Unifying model for random matrix theory in arbitrary space dimensions

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A sparse random block matrix model suggested by the Hessian matrix used in the study of elastic vibrational modes of amorphous solids is presented and analyzed. By evaluating some moments, benchmarked against numerics, differences in the eigenvalue spectrum of this model in different limits of space dimension d, and for arbitrary values of the lattice coordination number Z, are shown and discussed. As a function of these two parameters (and their ratio Z/d), the most studied models in random matrix theory (Erdos-Renyi graphs, effective medium, and replicas) can be reproduced in the various limits of block dimensionality d. Remarkably, the Marchenko-Pastur spectral density (which is recovered by replica calculations for the Laplacian matrix) is reproduced exactly in the limit of infinite size of the blocks, or $d \to \infty$, which clarifies the physical meaning of space dimension in these models. We feel that the approximate results for $d = 3$ provided by our method may have many potential applications in the future, from the vibrational spectrum of glasses and elastic networks to wave localization, disordered conductors, random resistor networks, and random walks.

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I. INTRODUCTION

The eigenvalue spectrum of sparse random matrices is a fascinating subject with widespread applications in physics, from the energy levels of nuclei, to random resistor networks, random walks, the electronic density of states of disordered conductors, and many other topics [1]. It was investigated for several decades, from pioneering works [2] to modern times [3].

In particular, random matrix theory has been applied extensively in recent years to the problem of the vibrational spectrum of glasses, where structural disorder leads to a number of puzzling effects in the vibrational density of states (DOS), such as the excess of soft low-energy modes (boson peak) with respect to Debye's ω^2 law [4–8]. This anomaly in the spectrum is related to well-known anomalies in the thermal properties at low temperatures [9]. This remains a famously unsolved problem because its mathematical description is plagued by the impossibility of analytically solving for the eigenvalue spectrum of the Hessian matrix of a disordered solid.

Recently, replica-symmetry breaking and allied techniques have been applied to the problem of vibrational eigenmodes of glasses, and produced results which recover the well-known Marchenko-Pastur (MP) distribution of eigenvalues of random Laplacian matrices [5]. The big question is about the applicability of these results: both MP and replica are generally thought to be valid for "high-dimensional" systems, but what this means, in practice or in quantitative terms, has remained unanswered. This is clearly a central point of paramount relevance in the current debate on the theoretical description of glasses.

In this work the correlations between blocks, existing in real disordered solids (due to excluded-volume, finite range of the bonds, and short-range order), are neglected and we study sparse block matrices where the blocks are independent identically distributed d-dimensional random matrices. This allows us to clarify that MP and replica results are exactly valid in the case of a random block Laplacian matrix where the dimension of the blocks is infinite. Furthermore, we show that while the lowest eigenvalue of the support is weakly dependent on the space dimension [which ensures that the $\sim(Z-2d)$ scaling of the boson peak frequency in jammed solids and some models of glasses is rather well captured by high-dimensional models [5,10]], conversely the shape of the eigenvalue distribution changes significantly with d and therefore high-dimensional methods such as MP and replica may not provide an accurate modeling of the vibrational DOS of disordered solids.

II. MODEL

In all models or random spring networks, the elastic energy is a quadratic function of the displacements of the particles from their instantaneous "frozen" positions. The stiffness matrix or Hessian matrix W is a Laplacian random symmetric matrix where each row is comprised of a small and random number of nonzero coefficients. The off-diagonal entries $W_{i,j}$, with $i < j$, are identical independent random variables, whereas the diagonal entries $W_{i,i} = -\sum_{j \neq i} W_{i,j}$. The latter requirement is dictated by enforcing mechanical equilibrium on every atom i in the lattice.

The most typical model is the study of the spectrum of the adjacency matrix or the Laplacian matrix of a Erdos-Renyi graph with N vertices in the limit of large order of the matrices (the large N limit).

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The only parameter in the model is the probability p/N of a link in the random graph to be present, whereas the dimension d of the space R^d of the amorphous material or the random spring model is absent.

In this work, we consider a block random matrix model which seems the simplest generalization of the above models, which retains a couple of physically relevant parameters.

We consider a real symmetric matrix M of dimension $Nd \times Nd$, where each row or column has N random block entries, each being a $d \times d$ matrix.

Every $d \times d$ off-diagonal block has probability $1 - Z/N$ of being a null matrix and a probability Z/N of being a rank one matrix, $X_{i,j} = X_{j,i} = (X_{i,j})^t = \hat{n}_{ij}\hat{n}_{ij}^t$, where $\hat{n}_{ij}$ is a d-dimensional random vector of unit length, chosen with uniform probability on the d-dimensional sphere. Furthermore, $\hat{n}_{ij}\hat{n}_{ij}^t$ is the usual matrix (or dyadic) product of a column vector times a row vector, which gives a rank-one matrix.

