Diffusion with finite-helicity field tensor: A mechanism of generating heterogeneity

N. Sato and Z. Yoshida
Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan

(Received 21 December 2017; revised manuscript received 11 February 2018; published 28 February 2018)

Topological constraints on a dynamical system often manifest themselves as breaking of the Hamiltonian structure; well-known examples are nonholonomic constraints on Lagrangian mechanics. The statistical mechanics under such topological constraints is the subject of this study. Conventional arguments based on *phase spaces*, *Jacobi identity*, *invariant measure*, or the *H theorem* are no longer applicable since all these notions stem from the symplectic geometry underlying canonical Hamiltonian systems. Remembering that Hamiltonian systems are endowed with field tensors (canonical 2-forms) that have zero helicity, our mission is to extend the scope toward the class of systems governed by finite-helicity field tensors. Here, we introduce a class of field tensors that are characterized by Beltrami vectors. We prove an H theorem for this Beltrami class. The most general class of energy-conserving systems are non-Beltrami, for which we identify the "field charge" that prevents the entropy to maximize, resulting in creation of heterogeneous distributions. The essence of the theory can be delineated by classifying three-dimensional dynamics. We then generalize to arbitrary (finite) dimensions.

DOI: 10.1103/PhysRevE.97.022145

I. INTRODUCTION

While Fick's law is amenable to the intuition telling that diffusion will gradually remove gradients in distributions, we do find many counterexamples where diffusion generates or sustains gradients (so-called "up-hill" or "inward" diffusion). Indeed, the theoretical guarantee for the minimization of gradients (or maximization of entropy) is rather limited; conventional arguments start from the identification of a phase space and an invariant measure (Liouville's theorem), by which one may construct an H theorem to give presumption of the ergodic hypothesis. Usually, these deductions rely on the Hamiltonian structure of underlying microscopic dynamics. Given a general (non-Hamiltonian) system, therefore, one should once abandon the hypothesis of maximum entropy, and study different conditions by which diffusion may diminish or generate gradients.

In this work, we propose a paradigm of dynamics by which the regime of the maximum-entropy law is extended beyond Hamiltonian systems; this regime is identified by the "Beltrami condition" that demands vanishing of what we will call "field charge." We will show that the field charge is the root cause of inhomogeneity that can persist against diffusion.

Before nailing down the target of analysis, we make a short review of the theories of up-hill diffusion. There are two different causes of such phenomena: one is the energy and the other is the geometry of space. If the energy of a system includes some term that works to attract particles, the "Boltzmann distribution" explains the heterogeneity in the thermal equilibrium. Gravitational systems are such examples. Chemical potentials also play a similar role in grand-canonical ensembles. However, our interest is in the second kind of systems where the energy is just simple (for example, a convex function) but the space is "distorted" by a set of geometrical constraints representing dynamical states that are not accessible. The set of possible states is then a subset, whose geometry is, in general, extremely complicated. These "topological constraints" limit the effective space of dynamics, resulting in heterogeneous distributions in the *a priori* space. For Hamiltonian systems, the Casimir invariants (which originate from the center of the Poisson algebra) foliate the phase space (such Hamiltonian systems are said noncanonical) [1–3]. The Boltzmann distribution on a Casimir leaf may be viewed as a grand-canonical distribution with a chemical potential multiplying on the Casimir invariant, i.e., a Casimir invariant may be regarded as an action variable [4,5]. In the self-organization of a magnetospheric plasma, the magnetic moment of charged particles plays the role of the Casimir invariant [6]. As far as the system is Hamiltonian, the effective phase space is a (locally) symplectic leaf, so that the standard methods of statistical mechanics are readily applicable. We can formulate a Fokker-Planck equation to simulate the diffusion in magnetospheric systems [7–9]. The key reserved for Hamiltonian systems is the "integrability" of the topological constraints, which, however, is no longer valid for non-Hamiltonian systems. This is the regime of our interest.

Here, we assume the constancy of energy in the autonomous limit, i.e., motion occurs in the direction perpendicular to the gradient of the energy. The statistical dynamics is driven by a white noise in the energy. When the topological constraints are *nonintegrable* (in the sense of Frobenius' theorem [10–12]), there is no way to construct symplectic leaves on which we can define a canonical phase space. Mathematically, the nonintegrability is equivalent to the failure of the Jacobi identity [13–15], with critical consequences for the dynamics [16]. Nonintegrable constraints occur, for example, in nonholonomic mechanical systems [17,18], such as the rolling of a disk without slipping on a horizontal surface. In addition to nonholonomic mechanics [18,19] and molecular dynamics [20–23], it will be shown that other systems, such as the $\boldsymbol{E}\times\boldsymbol{B}$ drift equation of plasma dynamics [24,25] and the

Landau-Lifshitz equation [26,27] for the magnetization of a ferromagnetic material, fall in this category.

The essence of the theory can be delineated by three-dimensional mechanics (the one-dimensional and two-dimensional cases are trivial since in the former the only variable is the constant energy while in the latter the Jacobi identity is always satisfied and dynamics is Hamiltonian). The velocity $\boldsymbol{v}$ of motion can be written as

$$\boldsymbol{v} = \boldsymbol{w} \times \nabla H, \tag{1}$$

where H is the energy, and $\boldsymbol{w}$ is a fixed vector such that the velocity is perpendicular to ∇H. The operation of $\boldsymbol{w}\times$ can be represented by an antisymmetric operator $\mathcal{J}$ that we call a field tensor. For (1) to be Hamiltonian, $\boldsymbol{w}$ must be "helicity free" ($\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = 0$) and, then, $\boldsymbol{w}$ is integrable; we may locally write $\boldsymbol{w} = \lambda \nabla C$ with some λ and C, and $C = \text{const}$ gives the Casimir leaves. The three-dimensional Lie-Poisson algebras are classified by the Bianchi classification; for the complete list of symplectic leaves, see [28]. However, the target of our study are systems where $\boldsymbol{w}$ has finite helicity. We define the "Beltrami class" by those $\boldsymbol{w}$ such that $\nabla \times \boldsymbol{w} = \gamma \boldsymbol{w}$ with $\gamma \neq 0$. In Secs. III and IX we will prove an H theorem for the Beltrami class. We will also show that the "field charge" that is measured by $\nabla \cdot [\boldsymbol{w} \times (\nabla \times \boldsymbol{w})]$ (hence, the Beltrami class is charge free) causes heterogeneity. Notice that the mechanism of creation of heterogeneity is totally different from the aforementioned ones operated by some attracting potential energy, or the foliation of the phase space. In Sec. IV we will give numerical demonstration of the effect of the field charge. We will then generalize the theory to arbitrary (>2) dimensions in Secs. V–IX.

II. CONSERVATIVE DYNAMICS IN THREE DIMENSIONS

The simpler and instructive three-dimensional (3D) case is first discussed. In its general form, the equation of motion of a 3D conservative system is given by (1). Here, $\boldsymbol{v} = \dot{\boldsymbol{x}}$ is the velocity in the Cartesian coordinate system $\boldsymbol{x} = (x,y,z)$ of $\mathbb{R}^3$, the vector field $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x})$ (assumed smooth and nonvanishing) serves as antisymmetric operator (to be defined later), and the real valued smooth function H represents the Hamiltonian function. Evidently, $\dot{H} = 0$. However, system (1) is not, in general, Hamiltonian. As already mentioned, the condition is given by the Jacobi identity, which demands that $\boldsymbol{w}$ has vanishing helicity density:

$$h = \boldsymbol{w} \cdot (\nabla \times \boldsymbol{w}) = 0. \tag{2}$$

The validity of (2), which determines whether $\boldsymbol{w}$ qualifies as a Poisson operator, is related to the existence of additional integral invariants and to the availability of an invariant measure. Indeed, the following conditions are locally equivalent: for some open set $U \subset \mathbb{R}^3$,

1. $\boldsymbol{w} \cdot (\nabla \times \boldsymbol{w}) = 0$ in U, (3a)
2. $\exists\, \lambda, C : U \to \mathbb{R} : \boldsymbol{w} = \lambda \nabla C$ in U, (3b)
3. $\exists\, g \neq 0, g : U \to \mathbb{R} : \nabla \cdot (g\boldsymbol{v}) = 0\ \forall\, H$ in U. (3c)

($1 \Rightarrow 2$) is the Frobenius integrability condition for the vector field $\boldsymbol{w}$ (see [10–12]). Then, locally we can find two functions λ and C such that $\boldsymbol{w} = \lambda \nabla C$. ($2 \Rightarrow 1$) is trivial since $\boldsymbol{w} \cdot (\nabla \times \boldsymbol{w}) = -(\lambda \nabla C \cdot \nabla \lambda \times \nabla C) = 0$. ($2 \Rightarrow 3$) can be verified by observing that

$$\nabla \cdot (g\boldsymbol{v}) = 0\ \forall\, H \iff \nabla H \cdot \nabla \times (g\boldsymbol{w}) = 0\ \forall\, H. \tag{4}$$

The implication follows by setting $g = \lambda^{-1}$. ($3 \Rightarrow 2$) If there is an invariant measure g for any H, then $\nabla \times (g\boldsymbol{w}) = \boldsymbol{0}$. Therefore, $\boldsymbol{w} = g^{-1} \nabla C$ on U.

The function C, called a Casimir invariant, is a constant of motion for any choice of H and poses an integrable topological constraint on the dynamics. If $\boldsymbol{w}$ cannot be expressed in terms of a Casimir invariant, the dynamics is still constrained by the condition $\boldsymbol{w} \cdot \boldsymbol{v} = 0$, which then represents a nonintegrable topological constraint.

