperature T. We note that up to now, we have been considering a system at equilibrium, within a single temperature T. Let us now allow the temperature, or equivalently β, to fluctuate following some distribution, say, $f(\beta)$, and seek a natural generalization of Eq. (5). In this case, the right-hand side of (5) becomes a conditional probability, $p(E_i \mid \beta)$, and as β varies adiabatically from cell to cell, the resulting probability for finding the system in the state E_i is obtained through marginalization (1),

$$\frac{\phi_i}{\psi_i} = B(\epsilon_i), \tag{6}$$

where

$$B(\epsilon_i) = \int_0^{\infty} d\beta f(\beta)e^{-\beta\epsilon_i}. \tag{7}$$

At this stage, one should point out that behind Eq. (7) there is the assumption that β fluctuates on a large spatiotemporal scale, which makes this approach a form of slow modulation (see Refs. [32,33] for an elaborate discussion). Equation (6) has a very simple and general form; the right-hand side, whose form is defined by the type of fluctuations, $f(\beta)$, defines the effective Boltzmann factor that would appear in the distribution number, and the functions ϕ_i and ψ_i appearing in the left-hand side define the form of extension beyond BE or FD statistics, due to the underlying exclusion principle governing the particle kinetics. A proper choice of ϕ_i and ψ_i allows us to construct superstatistics corresponding to Polychronakos or Haldane-Wu statistics, but before doing so, let us briefly discuss the classical case corresponding to the Maxwell-Boltzmann (MB) distribution. In this case, the transition probability does not depend on the occupational distribution of the arrival site, hence $\phi_i = n_i$ and $\psi_i = 1$. Equation (6) gives then the classical superstatistics [22]:

$$n_i = B(\epsilon_i), \tag{8}$$

where $B(\epsilon_i)$ is related to $f(\beta)$ through Eq. (7). Clearly, the form of the distribution $f(\beta)$ fixes the kind of superstatistics (8). A meaningful case is that when β follows a χ^2 distribution:

$$f(\beta) = \frac{1}{b\Gamma(c)}\left(\frac{\beta}{b}\right)^{c-1} e^{-\beta/b}. \tag{9}$$

The latter appears when many independent microscopic random variables are contributing to β in an *additive* way. In this case, the resulting superstatistics (8) corresponds to Tsallis statistics [22]:

$$B(\epsilon_i) = e_q(-\beta_0\epsilon_i) \quad e_q(x) \equiv [1 + (1-q)x]^{1/(1-q)}, \tag{10}$$

where β_0 is the average of β and $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$ is a universal parameter measuring the strength of fluctuations [22]. Of course, other forms of the distribution $f(\beta)$ lead to different superstatistics. In particular, instead of being a sum of many contributions, the random variable β may be generated by *multiplicative* random processes. In this case β follows a lognormal distribution,

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{\frac{-[\log(\beta/m)]^2}{2s^2}\right\}, \tag{11}$$

which leads to different superstatistics. This time, the superstatistics cannot be evaluated in closed form, but can be obtained numerically or evaluated in the case of sufficiently small energies as

$$B(\epsilon_i) = e^{-\beta_0\epsilon_i}\left[1 + \sum_{l=2}^{\infty} g_l(q)(\beta_0\epsilon_i)^l\right], \tag{12}$$

where the first correction terms read as [22]

$$g_2(q) = \frac{(q-1)}{2} \quad \text{and} \quad g_3(q) = -\frac{1}{6}(q^3 - 3q + 2); \tag{13}$$

the parameter q keeps the same definition: $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2$. In this sense, it constitutes a universal parameter measuring the strength of fluctuations regardless of the type of superstatistics. Note that according to its definition, one has $q \geqslant 1$. However, one can generalize the above superstatistics to cover the $q < 1$ case (see Ref. [34]). In the following, both cases will be considered.