In the formulation of the stiffness matrix W, the unit vector $\hat{n}_{ij}$ provides the direction between vertex i and vertex j (in a disordered solid or elastic network, between two atoms i and j). For more details on the Hessian matrix of disordered solids, see Refs. [11,12].

We study two prototypes of such block random matrices called the adjacency block matrix A and the Laplacian block matrix L,

$$A = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & \dots & X_{1,N} \\ X_{2,1} & 0 & X_{2,3} & \dots & X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ X_{N,1} & X_{N,2} & X_{N,3} & \dots & 0 \end{pmatrix}, \tag{1}$$

$$L = \begin{pmatrix} \sum_{j\neq 1} X_{1,j} & -X_{1,2} & -X_{1,3} & \dots & -X_{1,N} \\ -X_{2,1} & \sum_{j\neq 2} X_{2,j} & -X_{2,3} & \dots & -X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ -X_{N,1} & -X_{N,2} & -X_{N,3} & \dots & \sum_{j\neq N} X_{N,j} \end{pmatrix}. \tag{2}$$

In both the above matrices, the set of $X_{i,j}$, $i < j$ is a set of $N(N-1)/2$ independent identically distributed random matrices and each $X_{i,j}$ is a rank-one matrix and a projector.

The study of the spectral density of the matrices A, L, in the limit $N \to \infty$, with Z fixed and d fixed, is more difficult than the corresponding study with $d = 1$, the Erdos-Renyi graph, where all moments of both spectral functions are known [13], yet the spectral distributions are not known.

III. EVALUATION OF MOMENTS

Any symmetric matrix M of order N corresponds to a complete graph with N vertices where the nonoriented link (i,j) has the weight M_{ij} and $(M^k)_{ii}$ is evaluated as the sum of the contributions associated to all paths of k steps on the graph from vertex i to itself. We used this familiar technique to evaluate the limiting moments. However, in the present case, the contribution of each path is the product of matrices and the evaluation of moments of high order is laborious. We evaluated the first five limiting moments,

$$\mu_k = \lim_{N\to\infty} \frac{1}{N\,d}\langle \mathrm{Tr}\, A^k\rangle, \quad \mu_0 = 1, \quad \mu_{2k+1} = 0,$$
$$\nu_k = \lim_{N\to\infty} \frac{1}{N\,d}\langle \mathrm{Tr}\, L^k\rangle, \quad \nu_0 = 1,$$

which produce the following results:

$$\mu_2 = \frac{Z}{d}, \quad \mu_4 = \frac{Z}{d} + 2\left(\frac{Z}{d}\right)^2,$$
$$\mu_6 = \frac{Z}{d} + 6\left(\frac{Z}{d}\right)^2 + 5\left(\frac{Z}{d}\right)^3,$$
$$\mu_8 = \frac{Z}{d} + \left(\frac{Z}{d}\right)^2\left(12 + 2\frac{3}{d+2}\right) + 28\left(\frac{Z}{d}\right)^3 + 14\left(\frac{Z}{d}\right)^4,$$
$$\mu_{10} = \frac{Z}{d} + \left(\frac{Z}{d}\right)^2\left(20 + 10\frac{3}{d+2}\right) + \left(\frac{Z}{d}\right)^3\left(90 + 20\frac{3}{d+2}\right) + 120\left(\frac{Z}{d}\right)^4 + 42\left(\frac{Z}{d}\right)^5, \tag{3}$$

$$\nu_1 = \frac{Z}{d}, \quad \nu_2 = 2\frac{Z}{d} + \left(\frac{Z}{d}\right)^2,$$
$$\nu_3 = 4\frac{Z}{d} + 6\left(\frac{Z}{d}\right)^2 + \left(\frac{Z}{d}\right)^3,$$
$$\nu_4 = 8\frac{Z}{d} + \left(\frac{Z}{d}\right)^2\left(24 + \frac{3}{d+2}\right) + 12\left(\frac{Z}{d}\right)^3 + \left(\frac{Z}{d}\right)^4,$$
$$\nu_5 = 16\frac{Z}{d} + \left(\frac{Z}{d}\right)^2\left(80 + 10\frac{3}{d+2}\right) + \left(\frac{Z}{d}\right)^3\left(80 + 5\frac{3}{d+2}\right) + 20\left(\frac{Z}{d}\right)^4 + \left(\frac{Z}{d}\right)^5. \tag{4}$$

The paper by Bauer and Golinelli [13] studied the position and the height of the δ peaks in the spectral distributions of the Erdos-Renyi model. The position of the peaks is independent on the average connectivity Z, whereas the height of the peaks decreases with increasing Z. These authors and the other authors in Ref. [13] determined all the moments of the spectral distributions. The moments are polynomial in Z and give no hint for the δ peaks nor for the percolation transition. Our model

