Mechanism of Kuznetsov-Ma breathers

Li-Chen Zhao,[1,2,*] Liming Ling,[3] and Zhan-Ying Yang[1,2]
[1]*School of Physics, Northwest University, Xi'an 710069, China*
[2]*Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi'an 710069, China*
[3]*School of Mathematics, South China University of Technology, Guangzhou 510640, China*

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We discuss how to understand the dynamical process of Kuznetsov-Ma breather, based on some basic physical mechanisms. It is shown that the dynamical process of Kuznetsov-Ma breather involves at least two distinctive mechanisms: modulational instability and the interference effects between a bright soliton and a plane-wave background. Our analysis indicates that modulational instability plays dominant roles in the mechanism of Kuznetsov-Ma breather admitting weak perturbations, and the interference effect plays a dominant role for the Kuznetsov-Ma breather admitting strong perturbations. For intermediate cases, the two mechanisms are both greatly involved. These characters provide a possible way to understand the evolution of strong perturbations on a plane-wave background.

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I. INTRODUCTION

Recently, localized waves on plane-wave background (PWB) became intense since their dynamics is related with freak wave, mainly including Akhmediev breather (AB) [1], Peregrine rogue wave (RW) [2], and Kuznetsov-Ma breather (KMB) [3]. They have been excited in real nonlinear systems, such as optical laser field in fiber [4–6], and water wave tank [7]. The underlying mechanism of their dynamics has been paid much attention after investigating their dynamics through deriving analytical solutions. The mechanism mainly refers to how to understand the dynamical process of these localized waves in simple ways, based on some general physical properties. Modulational instability (MI), which is associated with the growth of weak perturbations on a PWB, has been seen as a mechanism of RWs and ABs [5]. Furthermore, baseband MI or MI with resonant perturbations is found to play an essential role in RW excitations [8–10], and much attention has been paid to the general nature of the nonlinear stage of MI and many efforts have been made to understand MI more systematically [11,12]. Furthermore, the underlying mechanisms for forming different spatial-temporal structures of fundamental RWs or ABs were uncovered very recently [13]. All these studies further deepen our understanding on RW and AB dynamics greatly. However, MI cannot explain the dynamical process of KMB well, partly because MI usually fails to explain the evolution of strong perturbations on a PWB [14]. Meanwhile, it should be noted that nonlinear MI has been proposed to predict and explain the evolution of strong localized perturbations with a continuous inverse scattering spectrum [11,15]. We mainly focus on how to understand the dynamical process of KMB, based on some general physical mechanisms.

KMB is generally a nonlinear superposition of a bright soliton (BS) and a PWB, since the bright soliton related term depends on the plane-wave term. Interestingly, it can be written in a linear superposition form of them at some special moments. Therefore the BS term could be seen as a "perturbation" term on the PWB. When the BS's amplitude is much smaller than the background's, the KMB's dynamics can be explained well by MI [10]. In a limit case, the soliton's amplitude tends to be zero, and KMB will tend to a RW. These cases for weak perturbations are surely explained well by linear stability analysis on a plane-wave background. However, the soliton's amplitude can be much larger than the background's for KMB, which makes the linear stability analysis usually not hold anymore. Moreover, we demonstrated that breathers could exist in the modulational stability (MS) regime which could not be reduced to RW anymore [16], and antidark soliton was reported to exist in the MI regime with some fourth-order effects [17]. These striking localized waves cannot be explained by MI. We would like to explain the mechanism of KMB to provide a reasonable understanding of them, since these localized waves are all related with KMB excitation.

In this paper, we discuss how to understand the dynamical process of KMB, based on some general physical mechanisms. An approximation form is introduced to describe the interference effects, which is one of the fundamental properties in both classical wave and quantum theory. The analysis results suggest that the dynamical process of KMB involves at least two distinctive mechanisms: MI and interference effects. Since the two mechanisms can be used to explain the dynamical characters of KMB well, they are seen as the mechanism of KMB. The interference effects are between a bright soliton and plane wave. For KMB admitting weak perturbation cases, the oscillation period can be explained by nonlinear interference effects, and the amplitude oscillation can be understood well by MI. For KMB admitting strong perturbation, the oscillation period and amplitude oscillation can both be explained by linear interference effects. The weak and strong perturbations are clarified by the ratio $p/s \ll 1$ and $p/s \gg 1$, respectively (p and s denote BS amplitude and PWB amplitude). For

