Emergence of nonwhite noise in Langevin dynamics with magnetic Lorentz force

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We investigate the low mass limit of Langevin dynamics for a charged Brownian particle driven by a magnetic Lorentz force. In the low mass limit, velocity variables relaxing quickly are coarse-grained out to yield effective dynamics for position variables. Without the Lorentz force, the low mass limit is equivalent to the high friction limit. Both cases share the same Langevin equation that is obtained by setting the mass to zero. The equivalence breaks down in the presence of the Lorentz force. The low mass limit cannot be achieved by setting the mass to zero. The limit is also distinct from the large friction limit. We derive the effective equations of motion in the low mass limit. The resulting stochastic differential equation involves a nonwhite noise whose correlation matrix has antisymmetric components. We demonstrate the importance of the nonwhite noise by investigating the heat dissipation by a driven Brownian particle, where the emergent nonwhite noise has a physically measurable effect.

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Recently, the dynamics of Brownian particles driven by velocity-dependent forces has attracted growing interest. The magnetic Lorentz force is a representative example of a velocity-dependent force [1–13]. It can be realized in experimental systems. In a superionic conductor, e.g., AgI, Ag^+ ions diffuse over the I^- ion background. The Lorentz force can be applied to Ag^+ ions with an external magnetic field [14,15]. The active matters are modeled with velocity-dependent forces. Such phenomenological forces are adopted in order to study the collective phenomena of active matters [16–22]. In stochastic thermodynamics, theoretical works focus on the extension of the entropy production, fluctuation theorems, fluctuation-dissipation relations, and the detailed balance to thermal systems driven by velocity-dependent forces [23–27]. In this paper, we investigate the low mass limit and the large friction limit of Langevin dynamics for a charged Brownian particle under a uniform external magnetic field. The magnetic Lorentz force is one of the fundamental forces. We will show that nonwhite noise emerges in the low mass limit in the presence of a magnetic Lorentz force.

Without velocity-dependent forces, the dynamics of a Brownian particle is described by the Langevin equation for its position $\boldsymbol{x}$ and velocity $\boldsymbol{v}$,

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{v}(t),\\ m\dot{\boldsymbol{v}}(t) &= \boldsymbol{f}(\boldsymbol{x}(t)) - \gamma\boldsymbol{v}(t) + \boldsymbol{\xi}(t),\end{aligned} \tag{1}$$

where $\boldsymbol{f}(\boldsymbol{x})$ is an external force, γ is a friction coefficient, and $\boldsymbol{\xi}(t)$ is the Gaussian white noise satisfying $\langle\xi_i(t)\rangle = 0$ and $\langle\xi_i(t)\xi_j(s)\rangle = 2\gamma T\delta_{ij}\delta(t-s)$ with the temperature T of the environment. We set the Boltzmann constant to unity. Since the last century, the Langevin equation has served as a framework for the study of the equilibrium and nonequilibrium dynamics of thermal systems [28–31]. It also plays a crucial role in the recent development of stochastic thermodynamics, the statistical physics theory at the level of microscopic stochastic trajectories [32,33].

In experimental situations, the damping force usually dominates the other forces [34,35]. Then, the velocity relaxes quickly in a time scale $\tau_r = m/\gamma$, and the inertia term $m\dot{\boldsymbol{v}}$ becomes negligible for $t \gg \tau_r$. The effective equations of motion in the limit are obtained in the following way: (i) One considers the Fokker-Planck (FP) equation for the probability distribution $P_t(\boldsymbol{x},\boldsymbol{v})$ corresponding to the Langevin equation (1). (ii) One then performs the $1/\gamma$ expansion to derive the effective FP equation for the coarse-grained probability distribution $Q_t(\boldsymbol{x}) \equiv \int d\boldsymbol{v}\, P_t(\boldsymbol{x},\boldsymbol{v})$. The expansion can be done systematically by using the Brinkman expansion [29,36] or the projection operator method [30]. (iii) The FP equation is transformed back to the Langevin equation. The resulting overdamped Langevin equation reads

$$\gamma\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{\xi}(t). \tag{2}$$

It has the same form as that obtained by setting $m = 0$ in (1). Namely, the systems in the large friction limit, in the low mass limit, and with zero mass are equivalent to each other. They all share the same Langevin equation (2). The overdamped limit of the Langevin equation with multiplicative noises was also studied [37–43].

The large friction limit and the low mass limit of Langevin dynamics has not been studied thoroughly in the presence of velocity-dependent forces. Some literature has studied the large friction dynamics of Lorentz force systems by setting m to zero [2,10–13]. We raise the question whether the equivalence between the low mass limit, the large friction limit, and the zero mass case is still valid in the presence of a magnetic Lorentz force, one of the simplest examples of velocity-dependent forces. We will derive the stochastic differential equation for motion in the low mass limit. It turns out that the low mass limit is singular. The dynamics in the low mass limit is different from that with zero mass and from that in the large friction limit. We discover that nonwhite noise emerges in the low mass limit. The nonwhite noise has an intriguing correlation property which has yet to be studied. Our work will open up a different avenue in the study of stochastic differential equations. It may also have an impact on experimental systems such as the superionic conduction mentioned earlier.

Suppose that the magnetic field is directed to the z direction, $\boldsymbol{B} = B_0\hat{z}$. The Lorentz force does not have a z component. Thus, we focus on the two-dimensional motion of the Brownian particle. The position and the velocity are denoted by the column vectors $\boldsymbol{x} = (x_1,x_2)^T$ and $\boldsymbol{v} = (v_1,v_2)^T$, where the superscript T stands for the transpose. The Langevin equation becomes $\dot{\boldsymbol{x}} = \boldsymbol{v}$ and

$$m\dot{\boldsymbol{v}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) - \mathsf{G}\boldsymbol{v}(t) + \boldsymbol{\xi}(t), \tag{3}$$

where the 2×2 matrix G is defined as

$$\mathsf{G} = \begin{pmatrix} \gamma & -B \\ B & \gamma \end{pmatrix}, \tag{4}$$

with $B = qB_0$. The external force $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))^T$ and the white noise $\boldsymbol{\xi}(t) = (\xi_1(t),\xi_2(t))^T$ are also denoted by two-dimensional column vectors.