To introduce a classification of conservative dynamics beyond Hamiltonian systems, we define the *field force*

$$\boldsymbol{b} = \boldsymbol{w} \times (\nabla \times \boldsymbol{w}), \tag{5}$$

and the *field charge*

$$\mathfrak{B} = 4\nabla \cdot \boldsymbol{b} = 4\nabla \cdot [\boldsymbol{w} \times (\nabla \times \boldsymbol{w})]. \tag{6}$$

This naming was chosen by analogy with electromagnetism: when $\boldsymbol{w}$ is the antisymmetric operator associated to the $\boldsymbol{E} \times \boldsymbol{B}$ drift motion [24] of a charged particle in a magnetic field $\boldsymbol{B}$ of constant strength, the vector $\boldsymbol{b}$ is the magnetic force $\boldsymbol{B} \times (\nabla \times \boldsymbol{B})$. In fact, the drifting velocity is given by $\boldsymbol{v} = \boldsymbol{E} \times \boldsymbol{B}/B^2$, with $\boldsymbol{E} = -\nabla \phi$ the electric field and ϕ the electrostatic potential. Hence, the antisymmetric operator is $\boldsymbol{w} = \boldsymbol{B}/B^2$, the Hamiltonian $H = \phi$, and the Jacobi identity holds when $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} = 0$. To understand the geometrical meaning of $\boldsymbol{b}$, the following vector identity $\boldsymbol{b} = \boldsymbol{w} \times (\nabla \times \boldsymbol{w}) = \nabla w^2/2 - (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}$ is useful. Using this formula for $\hat{\boldsymbol{w}} = \boldsymbol{w}/w$, we have $\hat{\boldsymbol{b}} = -(\hat{\boldsymbol{w}} \cdot \nabla)\hat{\boldsymbol{w}} = -\hat{\boldsymbol{k}}$, where $\hat{\boldsymbol{k}}$ is the curvature vector. Therefore, $\boldsymbol{b}$ is related to the curvature of $\boldsymbol{w}$. Furthermore, observe that the curl of a vector field $\boldsymbol{w}$ admits the decomposition

$$\nabla \times \boldsymbol{w} = \frac{\boldsymbol{b} \times \boldsymbol{w} + h\boldsymbol{w}}{w^2}. \tag{7}$$

Three-dimensional conservative systems are then classified according to Fig. 1. In the next section, the statistical relevance of this classification will be made clear.

POISSON (Hamiltonian)	$h = 0$ ($\boldsymbol{w} = \lambda \nabla C$)	$f^{eq} = f^{eq}(\lambda, C)$
BELTRAMI	$h \neq 0$, $\mathfrak{B} = 0$	$f^{eq} = \text{constant}$
ANTISYMMETRIC	$h \neq 0$, $\mathfrak{B} \neq 0$	$f^{eq} = f^{eq}[w, \mathfrak{B}]$

FIG. 1. Classification of 3D conservative dynamics. The right column shows the equilibrium distribution function f^{eq} of an ensemble of particles obeying (1) when ∇H is a white noise process (see Sec. III). The square brackets in the last column indicate that the dependence of f^{eq} on w and $\mathfrak{B}$ is not necessarily a functional one.

III. DIFFUSION IN THREE DIMENSIONS

To examine the properties of diffusion, consider the purely stochastic equation of motion with $\nabla H = \boldsymbol{\Gamma}$:

$$\boldsymbol{v} = \boldsymbol{w} \times \boldsymbol{\Gamma}, \tag{8}$$

where $\boldsymbol{\Gamma} = (\Gamma_x, \Gamma_y, \Gamma_z)$ is three-dimensional white noise. If this were a conventional diffusion process, one would expect the density distribution f of an ensemble of particles obeying (8) to become progressively flat. This is not necessarily the case. To see this, consider the Fokker-Planck equation (to be derived later) associated with the stochastic differential equation (8):

$$\frac{\partial f}{\partial t} = \frac{1}{2}\nabla \cdot [\boldsymbol{w} \times (\nabla \times f\boldsymbol{w})] = \frac{1}{2}\left(\Delta_\perp f + \nabla f \cdot \boldsymbol{b} + \frac{1}{4} f \mathfrak{B}\right). \tag{9}$$

Here, we introduced the *normal Laplacian* $\Delta_\perp f = \nabla \cdot [\boldsymbol{w} \times (\nabla f \times \boldsymbol{w})]$. The word *normal* refers to the fact that its value only depends on the component of ∇f perpendicular to $\boldsymbol{w}$, $\nabla_\perp f = \boldsymbol{w} \times (\nabla f \times \boldsymbol{w})/w^2$. In the following, we shall always assume f to be a classical solution to the diffusion equation that admits all necessary derivatives.

The stationary form of Eq. (9) is a nonelliptic partial differential equation (PDE) (see [29–31] for the definition of ellipticity). Hence, the existence of a unique solution is not trivial. As it will be shown in the following, the nature of the stationary solution changes depending on the geometric properties of $\boldsymbol{w}$.

For f to become flat, the diffusion process (8) must maximize Shannon's information entropy:

$$S = -\int_\Omega f \log f \, dV. \tag{10}$$

Here, $\Omega \subset \mathbb{R}^3$ is a smoothly bounded domain occupied by the statistical ensemble, and $dV = dx\,dy\,dz$ is the volume element in $\mathbb{R}^3$. However, for a given $\boldsymbol{w}$, S is not necessarily maximized. When $h = 0$ the system is Hamiltonian and from (3) it follows that the invariant measure is $\lambda^{-1} dV$ ($\lambda \neq 0$ since $\boldsymbol{w} \neq \boldsymbol{0}$). Then, as one may expect, the appropriate entropy is

$$\Sigma_\lambda = -\int_\Omega f \log(f\lambda)\, dV, \tag{11}$$

which is equivalent to S only if $\lambda = \text{const}$. In fact, using (9) and assuming the boundary condition $\boldsymbol{w} \times (\nabla \times f\boldsymbol{w}) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, with $\boldsymbol{n}$ the unit outward normal to $\partial\Omega$, it follows that

$$\frac{d\Sigma_\lambda}{dt} = \frac{1}{2}\int_\Omega f\lambda |\lambda \nabla C \times \nabla \log(f\lambda)|^2 \lambda^{-1} dV \geqslant 0. \tag{12}$$

Assuming that $f > 0$ in Ω and observing that $d\Sigma_\lambda/dt$ must vanish in the limit $t \to \infty$, one sees that

$$f^{eq} = \lim_{t\to\infty} f = \frac{A}{\lambda} \exp\{-\gamma \mathcal{F}(C)\} \text{ in } \Omega, \tag{13}$$

where $\mathcal{F}(C)$ is an arbitrary function of the Casimir invariant C determined by the initial conditions, $A > 0$ and $\gamma > 0$ real constants.

It is a pivotal point of this study the proof that the maximization of S for $h \neq 0$ depends on the behavior of the field charge $\mathfrak{B}$. Indeed, the following result holds:

Theorem III.1 Let $\boldsymbol{w}$ be a smooth vector field on a smoothly bounded domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$. Consider Eq. (9) for $f > 0$ in Ω with boundary conditions $\boldsymbol{b} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{w} \times (\nabla f \times \boldsymbol{w}) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Assume $\mathfrak{B} = 0$ and $h \neq 0$ in Ω. Then,

$$\lim_{t\to\infty} \nabla f = \boldsymbol{0} \text{ in } \Omega. \tag{14}$$

Proof. Using Eq. (9) and the boundary conditions, the rate of change in the entropy (10) reads as

$$\frac{dS}{dt} = \frac{1}{2}\int_\Omega f\left[-\frac{\mathfrak{B}}{4} + |\boldsymbol{w} \times \nabla \log f|^2\right] dV. \tag{15}$$

Since by hypothesis $\mathfrak{B} = 0$, we must have $\lim_{t\to\infty} \boldsymbol{w} \times \nabla f = \boldsymbol{0}$ in Ω. Furthermore, $h \neq 0$ implies that $\boldsymbol{w}$ is not integrable, i.e., there is no Casimir invariant C such that $\boldsymbol{w} = \lambda \nabla C$ for some function λ. Hence, if we could satisfy $\nabla f = \alpha \boldsymbol{w}$ in Ω for some function $\alpha \neq 0$, this would contradict the nonintegrability of $\boldsymbol{w}$. Therefore, $\nabla f = \boldsymbol{0}$ in Ω when $t \to \infty$. ■

The boundary conditions used to derive Eqs. (12) and (15) ensure the thermodynamical closure of the system by avoiding loss of probability through the boundaries and will be discussed in more detail later. It is also worth noticing that, if $\mathfrak{B} \neq 0$, $f = \text{const}$ is not a stationary solution of (9), as one can verify by substitution. Indeed, one obtains the condition $\mathfrak{B} = 0$. An operator $\boldsymbol{w}$ satisfying such property will be called a *Beltrami operator*. This name refers to the Beltrami condition $\boldsymbol{b} = \boldsymbol{w} \times (\nabla \times \boldsymbol{w}) = \boldsymbol{0}$, which describes vectors aligned with their own vorticity, resulting in $\mathfrak{B} = 0$.

The determination of the stationary solution to (9) in the remaining case where $h \neq 0$ and $\mathfrak{B}$ is allowed to take nonzero values in Ω requires the machinery of functional analysis and will not be discussed here as this mathematical issue goes beyond the scope of this paper. However, the special case in which the field force $\hat{\boldsymbol{b}} = \hat{\boldsymbol{w}} \times (\nabla \times \hat{\boldsymbol{w}})$ of the normalized vector $\hat{\boldsymbol{w}} = \boldsymbol{w}/w$ can be expressed by means of a scalar potential as $\hat{\boldsymbol{b}} = \nabla\zeta$ can be solved explicitly and provides a concrete example of how self-organization in non-Hamiltonian system is intrinsically different from the foliation by Casimir invariants obtained in (13). To see this, consider the entropy

$$\Sigma_\zeta = -\int_\Omega f[\log(fw) + \zeta]\, dV, \tag{16}$$

and assume the boundary condition $\boldsymbol{w} \times (\nabla \times f\boldsymbol{w}) \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Then, the rate of change in Σ_ζ takes the form

$$\frac{d\Sigma_\zeta}{dt} = \frac{1}{2}\int_\Omega f |\boldsymbol{w} \times \nabla[\zeta + \log(fw)]|^2\, dV \geqslant 0. \tag{17}$$

Since by hypothesis $h \neq 0$, it follows that

$$f^{eq} = \lim_{t\to\infty} f = \frac{A}{w} e^{-\zeta} \text{ in } \Omega. \tag{18}$$

Here, $A > 0$ is a real constant. Notice how f^{eq} is determined by the field charge $\hat{\mathfrak{B}} = \Delta\zeta$ and the strength $w = |\boldsymbol{w}|$.