*zhaolichen3@nwu.edu.cn

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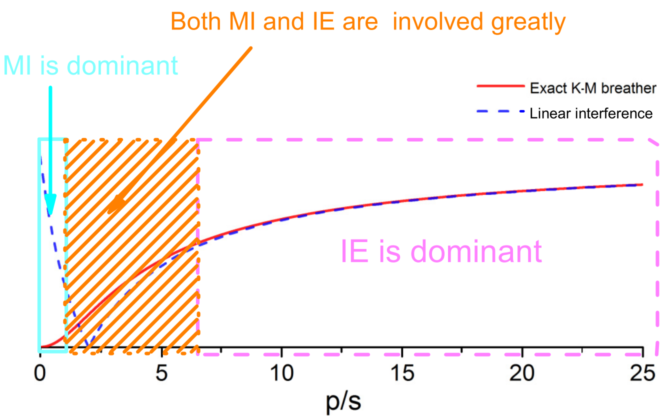

FIG. 1. A qualitative description on the mechanisms of KMB. MI plays a dominant role for KMB admitting weak perturbations, and the interference effects (IE) play a dominant role for KMB admitting strong perturbations. The weak and strong perturbations are clarified by the ratio $p/s \ll 1$ and $p/s \gg 1$, respectively (p and s denote perturbation amplitude and background amplitude, respectively). For intermediate cases (p is comparable with s), it is expected that the two mechanisms are both greatly involved. For the weak perturbation case and intermediate cases, the breathing period can be explained well by nonlinear interference effects.

intermediate cases, it is expected that the two mechanisms are both greatly involved. These understandings on the KMB mechanism are summarized in Fig. 1. Additionally, the interference effects can also be used to explain the antidark soliton in the MI regime and breather excitation in the MS regime reported previously [16,17].

II. ANALYSIS ON KUZNETSOV-MA BREATHER

For simplicity and without losing generality, we would like to begin with a generalized KMB solution which has been given widely for the well-known nonlinear Schrödinger equation $i\Psi_t + \frac{1}{2}\Psi_{xx} + |\Psi|^2\Psi = 0$. Similar discussion can be extended conveniently to KMB like a breather in other nonlinear systems. Its explicit form can be written as follows:

$$\Psi = \left[s - \frac{2(b^2 - s^2)\cos(\xi t) + i\xi\ \sin(\xi t)}{b\ \cosh(2x\sqrt{b^2 - s^2}) - s\ \cos(\xi t)}\right]e^{is^2 t}, \quad (1)$$

where $\xi = 2b\sqrt{b^2 - s^2}$. The parameter $b >= s$ determines the initial nonlinear localized wave's shape, and s is the background amplitude for localized waves. It should be noted that the solution mainly has two terms. The first term is a PWB, and the other term corresponds to a BS related term. Since the second term depends on the PWB generally, the KMB is a nonlinear superposition of BS and PWB. However, for $t = \frac{\pi + 2n\pi}{4b\sqrt{b^2 - s^2}}$ (n is an integer), the KMB can be written as

$$\Psi = se^{i\phi} - i2\sqrt{b^2 - s^2}\,\mathrm{sech}(2\sqrt{b^2 - s^2}x)e^{i\phi}, \quad (2)$$

where $\phi = s^2 \frac{\pi}{4b\sqrt{b^2 - s^2}}$. This can be seen as a linear superposition of a PWB and a BS. The BS's amplitude is $2\sqrt{b^2 - s^2}$, which can be varied conveniently to investigate the evolution of strong perturbations on PWB. If the soliton amplitude is much smaller than the PWB amplitude, the BS term can be seen as the perturbation term f_{pert} in the typical linear instability analysis on a PWB. This idea has been used to clarify the relations between MI and several nonlinear excitations [10]. When the amplitude of soliton perturbation tends to be zero, the KMB dynamical process tends to be a RW which admits rational amplification form [4–6]. Fourier analysis of the localized perturbation suggests that both RW and KMB admit a dominant wave vector at resonant one with the background [10,14] (the wave vector is called according to the spatial coordinate x). The MI analysis predicts the perturbations with resonant wave vector admit rational amplification form, namely, $1 + i2s^2 t$. Therefore, the amplification of KMB admitting weak soliton perturbations can be explained well by MI. However, when the soliton's amplitude is comparable with the background's, the linear stability does not hold anymore. For example, we demonstrated that breathers could exist in the MS regime which could not be reduced to RW anymore [16], and antidark soliton was reported to exist in the MI regime with some fourth-order effects [17]. The antidark soliton's existence indicates that strong perturbations are possible to be stable in the MI regime. The breather demonstrates striking oscillation characters in the MS regime. These localized waves' dynamical process cannot be explained at all, based on the MI mechanism. Therefore, we mainly try to find a possible way to understand these cases with strong perturbations in the following sections.