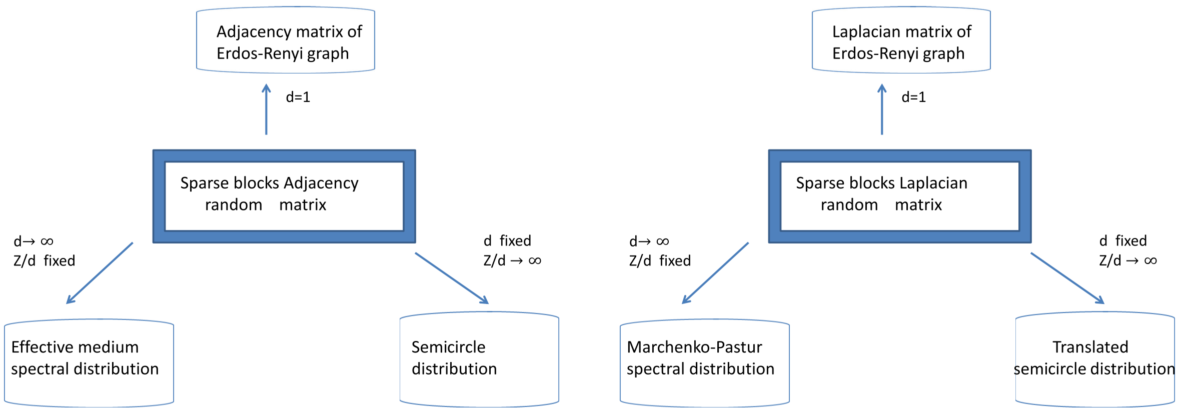

FIG. 1. Left side shows the relation of the adjacency block matrix with three simpler models in different limiting cases. The right side shows the parallel relations of the Laplacian block matrix.

is a generalization to arbitrary dimension of the Erdos-Renyi model. The moments we evaluated give no hint of the δ peaks nor of the percolation transition, just as in the $d = 1$ case.

IV. RESULTS AND DISCUSSION

The above evaluations are the main analytic task we performed. It involves identifying several nonequivalent classes of dominant paths, made of noncommuting sequences of blocks X_{ij}, which are dominant in the $N \to \infty$ limit, to evaluate their cardinality, to average over the random unit vectors in the R^d space, and to average over the probability of a block to be nonzero.

Equations (3) and (4) are displayed in a way to point out that the lowest moments are polynomials in the variable Z/d, whereas moments of higher order, starting with μ_8 and ν_4, have additional terms involving just the space dimension d.

We proceed to compare these moments, with the moments of three limiting cases, as it is schematically indicated in Fig. 1. Some relations are obvious but others are new and valuable.

First, in the $d \to 1$ limit our model reduces to the Erdos-Renyi graph. The moments of the spectral distributions of the adjacency matrix and Laplacian matrix were determined by recurrence relations at every order [13]. Those moments are reproduced by setting $d = 1$ in Eqs. (3) and (4) and this is merely a consistency check of our evaluations.

A second limiting case is shown in Fig. 1: the average connectivity Z is allowed to increase as the order N of the matrices increases: $Z/d \to \infty$ with d fixed. In this limit, the number of nonzero blocks in each row of the matrices increases in the $N \to \infty$ limit, still keeping $Z/N \to 0$. It is sometimes referred to as the dilute matrix limit. Many investigations found that in this limit the spectral distribution of the matrix is the same as a symmetric matrix with independent entries.

Let us consider the Wigner semicircle distribution and its well known moments (Catalan coefficients):

$$\rho(x) = \frac{\sqrt{4(Z/d) - x^2}}{2\pi(Z/d)}, \quad -2\sqrt{Z/d} \leqslant x \leqslant 2\sqrt{Z/d},$$
$$\mu_{2k} = \frac{(2k)!}{k!(k+1)!}\left(\frac{Z}{d}\right)^k. \tag{5}$$

These moments reproduce the highest powers of the polynomials of Eq. (3). Now let us consider the shifted semicircle distribution and the first five moments

$$\rho(x) = \frac{1}{4\pi(Z/d)}\sqrt{8(Z/d) - (x - Z/d)^2},$$
$$Z/d - 2\sqrt{2(Z/d)} \leqslant x \leqslant Z/d + 2\sqrt{2(Z/d)},$$
$$\nu_1 = \frac{Z}{d}, \quad \nu_2 = 2\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2,$$
$$\nu_3 = 6\left(\frac{Z}{d}\right)^2 + \left(\frac{Z}{d}\right)^3,$$
$$\nu_4 = 8\left(\frac{Z}{d}\right)^2 + 12\left(\frac{Z}{d}\right)^3 + \left(\frac{Z}{d}\right)^4,$$
$$\nu_5 = 40\left(\frac{Z}{d}\right)^3 + 20\left(\frac{Z}{d}\right)^4 + \left(\frac{Z}{d}\right)^5. \tag{6}$$

These moments reproduce the leading and the first nonleading powers of the polynomials of Eq. (4).