IV. NUMERICAL SIMULATION

It is now useful to make qualitative considerations on how the orbit of a conservative particle obeying (1) is modified by the introduction of random noise. First, consider the Euler rotation equation for a rigid body. In this case $\boldsymbol{w} = \boldsymbol{x}$, with

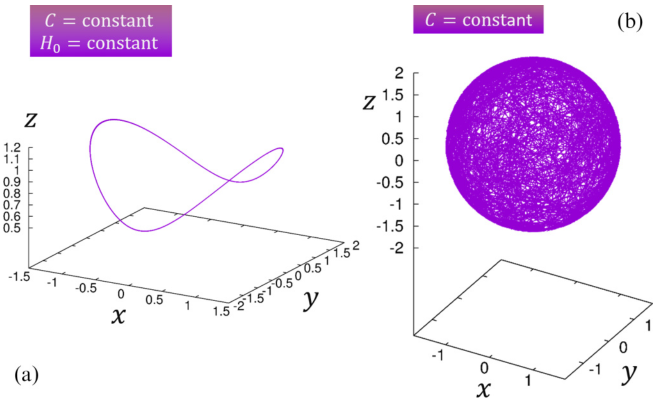

FIG. 2. (a) Numerical integration of the Euler rotation equation. The orbit is the intersection of the surfaces C and H_0. (b) Numerical integration of (19). If the Hamiltonian is perturbed $\nabla H = \nabla H_0 + \mathbf{\Gamma}$, the particle explores the surface C.

$\boldsymbol{x}$ the angular momentum, and the Hamiltonian is $H_0 = (x^2 I_x^{-1} + y^2 I_y^{-1} + z^2 I_z^{-1})/2$ with I_x, I_y, and I_z the momenta of inertia. $\boldsymbol{w}$ is a Poisson operator because the Jacobi identity is satisfied: $\boldsymbol{x} \cdot \nabla \times \boldsymbol{x} = 0$. As a consequence, the total angular momentum $C = \boldsymbol{x}^2/2$ is a Casimir invariant. The unperturbed orbit of the rigid body, given by the intersection of the integral surfaces H_0 and C, is given in Fig. 2(a). Now, we perturb the Hamiltonian H_0 so that the force acting on the particle becomes $\nabla H = \nabla H_0 + \mathbf{\Gamma}$. The resulting stochastic differential equation is

$$\boldsymbol{v} = \boldsymbol{x} \times (\nabla H_0 + \mathbf{\Gamma}). \tag{19}$$

Clearly, the energy H_0 is not anymore a constant of motion. However, the Casimir invariant C is unaffected by the perturbations. The result is a random process on the level set $C = \text{const}$ [see Fig. 2(b)].

Next, consider the antisymmetric operator $\boldsymbol{w} = (\cos z - \sin y, -\sin z, \cos y)$ with the same Hamiltonian H_0. One can check that $\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = \boldsymbol{w}^2$ so that no Casimir invariant exists. The unperturbed orbit is shown in Fig. 3(a). This time the trajectory is spiraling above the energy surface H_0. The absence of an invariant measure is also manifest. Again, perturb the Hamiltonian as $\nabla H = \nabla H_0 + \mathbf{\Gamma}$. The resulting orbit is shown in Fig. 3(b). Notice that no integral surface exists anymore.

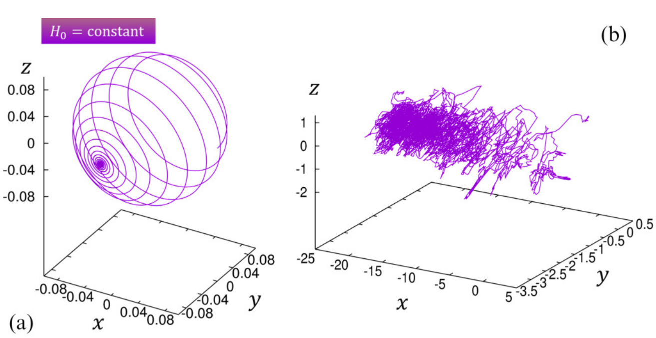

FIG. 3. Numerical integration of (1) for $\boldsymbol{w} = (\cos z - \sin y, -\sin z, \cos y)$. (a) The orbit explores the energy surface H_0 and falls toward a sink. (b) If the Hamiltonian is perturbed $\nabla H = \nabla H_0 + \mathbf{\Gamma}$, there are no integral surfaces.

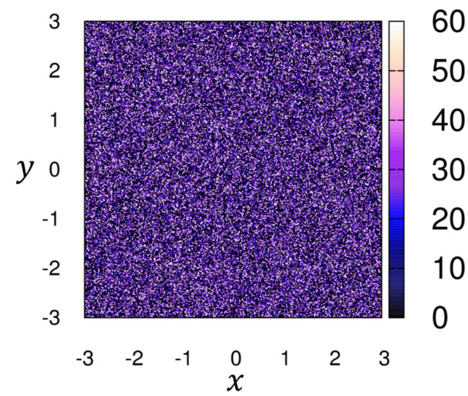

FIG. 4. Calculated equilibrium probability distribution f in the (x,y) plane at $z = 0$ with constant Poisson operator $\boldsymbol{w} = \partial_z$.

In the following part of this section, the analytical solution to the Fokker-Planck equation (9) is compared with the numerical integration of the stochastic equation (8) for different choices of $\boldsymbol{w}$. In each simulation, an ensemble of 8×10^6 particles is considered. The trajectory of each particle is tracked for the same period of time. Except when differently specified, the computational domain Ω is a cube in (x,y,z) space with sides of size 6 and centered at $\boldsymbol{x} = \mathbf{0}$. The boundary conditions are periodic (except when differently specified) with the period given by the sides of the cube. The initial condition is a flat (or Gaussian when so specified) probability distribution. All quantities are given in arbitrary units.

a. Uniform operator. The simplest possible situation is given by a uniform operator. We choose $\boldsymbol{w} = \partial_z$, with ∂_z the unit vector along the z axis. The helicity density $h = \boldsymbol{w} \cdot \nabla \times \boldsymbol{w}$ identically vanishes because $\nabla \times \boldsymbol{w} = \mathbf{0}$. Therefore, such $\boldsymbol{w}$ is a Poisson operator. The resulting dynamics $\boldsymbol{v} = \partial_z \times \mathbf{\Gamma}$ can be thought as the $\boldsymbol{E} \times \boldsymbol{B}$ motion of a charged particle in a constant magnetic field $B = w^{-1} = 1$ (remember that in the case of $\boldsymbol{E} \times \boldsymbol{B}$ drift $\boldsymbol{w} = \boldsymbol{B}/B^2$). It is also clear that the volume element $dx\,dy\,dz$ is an invariant measure for any choice of the Hamiltonian function, and that $\mathfrak{B} = 0$. The analytical form of the equilibrium probability distribution is then determined by observing that $\lambda = 1$ and $C = z$. Therefore, in light of (13),

$$f^{eq} = \lim_{t \to \infty} f = A \exp\{-\gamma \mathcal{F}(z)\} \text{ in } \Omega. \tag{20}$$

Furthermore, since the initial distribution is flat, the diffusion process $\boldsymbol{v} = \partial_z \times \mathbf{\Gamma}$, which is constrained in the (x,y) plane, cannot generate any inhomogeneity in the ∂_z direction. Hence, f must remain constant throughout the simulation. The result of the simulation is shown in Fig. 4.

b. Poisson operator on an invariant measure. Next, we consider the following Poisson operator:

$$\boldsymbol{w} = \nabla C = \nabla(z - \cos x - \cos y). \tag{21}$$

The Jacobi identity $h = 0$ is identically satisfied because $\nabla \times \boldsymbol{w} = \nabla \times \nabla C = \mathbf{0}$, also implying that $\boldsymbol{w}$ is a Beltrami operator since $\mathfrak{B} = 0$. If we interpret the resulting dynamics as the motion of a charged particle in the magnetic field $\boldsymbol{B} = \boldsymbol{w}/w^2$ (given the generality of $\boldsymbol{w}$, we do not

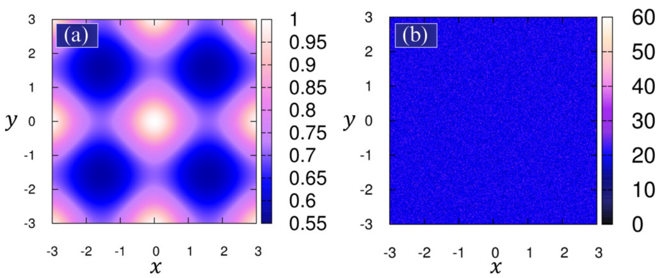

FIG. 5. (a) Magnetic field strength (22) in the (x,y) plane. (b) Calculated equilibrium probability distribution f in the (x,y) plane at $z=0$ with Poisson operator (21).

require $\nabla\cdot\boldsymbol{B}=0$ in these examples), the magnetic field strength is

$$B=(1+\sin x^2+\sin y^2)^{-1/2}. \tag{22}$$

See Fig. 5(a) for the plot of B. This time the Casimir invariant whose gradient spans the kernel of $\boldsymbol{w}$ is the function $C=z-\cos x-\cos y$. Using (3), we also know that $dx\,dy\,dz$ is an invariant measure for any choice of the Hamiltonian function. In light of (13), we expect the equilibrium probability distribution to be

$$f^{eq}=\lim_{t\to\infty}f=A\exp\{-\gamma\mathcal{F}(C)\}\text{ in }\Omega. \tag{23}$$

Let $f_0=f(t=0)$ be the (constant) value of the probability distribution at $t=0$ and $\Omega=[-\Delta x/2,\Delta x/2]\times[-\Delta y/2,\Delta y/2]\times[-\Delta z/2,\Delta z/2]$ the computational domain. Since the diffusion process cannot redistribute particles among different levels sets of C, the number of particles dN on each level set must be preserved, implying $dN(t=0)=f_0dC\int dx\wedge dy=f_0\Delta x\,\Delta y\,dC=dN(t\to\infty)=f^{eq}dC\int dx\wedge dy=f^{eq}\Delta x\,\Delta y\,dC$. But, then $f^{eq}=f_0=\text{const}$. Therefore, the distribution f must remain constant throughout the simulation. Figure 5(b) shows the results of the numerical simulation. In particular, notice that the distribution remains flat regardless of the fact that the random process is spatially inhomogeneous.

c. Poisson operator in arbitrary coordinates. Consider now the Poisson operator

$$\boldsymbol{w}=\lambda\nabla C=(\sqrt{1+\cos x^2})\nabla(z-\cos x-\cos y). \tag{24}$$

Here, $\lambda=\sqrt{1+\cos x^2}\neq0$ and $C=z-\cos x-\cos y$. The Jacobi identity is easily verified, $h=\lambda\nabla C\cdot\nabla\times\lambda\nabla C=0$, and C is a Casimir invariant. The corresponding magnetic field strength

$$B=[(1+\cos^2x)(1+\sin^2x+\sin^2y)]^{-1/2} \tag{25}$$

is shown in Fig. 6(a). According to (3), this time the invariant measure is given by the volume element $\lambda^{-1}dx\,dy\,dz$. In light of (13), we expect the solution to converge to a profile of the type $f\propto\lambda^{-1}$. Figure 6(b) shows a density plot of λ^{-1}. Figure 6(c) shows the result of the numerical simulation.

d. Beltrami operator. Next, consider the operator

$$\boldsymbol{w}=(\cos z+\sin z)\partial_x+(\cos z-\sin z)\partial_y. \tag{26}$$

One can verify that $h=\boldsymbol{w}^2=2\neq0$. Therefore, $\boldsymbol{w}$ is not a Poisson operator. Furthermore, the field force is $\boldsymbol{b}=\boldsymbol{w}\times\boldsymbol{w}=$

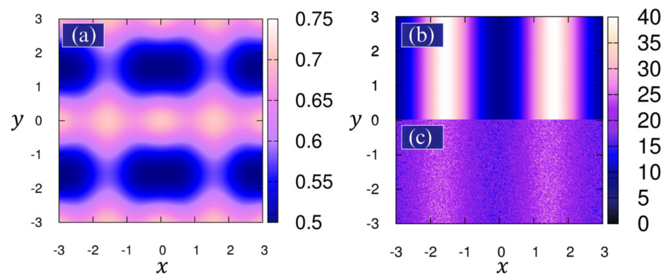

FIG. 6. (a) Magnetic field strength (25) in the (x,y) plane. (b) Spatial profile of λ^{-1} in the (x,y) plane. (c) Calculated equilibrium probability distribution f in the (x,y) plane at $z=0$ with Poisson operator (24). The scale at the right of (b) and (c) refers to plot (c).