We can see that even for the nonlinear superposition form, KMB tends to be a linear superposition form when the soliton amplitude is much larger than the background amplitude. This provides us with a hint to understand the dynamics of soliton-type perturbations with large amplitudes on a plane-wave background. We therefore introduce a linear superposition form of a BS with a generic form and a PWB according to the above linear form:

$$\Psi' = se^{is^2 t} - ip\ \mathrm{sech}(px)e^{ip^2 t/2}, \quad (3)$$

where p is the soliton amplitude and it corresponds to the amplitude $2\sqrt{b^2 - s^2}$ of the soliton perturbation term in the KMB solution. This can be seen as an approximation solution for strong BS type perturbations on a PWB. Since the linear coherent form of a soliton and a plane wave describes the well-known interference effects in both classical wave motion theory and quantum mechanics, the evolution of Ψ' can be understood well by interference effects between BS and PWB. In physical studies or theory, it is usual to explain one dynamical process from other much simpler and more general descriptions. Therefore, the linear interference effects, as an approximation solution, can be used to explain the oscillation of strong localized perturbations on a PWB. This way of understanding is similar to that of MI, as an approximation solution is used to explain the amplification of weak perturbations on a PWB. The linear superposition form is called linear interference effect in the following text. Moreover, it should be noted the linear superposition form does not hold anymore for the weak perturbation case and intermediate cases (for which perturbation amplitude p is comparable with the background amplitude s). In these cases, the superposition form is a nonlinear superposition form. The nonlinear superposition form can be used to describe the nonlinear interference process [18], and the dominant energy of perturbations should be used to analyze the interference period [10]. In the following, we show that the interference effects can be used to explain the dynamical process of KMB with strong perturbations well.

III. A DISCUSSION ON THE MECHANISM OF KUZNETSOV-MA BREATHER

We would like to explain the dynamical process of KMB from two aspects, which involve the soliton oscillation amplitude and the oscillation period, respectively. We calculate the maximum density value of KMB is $P_{\max} = (s + 2b)^2$, and the minimum density value is $P_{\min} = (s - 2b)^2$. To describe the oscillation amplitude with no singularity, we define a parameter $\eta = \frac{|P_{\min}-s^2|}{P_{\max}-s^2}$ to describe the breather's amplitude oscillation. The period of breathing is $T = \frac{2\pi}{2b\sqrt{b^2-s^2}}$. Then, we calculate the maximum density value of the introduced linear form Ψ' is $P'_{\max} = (s + p)^2$, and the minimum density value is $P'_{\min} = (s - p)^2$. Then the characterization parameter $\eta' = \frac{|P'_{\min}-s^2|}{P'_{\max}-s^2}$ describes the breather's amplitude oscillation for linear interference effects between BS and PWB. The period of breathing is $T' = \frac{2\pi}{|p^2/2-s^2|}$. Based on these defined parameters, we can compare the characters of KMB with the linear interference form in two aspects, to understand the dynamical process of KMB. Good agreement between them means that interference effects can be used to explain the dynamical process of KMB, and it can be seen as a mechanism of KMB. This point is similar to that MI has been seen as a mechanism of rogue wave and the Akhmediev breather [5,10,13].

Firstly, we investigate the amplitude oscillation behavior of the two forms through plotting the defined parameters η and η'. From the expressions of them, we can see that the ratio of BS amplitude p and PWB amplitude s plays an essential role for determining whether the soliton perturbation strength is weak or strong. Therefore, we plot the defined parameters η and η' vs the ratio p/s in Fig. 2. It is shown that the linear interference effects described line is quite different from the one described by KMB, for weak perturbations [$p/s \ll 1$)]. Namely, the amplitude amplification is much larger than the value expected by interference effects. The large amplification is surely induced by MI as AB and RW. But they agree with each other perfectly for strong perturbations ($p/s \gg 1$). Secondly, we show the relations between the oscillation period and the ratio p/s in Fig. 3. We can see that the linear interference effect can still describe the period of KMB with strong perturbations. But the linear interference effect fails to explain the dynamical characters for KMB with weak perturbations and intermediate cases (perturbation amplitude p is comparable with background amplitude s).