New and more relevant relations are related to the third limiting case: the limit $d \to \infty$, for Z/d fixed. Semerjian and Cugliandolo [14] evaluated the effective medium (EM) approximation for the spectral distribution of the ensemble of $N \times N$ real symmetric matrices where the diagonal elements vanish and the off-diagonal entry $J_{i,j}$, with $i < j$, is zero with probability $1 - p/N$ and it is one with probability p/N:

$$\rho(x) = \frac{\sqrt{3}}{2\pi}\left[-\left(\frac{p-1}{3x}\right)^2 - \frac{p+2}{6x} + \sqrt{\frac{(\lambda^2 - x^2)(x^2 + \alpha^2)}{27x^4}}\right]^{1/3} - \frac{\sqrt{3}}{2\pi}\left[-\left(\frac{p-1}{3x}\right)^2 - \frac{p+2}{6x} - \sqrt{\frac{(\lambda^2 - x^2)(x^2 + \alpha^2)}{27x^4}}\right]^{1/3}$$

where $-\lambda \leqslant x \leqslant \lambda$, and λ, α are functions of p.

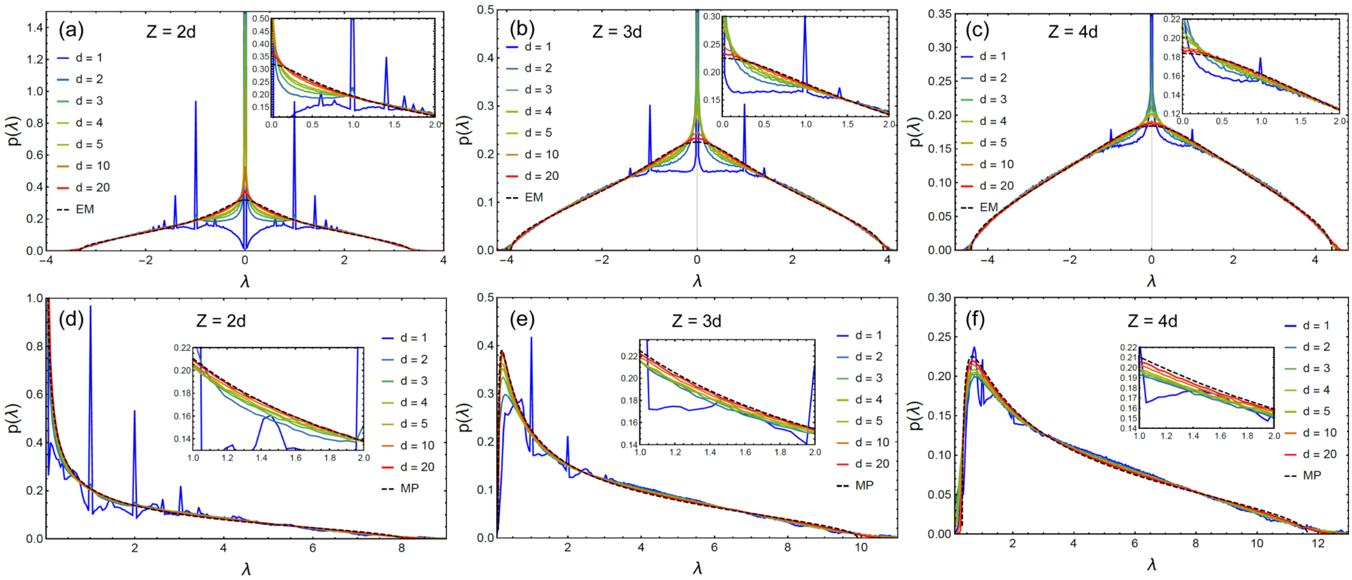

FIG. 2. (a)–(c) Plots of the eigenvalue spectra of the adjacency matrix obtained from our model systems for $d = 1,2,3,4,5,10,20$ and corresponding $N = 15000,7500,5000,4000,3000,2000,1000$. They approach the spectrum from effective medium theory, which appears as the infinite dimensional limit. (d)–(f) The spectra of the Laplace matrix for the same systems. As one can see they approach the Marchenko-Pastur distribution for infinite dimension.

We evaluated the moments of this spectral function from the Taylor expansion of the corresponding resolvent. One then obtains the moments in the table in Eq. (3) where the terms $\frac{3}{d+2}$ are absent and $p = Z/d$. That is, the limit $d \to \infty$ with Z/d fixed.