As we have previously outlined, a proper choice for the functions ϕ_i and ψ_i can lead to fractional superstatistics. In particular, one can obtain superstatistics corresponding to Polychronakos statistics. In this statistics, the particle kinetics is that the first particle in a system composed of many particles can occupy one of G states, the second particle one of $G - \gamma$ states, the third particle one of $G - 2\gamma$ states, and so on, where γ $(-1 \leqslant \gamma \leqslant 1)$ is a real parameter related to the exchange statistical parameter α appearing in the quantum phase $e^{i\pi\alpha}$. In this case, the function ψ_i depends on the particle distribution of the arrival site and on the parameter γ, in such a way that the transition probability is enhanced for $\gamma > 0$ (boson-like particle) and inhibited for $\gamma < 0$ (fermion-like particle). Then one has $\phi_i = n_i$ and $\psi_i = 1 + \gamma n_i$. Equation (6) gives then

$$n_i^P = \frac{1}{B^{-1}(\epsilon_i) - \gamma}. \tag{14}$$

In the absence of fluctuations, i.e., $f(\beta) \equiv \delta(\beta - \beta_0)$, $B^{-1}(\epsilon_i)$ reduces to $e^{\beta_0\epsilon_i}$ and Eq. (14) reduces to the usual Polychronakos statistics [15]. If β follows a χ^2 distribution (9), it reduces to the Polychronakos statistics *à la* Tsallis [16,21],

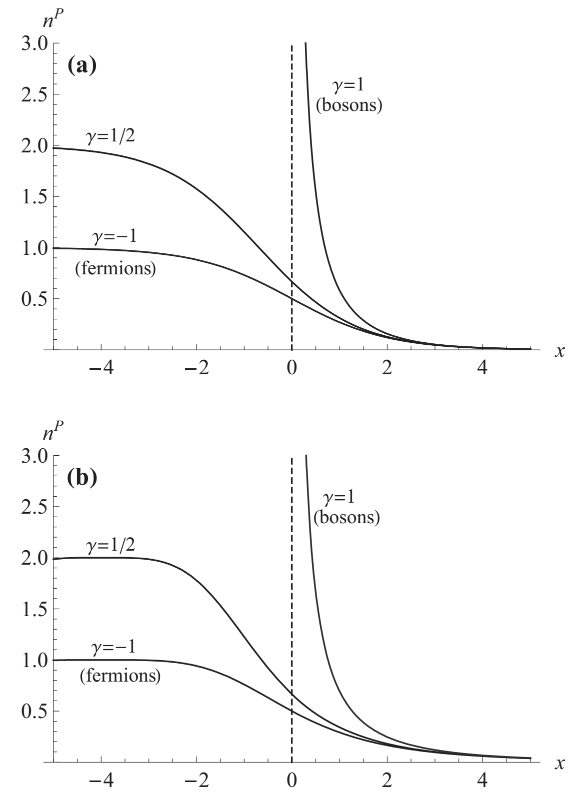

FIG. 1. Plot of the Polychronakos statistics against $x \equiv \beta_0(E_i - \mu)$ for different values of γ: (a) in the absence of fluctuations and (b) for a χ^2 distribution within $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2 = 1.2$.

$$n_i^P = \frac{1}{e_q(\beta_0\epsilon_i) - \gamma}, \tag{15}$$

within the change $q \to 2 - q$. In Fig. 1 we show the behavior of the statistics (15) with $x \equiv \beta_o(E_i - \mu)$, for different values of γ: (a) in absence of fluctuations and (b) with a fluctuation corresponding to the χ^2 distribution.

In a similar way, one can construct superstatistics corresponding to Haldane-Wu statistics. Consistent with Refs. [21,35], one has $\psi_i = [1 - gn_i]^g[1 + (1-g)n_i]^{1-g}$ with $\phi_i = n_i$. It follows from Eq. (6) that

$$n_i^{HW} = \frac{1}{w[B^{-1}(\epsilon_i)] + g}, \tag{16}$$

where $w[B^{-1}(\epsilon_i)]$ is the solution of the transcendental equation

$$w^g(1 + w)^{1-g} = B^{-1}(\epsilon_i). \tag{17}$$

Here the effect of the fluctuations appears in Eq. (17) satisfied by the Wu function $w(x)$. Equation (17) is highly nonlinear for arbitrary values of g, but it is easy to see that the distribution function (16) reduces to the BE superstatistics for $g = 0$ and FD superstatistics for $g = 1$ [36]. Classical superstatistics are recovered when $B^{-1}(\epsilon_i)$ is very large such that $w(x) \simeq x$. Equation (17) can be solved analytically for some values of g beyond Bose and Fermi limits. The simplest result is obtained