$\mathbf{0}$. This means that $\boldsymbol{w}$ is a Beltrami operator. The corresponding magnetic field strength is constant: $B=w^{-1}=1/\sqrt{2}$. By Theorem III.1, $\nabla f=\mathbf{0}$ in Ω when $t\to\infty$. This is confirmed by the simulation, Fig. 7.

e. Antisymmetric operator with $\hat{\boldsymbol{b}}=\nabla\zeta$. Consider the operator:

$$\boldsymbol{w}=-y\partial_x+x\partial_y+r\partial_z. \tag{27}$$

Here, $r=\sqrt{x^2+y^2}$. The helicity density is $h=r$ and the field charge is $\mathfrak{B}=6$, implying that $\boldsymbol{w}$ is neither Poisson nor Beltrami. The magnetic field strength is $B=w^{-1}=1/\sqrt{2}r$. The normalized vector is $\hat{\boldsymbol{w}}=r^{-1}\boldsymbol{w}/\sqrt{2}$, while the field force of $\hat{\boldsymbol{w}}$ reads as $\hat{\boldsymbol{b}}=\hat{\boldsymbol{w}}\times(\nabla\times\hat{\boldsymbol{w}})=\nabla(\log r)/2$. Hence, Eq. (18) applies with $2\zeta=\log r$, leading to

$$f^{eq}=\lim_{t\to\infty}f=\frac{A}{\sqrt{2}}r^{-3/2}\text{ in }\Omega. \tag{28}$$

Figure 8 shows the result of the corresponding numerical simulation. In this case, no boundary conditions are assumed (the trajectories are followed as far as they go). The initial condition is the flat distribution of Fig. 3. Notice how the density distribution progressively approaches the profile of (28).

f. Antisymmetric operator. Consider the operator

$$\boldsymbol{w}=\partial_x+(\sin x+\cos y)\partial_y+(\cos x)\partial_z. \tag{29}$$

The helicity density is $h=1+\sin x\cos y\geqslant0$, meaning that the Jacobi identity is violated almost everywhere. Furthermore, the field charge is given by $\mathfrak{B}=-4\sin x\cos y$, which is finite

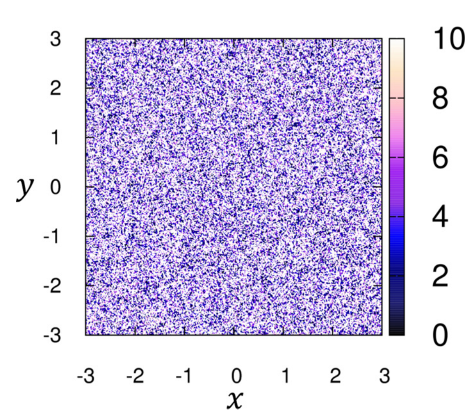

FIG. 7. Calculated equilibrium probability distribution f in the (x,y) plane at $z=0$ with Beltrami operator (26).

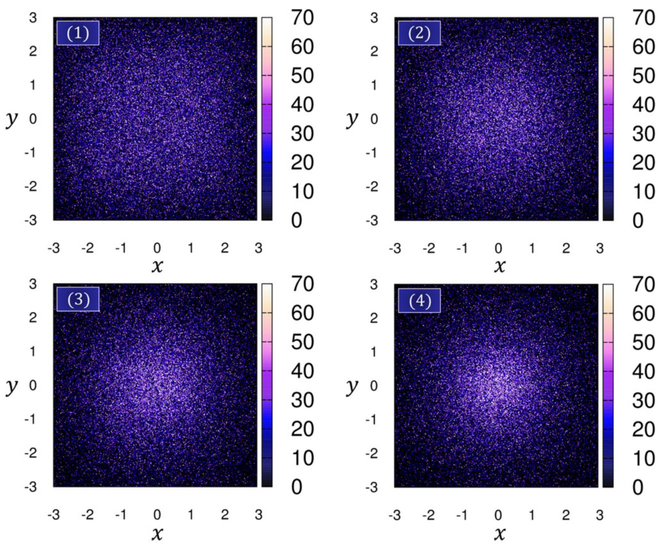

FIG. 8. Calculated equilibrium probability distribution f in the (x,y) plane with antisymmetric operator (27). Each plot number i corresponds to the instant $t = i\Delta t$, where Δt is a fixed time interval.

except in a set of measure zero. Therefore, this operator is neither a Poisson operator nor a Beltrami operator in the chosen coordinate system. The corresponding magnetic field strength is

$$B = w^{-1} = [1 + (\sin x + \cos y)^2 + \cos^2 x]^{-1/2}. \quad (30)$$

A density plot of B is given in Fig. 9(a). The result of the corresponding numerical simulation is given in Fig. 9(b). Notice that there is a similarity between the profile of magnetic field strength $B = w^{-1}$ and that of the equilibrium probability distribution f. This is in agreement with the behavior $f^{eq} \propto Be^{-\zeta}$ obtained in Eq. (18) for the special case $\hat{\boldsymbol{b}} = \nabla\zeta$.

g. Antisymmetric operator with unit norm. In the previous paragraph, we analyzed an antisymmetric operator and observed that the profile of the probability distribution resembled that of the magnetic field strength $B = w^{-1}$. To understand the role of the field charge in determining the probability distribution, we consider the antisymmetric operator

$$\hat{\boldsymbol{w}} = \frac{1}{\sqrt{1+\cos^2 x}}(\cos y, \cos x, \sin y). \quad (31)$$

Observe that $B = \hat{w}^{-1} = 1$ (and thus $\boldsymbol{B} = \hat{\boldsymbol{w}}$). One can check that the Jacobi identity is not satisfied and thus $\hat{\boldsymbol{w}}$ is not a Poisson operator. The field charge $\hat{\mathfrak{B}}$ of the operator $\hat{\boldsymbol{w}}$ does

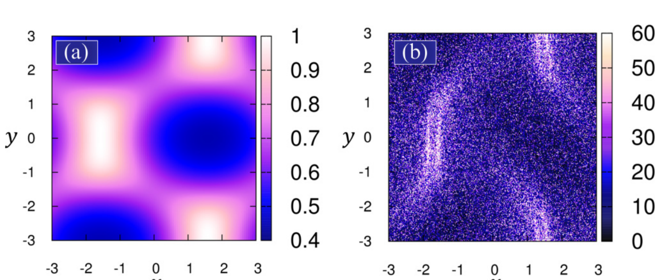

FIG. 9. (a) Magnetic field strength (30) in the (x,y) plane. (b) Calculated equilibrium probability distribution f in the (x,y) plane at $z = 0$ with antisymmetric operator (29).

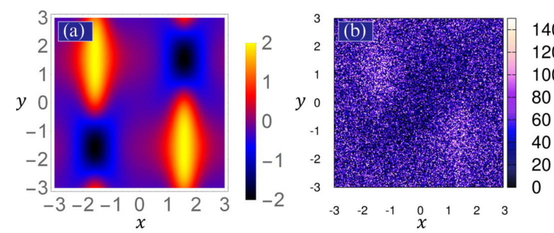

FIG. 10. (a) Plot of $\hat{\mathfrak{B}}$ for $\hat{\boldsymbol{w}}$ given by Eq. (31). (b) Calculated equilibrium probability distribution f in the (x,y) plane at $z = 0$ with antisymmetric operator (31).

not vanish (the lengthy expression of $\hat{\mathfrak{B}}$ is omitted). Therefore, $\hat{\boldsymbol{w}}$ is not a Beltrami operator in the chosen coordinate system.