Finally, the same limit, $d \to \infty$, with Z/d fixed, performed on the table in Eq. (4) leads to

$$\begin{aligned} \nu_1 &= \frac{Z}{d}, \quad \nu_2 = 2\,\frac{Z}{d} + \left(\frac{Z}{d}\right)^2, \\ \nu_3 &= 4\,\frac{Z}{d} + 6\left(\frac{Z}{d}\right)^2 + \left(\frac{Z}{d}\right)^3, \\ \nu_4 &= 8\,\frac{Z}{d} + 24\left(\frac{Z}{d}\right)^2 + 12\left(\frac{Z}{d}\right)^3 + \left(\frac{Z}{d}\right)^4, \\ \nu_5 &= 16\,\frac{Z}{d} + 80\left(\frac{Z}{d}\right)^2 + 80\left(\frac{Z}{d}\right)^3 + 20\left(\frac{Z}{d}\right)^4 + \left(\frac{Z}{d}\right)^5. \end{aligned} \tag{7}$$

The moments $\int_a^b dx\, x^k \rho_{MP}(x)$ of the Marchenko-Pastur distribution

$$\rho_{MP}(x) = \frac{\sqrt{(b-x)(x-a)}}{4\pi\, x}, \quad 0 \leqslant a \leqslant x \leqslant b,$$

with the following definition of parameters:

$$a = (\sqrt{p} - \sqrt{2})^2, \quad b = (\sqrt{p} + \sqrt{2})^2,$$

where $p = Z/d$ reproduces the above Eq. (7). The relations we observe from the evaluated moments in the limit $d \to \infty$ with the ratio Z/d fixed are new and possibly the most intriguing analytic result of this article. They agree with the usual understanding that the symmetric replica approach is correct in a space with large dimensionality. Certainly it would be valuable to prove that in this limit the random block matrix reproduces the results of the symmetric replica approach.

It is important to support the analytic evaluations of few moments with the full numerical evaluation of the spectral distributions. Large $Nd \times Nd$ block-adjacency matrices and block-Laplacian matrices, with $N = 1000$–15000 and $d = 1,2,3,4,5,10,20$, were generated according to the probability distribution of our model and the eigenvalues were numerically evaluated. The obtained spectral distributions are in Fig. 2. They support the conjectured limits indicated in Fig. 1 and the emerging unifying picture.

The plots in the upper row of Fig. 2 describe the spectral distribution of the adjacency matrix. The Dirac δ peaks well studied in the $d = 1$ case are less prominent with increasing values of the space dimension and are absent in the effective medium approximation, which is reproduced by the $d \to \infty$ limit of the present block model.

The plots in the lower row of Fig. 2 describe the spectral distribution of the Laplacian matrix, which has a greater physical interest. As the d parameter varies from 1 to ∞, the latter reproducing the Marchenko-Pastur distribution, the general shape of the distribution does not change in a qualitative way. Still the quantitative difference, in the region of eigenvalues close to the peak of the distribution, is a sizable effect. For $d = 3$ this difference may be confronted with experimental findings.

V. CONCLUSIONS

In conclusion, the analytic evaluations of a few limiting moments and the numerical simulations support the conjecture of the relations schematically indicated in Fig. 1 among different random matrix models. Since in the traditional models of disordered systems through random matrices and replica approach, the space dimension does not enter in the formulation of the model, the argument that the effective medium approximation (for the adjacency matrix) and the Marchenko-Pastur distribution (for the Laplacian matrix) are valid for infinite space dimension is rather indirect and not well defined. The proposed relations and the systematic numerical results presented in this work substantiate these arguments by clarifying the role of space dimension for the various random matrix models and suggest new ways to investigate disordered systems in finite space dimension.

With respect to the theory of random matrices, the present model explores ensembles of block random matrices with two different probabilities: the probability of independent identically distributed blocks to occur and the probability of the entries in the blocks. This structure is very promising and of great relevance for physics applications.

The conjectured relations schematically indicated in Fig. 1 indicate that this structure interpolates among all best studied spectral distributions.

We are also confident that the limiting moments evaluated here will be useful in the search for suitable approximate analytic representations of the eigenvalue distributions of physical models in finite space dimensions.

It seems proper to make some connection to the current research on sparse random matrices where the matrix entries are random blocks [15]. Perhaps the most relevant model is the stochastic block model, well known in the study of social and biological networks. The model describes a complex network with n nodes, partitioned into communities or blocks, often of equal size. If two nodes belong to different communities, there is an edge with probability which may depend on the chosen pair of communities. If two nodes belong to the same community there is an edge with different probability. The model is flexible enough to properly describe many nontrivial types of structures.

The stochastic block model is substantially different from the random block matrix of the present paper, which is a straightforward picture of the Hessian of a system of points (e.g., atoms or particles) connected by springs. However, the cavity method familiar in statistical physics was used in several block models, suggesting the possibility of using it also in our model [15].