The equations of motion in the low mass limit may be obtained indirectly by using the FP equation representation. This method works well for systems without velocity-dependent force [29]. The probability distribution $P_t(\boldsymbol{v},\boldsymbol{x})$ satisfies the FP or Kramer equation

$$\partial_t P_t(\boldsymbol{x},\boldsymbol{v}) = (L_{\rm rev} + L_{\rm irr})P_t(\boldsymbol{x},\boldsymbol{v}), \tag{5}$$

where $L_{\rm rev}$ ($L_{\rm irr}$) is the reversible (irreversible) part of the time evolution operator. They are given by

$$L_{\rm rev} = -\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} - \frac{\boldsymbol{f}}{m}\cdot\nabla_{\boldsymbol{v}} - \frac{q}{m}\nabla_{\boldsymbol{v}}\cdot(\boldsymbol{v}\times\boldsymbol{B}),$$
$$L_{\rm irr} = \frac{\gamma}{m}\nabla_{\boldsymbol{v}}\cdot\left(\boldsymbol{v} + \frac{T}{m}\nabla_{\boldsymbol{v}}\right). \tag{6}$$

We use the shorthand notation ∂_α for the partial derivative with respective to a variable α. When one takes the cross product, a two-dimensional vector should be regarded as a three-dimensional one with a null z component.

Following the standard procedure [29], we first rewrite (5) in terms of $\bar{P}_t(\boldsymbol{x},\boldsymbol{v}) = [\psi_0(v_1)\psi_0(v_2)]^{-1}P_t(\boldsymbol{x},\boldsymbol{v})$ and $\bar{L}_{\rm rev,irr} = [\psi_0(v_1)\psi_0(v_2)]^{-1}L_{\rm rev,irr}[\psi_0(v_1)\psi_0(v_2)]^1$ with $\psi_0(v) \equiv (2\pi T/m)^{-1/4}e^{-mv^2/(4T)}$. Then, the transformed distribution is expanded as

$$\bar{P}_t(\boldsymbol{x},\boldsymbol{v}) = \sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}(\boldsymbol{x},t)\psi_{n_1}(v_1)\psi_{n_2}(v_2) \tag{7}$$

in terms of the orthonormal basis functions $\psi_n(v) \equiv \left(-\sqrt{\frac{T}{m}}\partial_v + \frac{1}{2}\sqrt{\frac{m}{T}}v_i\right)^n\psi_0(v)/\sqrt{n!}$. The FP equation yields the coupled differential equations for the coefficients $\{c_{n_1,n_2}\}$, called the Brinkman's hierarchy [29]. Among all the coefficients, $c_{0,0}(\boldsymbol{x},t)$ is the most important one since it is equal to the marginal distribution $Q_t(\boldsymbol{x}) = \int d\boldsymbol{v}P_t(\boldsymbol{x},\boldsymbol{v})$. Orthonormality of $\{\psi_n(v)\}$ ensures the equality $c_{0,0}(\boldsymbol{x},t) = Q_t(\boldsymbol{x})$. In the low mass limit, the hierarchy is closed within the set of three coefficients $\{c_{0,0},c_{1,0},c_{0,1}\}$. Introducing the notation $\boldsymbol{c} = (c_{1,0},c_{0,1})^T$, it becomes $\partial_t c_{0,0} = -\nabla_{\boldsymbol{x}}\cdot(\sqrt{\frac{T}{m}}\boldsymbol{c})$ and $\sqrt{\frac{T}{m}}\boldsymbol{c} = \mathsf{G}^{-1}(\boldsymbol{f} - T\nabla_{\boldsymbol{x}})c_{0,0} + O(m)$. Combining these equations, we obtain the effective FP equation

$$\partial_t Q_t(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\cdot\boldsymbol{J}, \tag{8}$$

with the probability current

$$\boldsymbol{J} = [\mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x}) - T\mathsf{G}^{-1}\nabla_{\boldsymbol{x}}]Q_t(\boldsymbol{x}). \tag{9}$$

The first term represents the drift current and the second term the diffusion current. Details of the derivation are presented in Appendix A.

The diffusion current has an abnormal form. For the Langevin system, the diffusion current is given by the product of a *symmetric* diffusion matrix and the gradient of the probability distribution [29–31]. By contrast, the matrix G^{-1} has *antisymmetric* components $(\mathsf{G}^{-1})_{12} = -(\mathsf{G}^{-1})_{21}$. Such a diffusion current cannot be realized by any Langevin system. As a remedy, one may replace the probability current $\boldsymbol{J}$ with $\boldsymbol{J}_s = \mathsf{G}^{-1}\boldsymbol{f} - T\mathsf{G}_s^{-1}\nabla_{\boldsymbol{x}}Q$ using the symmetrized matrix $\mathsf{G}_s^{-1} \equiv [\mathsf{G}^{-1} + (\mathsf{G}^{-1})^T]/2$. Noting that $\nabla_{\boldsymbol{x}}\cdot\mathsf{G}^{-1}\nabla_{\boldsymbol{x}} = \nabla_{\boldsymbol{x}}\cdot\mathsf{G}_s^{-1}\nabla_{\boldsymbol{x}}$, one finds that the symmetrized current leaves the FP equation (8) unchanged. The symmetrized FP equation is equivalent to the Langevin equation

$$\dot{\boldsymbol{x}}(t) = \mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{\zeta}(t), \tag{10}$$

where $\boldsymbol{\zeta}(t)$ is the white noise satisfying $\langle\boldsymbol{\zeta}(t)\rangle = 0$ and $\langle\boldsymbol{\zeta}(t)\boldsymbol{\zeta}(s)^T\rangle = 2T\mathsf{G}_s^{-1}\delta(t-s)$.

We notice the equality $\mathsf{G}_s^{-1} = \gamma\mathsf{G}^{-1}(\mathsf{G}^{-1})^T$ for the specific matrix G in (4). It implies that the noise $\boldsymbol{\zeta}(t)$ has the same statistical property as $\mathsf{G}^{-1}\boldsymbol{\xi}(t)$ with the white noise $\boldsymbol{\xi}(t)$ in (3). Thus, the effective Langevin equation (10) is equivalent to the one obtained by setting m to zero from the original Langevin equation (3). One may be tempted to conclude that the low mass limit is also equivalent to the mass zero system in the presence of the Lorentz force. However, the Langevin equation (10) does not reproduce the probability current (9). Furthermore, as will be shown later, the dissipations in the system (10) and (3) are different from each other in the $m \to 0$ limit. These observations strongly suggest that the Langevin equation in (10) is not the proper low mass limit.