The density profile obtained from the numerical simulation is shown in Fig. 10(b). Regardless of the fact that $B = \hat{w}^{-1} = 1$, a heterogeneous structure is self-organized. The determinant of this structure is the nonvanishing field charge $\hat{\mathfrak{B}}$. In fact, there is a strong similarity between the profile of the probability distribution and that of $\hat{\mathfrak{B}}$ [compare Fig. 10(b) with Fig. 10(a)].

h. Landau-Lifshitz equation. The last case we consider is the Landau-Lifshitz equation describing the time evolution of the magnetization $\boldsymbol{x}$ in a ferromagnet [specifically, we study Eq. (35) of [26]]. Without entering into details, the Hamiltonian of the system, physically corresponding to the total magnetization, is given by $H_0 = \boldsymbol{x}^2/2$. Therefore, in this simulation the perturbed Hamiltonian H is such that $\nabla H = \nabla H_0 + \boldsymbol{\Gamma}$. The relevant operator is

$$\boldsymbol{w} = \gamma\mathcal{H} - \frac{\sigma}{\boldsymbol{x}^2}\mathcal{H}\times\boldsymbol{x}. \quad (32)$$

Here, γ is a physical constant, σ the so called damping parameter, and $\mathcal{H}$ the effective magnetic field. The effective magnetic field $\mathcal{H}$ is chosen to be $\mathcal{H} = (c,0,z)$, where c represents a constant external magnetic field. Then, Eq. (32) can be rewritten as

$$\boldsymbol{w} = \left(\gamma c + \sigma\frac{zy}{\boldsymbol{x}^2}\right)\partial_x + \sigma\frac{z(c-x)}{\boldsymbol{x}^2}\partial_y + \left(\gamma z - \sigma\frac{cy}{\boldsymbol{x}^2}\right)\partial_z. \quad (33)$$

One can verify that this operator violates the Jacobi identity and that the field charge does not vanish. Therefore, $\boldsymbol{w}$ is not a Poisson operator, nor a Beltrami operator. In Fig. 11 the results of the numerical simulation are shown. This time,

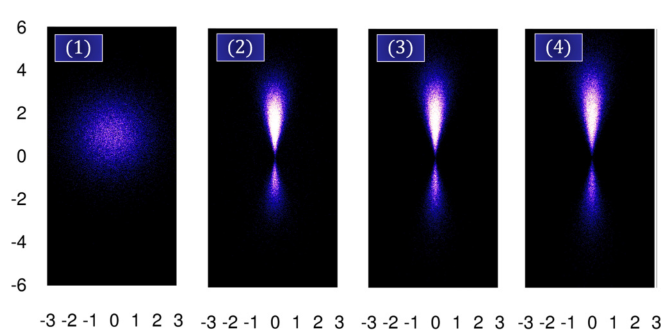

FIG. 11. Time evolution of the probability distribution f in the (x,z) plane at $y = 0$. Each plot number i corresponds to the instant $t = i\Delta t$, where Δt is a fixed time interval.

the initial condition is a Maxwell-Boltzmann distribution centered at $\boldsymbol{x} = (0,0,z_0)$. Furthermore, the trajectory of each magnetization is followed as far as it goes, i.e., no boundary conditions are used. Notice how the probability distribution becomes strongly anisotropic, with preferential alignment of the magnetization along the z axis (representing the direction of easiest magnetization of the ferromagnetic crystal).

V. CONSERVATIVE DYNAMICS AND TOPOLOGICAL CONSTRAINTS

The remaining part of this paper is devoted to the generalization of the theory to n dimensions. The key idea is that, by invoking change of coordinates, the classification of $\boldsymbol{w}$ in terms of h and $\mathfrak{B}$ developed in Sec. II can be generalized to include operators that satisfy the criterion $\mathfrak{B} = 0$ in different reference frames. We will see that in this way the Poisson operator of Hamiltonian systems defines a subclass of Beltrami operators. This requires a coordinate free formulation. For this reason, the formalism of differential geometry will be used.

In this section we review the concepts of antisymmetric operators, Poisson operators, and topological constraints, and introduce the mathematical notation used in the rest of the paper.

Let $\mathcal{M}$ be a smooth manifold of dimension n. An *antisymmetric operator* is a bivector field $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$, where $\bigwedge^2 T\mathcal{M}$ represents the set of antisymmetric matrices defined on the tangent space $T\mathcal{M}$ to $\mathcal{M}$. Let $(x^1,\dots,x^n)$ be a coordinate system on $\mathcal{M}$. Consider the tangent basis $(\partial_1,\dots,\partial_n)$. We have ($\wedge$ is the wedge product)

$$\mathcal{J} = \sum_{i<j} \mathcal{J}^{ij}\partial_i \wedge \partial_j = \frac{1}{2}\mathcal{J}^{ij}\partial_i \wedge \partial_j, \quad \mathcal{J}^{ij} = -\mathcal{J}^{ji}. \quad (34)$$

Here and throughout this study we shall assume $\mathcal{J}^{ij} \in C^\infty(\mathcal{M})$, except when differently specified. The matrix $\mathcal{J}^{ij}$ is antisymmetric and defines an antisymmetric bilinear inner product on pairs of functions $f,g \in C^\infty(\mathcal{M})$ called antisymmetric bracket:

$$\{f,g\} = \mathcal{J}(df,dg) = -\mathcal{J}(dg,df) = \mathcal{J}^{ij} f_i g_j. \quad (35)$$

In this notation, lower indices applied to a function indicate derivation, i.e., $f_i = \partial f/\partial x^i$.

An antisymmetric operator $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ and a Hamiltonian function $H \in C^\infty(\mathcal{M})$ define a *conservative vector field* $X \in T\mathcal{M}$ as

$$X = \mathcal{J}(dH) = \mathcal{J}^{ij} H_j \partial_i. \quad (36)$$

For the 3D case, one can verify that by setting

$$\begin{aligned}\mathcal{J} &= \mathcal{J}^{zy}\partial_z \wedge \partial_y + \mathcal{J}^{xz}\partial_x \wedge \partial_z + \mathcal{J}^{yx}\partial_y \wedge \partial_x \\ &= w_x\partial_z \wedge \partial_y + w_y\partial_x \wedge \partial_z + w_z\partial_y \wedge \partial_x,\end{aligned} \quad (37)$$

we have in a unique manner $X = \mathcal{J}(dH) = \boldsymbol{w} \times \nabla H$. Thanks to antisymmetry, a conservative vector field X always preserves the Hamiltonian H along the flow.

The antisymmetric bracket defined by $\mathcal{J}$ is called a Poisson bracket if it satisfies the Jacobi identity

$$h = \mathcal{J}^{im}\frac{\partial \mathcal{J}^{jk}}{\partial x^m} + \mathcal{J}^{jm}\frac{\partial \mathcal{J}^{ki}}{\partial x^m} + \mathcal{J}^{km}\frac{\partial \mathcal{J}^{ij}}{\partial x^m} = 0, \quad (38)$$

$\forall\, i,j,k = 1,\dots,n$. In this case, $\mathcal{J}$ is called a *Poisson operator* and the associated vector field X a *noncanonical Hamiltonian vector field*.

If $\mathcal{J}$ is invertible (and therefore $n = 2m$, $m \in \mathbb{N}$) with inverse $\omega \in \bigwedge^2 T^*\mathcal{M}$, the Jacobi identity is equivalent to demanding that $d\omega = 0$ (remember that a dual definition of Hamiltonian system can be given in terms of the symplectic 2-form ω as $i_X\omega = -dH$, with $d\omega = 0$). Canonical Hamiltonian systems correspond to a special class of Poisson operators called symplectic operators (or simplectic matrices):

$$\mathcal{J}_c = \sum_{i=1}^{m} \partial_{m+i} \wedge \partial_i. \quad (39)$$

The vector field $X = \mathcal{J}_c(dH)$ is then called a canonical Hamiltonian vector field. In light of Darboux's theorem [32,33], given a constant rank Poisson operator $\mathcal{J}$ of dimension $n = 2m + r$ ($2m$ is the rank), one can always find a local coordinate change by which $\mathcal{J}$ is expressed in the form (39).

In general, an antisymmetric operator $\mathcal{J}$ needs not to be invertible, i.e., its rank can be smaller than its dimension, $\text{rank}(\mathcal{J}) \leqslant \dim(\mathcal{J})$. When this happens, $\mathcal{J}$ has a nontrivial kernel, $\ker(\mathcal{J}) = \{\theta \in T^*\mathcal{M} : \mathcal{J}(\theta) = 0\}$. Clearly, we must have $\dim(\mathcal{J}) = \text{rank}(\mathcal{J}) + \dim(\ker(\mathcal{J}))$. Notice that any 1-form $\theta \in \ker(\mathcal{J})$ is orthogonal to the conservative vector field $X = \mathcal{J}(dH)$ for any choice of H:

$$\theta(X) = i_X\theta = \theta_i \mathcal{J}^{ij} H_j = 0 \quad \forall\, H. \quad (40)$$

This condition represents a geometrical constraint that is independent of the properties of matter (which are encoded in H), i.e., it defines a *topological constraint*. A collection of r constraints on a $2m + r$ dimensional manifold $\mathcal{M}^{2m+r}$ defines a $2n$ dimensional distribution $\Delta_{2m} = \{X \in T\mathcal{M}^{2m+r} : \theta_i(X) = 0 \;\; \forall\, i = 1,\dots,r\}$. As a consequence of Darboux's theorem, the distribution Δ_{2m} associated to a Poisson operator $\mathcal{J}$ of dimension $2m + r$, with $\dim(\ker(\mathcal{J})) = r$, is always integrable in the sense of Frobenius' theorem [10–12], i.e., there exists r scalar functions C^i called Casimir invariants such that $\mathcal{J}(dC^i) = 0$ and therefore $\Delta_{2m} = \{X \in T\mathcal{M}^{2m+r} : dC^i(X) = 0 \;\; \forall\, i = 1,\dots,r\}$. The word "invariant" refers to the fact that the C^i's are constants of motion that do not depend on the specific choice of H.

VI. GEOMETRICAL CLASSIFICATION OF ANTISYMMETRIC OPERATORS

The objective of this section is to produce a geometrical classification of antisymmetric operators that is relevant from the standpoint of statistical mechanics. For this purpose, we need a representation of antisymmetric operators in terms of differential forms.

Let $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ be an antisymmetric operator. Let $vol^n = g\,dx^1 \wedge \dots \wedge dx^n$ be a volume element on $\mathcal{M}$, with $g \neq 0$ and $g \in C^\infty(\mathcal{M})$. The *covorticity* $n-2$ form with respect to vol^n

is defined as

$$\mathcal{J}^{n-2} = i_{\mathcal{J}} vol^n = \sum_{i<j} (-1)^{i+j-1} g \mathcal{J}^{ij} \left(i_{\partial_i \wedge \partial_j} dx^i \wedge dx^j \right) \wedge dx^{n-2}_{ij} = 2 \sum_{i<j} (-1)^{i+j-1} g \mathcal{J}^{ij} dx^{n-2}_{ij}. \quad (41)$$

In this notation, $dx^{n-2}_{ij} = dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n$. Next, it is useful to define the *cocurrent* $n-1$ form of $\mathcal{J}$ with respect to the volume form vol^n on $\mathcal{M}$ as

$$\mathcal{O}^{n-1} = d\mathcal{J}^{n-2}. \quad (42)$$

In the same way the closeness of the 2-form ω defines Hamiltonian mechanics, the closeness of the $n-2$ form $\mathcal{J}^{n-2}$ is a powerful condition. Indeed, we can show the following:

The conservative vector field $X = \mathcal{J}(dH)$ admits an invariant measure vol^n for any choice of the Hamiltonian H if and only if $\mathcal{O}^{n-1} = 0$ on the volume form vol^n:

$$\mathfrak{L}_X vol^n = 0 \ \forall \ H \iff \mathcal{O}^{n-1} = 0 \text{ on } vol^n. \quad (43)$$

To see this, note that from (42) we have

$$\mathcal{O}^{n-1} = 2(-1)^j \frac{\partial (g \mathcal{J}^{ij})}{\partial x^i} dx^{n-1}_j. \quad (44)$$

On the other hand,

$$\mathfrak{L}_X vol^n = \frac{\partial (g \mathcal{J}^{ij})}{\partial x^i} H_j dx^1 \wedge \cdots \wedge dx^n. \quad (45)$$

Hence, (45) vanishes for any H if and only if $\mathcal{O}^{n-1} = 0$.