APPENDIX A: DEFINITION OF THE MODEL

It is useful to recall the well known correspondence between any real symmetric matrix M of order N and the corresponding nondirected graphs G with N vertices. Between a generic pair of vertices (i,j) of the graph there is a link, or edge, with the weight $M_{i,j}$. The edge is absent if the corresponding matrix entry is zero. Edges where the extrema of the edge is the same vertex correspond to the diagonal entries of the matrix. The matrix element of a power of the matrix, say $(M^k)_{i,j}$, may be evaluated as the sum of the contributions of weighted paths of k steps from vertex i and vertex j on the graph:

$$(M^k)_{i,j} = \sum_{s_1=1,\dots,N,\dots,s_{k-1}=1,\dots,N} M_{i,s_1} M_{s_1,s_2} \cdots M_{s_{k-1},j}.$$

The sparse random block matrix we study in this work is an ensemble of real symmetric matrices M of dimension $Nd \times Nd$.

The generic matrix of the ensemble is a block matrix, with N blocks in each row and column. Each block $X_{i,j}$ is a real symmetric matrix of order $d \times d$:

$$M = \begin{pmatrix} X_{1,1} & X_{1,2} & X_{1,3} & \dots & X_{1,N} \\ X_{2,1} & X_{2,2} & X_{2,3} & \dots & X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ X_{N,1} & X_{N,2} & X_{N,3} & \dots & X_{N,N} \end{pmatrix}. \quad \text{(A1)}$$

The blocks $X_{i,j}$ are independent identically distributed random matrices.

The graph corresponding to the matrix M has N vertices; the weight of the (nondirected) edge connecting the pair of vertices (i,j) is a $d \times d$ matrix $X_{i,j} = X_{j,i} = X^t_{i,j}$. It is still useful to evaluate elements of powers of the matrix in terms of the weighted paths connecting the vertices. Since the weight of a path is a product of noncommuting blocks, the order of them is relevant.

The adjacency matrix has a zero $d \times d$ block on the diagonal entries:

$$A = \begin{pmatrix} 0 & X_{1,2} & X_{1,3} & \dots & X_{1,N} \\ X_{2,1} & 0 & X_{2,3} & \dots & X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ X_{N,1} & X_{N,2} & X_{N,3} & \dots & 0 \end{pmatrix}. \quad \text{(A2)}$$

One easily evaluates traces of powers in terms of classes of nonequivalent paths [16]. Since the blocks are independent identically distributed random matrices, it is sufficient to record when a block has previously appeared in a path. Then X_1 stands for any of the $N(N-1)/2$ blocks $X_{i,j}$, X_2 stands for any block, different from X_1, etc. For instance,

$$\frac{1}{N(N-1)} \mathrm{Tr}\, A^4 = \mathrm{Tr}_d X_1^4 + 2\,(N-2)\,\mathrm{Tr}_d X_1^2 X_2^2 + (N-2)(N-3)\,\mathrm{Tr}_d X_1 X_2 X_3 X_4. \quad \text{(A3)}$$

The analogous evaluation for the Laplacian block matrix L is more involved:

$$L = \begin{pmatrix} \sum_{j\neq 1} X_{1,j} & -X_{1,2} & -X_{1,3} & \dots & -X_{1,N} \\ -X_{2,1} & \sum_{j\neq 2} X_{2,j} & -X_{2,3} & \dots & -X_{2,N} \\ \dots & \dots & \dots & \dots & \dots \\ -X_{N,1} & -X_{N,2} & -X_{N,3} & \dots & \sum_{j\neq N} X_{N,j} \end{pmatrix}, \quad \text{(A4)}$$

$$\frac{1}{N(N-1)} \mathrm{Tr}\, L^4 = 8\,\mathrm{Tr}_d X_1^4 + 16\,(N-2)\,\mathrm{Tr}_d X_1^3 X_2 + 8\,(N-2)\mathrm{Tr}_d X_1^2 X_2^2 + (N-2)\,\mathrm{Tr}_d X_1 X_2 X_1 X_2 + 8(N-2)(N-4) \times \mathrm{Tr}_d X_1^2 X_2 X_3 + 2\,(N-2)(2N-5)\,\mathrm{Tr}_d X_1 X_2 X_1 X_3 + (N-2)(N-3)(N-7)\,\mathrm{Tr}_d X_1 X_2 X_3 X_4. \quad \text{(A5)}$$

Each block X is the null matrix $d \times d$, with probability $1-(Z/N)$ or it is a rank-one random matrix $X = \hat{n}\hat{n}^t$, with probability Z/N, where $\hat{n}$ is a random vector of length one, chosen with uniform probability in R^d.

Then, for instance, $\mathrm{Tr}_d X_1X_2X_3X_4 = 0$ with probability $1-(Z/N)^4$ or $(\hat{n}_1\hat{n}_2)(\hat{n}_2\hat{n}_3)(\hat{n}_3\hat{n}_4)(\hat{n}_4\hat{n}_1)$ with probability $(Z/N)^4$. And $\mathrm{Tr}_d X_1X_2X_1X_3 = 0$ with probability $1-(Z/N)^3$ or $(\hat{n}_1\hat{n}_2)^2(\hat{n}_3\hat{n}_1)^2$ with probability $(Z/N)^3$. The expected number of nonzero $d \times d$ blocks in each row or column of the adjacency matrix is $\frac{N-1}{N}Z$; then Z is the average connectivity of the large graph (or the average degree of the vertices).