These suggest that the dynamics of KMBs with strong perturbations can be understood by linear interference effects between BS and PWB. To support this point, we plot the evolution dynamics of KMB and the linear superposition form with an identical large ratio value p/s in Fig. 4. The two dynamics indeed agree with each other perfectly for strong perturbation cases. The linear interference form describes the dynamics of

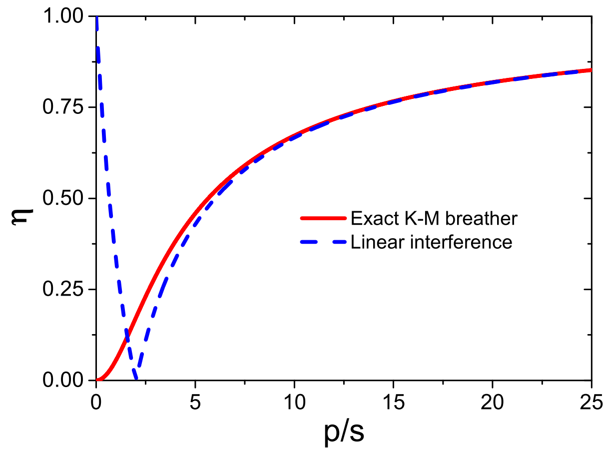

FIG. 2. The oscillation amplitude characterization parameter η vs p/s (the ratio of perturbation soliton amplitude and background amplitude). The solid red line and blue dashed line correspond to the results of KMB and the linear interference case, respectively. It is shown that the amplitude oscillation behavior of KMB does not agree with linear interference effects for weak soliton perturbations and the intermediate cases, but the behavior agrees perfectly with the prediction of linear interference effects for strong soliton perturbations.

KMB better with stronger BS type perturbations. For weak perturbations, the linear interference effects fail to explain the KMB dynamics. These cases can be understood well by linear stability analysis. The amplifications of perturbations are described well by the MI mechanism. But the MI fails to explain the oscillation period of KMB. The oscillation period can be understood from nonlinear interference effects between PWB and a weak perturbation. The nonlinear interference effects refer to the nonlinear superposition form of a plane wave and soliton type perturbation. In the nonlinear case, the period

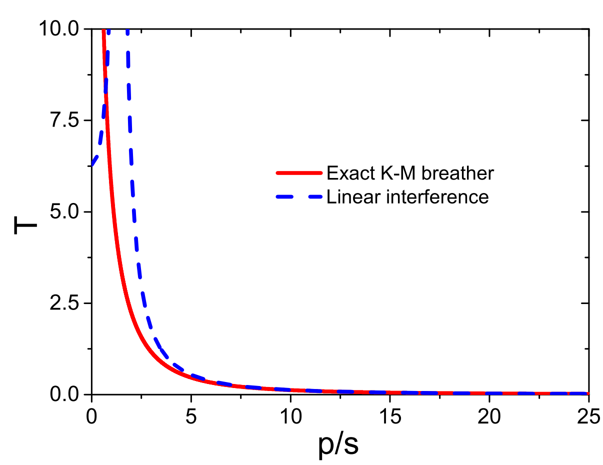

FIG. 3. The oscillation period T vs p/s (the ratio of perturbation soliton amplitude and background amplitude). The solid red line and blue dashed line correspond to the results of KMB and the linear interference case, respectively. It is shown that the amplitude oscillation behavior of KMB does not agree with linear interference effects for weak soliton perturbations and the intermediate cases, but the behavior agrees perfectly with the prediction of linear interference effects for strong soliton perturbations.

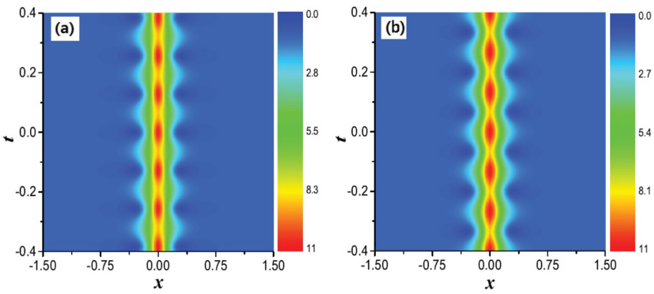

FIG. 4. The evolution of KMB (a) and the linear interference form (b) with an identical strong perturbation strength $p/s = 4\sqrt{6}$. It is shown that their evolution processes agree well for large perturbation soliton amplitude, which indicates the linear interference effect can be used to explain the dynamical process of KMB with strong perturbation cases. The plane-wave background amplitude is $s = 1$.

is determined by the evolution energy difference between the background and the perturbation's dominant energy [10]. In this way, the oscillation period can be estimated to be identical with the precise one of the KMB solution. This is helpful to understand the recent discussion on the spectral stability of rogue wave based on Floquet analysis of KMB [19].