As the FP equation approach fails, we derive the low mass limit directly from the equations of motion. We start with the formal solution

$$\boldsymbol{v}(t) = \frac{1}{m}\int_0^t dt' e^{-\mathsf{G}(t-t')/m}[\boldsymbol{f}(\boldsymbol{x}(t')) + \boldsymbol{\xi}(t')] \tag{11}$$

of the Langevin equation (3). We omitted the transient term $e^{-\frac{\mathsf{G}}{m}t}\boldsymbol{v}(0)$ because it is negligible for finite t in the small m limit. The transient term will always be neglected. The formal solution leads to the stochastic integrodifferential equation for $\boldsymbol{x}(t)$,

$$\dot{\boldsymbol{x}}(t) = \frac{1}{m}\int_0^t dt'\, e^{-\mathsf{G}(t-t')/m}\boldsymbol{f}(\boldsymbol{x}(t')) + \boldsymbol{\eta}_m(t), \tag{12}$$

where the noise is given by

$$\boldsymbol{\eta}_m(t) = \frac{1}{m}\int_0^t dt'\, e^{-\mathsf{G}(t-t')/m}\boldsymbol{\xi}(t'). \tag{13}$$

We first reveal the statistical property of the noise. The noise $\boldsymbol{\eta}_m$ is Gaussian distributed with $\langle\boldsymbol{\eta}_m(t)\rangle = 0$ and $\langle\boldsymbol{\eta}_m(t)\boldsymbol{\eta}_m(s)^T\rangle = \mathsf{C}_m(t,s)$, where the correlation matrix is given by

$$\begin{aligned}\mathsf{C}_m(t,s) &= \frac{T}{m}e^{-\frac{1}{m}(\mathsf{G}t+\mathsf{G}^T s)+\frac{1}{m}(\mathsf{G}+\mathsf{G}^T)\min(t,s)} \\ &= \begin{cases} \frac{T}{m}e^{-\frac{\mathsf{G}}{m}(t-s)} & \text{if } t \geqslant s, \\ \frac{T}{m}e^{-\frac{\mathsf{G}^T}{m}(s-t)} & \text{if } t < s. \end{cases}\end{aligned} \tag{14}$$

As it depends on $(t-s)$, we will use the notation $\mathsf{C}_m(t-s)$ for the correlation matrix. It satisfies $\mathsf{C}_m(-u) = \mathsf{C}_m(u)^T$. The elements are given by

$$\mathsf{C}_m(u) = \frac{T}{m}e^{-\frac{\gamma}{m}|u|}\begin{pmatrix} \cos\left(\frac{B}{m}u\right) & \sin\left(\frac{B}{m}u\right) \\ -\sin\left(\frac{B}{m}u\right) & \cos\left(\frac{B}{m}u\right)\end{pmatrix}. \tag{15}$$

The magnetic field generates oscillating antisymmetric off-diagonal components.

The correlation functions oscillate with an amplitude decaying exponentially. As m decreases, they become singular with diverging oscillation frequency B/m, vanishing decay time m/γ, and diverging amplitude T/m. In order to extract the limiting behavior, we consider the integral $I_\alpha \equiv \frac{1}{m}\int_0^\infty du u^\alpha e^{-\frac{\gamma+iB}{m}u}$ for $\alpha \geqslant 0$. A straightforward algebra yields that

$$I_\alpha = \frac{\Gamma(1+\alpha)}{(\gamma+iB)^{1+\alpha}}m^\alpha, \tag{16}$$

with the gamma function $\Gamma(z) = \int_0^\infty dx x^{z-1}e^{-x}$. Thus, for any function $h(u)$ having a nonsingular expansion around $u=0$, we have $\frac{1}{m}\int_0^\infty du\, h(u)e^{-\frac{\gamma+iB}{m}u} = I_0\sum_{\alpha=0}^\infty (mI_0)^\alpha \frac{d^\alpha h(u)}{du^\alpha}\big|_{u=0}$. It can be approximated by the limiting value $I_0 h(0)$ when the first term with $\alpha=0$ is dominant over the other terms with $\alpha>0$. This approximation is valid in the regime $|mI_0| = m/\sqrt{\gamma^2+B^2} \ll \tau$ with the characteristic time scale τ of $h(u)$, where the low mass limit provides a leading-order contribution.

The low mass limit yields that

$$\begin{aligned}\lim_{m\to0}\int_0^\infty du h(u)\mathsf{C}_m(u) &= h(0)T\mathsf{G}^{-1}, \\ \lim_{m\to0}\int_{-\infty}^0 du h(u)\mathsf{C}_m(u) &= h(0)T(\mathsf{G}^{-1})^T.\end{aligned} \tag{17}$$

The second equality comes from the symmetry property $\mathsf{C}_m(-u) = \mathsf{C}_m(u)^T$. We introduce the notations $\delta_\pm(u)$ as the variants of the Dirac δ function. They are equal to zero for $u \neq 0$ while $\int_0^\infty du\,\delta_+(u) = \int_{-\infty}^0 du\,\delta_-(u) = 1$ and $\int_0^\infty du\,\delta_-(u) = \int_{-\infty}^0 du\,\delta_+(u) = 0$. Then, the correlation matrix in the $m\to0$ limit ($m/\sqrt{\gamma^2+B^2} \ll \tau$ to be more precise) is represented as

$$\mathsf{C}(u) \equiv \lim_{m\to0}\mathsf{C}_m(u) = T\mathsf{G}^{-1}\delta_+(u) + T(\mathsf{G}^{-1})^T\delta_-(u). \tag{18}$$

We next consider the first term on the right-hand side of (12). When one changes the integration variable from t' to $u=(t-t')$, it is written as $\frac{1}{T}\int_0^t du\,\mathsf{C}_m(u)\boldsymbol{f}(\boldsymbol{x}(t-u))$. It converges to $\mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x}(t))$ from (17). Therefore, we finally obtain the effective equations of motion in the low mass limit,

$$\dot{\boldsymbol{x}}(t) = \mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{\eta}(t), \tag{19}$$

where the noise $\boldsymbol{\eta}(t)$ has the correlation matrix C in (18). It is a nonwhite noise whose correlation matrix C is nonsymmetric. The antisymmetric components of C make (19) fundamentally different from (10).