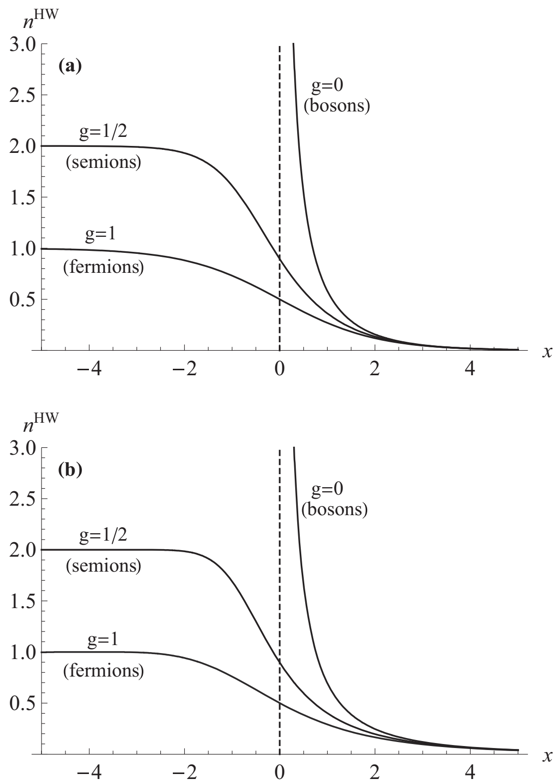

FIG. 2. Plot of the Haldane-Wu statistics against $x \equiv \beta_0(E_i - \mu)$ for different values of g: (a) in the absence of fluctuations and (b) for a χ^2 distribution within $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2 = 1.2$.

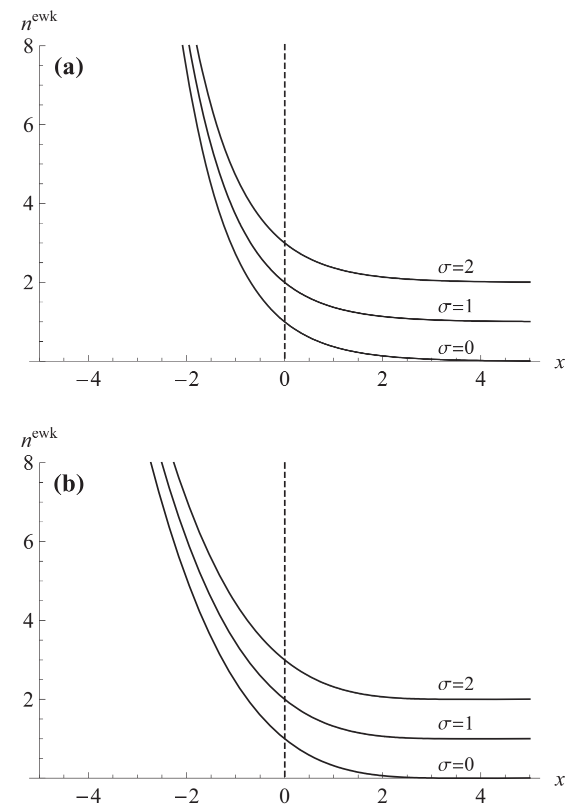

FIG. 3. Plot of ewkons statistics against $x \equiv \beta_0(E_i - \mu)$ for different values of σ: (a) in the absence of fluctuations and (b) for a χ^2 distribution within $q \equiv \langle\beta^2\rangle/\langle\beta\rangle^2 = 1.2$.