A. Measure preserving operator

Equation (43) introduces a notion of invariant measure that does not depend on the specific choice of the Hamiltonian H, but only on the geometrical properties of the operator $\mathcal{J}$. To know whether a certain operator $\mathcal{J}$ admits this kind of Hamiltonian-independent invariant measure, it is therefore sufficient to determine whether a metric g can be found such that $\mathcal{O}^{n-1} = 0$.

It is now natural to define the *measure preserving operator*: an antisymmetric operator $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ will be called measure preserving if there exists a volume form vol^n on $\mathcal{M}$ such that $\mathcal{O}^{n-1} = 0$. Evidently, an antisymmetric operator can be measure preserving without satisfying the Jacobi identity (38), i.e., without being a Poisson operator. Furthermore, notice that a constant rank Poisson operator is measure preserving. The proof of this statement, which is omitted, can be obtained by applying Darboux's theorem.

In the next part of this study it will be shown that on the invariant measure defined by a measure preserving operator the standard results of statistical mechanics can be recovered. Because of the special properties of the measure preserving operator, it is useful to determine whether a general antisymmetric operator can be transformed to a measure preserving form. On this regard, the following extension method applies:

Let $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ be an antisymmetric operator on a smooth manifold $\mathcal{M}$ of dimension n. Let x^{n+1} be a new variable with domain $\mathcal{D} \subset \mathbb{R}$. Then, the $n+1$ dimensional antisymmetric operator on $\bigwedge^2 T(\mathcal{M} \times \mathcal{D})$,

$$\mathfrak{J} = \mathcal{J} + x^{n+1} \frac{\partial \mathcal{J}^{ij}}{\partial x^i} \partial_j \wedge \partial_{n+1}, \quad (46)$$

is measure preserving.

To prove the statement, it is sufficient to show that on the volume form $vol^{n+1} = dx^1 \wedge \cdots \wedge dx^n \wedge dx^{n+1}$, the cocurrent $\mathcal{O}^n = d\mathfrak{J}^{n-1}$ vanishes. Recalling the condition given by Eq. (44), it follows that

$$\sum_{i=1}^{n+1} \frac{\partial \mathfrak{J}^{ij}}{\partial x^i} = \frac{\partial \mathfrak{J}^{n+1,j}}{\partial x^{n+1}} + \sum_{i=1}^{n} \frac{\partial \mathfrak{J}^{ij}}{\partial x^i} = x^{n+1} \sum_{i,k=1}^{n} \frac{\partial^2 \mathcal{J}^{ki}}{\partial x^i \partial x^k} = 0, \quad (47)$$

as desired. Observe that the extended system $X^{n+1} = \mathfrak{J}(dH)$ preserves the form of the original equations of motion $X^n = \mathcal{J}(dH)$ for the original n variables because the Hamiltonian H does not depend on the new variable x^{n+1}, i.e., $H_{n+1} = 0$.

Finally, if the operator $\mathcal{J}$ is invertible with inverse ω, the measure preserving condition $\partial_i (g \mathcal{J}^{ij}) = 0$ can be cast in a metric independent fashion. First, multiply by ω^{kj}:

$$\omega^{kj} \frac{\partial (g \mathcal{J}^{ij})}{\partial x^i} = g \left[\omega^{kj} \frac{\partial \mathcal{J}^{ij}}{\partial x^i} - \frac{\partial \log g}{\partial x^k} \right] = 0. \quad (48)$$

Define the 1-form $\mathfrak{A} = \omega^{kj} \frac{\partial \mathcal{J}^{ij}}{\partial x^i} dx^k$. Then, Eq. (48) reads as $\mathfrak{A} = d \log g$. If Ω is an open ball of $\mathbb{R}^n$ or a star-shaped open set about $\mathbf{0}$, Poincaré's lemma applies, and Eq. (48) can be satisfied by demanding that $d\mathfrak{A} = 0$ or explicitly

$$\left(\frac{\partial \omega^{kj}}{\partial x^m} - \frac{\partial \omega^{mj}}{\partial x^k} \right) \frac{\partial \mathcal{J}^{ij}}{\partial x^i} + \omega^{kj} \frac{\partial^2 \mathcal{J}^{ij}}{\partial x^i \partial x^m} - \omega^{mj} \frac{\partial^2 \mathcal{J}^{ij}}{\partial x^i \partial x^k} = 0. \quad (49)$$

Therefore, by checking the identity (49) on the domain Ω above, it is possible to establish whether there exists a coordinate system where an invertible operator is measure preserving.

B. Beltrami operator

The remaining task is the generalization of the concept of field charge to arbitrary dimensions n. By consistency with Eq. (6), the field charge of a general antisymmetric operator $\mathcal{J}$ must be a 0-form. Furthermore, since $\mathfrak{B}$ is the divergence of the vector $\boldsymbol{b}$, the generalization of $\boldsymbol{b}$ must be an $n-1$ form. Hence, it is natural to define the *field force* $n-1$ form of $\mathcal{J}$ as

$$b^{n-1} = \mathcal{J}^{n-2} \wedge * d\mathcal{J}^{n-2} = 4 \sum_{i<j} (-1)^{i+j+k-1} g \mathcal{J}^{ij} \frac{\partial (g \mathcal{J}^{lk})}{\partial x^l} dx^{n-2}_{ij} \wedge * dx^{n-1}_k. \quad (50)$$

Then, the *field charge* of $\mathcal{J}$ will be

$$\mathfrak{B} = * db^{n-1}. \quad (51)$$

In $\mathbb{R}^n$, this gives $\mathfrak{B} = 4\partial_i (\mathcal{J}^{ij} \partial_l (\mathcal{J}^{lj}))$. One can check that these definitions correctly reproduce those of the case $n = 3$ of $\mathbb{R}^3$.

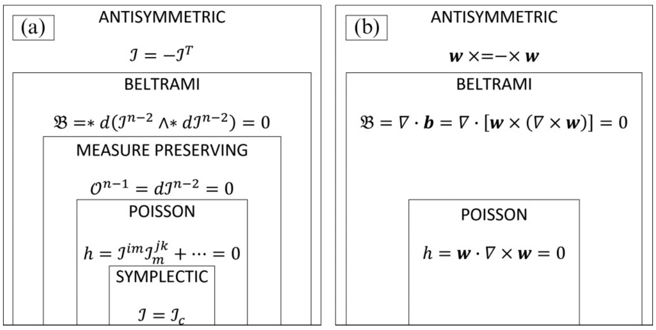

FIG. 12. (a) The hierarchical structure of antisymmetric operators. Each box is named by the corresponding operator. (b) The hierarchical structure of antisymmetric operators for $n = 3$. Notice that measure preserving operators do not appear because they degenerate to Poisson operators when $n = 3$. Specifically, the measure preserving condition $\nabla \times (g\boldsymbol{w}) = \boldsymbol{0}$ reduces to the integrability condition for $\boldsymbol{w}$ [see (3)]. Similarly, the symplectic operator does not appear because canonical pairs cannot be defined in odd dimensions.

Now, we can introduce the notion of Beltrami operator: let $\mathcal{J}$ be an antisymmetric operator. If a volume form $vol^n = g dx^1 \wedge \cdots \wedge dx^n$ can be found such that the field charge is zero, i.e., $\mathfrak{B} = * db^{n-1} = 0$, $\mathcal{J}$ is called a *Beltrami operator* on vol^n. If the field force $n-1$ form is zero, i.e., $b^{n-1} = 0$, $\mathcal{J}$ is called a *strong Beltrami operator* on vol^n.

Suppose that $\mathcal{J}$ is a measure preserving operator with invariant measure vol^n. Evidently, such $\mathcal{J}$ is a strong Beltrami operator on the invariant measure, i.e., $b^{n-1} = 0$ on vol^n. This is because a measure preserving operator satisfies $d\mathcal{J}^{n-2} = 0$ on the metric of the invariant measure [recall Eq. (43)]. Therefore, the corresponding field force $n-1$ form $b^{n-1} = \mathcal{J}^{n-2} \wedge * d\mathcal{J}^{n-2}$ identically vanishes.

Figure 12(a) summarizes the geometrical categorization of antisymmetric operators developed in the present section. Figure 12(b) shows a similar summary for the special and instructive case $n = 3$.

VII. FOKKER-PLANCK EQUATION

Consider now an ensemble of particles with an antisymmetric operator $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ and a Hamiltonian function $H_0 \in C^\infty(\mathcal{M})$. In order to construct the evolution equation for the corresponding probability distribution f, we must first obtain the stochastic differential equations governing particle dynamics. The motion of a single particle is described by the differential equation

$$X_0 = \mathcal{J}(dH_0). \tag{52}$$

First, assume that all the particles in the ensemble are not interacting, each of them obeying Eq. (52). Then, if we switch on some interaction, the energy H_0 will change according to $H = H_0(\boldsymbol{x}) + H_I(\boldsymbol{x},t)$ where H is the new Hamiltonian function accounting for the interaction energy $H_I(\boldsymbol{x},t)$. We take H_I, and thus H, to be $C^\infty(\mathcal{M} \times \mathbb{R}_{\geqslant 0})$. The interaction is therefore represented by the vector field X_I with components $X_I^i = \mathcal{J}^{ij} H_{Ii}$. To complete the description of particle dynamics, we further assume that all perturbations caused by H_I are counterbalanced by a *friction force*:

$$X_F^i = -\gamma^{ij} H_{0j} = -\tfrac{1}{2}\beta \mathcal{J}^{ik}\mathcal{J}^{jk} H_{0j} = \tfrac{1}{2}\beta \mathcal{J}^{ik} X_0^k. \tag{53}$$

Here, $\gamma^{ij} = \frac{1}{2}\beta \mathcal{J}^{ik}\mathcal{J}^{jk}$ is the friction coefficient with $\beta \in \mathbb{R}$ a spatial constant. Since the gradient of the Hamiltonian physically represents force, Eq. (53) leads to a total force $-H_{0i} - H_{Ii} - \frac{1}{2}\beta X_0^i$ where the friction term is proportional to the velocity as in the usual definition.