By analogy with the Wiener process $\boldsymbol{W}(t) = \int_0^t dt'\boldsymbol{\xi}(t')$ with white noise $\boldsymbol{\xi}(t)$, one may consider the time-integrated quantity $\boldsymbol{\Omega}(t) = \int_0^t dt'\boldsymbol{\eta}(t')$. It will be called the Ω process. The statistical properties of the Ω process are summarized as

$$\langle\boldsymbol{\Omega}(t)\boldsymbol{\Omega}(s)^T\rangle = \frac{2\gamma T}{\gamma^2+B^2}\min(t,s)\mathsf{I}, \tag{20}$$

$$\langle\boldsymbol{\Omega}(t)\boldsymbol{\eta}(s)^T\rangle = \begin{cases} \frac{2\gamma T}{\gamma^2+B^2}\mathsf{I}, & \text{if } t > s, \\ T(\mathsf{G}^{-1})^T, & \text{if } t = s, \\ 0, & \text{otherwise},\end{cases} \tag{21}$$

with the identity matrix I. These are derived by taking the $m\to0$ limit of the corresponding quantities with $\boldsymbol{\eta}_m$.

Because of the nonwhite noise $\boldsymbol{\eta}(t)$, the stochastic equation (19) does not have a corresponding FP equation. Nevertheless, the equation governing the time evolution of the probability distribution can be derived by using the functional derivative method [37,44–46]. The probability distribution is given by $Q_t(\boldsymbol{x}) = \langle\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle$, where $\boldsymbol{x}(t)$ is a functional of the noise $\{\boldsymbol{\eta}(s)|0<s<t\}$ and $\langle\ \rangle$ denotes the average over the noise realizations. The time derivative of $Q_t(\boldsymbol{x})$ involves $\partial_t\delta(\boldsymbol{x}(t)-\boldsymbol{x}) = [\dot{\boldsymbol{x}}(t)\cdot\nabla_{\boldsymbol{x}(t)}]\delta(\boldsymbol{x}(t)-\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\cdot[\dot{\boldsymbol{x}}(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})]$, where the last equality is obtained by using the property of the δ function. Thus, the time evolution of $Q_t(\boldsymbol{x})$ is governed by $\partial_t Q_t(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\cdot\boldsymbol{J}(\boldsymbol{x},t)$ with $\boldsymbol{J}(\boldsymbol{x},t) = \langle\dot{\boldsymbol{x}}(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle$. Eliminating $\dot{\boldsymbol{x}}(t)$ using (19), one obtains

$$\boldsymbol{J}(\boldsymbol{x},t) = \mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x})Q_t(\boldsymbol{x}) + \langle\boldsymbol{\eta}(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle. \tag{22}$$

In order to evaluate $\langle\boldsymbol{\eta}(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle$, we use the Novikov relation [47]

$$\langle\eta_i(t)F[\boldsymbol{\eta}]\rangle = \sum_j\int_0^t ds C_{ij}(t-s)\left\langle\frac{\delta F[\boldsymbol{\eta}]}{\delta\eta_j(s)}\right\rangle \tag{23}$$

for any functional $F[\boldsymbol{\eta}]$ with the noise-noise correlation matrix C_{ij}. Taking $F[\boldsymbol{\eta}] = \delta(\boldsymbol{x}(t)-x)$ and noting that $\boldsymbol{x}(t)$ is a functional of $\boldsymbol{\eta}$, we have

$$\begin{aligned}\langle\eta_i(t)\delta(\boldsymbol{x}(t)-\boldsymbol{x})\rangle = -\sum_{j,k}\int_0^t ds C_{ij}(t-s) \\ \times\frac{\partial}{\partial x_k}\left\langle\frac{\delta x_k(t)}{\delta\eta_j(s)}\delta(\boldsymbol{x}(t)-\boldsymbol{x})\right\rangle.\end{aligned} \tag{24}$$

It is a formidable task to find a closed form expression for the functional derivative $\delta x_k(t)/\delta\eta_j(s)$ at arbitrary values of t and s. Fortunately, owing to the property of C in (18), it suffices to consider the functional derivative at $s=t^-$. It is given by $\lim_{s\to t^-}\delta x_k(t)/\delta\eta_j(s) = \delta_{jk}$. Consequently, the probability current in (22) is the same as that in (9). It confirms that the stochastic differential equations (19) are indeed the proper equations of motion in the low mass limit.

We add a remark on the large friction limit. In the large γ limit, the quantity in (16) is given by $I_\alpha = \Gamma(1+\alpha)m^\alpha/\gamma^{1+\alpha}[1+O(B/\gamma)]$. It yields that $\mathsf{C}_m(u) = \frac{T}{\gamma}[\delta_+(u)+\delta_-(u)]\mathsf{I} + O(\gamma^{-2})$. Thus, in the leading order in $1/\gamma$, the equations of motion are given by $\gamma\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) + \boldsymbol{\xi}(t)$ with

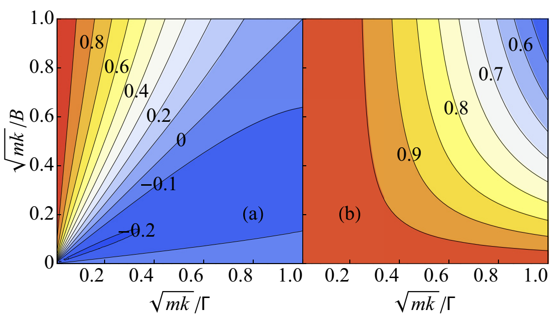

FIG. 1. Contour plot for $w_{\rm zm}/w_{\rm fm}$ in (a) and $w_{\rm lm}/w_{\rm fm}$ in (b) as varying γ and B with fixed $k = m = \epsilon = 1$.

white noise $\boldsymbol{\xi}(t)$ with a variance $2\gamma T$. The Lorentz force contributes as a $O(B/\gamma^2)$ correction, and is discarded in the leading order. It shows that the large friction limit is different from the low mass limit.