In fact, the MI or linear interference effect alone cannot explain the dynamical process of KMB for which the perturbation amplitude p is comparable with the background amplitude s. This can be seen from the results in Figs. 2 and 3. MI can be used to understand the amplification of amplitude, and interference effect can be used to explain the temporal oscillation. But the breathing period does not agree with the one calculated by linear interference form anymore, since the linear superposition form does not hold in this case. The period can be understood well with the aid of the nonlinear interference effects (the dominant energy of perturbations is also needed). Based on the above discussions, we summarize the understanding on the dynamical process of KMB in Fig. 1.

The interference effects can not only explain the dynamics of the KMB with strong BS perturbation cases, but also can be used to understand the antidark soliton and W-shaped soliton with nonrational form obtained in a nonlinear Schrödinger equation with some fourth-order effects [17]. The fourth-order effects make the soliton excitation admit identical evolution energy with the PWB, but these cases do not exist for the typical nonlinear Schrödinger equation and Hirota equation [10,20]. Since they are reduced from a generic KMB, when the BS term admits identical energy with the PWB, the oscillation behavior will disappear and the breather will become an antidark soliton or a W-shaped soliton. Moreover, soliton excitation usually exists and RW or breather usually do not exist in the MS regime. But the linear interference effect would make breatherlike excitation exist in the MS regime, such as the breather-II obtained in mixed coupled nonlinear Schrödinger equations [16]. By the way, it should be noted that the interference-induced breatherlike excitations in the MS regime cannot reduce to be RW anymore, in contrast to the breathers in the MI regime [4–6]. This is because the breather-II is purely induced by the interference effects.

IV. CONCLUSION AND DISCUSSION

In summary, MI and interference effects can be used to understand the dynamical process of KMB. MI plays a dominant role for KMB admitting weak perturbations, and linear interference effects play a dominant role for KMB admitting strong perturbations. The weak and strong perturbations are clarified by the ratio $p/s \ll 1$ and $p/s \gg 1$, respectively. For intermediate cases, it is expected that the two mechanisms are both greatly involved. Maybe it is still needed to develop some proper ways to distinguish them or clarify the quantitative effects of them on the KMB dynamics. The results here can be extended to explain the dynamics of KMB and antidark soliton in a three-wave resonant system [21], scalar nonlinear Schrödinger equation with high-order effects [22–24], and other types of nonlinear models [25–28].

An approximation form is introduced to describe the linear interference effects. The reasonability of the introduced form is supported directly by the fact that Ψ with b tends to be infinity will become the form Ψ'. However, it should be emphasized that the localized perturbation form is chosen to be a sech soliton type, but it is not a generic form as for the Fourier perturbation modes in the linear stability analysis case. Very recently, it was shown that many different localized perturbations could evolve to be KMB in a microfabricated optomechanical array [29]. Therefore, a more generic localized form should be introduced to explain the evolution process of strong perturbations better. Recently, many different types of strong localized perturbations on plane-wave background were discussed in detail, which demonstrated the evolutions process involving KMBs with many different periods [11,15,30]. The KMBs' properties and numbers can be further evaluated by calculating the inverse scattering technique eigenvalues with the initial conditions, since all eigenvalues corresponding to KMB and other nonlinear modes are contained in the initial conditions [31,32]. If the inverse scattering eigenvalues with the initial perturbations admit a continuous spectrum, the evolution of perturbation was discussed in [11]. If the inverse scattering eigenvalues with the initial perturbations admit a discrete spectrum, the evolution of perturbations should correspond to the cases discussed in [12,33]. Obviously, the inverse scattering eigenvalues for the cases of soliton type perturbations discussed here also admit the discrete spectrum (the profile of soliton is not arbitrary for the discrete spectrum, which is different from the sech-shaped ones discussed in [30]). But this is a qualitative understanding on nonlinear MI. A unified quantitative discussion is still needed.