In summary, the equation of motion governing the dynamics of a particle in the ensemble is

$$\begin{aligned} X &= X_0 + X_I + X_F \\ &= \left[\left(\mathcal{J}^{ij} - \tfrac{1}{2}\beta \mathcal{J}^{ik}\mathcal{J}^{jk}\right) H_{0j} + \mathcal{J}^{ij}\Gamma_j\right]\partial_i. \end{aligned} \tag{54}$$

In the last passage, we made the substitution $\mathcal{J}^{ij} H_{Ij} = \mathcal{J}^{ij}\Gamma_j$. Here, we assumed that the jth component of the gradient of H_I is represented by Gaussian white noise Γ_j, i.e., $H_{Ij} = \Gamma_j$. We will justify this assumption later.

In the following, we will need a slightly more general form of Eq. (54). Indeed, in Eq. (54) white noise is applied in the same coordinate system $\boldsymbol{x} = (x^1, \ldots, x^n)$ used to describe the dynamics. However, we want to be able to perturb the ensemble in a different coordinate system, say $\boldsymbol{y} = (y^1, \ldots, y^n)$. Restricting to the cases in which the map $\mathcal{T} : \boldsymbol{x} \to \boldsymbol{y}$ is a diffeomorphism, we introduce the tensor $R_j^m = \partial y^m / \partial x^j$ and generalize Eq. (54):

$$X = \left[\left(\mathcal{J}^{ij} - \tfrac{1}{2}\beta \mathcal{J}^{ir} R_r^k \mathcal{J}^{js} R_s^k\right) H_{0j} + \mathcal{J}^{ij} R_j^r \Gamma_r\right]\partial_i. \tag{55}$$

Here, the friction coefficient is $\gamma^{ij} = \frac{1}{2}\beta \mathcal{J}^{ir} R_r^k \mathcal{J}^{js} R_s^k$ and we used the formula $H_{Ij} = R_j^r \Gamma_r$. Now, white noise is applied in the new coordinates $\boldsymbol{y}$ since $\partial H_I / \partial y^r = \Gamma_r$.

Observe that Eq. (55) is now a stochastic differential equation. Therefore, by application of the standard procedure (see, for example, [7,34,35]), we can derive the corresponding Fokker-Planck equation for the probability distribution f on the volume element $vol^n = dx^1 \wedge \cdots \wedge dx^n$. We have

$$\begin{aligned} \frac{\partial f}{\partial t} = \frac{\partial}{\partial x^i}\Bigg[&-\left(\mathcal{J}^{ij} - \frac{1}{2}\beta \mathcal{J}^{ir} R_r^k \mathcal{J}^{js} R_s^k\right) H_{0j} f \\ &+ \frac{1}{2}\frac{\partial}{\partial x^j}\left(\mathcal{J}^{ir} R_r^k \mathcal{J}^{js} R_s^k f\right) - \alpha \frac{\partial \mathcal{J}^{ir} R_r^k}{\partial x^j} \mathcal{J}^{js} R_s^k f\Bigg]. \end{aligned} \tag{56}$$

Finally, we must assign a specific value to the parameter $\alpha \in [0,1]$ (which defines the stochastic integral [7,34,35]) for the stochastic differential equation (55) and for the Fokker-Planck equation (56) to make mathematically sense. Assuming that the white noise $\boldsymbol{\Gamma}$ appearing in the equations is the limiting representation of a continuous perturbation, we take the value $\alpha = \frac{1}{2}$ (corresponding to the Stratonovich definition of the stochastic integral). When $\alpha = \frac{1}{2}$, Eq. (56) reduces to

$$\begin{aligned} \frac{\partial f}{\partial t} = \frac{\partial}{\partial x^i}\Bigg[&-\left(\mathcal{J}^{ij} - \frac{1}{2}\beta \mathcal{J}^{ir} R_r^k \mathcal{J}^{js} R_s^k\right) H_{0j} f \\ &+ \frac{1}{2}\mathcal{J}^{ir} R_r^k \frac{\partial}{\partial x^j}\left(\mathcal{J}^{js} R_s^k f\right)\Bigg]. \end{aligned} \tag{57}$$

Observe that the matrix R_r^k can be interpreted as the square root of a generalized diffusion parameter.

VIII. *H* THEOREM FOR MEASURE PRESERVING OPERATORS

The derived Fokker-Planck equation (56) shows that the behavior of the probability distribution f depends on three factors: the energy H representing the properties of matter, the metric of space characterized by the operator $\mathcal{J}$, and the type of perturbations described by the tensor R^k_r and the parameter α (notice that physically R^k_r accounts for the spatial properties and α for the type of time evolution of the perturbations). In this section, we examine the form of $f^{eq} = \lim_{t\to\infty} f$. It is convenient to define the concept of Fokker-Planck velocity Z. Since the probability $f\, vol^n$ enclosed in each volume element must be preserved along the trajectories, if $Z \in T\mathcal{M}$ is the dynamical flow generating the evolution of such probability, we must have the following conservation law:

$$(\partial_t + \mathfrak{L}_Z) f\, vol^n = \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial x^i}(f Z^i)\right] vol^n = 0. \tag{58}$$

Comparing this equation with the Fokker-Planck equation (56), wee see that

$$Z^i = \left(\mathcal{J}^{ij} - \frac{1}{2}\beta \mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s\right) H_{0j} - \frac{1}{2f}\frac{\partial}{\partial x^j}\left(\mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s f\right) + \alpha \frac{\partial \mathcal{J}^{ir} R^k_r}{\partial x^j}\mathcal{J}^{js} R^k_s. \tag{59}$$

The quantity Z is called the Fokker-Planck velocity of the system.

We anticipated that, in the absence of canonical phase space, the form of f^{eq} departs from the standard Maxwell-Boltzmann distribution and takes a form depending on the operator $\mathcal{J}$. On this regard, the following convergence theorem for measure preserving operators holds.

Assume the following conditions:

(i) $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ is a measure preserving operator of C^2 class.

(ii) $\boldsymbol{x} = (x^1, \ldots, x^n)$ is a coordinate system on $\mathcal{M}$ endowed with the invariant measure, i.e., $\partial_i \mathcal{J}^{ij} = 0\ \forall\ j = 1, \ldots, n$.

(iii) Let W_i, $i = 1, \ldots, n$, be n Wiener processes, with $dW_i = \Gamma_i dt$ and $\alpha = \frac{1}{2}$ (Stratonovich stochastic integral).

(iv) Define $R^j_k = \partial_k y^j$, $j,k = 1, \ldots, n$, where $\boldsymbol{y} = (y^1, \ldots, y^n)$ is a coordinate system such that the map $\mathcal{T} : \boldsymbol{x} \to \boldsymbol{y}$ is a diffeomorphism.

(v) Let the equations of motion be

$$X^i = (\mathcal{J}^{ij} - \gamma^{ij}) H_{0j} + \mathcal{J}^{ik} R^j_k \Gamma_j, \tag{60}$$

where the function $H(\boldsymbol{x},t) = H_0(\boldsymbol{x}) + y^i \Gamma_i(t)$ is the Hamiltonian of the system, $H_0 \in C^2(\mathcal{M})$, and $\gamma^{ij} = \frac{1}{2}\beta \mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s$ is the friction coefficient with $\beta \in \mathbb{R}$ a spatial constant.

(vi) The corresponding transport equation for the probability distribution $f > 0$ on a smoothly bounded domain $\Omega \subset \mathcal{M}$ with volume element $vol^n = dx^1 \wedge \cdots \wedge dx^n$ is given by Eq. (57). Suppose that on the boundary $\partial\Omega$ the conditions $Z \cdot N = 0$ and $X_0 \cdot N = 0$ hold, with Z the Fokker-Planck velocity such that $\partial_t f = -\partial_i (f Z^i)$, $X_0 = \mathcal{J}^{ij} H_{0j} \partial_i$, and N the outward normal to $\partial\Omega$.

Then, the solution to (57) is such that

$$\lim_{t\to\infty} \mathcal{J}(d \log f + \beta d H_0) = 0 \text{ in } \Omega, \tag{61}$$

for any choice of the coordinates y^j, $j = 1, \ldots, n$.

Let us prove this statement. Recalling the expression of the Fokker-Planck velocity Z [Eq. (59)] and setting $\alpha = \frac{1}{2}$ we obtain

$$Z^i = (\mathcal{J}^{ij} - \gamma^{ij}) H_{0j} - \frac{1}{2}\mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s \frac{\partial \log f}{\partial x^j}. \tag{62}$$

In going from (59) to this expression, we used the fact that $\mathcal{J}$ is measure preserving ($\partial_i \mathcal{J}^{ij} = 0$, $j = 1, \ldots, n$) and that the matrix $R^k_{sj} = \partial^2 y^k / \partial x^s \partial x^j$ is symmetric so that $\mathcal{J}^{sj} R^k_{sj} = 0$, $k = 1, \ldots, n$. Consider now the following entropy functional:

$$S = -\int_\Omega f \log f\ vol^n. \tag{63}$$

The rate of change of S is

$$\begin{aligned}\frac{dS}{dt} &= -\int_\Omega \frac{\partial f}{\partial t}(1 + \log f)\, vol^n \\ &= \int_\Omega f \frac{\partial Z^i}{\partial x^i}\, vol^n + \int_{\partial\Omega} f \log f\ Z^i N_i\, dS^{n-1} \\ &= -\int_\Omega f_i Z^i\ vol^n.\end{aligned} \tag{64}$$

Here, we used the fact that $Z^i N_i$ vanish on the boundary $\partial\Omega$. In this notation $N = N_i \partial_i$ is the outward normal to the bounding surface $\partial\Omega$ with surface element dS^{n-1}. Substituting (62) in (64) we get

$$\frac{dS}{dt} = \frac{1}{2}\int_\Omega f_i \mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s \left(\frac{\partial \log f}{\partial x^j} + \beta H_{0j}\right) vol^n. \tag{65}$$

Here, we used the fact that $\mathcal{J}$ is measure preserving and thus the term involving $f_i \mathcal{J}^{ij} H_{0j} = \frac{\partial}{\partial x^i}(f X^i_0)$ can be written as a vanishing surface integral. Consider now conservation of total energy $E = \int_\Omega f H_0\, vol^n$:

$$\begin{aligned}\frac{dE}{dt} &= \int_\Omega f \mathcal{J}^{ij} H_{0j} H_{0i}\, vol^n \\ &\quad - \frac{1}{2}\int_\Omega f \mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s \left(\frac{\partial \log f}{\partial x^j} + \beta H_{0j}\right) H_{0i}\, vol^n \\ &= 0.\end{aligned} \tag{66}$$

Again, we used the fact that surface integrals vanish and the antisymmetry of $\mathcal{J}$. This implies

$$\int_\Omega \mathcal{J}^{ir} R^k_r \mathcal{J}^{js} R^k_s f_j H_{0i}\, vol^n = -\beta \int_\Omega f \left(\mathcal{J}^{ir} R^k_r H_{0i}\right)^2 vol^n. \tag{67}$$

Observe that (67) defines the spatial constant β at each time t. Substituting this result in (65) and after some manipulations we obtain

$$\frac{dS}{dt} = \frac{1}{2}\int_\Omega f \left[\mathcal{J}^{ir} R^k_r \left(\frac{\partial \log f}{\partial x^i} + \beta H_{0i}\right)\right]^2 vol^n. \tag{68}$$

In the limit of thermodynamic equilibrium, we must have $\lim_{t\to\infty} dS/dt = 0$. Thus, for all nonzero f one arrives at the result (61). Notice that the matrix R^k_r could be removed because the transformation $\mathcal{T}: \boldsymbol{x} \to \boldsymbol{y}$ is a diffeomorphism and is therefore invertible.