We demonstrate the crucial role of nonwhite noise $\boldsymbol{\eta}(t)$ with a linear system. Consider a two-dimensional motion of a Brownian particle of charge q in the xy plane. It is trapped by a conservative harmonic force $\boldsymbol{f}_c(\boldsymbol{x}) = -k\boldsymbol{x}$ and driven by a nonconservative rotating force $\boldsymbol{f}_{nc}(\boldsymbol{x}) = \epsilon \boldsymbol{x} \times \hat{\boldsymbol{z}}$. The uniform magnetic field $\boldsymbol{B} = B_0\hat{\boldsymbol{z}}$ is applied to the z direction. The Langevin equation reads $\dot{\boldsymbol{x}}(t) = \boldsymbol{v}$ and

$$m\dot{\boldsymbol{v}}(t) = -\mathsf{K}\boldsymbol{x}(t) - \mathsf{G}\boldsymbol{v}(t) + \boldsymbol{\xi}(t), \tag{25}$$

where the force matrix K is given by $\mathsf{K} = \begin{pmatrix} k & -\epsilon \\ \epsilon & k \end{pmatrix}$ and the matrix G is given in (4) with $B = qB_0$. The nonconservative force performs a work on the particle and the injected energy is dissipated into the heat bath as a heat. The linear system has been studied extensively for its nontrivial steady state properties and nonequilibrium fluctuation theorems of the work and heat [48–52].

We focus on the average power $w = \langle \boldsymbol{f}_{nc} \cdot \boldsymbol{v}\rangle_s = \epsilon\langle(\boldsymbol{x} \times \hat{\boldsymbol{z}}) \cdot \boldsymbol{v}\rangle_s = -\epsilon\langle(x_1v_2 - x_2v_1)\rangle_s$ of the work done by the nonconservative force $\boldsymbol{f}_{nc}$ in the steady state. $\langle\ \rangle_s$ denotes the steady state average. We calculate the power for three systems: $w_{\rm fm}$ from the original Langevin equation in (25) with finite m, $w_{\rm zm}$ from (10) where m is set to zero, and $w_{\rm lm}$ from the low mass limit in (19). They are given by

$$w_{\rm fm} = \frac{2\epsilon^2 T}{\gamma k + \epsilon B - m\epsilon^2/\gamma}, \tag{26}$$

$$w_{\rm zm} = \frac{2\epsilon^2 T}{\gamma k + \epsilon B} - \frac{2\epsilon BT}{\gamma^2 + B^2}, \tag{27}$$

$$w_{\rm lm} = \frac{2\epsilon^2 T}{\gamma k + \epsilon B}, \tag{28}$$

whose derivations are presented in Appendix B. They are compared in Fig. 1. The ratio $w_{\rm lm}/w_{\rm fm}$ approaches unity as m decreases irrespective of the relative strength of γ and B, while $w_{\rm zm}/w_{\rm fm}$ is singular in the low mass limit. It demonstrates that nonwhite noise is essential for the proper prediction of the dissipation in the low mass limit $m/\sqrt{\gamma^2 + B^2} \ll \tau \simeq \sqrt{m/K}$. The difference $w_{\rm lm} - w_{\rm zm} = \frac{2\epsilon BT}{\gamma^2+B^2}$ represents a physical effect due to the emergence of nonwhite noise.

We can pinpoint the origin for the discrepancy between $w_{\rm zm}$ and $w_{\rm lm}$. The equations of motion (19), $\dot{\boldsymbol{x}}(t) = -\mathsf{A}\boldsymbol{x} + \boldsymbol{\eta}(t)$ with $\mathsf{A} \equiv \mathsf{G}^{-1}\mathsf{K}$, have the formal solution $\boldsymbol{x}(t) = \int_0^t ds\, e^{-\mathsf{A}(t-s)}\boldsymbol{\eta}(s)$. The power involves the correlation matrix $\langle\boldsymbol{x}(t)\dot{\boldsymbol{x}}(t)^T\rangle_s = -\langle\boldsymbol{x}(t)\boldsymbol{x}(t)^T\rangle_s\mathsf{A}^T + \langle\boldsymbol{x}(t)\boldsymbol{\eta}(t)^T\rangle_s$. We can use the formal solution to evaluate the correlation functions in terms of the noise-noise correlation matrix C. Especially, the second term becomes $\langle\boldsymbol{x}(t)\boldsymbol{\eta}(t)^T\rangle_s = \int_0^t ds e^{-\mathsf{A}(t-s)}\mathsf{C}(s-t) = T(\mathsf{G}^{-1})^T$ using (18). On the contrary, if one adopts the equations of motion (10), one obtains $\langle\boldsymbol{x}(t)\boldsymbol{\zeta}(t)^T\rangle_s = T(\mathsf{G}_s^{-1})^T$, which misses the antisymmetric component of G^{-1}. It makes $w_{\rm zm}$ deviate from $w_{\rm lm} = \lim_{m\to 0} w_{\rm fm}$.

In summary, we have discovered a different type of stochastic dynamics from the low mass limit, valid in the regime $m/\sqrt{\gamma^2 + B^2} \ll \tau$, of Langevin dynamics in the presence of a magnetic Lorentz force. One cannot obtain the limiting dynamics by setting the mass to zero. The low mass limit is also different from the large friction limit. The stochastic dynamics in the low mass limit is characterized by nonwhite noise whose correlation matrix has antisymmetric components. The importance of the noise correlation is demonstrated in a linear driven system. The dissipation is correctly accounted for by nonsymmetric noise correlations. Our discovery will be relevant for the study of driven charged Brownian particles. Experiments using charged colloidal particles may be useful for observing the properties of nonwhite noise. The stochastic noise $\boldsymbol{\eta}$ and the corresponding Ω process are different from the white noise and the Wiener process. It will be interesting to study the extent to which the Ω process and the Wiener process differ. Finding a numerical algorithm that generates the Ω process may help one understand the differences between the two processes. It requires one to understand the discretization scheme of nonwhite noise, which is left for a future work. We hope that our paper triggers a thorough and rigorous study on the properties of nonwhite noise and the associated stochastic differential equation.