Finally the average over the uniform probability of the direction of all the random vectors $\hat{n}_j$ involves integrals for each of them over the unit sphere in R^d. Let us consider two vectors $\vec{p},\vec{y} \in R^d$ and the integral

$$I_d(\vec{p}) = \int_{S_d} e^{(\vec{p}\cdot\vec{y})} \prod_{j=1}^{d} dy_j$$

where S_d is the surface $\sum_{j=1}^d y_j^2 = R^2$. We define the spherical average over the vector $\vec{y}$ with the normalization

$$\langle e^{(\vec{p}\cdot\vec{y})}\rangle_y = \frac{I_d(\vec{p})}{I_d(0)} = \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{d}{2}\right)}{m!\,\Gamma\left(m+\frac{d}{2}\right)}\left[\frac{(\vec{p}\cdot\vec{p})(\vec{y}\cdot\vec{y})}{4}\right]^m. \tag{A6}$$

After expanding the left side and by considering unit vectors, one has

$$\frac{1}{(2m)!}\langle(\vec{p}\cdot\vec{y})^{2m}\rangle_y = \frac{\Gamma\left(\frac{d}{2}\right)}{m!\,\Gamma\left(m+\frac{d}{2}\right)4^m},$$

then

$$\langle(\hat{n}_1\hat{n}_2)^2\rangle_d = \frac{1}{d}, \quad \langle(\hat{n}_1\hat{n}_2)^4\rangle_d = \frac{3}{d(d+2)},$$
$$\langle(\hat{n}_1\hat{n}_2)^6\rangle_d = \frac{15}{d(d+2)(d+4)}.$$

To deal with one more external vectors, one rewrites Eq. (A6) with the replacement $\vec{p} = \vec{a}+\vec{b}$:

$$\langle e^{(\vec{a}\cdot\vec{y})}e^{(\vec{b}\cdot\vec{y})}\rangle_y = \frac{I_d(\vec{a},\vec{b})}{I_d(0)} = \sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{d}{2}\right)}{m!\,\Gamma\left(m+\frac{d}{2}\right)}\frac{(\vec{y}\cdot\vec{y})^m}{4^m} \times [(\vec{a}\cdot\vec{a})+(\vec{b}\cdot\vec{b})+2(\vec{a}\cdot\vec{b})]^m.$$

Then, for unit vectors, $\langle(\vec{a}\cdot\vec{y})(\vec{b}\cdot\vec{y})\rangle_y = \frac{1}{d}[1+(\vec{a}\cdot\vec{b})]$. Many averages are chains

$$\langle(\hat{n}_1\hat{n}_2)(\hat{n}_2\hat{n}_3)(\hat{n}_3\hat{n}_1)\rangle = \langle(\hat{n}_1\hat{n}_2)\langle(\hat{n}_2\hat{n}_3)(\hat{n}_3\hat{n}_1)\rangle_{\hat{n}_3}\rangle_{\hat{n}_2}$$
$$= \left\langle(\hat{n}_1\hat{n}_2)\frac{1}{d}[1+(\hat{n}_1\hat{n}_2)]\right\rangle_{\hat{n}_2} = \frac{1}{d^2},$$
$$\langle(\hat{n}_1\hat{n}_2)(\hat{n}_2\hat{n}_3)(\hat{n}_3\hat{n}_4)(\hat{n}_4\hat{n}_1)\rangle_d = \frac{1}{d^3}.$$

By this method we find from Eqs. (A3) and (A5)

$$\lim_{N\to\infty}\frac{\langle\mathrm{Tr}A^4\rangle}{Nd} = \frac{Z}{d} + 2\left(\frac{Z}{d}\right)^2,$$
$$\lim_{N\to\infty}\frac{\langle\mathrm{Tr}L^4\rangle}{Nd} = 8\frac{Z}{d} + \left(\frac{Z}{d}\right)^2\left(24+\frac{3}{d+2}\right) + 12\left(\frac{Z}{d}\right)^3 + \left(\frac{Z}{d}\right)^4.$$