for $g = 1/2$ corresponding to so-called *semions:*

$$n_i^{HW} = \frac{1}{\sqrt{1/4 + B^{-2}(\epsilon_i)}}. \quad (18)$$

It is interesting to note that Eq. (17) preserves the duality property that relates the statistics for g and $1/g$ [37]. In the case of superstatistics, the latter reads as

$$1 - g n_g[B^{-1}(\epsilon_i)] = \frac{1}{g} n_{1/g}[B^{1/g}(\epsilon_i)]. \quad (19)$$

In Fig. 2 we show the behavior of the statistics (16) with $x \equiv \beta_o(E_i - \mu)$, for different values of g; (a) in the absence of fluctuations and (b) with a fluctuation corresponding to the χ^2 distribution.

Beyond fractional statistics, extending BE and FD statistics, one may also consider extensions of the classical statistics. In this vein, ewkons statistics has been introduced recently from the assumption that noninteracting particles imply a free diffusion coefficient in energy space [30]. Ewkons statistics reads as

$$n_i^{ewk} = \sigma + e^{-\beta\epsilon_i}. \quad (20)$$

The latter is equal to the MB distribution displaced by some fixed quantity σ, with σ a positive integer. Particles obeying such statistics have the propriety to spread in whole space, assuming a homogeneous number of states per unit volume, and have a negative relation between pressure and energy density, two features that make them suitable to describe dark energy [31]. Like classical particles, ewkons are not subject to an exclusion principle, and there is no maximum occupation number of a state, that is $\psi_i = 1$. Furthermore, ewkons have a nonvacuum ground state, and their kinetics is such that any energy level is occupied by at least σ particles. Accordingly, the transition probability is inhibited by subtracting some quantity σ, that is, $\phi_i = n_i - \sigma$. Ewkons superstatistics follows from Eq. (6) as

$$n_i^{ewk} = \sigma + B(\epsilon_i). \quad (21)$$

Figure 3 displays the behavior of ewkons superstatistics (21) with $x \equiv \beta_o(E_i - \mu)$, for different values of σ: (a) in absence of fluctuations and (b) with a fluctuation corresponding to the χ^2 distribution.

III. VIRIAL AND CLUSTER EXPANSIONS

Consider a two-dimensional system of density ρ_2, $\rho_2 = N/A$ where N is the number of particles and A the area. The virial expansion for the equation of state of such a system reads as

$$\frac{p}{T} = \rho_2[1 + b_2(\rho_2\lambda^2) + b_3(\rho_2\lambda^2)^2 + \cdots], \quad (22)$$

where p and T stand for the pressure and the temperature, respectively, and $\lambda \equiv \sqrt{2\pi\hbar^2/mT}$ is the thermal de Broglie wavelength of a particle of mass m. The dimensionless factors b_i appearing in the expansion (22) are the virial coefficients. For a system of bosons or fermions, the second virial coefficients are known as [38]

$$b_2^F = \tfrac{1}{4} \quad \text{and} \quad b_2^B = -\tfrac{1}{4}, \tag{23}$$

while for an ideal anyon gas, the virial coefficients are functions of the statistical parameter α:

$$\begin{aligned} b_2^{\text{anyon}} &= -\tfrac{1}{4}(1-4\alpha+2\alpha^2),\\ b_3^{\text{anyon}} &= \tfrac{1}{36}+\frac{\sin^2\pi\alpha}{12\pi^2}+c_3\sin^4\pi\alpha\\ c_3 &= -(1.652\pm 0.012)\times 10^{-5}, \end{aligned}\tag{24}$$

where the expression for the second coefficient b_2^{anyon} is exact while b_3^{anyon} is obtained through numerical calculations [39]. To seek for the fractional statistics that describes the best a system of anyons, one can proceed to a comparison between the virial coefficients corresponding to different statistics and b_i^{anyons}. In this respect, both Polychronakos statistics and Haldane-Wu statistics allow us to establish a correspondence with the second virial coefficient of anyons as

$$\begin{aligned} b_2^P &= -\tfrac{1}{4}|\gamma| \quad \text{with}\\ \gamma &= 4\alpha-2\alpha^2-1 \quad \text{(Polychronakos)},\\ b_2^{HW} &= \tfrac{1}{4}(2g-1) \quad \text{with}\\ g &= 2\alpha-\alpha^2 \quad \text{(Haldane-Wu)}. \end{aligned}\tag{25}$$