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APPENDIX A: LOW MASS LIMIT OF THE FOKKER-PLANCK EQUATION

The Fokker-Planck (FP) equation for the probability distribution $P_t(\boldsymbol{x},\boldsymbol{v})$ is given by

$$\partial_t P_t(\boldsymbol{x},\boldsymbol{v}) = (L_{\rm rev} + L_{\rm irr})P_t(\boldsymbol{x},\boldsymbol{v}), \tag{A1}$$

where $L_{\rm rev}$ ($L_{\rm irr}$) is the reversible (irreversible) part of the time evolution operator $L = L_{\rm rev} + L_{\rm irr}$. They are given by

$$L_{\rm rev} = -\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} - \frac{\boldsymbol{f}(\boldsymbol{x})}{m}\cdot\nabla_{\boldsymbol{v}} - \frac{q}{m}\nabla_{\boldsymbol{v}}\cdot(\boldsymbol{v}\times\boldsymbol{B}),$$

$$L_{\rm irr} = \frac{\gamma}{m}\nabla_{\boldsymbol{v}}\cdot\left(\boldsymbol{v} + \frac{T}{m}\nabla_{\boldsymbol{v}}\right). \tag{A2}$$

Following the procedure in Ref. [29], we introduce the transformed probability distribution $\bar{P}_t(\boldsymbol{x},\boldsymbol{v}) = P_t(\boldsymbol{x},\boldsymbol{v})\rho(\boldsymbol{v})^{-1/2}$ and the operators $\bar{L}_{\rm rev,irr} = \rho(\boldsymbol{v})^{-1/2}L_{\rm rev,irr}\rho(\boldsymbol{v})^{1/2}$ with $\rho(\boldsymbol{v}) \equiv (2\pi T/m)^{-1/2}e^{-m\boldsymbol{v}^2/(2T)}$. The transformed operator $\bar{L}_{\rm irr}$ becomes Hermitian and given by

$$\bar{L}_{\rm irr} = -\frac{\gamma}{m}(b_1^\dagger b_1 + b_2^\dagger b_2), \tag{A3}$$

where $b_i = \sqrt{\frac{T}{m}}\partial_{v_i} + \frac{1}{2}\sqrt{\frac{m}{T}}v_i$ and $b_i^\dagger = -\sqrt{\frac{T}{m}}\partial_{v_i} + \frac{1}{2}\sqrt{\frac{m}{T}}v_i$ are the lowering and raising operators satisfying the commutation relations $[b_i,b_j^\dagger] = \delta_{ij}$ and $[b_i,b_j] = [b_i^\dagger,b_j^\dagger] = 0$. The operator $\bar{L}_{\rm rev}$ is given by

$$\bar{L}_{\rm rev} = -\sqrt{\frac{T}{m}}\nabla_{\boldsymbol{x}}\cdot(\boldsymbol{b}+\boldsymbol{b}^\dagger) + \frac{\boldsymbol{f}(\boldsymbol{x})}{\sqrt{mT}}\cdot\boldsymbol{b}^\dagger + \frac{q}{m}\boldsymbol{B}\cdot(\boldsymbol{b}^\dagger\times\boldsymbol{b}), \tag{A4}$$

where $\boldsymbol{b} = (b_1,b_2)^T$ and $\boldsymbol{b}^\dagger = (b_1^\dagger,b_2^\dagger)^T$ denote the column vectors of the lowering and raising operators. With the transformed operators, the FP equation (A1) becomes

$$\partial_t\bar{P}_t(\boldsymbol{x},\boldsymbol{v}) = (\bar{L}_{\rm rev} + \bar{L}_{\rm irr})\bar{P}_t(\boldsymbol{x},\boldsymbol{v}). \tag{A5}$$

We now consider the expansion

$$\bar{P}_t(\boldsymbol{x},\boldsymbol{v}) = \sum_{n_1,n_2=0}^{\infty} c_{n_1,n_2}(\boldsymbol{x},t)\psi_{n_1}(v_1)\psi_{n_2}(v_2) \tag{A6}$$

in terms of the eigenfunctions $\{\psi_n(v_i)\}$ of $\bar{L}_{\rm irr}$. They are given by

$$\begin{aligned}\psi_0(v_i) &= (2\pi T/m)^{-1/4}e^{-mv_i^2/(4T)},\\ \psi_n(v_i) &= (b_i^\dagger)^n\psi_0(v_i)/\sqrt{n!}\\ &= H_n(v_i/\sqrt{2T/m})\psi_0(v_i)/\sqrt{n!2^n}\quad (n\geqslant 1),\end{aligned} \tag{A7}$$

with the Hermite polynomial H_n. They are orthonormal basis functions such that $\int_{-\infty}^{\infty}dv\psi_l(v)\psi_n(v) = \delta_{l,n}$. Note that $\rho(\boldsymbol{v})^{1/2} = \psi_0(v_1)\psi_0(v_2)$. Inserting this expansion into Eq. (A5) and using the algebra of the lowering and raising operators, one can extract the hierarchy of differential equations for $c_{n_1,n_2}(\boldsymbol{x},t)$,

$$\begin{aligned}\partial_t c_{n_1,n_2} = &-\frac{\gamma}{m}(n_1+n_2)c_{n_1,n_2} - \sqrt{n_1+1}D_1c_{n_1+1,n_2}\\ &-\sqrt{n_1}\hat{D}_1c_{n_1-1,n_2} - \sqrt{n_2+1}D_2c_{n_1,n_2+1}\\ &-\sqrt{n_2}\hat{D}_2c_{n_1,n_2-1} + \frac{B}{m}\sqrt{n_1(n_2+1)}c_{n_1-1,n_2+1}\\ &-\frac{B}{m}\sqrt{(n_1+1)n_2}c_{n_1+1,n_2-1},\end{aligned} \tag{A8}$$

where $D_i = \sqrt{\frac{T}{m}}\partial_{x_i}$ and $\hat{D}_i = \sqrt{\frac{T}{m}}\partial_{x_i} - \frac{1}{\sqrt{mT}}f_i$.