APPENDIX B: MOMENTS OF THE LIMITING MODELS

1. Simple random graph

For a simple (that is, no multiple edges, no edge with just one vertex) random graph, where the probability of any edge is Z/N, the moments of the spectral distribution of the adjacency matrix and the Laplacian matrix were evaluated in the $N\to\infty$ limit and fixed average connectivity Z at every order [13]. We report here the first few moments, from Tables 1 and 2 of Bauer and Golinelli [13]. For the adjacency matrix we have

$$\mu_k = \lim_{N\to\infty}\frac{1}{N}\langle\mathrm{Tr}\,A^k\rangle, \quad \mu_0 = 1, \quad \mu_{2k+1} = 0,$$

which produces

$$\mu_2 = Z, \quad \mu_4 = Z + 2Z^2,$$
$$\mu_6 = Z + 6Z^2 + 5Z^3,$$
$$\mu_8 = Z + 14Z^2 + 28Z^3 + 14Z^4,$$
$$\mu_{10} = Z + 30\,Z^2 + 110\,Z^3 + 120\,Z^4 + 42\,Z^5,$$
$$\mu_{12} = Z + 62\,Z^2 + 375\,Z^3 + 682\,Z^4 + 495\,Z^5 + 132\,Z^6,$$

while for the Laplacian matrix we have

$$\nu_k = \lim_{N\to\infty}\frac{1}{N}\langle\mathrm{Tr}\,L^k\rangle, \quad \nu_0 = 1, \tag{B1}$$

which produces

$$\nu_1 = Z, \quad \nu_2 = 2\,Z + Z^2,$$
$$\nu_3 = 4\,Z + 6\,Z^2 + Z^3,$$
$$\nu_4 = 8\,Z + 25\,Z^2 + 12\,Z^3 + Z^4,$$
$$\nu_5 = 16\,Z + 90\,Z^2 + 85\,Z^3 + 20\,Z^4 + Z^5,$$
$$\nu_6 = 32\,Z + 301\,Z^2 + 476\,Z^3 + 215\,Z^4 + 30\,Z^5 + Z^6.$$

2. Effective medium approximation

In the same model, the spectral distribution of the adjacency matrix in the effective medium (EM) approximation is

$$\rho^{\mathrm{EM}}(x) = -\frac{1}{\pi}\mathrm{Im}\,g(x+i\epsilon), \quad g(z) = \int\frac{\rho^{\mathrm{EM}}(x)}{z-x}\,dx,$$

$$\rho^{\mathrm{EM}}(x) = \frac{\sqrt{3}}{2\pi}\left[\left(\frac{p-1}{3x}\right)^2 + \frac{p+2}{6x} + \sqrt{\frac{(\lambda^2-x^2)(x^2-\alpha^2)}{27x^4}}\right]^{1/3} - \frac{\sqrt{3}}{2\pi}\left[\left(\frac{p-1}{3x}\right)^2 + \frac{p+2}{6x} - \sqrt{\frac{(\lambda^2-x^2)(x^2-\alpha^2)}{27x^4}}\right]^{1/3},$$

where $-\lambda \leqslant x \leqslant \lambda$,

$$\lambda = \sqrt{\frac{-p^2+20p+8+\sqrt{p(p+8)^3}}{8}}, \quad \alpha^2 = \frac{p^2-20p-8+\sqrt{p(p+8)^3}}{8}.$$

It is difficult to evaluate the moments $\mu_{2k} = \int_{-\lambda}^{\lambda} x^{2k}\,\rho^{\mathrm{EM}}(x)\,dx$ by analytic integration, but the first few moments are easily obtained from the series solution of the cubic

$$[g(z)]^3 + \frac{p-1}{z}[g(z)]^2 - g(z) + \frac{1}{z} = 0,$$

$$g(z) = \sum_{k=0}^{\infty}\frac{\mu^{2k}}{z^{2k+1}} = \frac{1}{z} + \frac{p}{z^3} + \frac{p+2p^2}{z^5} + \frac{p+6p^2+5p^3}{z^7} + \frac{p+12p^2+28p^3+14p^4}{z^9}$$
$$+ \frac{p+20p^2+90p^3+120p^4+42p^5}{z^{11}} + \frac{p+30p^2+220p^3+550p^4+495p^5+132p^6}{z^{13}}$$
$$+ \frac{p+42p^2+455p^3+1820p^4+3003p^5+2002p^6+429p^7}{z^{15}} + O(z^{-17}).$$

3. Marchenko-Pastur distribution

The Marchenko-Pastur distribution reads as

$$\rho_{MP}(x) = \frac{\sqrt{(b-x)(x-a)}}{4\pi\, x}, \quad 0 \leqslant a \leqslant x \leqslant b, \quad a = (\sqrt{p}-\sqrt{2})^2, \quad b = (\sqrt{p}+\sqrt{2})^2.$$

The moments $\nu_k = \int_a^b dx\, x^k\, \rho_{MP}(x)$ are well known and are given by

$$\nu_k = \int_a^b dx\, x^k\, \frac{\sqrt{(b-x)(x-a)}}{4\pi\, x} = \frac{(2p)^{(k+1)/2}2^{k-1}}{\pi}\int_{-1}^{1}\left(t+\frac{p+2}{\sqrt{8p}}\right)^{k-1}\sqrt{1-t^2}\,dt$$
$$= p(p+2)^{k-1}\,{}_2F_1\left(\frac{1-k}{2}, 1-\frac{k}{2}; 2; \frac{8p}{(p+2)^2}\right).$$

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