However, none of the above-mentioned statistics can reproduce correctly the third virial coefficient [38], which suggests therefore the introduction of a more general class of fractional statistics. Such a generalization has been done phenomenologically by Rovenchak [17], with the Tsallis q-exponential standing instead of the conventional one in the expressions for occupation numbers. The obtained statistics allows us to reproduce correctly the third virial coefficient, within a difference only in the fourth one, leading to small correction in the equation of state. In this section, we establish the virial coefficients corresponding to the fractional superstatistics (14) and (16) in a very general form, as a function of the distribution $f(\beta)$, within special attention given to the correspondence that can be established for the χ^2 distribution (9) and the lognormal distribution (11).

The virial coefficients of the different statistics can be determined through the cluster integrals B_l appearing in the expansion of the grand-canonical partition function Ξ as a series over the fugacity $z \equiv e^{\mu/T}$:

$$\frac{1}{A}\ln\Xi = \sum_{l=1}^{\infty} B_l z^l. \tag{26}$$

By virtue of the relation linking the pressure and the density to the grand-canonical partition function, the virial coefficients are linked to the cluster integrals B_l in the following way [40]:

$$\begin{aligned} b_2\lambda^2 &= -\frac{B_2}{B_1^2} \quad b_3\lambda^4 = -2\frac{B_3}{B_1^3}+4\frac{B_2^2}{B_1^4},\\ b^4\lambda^6 &= -3\frac{B_4}{B_1^4}+18\frac{B_2B_3}{B_1^5}-20\frac{B_2^3}{B_1^6},\\ &\cdots . \end{aligned}\tag{27}$$

Having in mind that the total number of particles and the density are given as

$$N = \sum_j g_j n_j, \quad \frac{N}{A} = \frac{1}{A}\sum_j g_j n_j = \sum_l l B_l z^l, \tag{28}$$

where the sum runs over all the energy levels with degeneracies g_j, one can establish the cluster integrals B_l for a particular statistics n_j and obtain immediately the virial coefficients through Eq. (27). At this stage, a crucial point is the definition of the fugacity z over which the expansion is made. Following Ref. [17], we define z such that the occupation numbers (14) and (16) read as

$$n_j^P = \frac{1}{z^{-1}B^{-1}(E_i)-\gamma}, \quad n_j^{HW} = \frac{1}{w[z^{-1}B^{-1}(E_i)]+g}. \tag{29}$$

Note that in general z is different from $B(\mu)$ since for an arbitrary distribution, $B^{-1}(E_i-\mu) \neq B^{-1}(E_i)B(\mu)$. The expansion of the statistics (29) over z is then given as

$$\begin{aligned} n_j^P &= \sum_{l=1}^{\infty} B^l(E_i)\gamma^{l-1}z^l = B(E_i)z + B^2(E_i)\gamma z^2\\ &\quad + B^3(E_i)\gamma^2 z^3 + \cdots, \end{aligned}\tag{30}$$

$$\begin{aligned} n_j^{HW} &= \sum_{m=0}^{\infty}(-1)^m\frac{\Gamma[g(m+1)]}{m!\Gamma[g(m+1)-m]}B^{m+1}(E_i)z^{m+1}\\ &= B(E_i)z-(2g-1)B^2(E_i)z^2+\frac{(3g-2)(3g-1)}{2!}\\ &\quad \times B^3(E_i)z^3 \mp \cdots; \end{aligned}\tag{31}$$