We are interested in the marginal distribution $Q_t(\boldsymbol{x}) = \int d\boldsymbol{v}P_t(\boldsymbol{x},\boldsymbol{v}) = \int dv_1\int dv_2\bar{P}_t(\boldsymbol{x},\boldsymbol{v})\psi_0(v_1)\psi_0(v_2)$. Orthonormality of $\{\psi_n(v)\}$ yields that $Q_t(\boldsymbol{x}) = c_{0,0}(\boldsymbol{x},t)$. Its time evolution is governed by

$$\partial_t c_{0,0} = -D_1c_{1,0} - D_2c_{0,1}, \tag{A9}$$

where $c_{1,0}$ and $c_{0,1}$ are governed by

$$\partial_t c_{1,0} = -\frac{\gamma}{m}c_{1,0} + \frac{B}{m}c_{0,1} - \hat{D}_1c_{0,0} - \sqrt{2}D_1c_{2,0} - D_2c_{1,1},$$

$$\partial_t c_{0,1} = -\frac{\gamma}{m}c_{0,1} - \frac{B}{m}c_{1,0} - \hat{D}_2c_{0,0} - \sqrt{2}D_1c_{0,2} - D_1c_{1,1}. \tag{A10}$$

Now we take the low mass limit by requiring that $\frac{B}{m}c_{n_1,n_2} \sim \frac{\gamma}{m}c_{n_1,n_2} \gg \partial_t c_{n_1,n_2}$; it is equivalent to neglecting $\partial_t c_{n_1,n_2}$ for $n_1+n_2>0$ in the hierarchy of Eq. (A8). The power counting yields that $c_{n_1,n_2} = O(m^{(n_1+n_2)/2})$. Thus, up to leading order in m, (A10) becomes

$$\mathsf{G}\boldsymbol{c} = \sqrt{mT}\nabla_{\boldsymbol{x}}c_{0,0} - \sqrt{\frac{m}{T}}\boldsymbol{f}(\boldsymbol{x})c_{0,0}, \tag{A11}$$

where $\boldsymbol{c} = (c_{1,0},c_{0,1})^T$ and

$$\mathsf{G} = \begin{pmatrix}\gamma & -B\\ B & \gamma\end{pmatrix}. \tag{A12}$$

We ignored the higher-order terms $\partial_t c_{1,0}$, $\partial_t c_{0,1}$, $c_{2,0}$, $c_{0,2}$, and $c_{1,1}$. Inserting $\boldsymbol{c}$ into (A9) and identifying $c_{0,0} = Q_t(\boldsymbol{x})$, we obtain that

$$\frac{\partial}{\partial t}Q_t(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\cdot\boldsymbol{J}, \tag{A13}$$

with the probability current

$$\boldsymbol{J} = [\mathsf{G}^{-1}\boldsymbol{f}(\boldsymbol{x}) - T\mathsf{G}^{-1}\nabla_{\boldsymbol{x}}]Q_t(\boldsymbol{x}). \tag{A14}$$

APPENDIX B: AVERAGE POWER OF THE WORK DONE BY NONCONSERVATIVE FORCE

In this Appendix, we calculate the average power $w = \langle\boldsymbol{f}_{nc}\cdot\dot{\boldsymbol{x}}\rangle_s$ of the work done by a nonconservative force $\boldsymbol{f}_{nc} = \epsilon\boldsymbol{x}\times\hat{\boldsymbol{z}}$ in the steady state. First, we consider the Langevin equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t)$ and

$$m\dot{\boldsymbol{v}}(t) = -\mathsf{K}\boldsymbol{x}(t) - \mathsf{G}\boldsymbol{v}(t) + \boldsymbol{\xi}(t) \tag{B1}$$

with finite m, where the white noise satisfies $\langle\boldsymbol{\xi}(t)\rangle = 0$ and $\langle\boldsymbol{\xi}(t)\boldsymbol{\xi}^T(s)\rangle = 2\gamma T\delta(t-s)\mathsf{I}$ with the identity matrix I. The matrices K and G are given by

$$\mathsf{K} = \begin{pmatrix}k & -\epsilon\\ \epsilon & k\end{pmatrix},\quad \mathsf{G} = \begin{pmatrix}\gamma & -B\\ B & \gamma\end{pmatrix}. \tag{B2}$$

This system falls into the class of the multivariate Ornstein-Uhlenbeck process. To make it clear, we introduce column vectors $\boldsymbol{q}(t) = (x_1(t),x_2(t),v_1(t),v_2(t))^T$ and $\tilde{\boldsymbol{\xi}}(t) = (0,0,m^{-1}\xi_1(t),m^{-1}\xi_2(t))^T$, and a matrix

$$\mathsf{F} = \frac{1}{m}\begin{pmatrix}0 & -m\mathsf{I}\\ \mathsf{K} & \mathsf{G}\end{pmatrix}. \tag{B3}$$

Then, the Langevin equation (B1) is written as $\dot{\boldsymbol{q}}(t) = -\mathsf{F}\boldsymbol{q}(t) + \tilde{\boldsymbol{\xi}}(t)$. The noise $\tilde{\boldsymbol{\xi}}(t)$ satisfies $\langle\tilde{\boldsymbol{\xi}}(t)\tilde{\boldsymbol{\xi}}^T(s)\rangle = \delta(t-s)\mathsf{D}_{\rm fm}$ with the diffusion matrix $\mathsf{D}_{\rm fm} = \frac{2\gamma T}{m^2}{\rm diag}\{0,0,1,1\}$.

The power $w_{\rm fm} = -\epsilon\langle(x_1v_2 - v_1x_2)\rangle_s$ is determined by the moments $\langle x_iv_j\rangle_s$ in the steady state. We define the covariance matrix as $\Sigma(t) = \langle\boldsymbol{q}(t)\boldsymbol{q}(t)^T\rangle$. During the infinitesimal time interval dt, it changes by the amount of

$$\begin{aligned} d\Sigma(t) &= -[\mathsf{F}\Sigma(t) + \Sigma(t)\mathsf{F}^T]dt \\ &\quad + \int_t^{t+dt} dt' \int_t^{t+dt} dt'' \langle \tilde{\boldsymbol{\xi}}(t')\tilde{\boldsymbol{\xi}}^T(t'')\rangle \\ &= [-\mathsf{F}\Sigma(t) - \Sigma(t)\mathsf{F}^T + \mathsf{D}_{\rm fm}]dt. \end{aligned} \tag{B4}$$

Hence, the steady state covariant matrix Σ_s should satisfy $\mathsf{F}\Sigma_s + \Sigma_s\mathsf{F}^T = \mathsf{D}_{\rm fm}$ [30]. This relation provides the coupled linear equations for the elements of Σ_s. The solution is given by

$$\Sigma_s = \frac{T}{m}{\rm diag}\{0,0,1,1\} + \frac{T}{\gamma k + \epsilon B - m\epsilon^2/\gamma} \times \begin{pmatrix} \gamma & 0 & 0 & -\epsilon \\ 0 & \gamma & \epsilon & 0 \\ 0 & \epsilon & \epsilon^2/\gamma & 0 \\ -\epsilon & 0 & 0 & \epsilon^2/\gamma \end{pmatrix}. \tag{B5}$$