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[1] N. Akhmediev and V. I. Korneev, Theor. Math. Phys. **69**, 1089 (1986).
[2] D. H. Peregrine, J. Aust. Math. Soc. Ser. B **25**, 16 (1983).
[3] E. Kuznetsov, Sov. Phys. Dokl. **22**, 507 (1977); Y. C. Ma, Stud. Appl. Math. **60**, 43 (1979).
[4] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev, and J. M. Dudley, Nat. Phys. **6**, 790 (2010).
[5] J. M. Dudley, G. Genty, F. Dias, B. Kibler, and N. Akhmediev, Opt. Express **17**, 21497 (2009).
[6] B. Kibler, J. Fatome, C. Finot, G. Millot, G. Genty, B. Wetzel, N. Akhmediev, F. Dias, and J. M. Dudley, Sci. Rep. **2**, 463 (2012).
[7] A. Chabchoub, N. Hoffmann, M. Onorato, and N. Akhmediev, Phys. Rev. X **2**, 011015 (2012).
[8] F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, and S. Wabnitz, Phys. Rev. Lett. **113**, 034101 (2014).
[9] F. Baronio, S. Chen, P. Grelu, S. Wabnitz, and M. Conforti, Phys. Rev. A **91**, 033804 (2015).
[10] L.-C. Zhao and L. Ling, J. Opt. Soc. Am. B **33**, 850 (2016).
[11] G. Biondini and D. Mantzavinos, Phys. Rev. Lett. **116**, 043902 (2016).
[12] V. E. Zakharov and A. A. Gelash, Phys. Rev. Lett. **111**, 054101 (2013).
[13] L. Ling, L.-C. Zhao, Z.-Y. Yang, and B. Guo, Phys. Rev. E **96**, 022211 (2017).
[14] G. Biondini and E. Fagerstrom, SIAM J. Appl. Math. **75**, 136 (2015).
[15] G. Biondini, S. Li, and D. Mantzavinos, Phys. Rev. E **94**, 060201(R) (2016).
[16] L. Ling, L.-C. Zhao, and B. Guo, Commun. Nonlinear Sci. Numer. Simulat. **32**, 285 (2016).
[17] L. Duan, L.-C. Zhao, W.-H. Xu, C. Liu, Z.-Y. Yang, and W.-L. Yang, Phys. Rev. E **95**, 042212 (2017).
[18] L.-C. Zhao, L. Ling, Z.-Y. Yang, and J. Liu, Nonlinear Dyn. **83**, 659 (2016).
[19] J. Cuevas-Maraver, P. G. Kevrekidis, D. J. Frantzeskakis, N. I. Karachalios, M. Haragus, and G. James, Phys. Rev. E **96**, 012202 (2017).
[20] C. Liu, Z.-Y. Yang, L.-C. Zhao, and W.-L. Yang, Phys. Rev. E **91**, 022904 (2015).
[21] A. Degasperis, M. Conforti, F. Baronio, and S. Wabnitz, Phys. Rev. Lett. **97**, 093901 (2006); F. Baronio, M. Conforti, A. Degasperis, and S. Lombardo, *ibid.* **111**, 114101 (2013).
[22] R. Hirota, J. Math. Phys. **14**, 805 (1973).
[23] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[24] X. Y. Wen, Y. Yang, and Z. Yan, Phys. Rev. E **92**, 012917 (2015).
[25] K. Porsezian and K. Nakkeeran, Phys. Rev. Lett. **76**, 3955 (1996); K. Nakkeeran, K. Porsezian, P. S. Sundaram, and A. Mahalingam, *ibid.* **80**, 1425 (1998).
[26] Y. Zhang and W. X. Ma, Appl. Math. Comput. **256**, 252 (2015).
[27] J. Chen, Y. Chen, B. F. Feng, and K. Maruno, Phys. Lett. A **379**, 1510 (2015).
[28] J. He, S. Xu, K. Porsezian, Y. Cheng, and P. T. Dinda, Phys. Rev. E **93**, 062201 (2016); C. Li, J. He, and K. Porsezian, *ibid.* **87**, 012913 (2013).
[29] H. Xiong, J. H. Gan, and Y. Wu, Phys. Rev. Lett. **119**, 153901 (2017).
[30] G. Biondini, S. Li, D. Mantzavinos, and S. Trillo, arXiv:1710.05068.
[31] J. M. Soto-Crespo, N. Devine, and N. Akhmediev, Phys. Rev. Lett. **116**, 103901 (2016).
[32] N. Akhmediev, J. M. Soto-Crespo, and N. Devine, Phys. Rev. E **94**, 022212 (2016).
[33] C. Liu, Y. Ren, Z.-Y. Yang, and W.-L. Yang, Chaos **27**, 083120 (2017).