(see Ref. [41]). Assuming that the energy gaps are small compared to the average energy, the sum over states is transformed into an integral,

$$\sum_j \cdots \to \int_0^{\infty} d\varepsilon G(\varepsilon)\cdots, \tag{32}$$

where $G(\varepsilon) = mA/2\pi\hbar^2$ is the density of states of a two-dimensional ideal gas. Cluster integrals B_l are then straightforward to calculate from the expansions (30) and (31), giving in turn the virial coefficients through Eqs. (28) and (29). One obtains the general form for the second and third virial coefficients for Polychronakos superstatistics:

$$\begin{aligned} b_2 &= -\frac{\gamma}{2}\frac{\int_0^{\infty}B^2(x)\,dx}{\left[\int_0^{\infty}B(x)\,dx\right]^2},\\ b_3 &= -\frac{2}{3}\gamma^2\frac{\int_0^{\infty}B^3(x)\,dx}{\left[\int_0^{\infty}B(x)\,dx\right]^3}+\gamma^2\frac{\left[\int_0^{\infty}B^2(x)\,dx\right]^2}{\left[\int_0^{\infty}B(x)\,dx\right]^4}, \end{aligned}\tag{33}$$

and Haldane-Wu superstatistics:

$$\begin{aligned} b_2 &= \frac{(2g-1)}{2}\frac{\int_0^{\infty}B^2(x)\,dx}{\left[\int_0^{\infty}B(x)\,dx\right]^2},\\ b_3 &= -\frac{(3g-2)(2g-3)}{3}\frac{\int_0^{\infty}B^3(x)\,dx}{\left[\int_0^{\infty}B(x)\,dx\right]^3}\\ &\quad +(1-2g)^2\frac{\left[\int_0^{\infty}B^2(x)\,dx\right]^2}{\left[\int_0^{\infty}B(x)\,dx\right]^4}. \end{aligned}\tag{34}$$

Note that, for an arbitrary distribution $f(\beta)$, the evaluation of the integrals appearing the coefficients (33) and (34) is not straightforward and may necessitate a numerical estimation. However, in the case of χ^2 superstatistics, they can be obtained analytically, and the virial coefficients may be written in a very simple form as follows:

$$b_2 = -\frac{\gamma q^2}{2(1+q)}, \quad b_3 = \gamma^2 q^4 \left[\frac{1}{(1+q)^2} - \frac{1}{3q(2+q)}\right] \quad \text{(Polychronakos)},$$
$$b_2 = \frac{2g-1}{2}\,\frac{q^2}{1+q}, \quad b_3 = q^4 \left[\frac{(2g-1)^2}{(1+q)^2} - \frac{(3g-2)(3g-1)}{3q(2+q)}\right] \quad \text{(Haldane-Wu)}, \tag{35}$$

which corresponds to the virial coefficients established by Rovenchak [16] in the context of Tsallis statistics. One can check that the coefficients (35) allow us to establish a partial correspondence with the virial coefficients of an ideal anyon gas (see Ref. [16], Table 1). In the case of lognormal superstatistics, the coefficients (33) and (34) cannot be obtained in closed form, but a numerical evaluation allows us to establish such a correspondence between the parameters (γ, q) of the Polychronakos superstatistics and (g, q) of the Haldane-Wu superstatistics, by equating the coefficients (33) and (34) and the virial coefficients of an ideal anyon gas (24). The results are displayed in Table I.

IV. A POSSIBLE GENERALIZATION

To obtain the fractional superstatistics (14) and (16), the fluctuating quantity was identified with the inverse temperature β. In other words, we were considering a nonequilibrium system with a long-term stationary state composed of smaller parts that are temporarily in thermal equilibrium. In many systems however, another quantity may exhibit spatiotemporal fluctuations, or even more than one quantity at the same time. In practice, such a fluctuating quantity may be the chemical potential, an effective friction constant, a changing mass parameter, a changing amplitude of Gaussian white noise, the fluctuating energy dissipation in turbulent flows [33], and so on. In the context of fractional statistics, it is natural to ask *what kind of statistics may emerge from a fluctuating statistical parameter?* Note that some alternative generalizations of fractional statistics, entering through modifications of the statistical parameter itself, have been addressed in the literature, for example, by considering a complex-valued parameter [42–44]. Let us address here the simple case of the Polychronakos statistics where the parameter $\gamma \in [-1,1]$ is subject to fluctuations according to some distribution, say, $h(\gamma)$. As γ varies slowly from cell to cell, the resulting distribution arises out of the occupational numbers associated with the cells that are averaged over the various fluctuating γ,