Therefore, the average power is given by

$$w_{\rm fm} = \frac{2\epsilon^2 T}{\gamma k + \epsilon B - m\epsilon^2/\gamma}. \tag{B6}$$

Second, we consider the system where the mass m is set to zero. The Langevin equation is given by

$$\dot{\boldsymbol{x}}(t) = -\mathsf{A}\boldsymbol{x}(t) + \boldsymbol{\zeta}(t), \tag{B7}$$

where $\mathsf{A} \equiv \mathsf{G}^{-1}\mathsf{K}$ and the noise $\boldsymbol{\zeta}(t) = \mathsf{G}^{-1}\boldsymbol{\xi}(t)$ satisfies $\langle\boldsymbol{\zeta}(t)\rangle = 0$ and $\langle\boldsymbol{\zeta}(t)\boldsymbol{\zeta}^T(s)\rangle = \delta(t-s)\mathsf{D}_{\rm zm}$ with the diffusion matrix

$$\mathsf{D}_{\rm zm} = 2\gamma T\mathsf{G}^{-1}(\mathsf{G}^{-1})^T = \frac{2\gamma T}{\gamma^2 + B^2}\mathsf{I}. \tag{B8}$$

For the average power $w_{\rm zm}$, one needs to evaluate the steady state moments covariance matrix $\langle\boldsymbol{x}\dot{\boldsymbol{x}}^T\rangle_s = \lim_{t\to\infty}\langle\boldsymbol{x}(t)\dot{\boldsymbol{x}}^T(t)\rangle$. Eliminating $\dot{\boldsymbol{x}}(t)$ using the equations of motion, one obtains

$$\langle\boldsymbol{x}\dot{\boldsymbol{x}}^T\rangle_s = -\langle\boldsymbol{x}\boldsymbol{x}^T\rangle_s\mathsf{A}^T + \langle\boldsymbol{x}\boldsymbol{\zeta}^T\rangle_s, \tag{B9}$$

with $\langle\boldsymbol{x}\boldsymbol{x}^T\rangle_s \equiv \lim_{t\to\infty}\langle\boldsymbol{x}(t)\boldsymbol{x}^T(t)\rangle$ and $\langle\boldsymbol{x}\boldsymbol{\zeta}^T\rangle_s \equiv \lim_{t\to\infty}\langle\boldsymbol{x}(t)\boldsymbol{\zeta}^T(t)\rangle$. The Langevin equation has the solution $\boldsymbol{x}(t) = e^{-\mathsf{A}t}\boldsymbol{x}(0) + \int_0^t ds e^{-\mathsf{A}(t-s)}\boldsymbol{\zeta}(s)$. Inserting this solution into each term on the right-hand side of (B9), we obtain

$$\begin{aligned} \langle\boldsymbol{x}\boldsymbol{x}^T\rangle_s &= \lim_{t\to\infty}\int_0^t dt'\int_0^t dt'' e^{-\mathsf{A}(t-t')}\langle\boldsymbol{\zeta}(t')\boldsymbol{\zeta}^T(t'')\rangle e^{-\mathsf{A}^T(t-t'')} \\ &= \mathsf{D}_{\rm zm}(\mathsf{A}+\mathsf{A}^T)^{-1} = \frac{\gamma T}{\gamma k + \epsilon B}\mathsf{I}, \\ \langle\boldsymbol{x}\boldsymbol{\zeta}^T\rangle_s &= \lim_{t\to\infty}\int_0^t ds e^{-\mathsf{A}(t-s)}\langle\boldsymbol{\zeta}(s)\boldsymbol{\zeta}^T(t)\rangle = \frac{1}{2}\mathsf{D}_{\rm zm} \\ &= \frac{\gamma T}{\gamma^2 + B^2}\mathsf{I}. \end{aligned} \tag{B10}$$

Therefore, the covariance matrix is given by

$$\langle\boldsymbol{x}\dot{\boldsymbol{x}}^T\rangle_s = \frac{\gamma T(\gamma\epsilon - kB)}{(\gamma k + \epsilon B)(\gamma^2 + B^2)}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{B11}$$

and the average power is given by

$$w_{\rm zm} = \frac{2\epsilon^2 T}{\gamma k + \epsilon B} - \frac{2\epsilon BT}{\gamma^2 + B^2}. \tag{B12}$$

Finally, we consider the equation of motion

$$\dot{\boldsymbol{x}}(t) = -\mathsf{A}\boldsymbol{x}(t) + \boldsymbol{\eta}(t) \tag{B13}$$

in the low mass limit. It has the same form as (B7) but with different noise statistics characterized by $\langle\boldsymbol{\eta}(t)\rangle = 0$ and

$$\langle\boldsymbol{\eta}(t)\boldsymbol{\eta}^T(s)\rangle = T\mathsf{G}^{-1}\delta_+(t-s) + T(\mathsf{G}^{-1})^T\delta_-(t-s). \tag{B14}$$

The covariance matrix $\langle\boldsymbol{x}\dot{\boldsymbol{x}}\rangle_s$ can be obtained by using the same formulas (B9) and (B10) with the noise correlator in (B14). One can easily derive that

$$\langle\boldsymbol{x}\boldsymbol{x}^T\rangle_s = \frac{\gamma T}{\gamma k + \epsilon B}\mathsf{I} \quad \text{and} \quad \langle\boldsymbol{x}\boldsymbol{\eta}^T\rangle_s = T(\mathsf{G}^{-1})^T. \tag{B15}$$

Therefore, the covariance matrix is given by

$$\langle\boldsymbol{x}\dot{\boldsymbol{x}}^T\rangle_s = \frac{\epsilon T}{\gamma k + \epsilon B}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{B16}$$

and the average power is given by

$$w_{\rm lm} = \frac{2\epsilon^2 T}{\gamma T + \epsilon B}. \tag{B17}$$

Notice that $\langle\boldsymbol{x}\boldsymbol{\zeta}^T\rangle_s$ of the system with $m=0$ and $\langle\boldsymbol{x}\boldsymbol{\eta}^T\rangle_s$ of the system in the low mass limit are different. It leads to the difference between $w_{\rm zm}$ and $w_{\rm lm}$.

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