$$n^P = \int_{-1}^{+1} \frac{d\gamma\, h(\gamma)}{e^{\beta\epsilon_i} - \gamma}, \tag{36}$$

where the function $h(\gamma)$ must be a normalized probability density that reduces to $\delta(\gamma - \gamma_0)$ in some limit, in which case Eq. (36) reduces to the usual Polychronakos statistics. Let us consider the simple case where γ follows a two-level distribution for which the occupational number (36) is straightforward to deduce. In this very simple model, the parameter γ can switch between two discrete values γ_1 and γ_2 with probabilities a_1 and a_2:

$$h(\gamma) \equiv a_1\delta(\gamma-\gamma_1) + a_2\delta(\gamma-\gamma_2), \tag{37}$$

where $a_1 + a_2 = 1$. In this case, the statistics (36) reads as a mixture of two fractional statistics with different parameters:

$$n^P = \frac{a_1}{e^{\beta\epsilon_i} - \gamma_1} + \frac{a_2}{e^{\beta\epsilon_i} - \gamma_2}, \tag{38}$$

which reduces to the Polychronakos statistics if $a_1 = 0$ or $a_2 = 0$. The form of the statistics (38), as a sum of different statistics, is reminiscent of the Gentile statistics [45] introduced in 1940. As a final remark to close this section, we note that in the case of a fluctuating β, the distribution $f(\beta)$ can be obtained by maximizing the entropy under appropriate constraints [46,47]. A similar approach can be done to determine the possible distributions $h(\gamma)$, considering an adequate set of constraints. This, however, is out of the scope of the present work.

TABLE I. Dependence of the parameters of the lognormal fractional superstatistics on the statistical parameter α.

	Polychronakos		Haldane-Wu	
α	γ	q	g	q
0.00	1.00000	1.00000	0.00000	1.00000
0.10	0.32995	0.80307	0.31097	0.82656
0.20	0.00379	1.70443	0.37630	0.17609
0.30	0.00365	1.64577	0.50856	0.18452
0.40	− 0.00117	1.65859	0.62007	0.18409
0.50	− 0.00978	1.68433	0.67562	0.86253
0.60	− 0.44921	1.40585	0.73963	0.23038
0.70	− 0.73162	1.31812	0.76343	0.24905
0.80	− 0.79936	0.92959	0.78294	0.25727
0.90	− 0.94408	0.97738	0.87794	0.19115
1.00	− 1.00000	1.00000	1.00000	1.00000

V. SUMMARY

In summary, through a semiclassical kinetic approach that accounts for temperature fluctuations, a class of fractional statistics was derived. The latter are the superstatistical analogues of the Polychronakos and the Haldane-Wu statistics. Superstatistics of "ewkons," introduced recently as a plausible model for dark energy, was also obtained within the same approach. In contrast to the one-parameter generalizations of fractional statistics previously reported in the literature, the peculiar feature of such statistics is that the generalization is made through a distribution function characterizing temperature fluctuations. A particular case is when the inverse temperature follows a χ^2 distribution, which results in the Tsallis form of fractional statistics known in the literature [16,20,21]. Special attention was given to the examination of superstatistics that follows from the χ^2 distribution and the log-normal distribution, emerging respectively from additive and multiplicative random processes. The virial coefficients corresponding to these superstatistics were derived and compared to those of an ideal anyon gas. In addition, a further generalization was suggested by allowing the statistical parameter to fluctuate, which may result in another form of fractional statistics, reminiscent of Gentile statistics. The obtained statistics are expected to model systems obeying fractional statistics in a fluctuating environment. Potential application areas include anyons and dark energy models, exhibiting spatiotemporal fluctuations. Furthermore, the present approach opens some prospects for further studies on a more fundamental ground. A closer look at superstatistics characterized by another type of fluctuations, such as the inverse χ^2 distribution or the F distribution, seems worthwhile to pursue.

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