Let us make some considerations on the meaning and the physical implications of this result. The reason why Eq. (61) holds is that $\mathcal{J}$ is measure preserving and f is the probability distribution *on* the invariant measure. Only in such coordinate system Shannon's entropy (63) has proper physical meaning, i.e., the entropy production represented by Eq. (68) has a definite sign and therefore an extremum principle (maximum entropy) applies. If g is the Jacobian of the coordinate change sending the invariant measure vol^n to a different reference system $vol_c^n = g^{-1} vol^n$, the probability distribution in the new frame is $u = fg$. Here, the letter c stands for Cartesian since usually one is interested in the probability distribution observed in the Cartesian coordinate system of the laboratory frame. Define Shannon's entropy for the new distribution u as $S_c = -\int_\Omega u \log u \, vol_c$. Then, the thermodynamically consistent entropy Σ and the information measure S_c are related as

$$\Sigma = S_c + \langle \log g \rangle, \tag{69}$$

where the angle brackets stand for ensemble average.

It is useful to add some considerations on the boundary conditions $Z \cdot N = 0$ and $X_0 \cdot N = 0$ on $\partial\Omega$. Physically, they express the fact that probability does not escape from the domain Ω, and therefore the system can be considered as thermodynamically closed. The condition $X_0 \cdot N = 0$ can be thought as a definition of the boundary itself, and can be satisfied, for example, by taking an Hamiltonian H_0 that is constant on the boundary $H_{0i} = 0$ on $\partial\Omega$. The condition $Z \cdot N = 0$ is rather a boundary condition for f. If $H_{0i} = 0$ on $\partial\Omega$, one can use the Neumann boundary condition $df = 0$ on $\partial\Omega$.

If the matrix $\mathcal{J}$ is invertible, Eq. (61) becomes

$$f^{eq} = \lim_{t\to\infty} f = A \exp\{-\beta H_0\} \text{ in } \Omega, \tag{70}$$

where $A \in \mathbb{R}_{>0}$ is a normalization constant. Thereby, we can rephrase the result (61) in the following way: if the metric of space if current free, i.e., $\mathcal{O}^{n-1} = 0$, and space is completely accessible, i.e., $\ker(\mathcal{J}) = 0$, the standard result of statistical mechanics apply on the invariant measure. The effect of a nontrivial kernel $\ker(\mathcal{J}) \neq 0$ can be understood with the next corollary of theorem (61).

Assume the hypothesis used to derive (61). In addition, assume that $\mathcal{J}$ has constant rank $2m = n - r$ and that it is a Poisson operator. Then, for every point $\boldsymbol{x} \in \Omega$ there exists a neighborhood $U \subset \Omega$ of $\boldsymbol{x}$ such that

$$f^{eq} = \lim_{t\to\infty} f = A \exp\{-\beta H_0 - \gamma \mathcal{F}(\boldsymbol{C})\} \text{ in } U, \tag{71}$$

where $\gamma \in \mathbb{R}$ is a constant and $\mathcal{F}(\boldsymbol{C})$ an arbitrary function of the r Casimir invariants $\boldsymbol{C} = (C^1, \ldots, C^r)$ whose gradients span the kernel of $\mathcal{J}$, i.e., $\mathcal{J}(dC^i) = 0$.

This result is a consequence of Darboux's theorem, according to which $\forall\, \boldsymbol{x} \in \Omega$ there exists a neighborhood $U \subset \Omega$ of $\boldsymbol{x}$ where we can find coordinates $(u^1, \ldots, u^{2m}, C^1, \ldots, C^r)$ such that the C^i are Casimir invariants. Thus, the local solution to Eq. (61) is of the form (71).

In the case of a noncanonical Hamiltonian system, we see that statistical equilibrium, which is achieved on the invariant measure assigned by Liouville's theorem, is determined by the energy H_0 and the Casimir invariants C^i. In this way, the functions C^i impart a nontrivial structure to the probability distribution f. This type of self-organization is caused by the existence of inaccessible regions in the phase space, which are mathematically represented by the fact that motion is restricted on the level sets of the Casimir invariants.

The last remark concerns the white noise assumption. This assumption must be justified on a case by case basis by showing that the perturbations affecting a certain ensemble statistically behave as Gaussian white noise in some appropriate coordinate system $\boldsymbol{y}$ (in the sense that the gradient $\partial H_I/\partial y^r$ of the interaction Hamiltonian H_I with respect to the coordinates $\boldsymbol{y}$ can be considered as Gaussian white noise). In practice, using the invariant measure provided by the measure preserving operator, one invokes the ergodic hypothesis by which ensemble and time averages can be interchanged. Then, fluctuations with vanishing ensemble averages can be conveniently represented as white noise processes of zero time average. Finally, notice that Eq. (61) does not depend on the specific coordinates $\boldsymbol{y}$. This means that, regardless of the coordinate frame where a system is perturbed, statistical equilibrium is achieved on the invariant measure determined by $\mathcal{J}$.

IX. DIFFUSION WITH BELTRAMI OPERATORS

We now move to operators that are not endowed with an invariant measure. Specifically, we generalize Eq. (15) to nD. In this case we are interested in pure diffusion, i.e., $H_0 = 0$. Then, from Eq. (55), the relevant equation of motion reads as

$$X = \left(\mathcal{J}^{ij} R_j^r \Gamma_r\right)\partial_i. \tag{72}$$

To further simplify the problem, set $R_j^r = \delta_j^r$. Recalling the transport equation (56) and putting $\alpha = \frac{1}{2}$, we arrive at the corresponding diffusion equation

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x^i}\left[\mathcal{J}^{ik}\frac{\partial(\mathcal{J}^{jk} f)}{\partial x^j}\right] = \frac{1}{2}\left(\Delta_\perp f + b^i f_i + \frac{1}{4} f \mathfrak{B}\right). \tag{73}$$

Here, $\Delta_\perp f = \partial_i(\mathcal{J}^{ik}\mathcal{J}^{jk} f_j)$ is the n-dimensional normal Laplacian and $b^i = \mathcal{J}^{ik}\frac{\partial \mathcal{J}^{jk}}{\partial x^j}$. We have the following:

Assume that $\mathcal{J} \in \bigwedge^2 T\mathcal{M}$ is a Beltrami operator ($\mathfrak{B} = 0$) on $vol^n = dx^1 \wedge \cdots \wedge dx^n$. Consider the diffusion equation (73) for the probability distribution $f > 0$ on a smoothly bounded domain $\Omega \subset \mathcal{M}$. Assume the boundary conditions $Z \cdot N = 0$ and $\boldsymbol{b} \cdot N = 0$ on $\partial\Omega$, where Z is the Fokker-Planck velocity such that $\partial_t f = -\partial_i(f Z^i)$, $\boldsymbol{b} = \mathcal{J}^{ik}\mathcal{J}_j^{jk}\partial_i$, and N the outward normal to $\partial\Omega$. Then,

$$\lim_{t\to\infty} \mathcal{J}(d \log f) = 0 \text{ in } \Omega. \tag{74}$$

The proof can be given as follows. Consider the entropy functional

$$S = -\int_\Omega f \log f \, vol^n. \tag{75}$$

Following the same calculation of Eq. (64), the rate of change in S is

$$\frac{dS}{dt} = \frac{1}{2}\int_\Omega \left[-\frac{f}{4}\mathfrak{B} + f|\mathcal{J}(d\log f)|^2 \right] vol^n$$
$$= \frac{1}{2}\int_\Omega f|\mathcal{J}(d\log f)|^2\, vol^n. \tag{76}$$

Here, we used the boundary conditions to eliminate surface integrals and the vanishing of $\mathfrak{B}$. Then, since by hypothesis $f > 0$, one arrives at (74).

As for theorem (61), the physical meaning of the requirements $Z \cdot N = 0$ and $\boldsymbol{b} \cdot N = 0$ on $\partial\Omega$ is that probability does not escape from the boundaries. If the diffusion equation is written in terms of the Cartesian coordinate system of $\mathbb{R}^n$, the vector $\boldsymbol{b}$ corresponds to the field force $n-1$ form (50) and, in $\mathbb{R}^3$, one obtains $\boldsymbol{b} = \boldsymbol{w} \times (\nabla \times \boldsymbol{w})$. $\boldsymbol{b}$ acts as an effective drift. Indeed, from Eq. (73), one sees that the Fokker-Plack velocity Z can be decomposed as $2Z^i = f^{-1}\mathcal{J}^{ik}\frac{\partial(\mathcal{J}^{jk} f)}{\partial x^j} = b^i + \mathcal{J}^{ik}\mathcal{J}^{jk}\frac{\partial \log f}{\partial x^j}$. Thus, $\boldsymbol{b} \cdot N = 0$ on $\partial\Omega$ means that the boundary must be chosen so that the drift $\boldsymbol{b}$ does not transport any probability out of the domain Ω. The second condition $Z \cdot N = 0$ can be intended as a boundary condition for the probability distribution f. A possible way to satisfy these conditions is, for example, to assume that $\mathcal{J}$ is a strong Beltrami operator in a Cartesian coordinate system so that $\boldsymbol{b} = \boldsymbol{0}$, and set $\nabla f = \boldsymbol{0}$ on $\partial\Omega$.

Equation (74) says that the flat distribution $f =$ const can be obtained even if no invariant measure exists. In other words, the Beltrami operator is the largest class of antisymmetric operators such that the diffusion equation (73) admits the solution $f =$ const. As already noted in Sec. III, this fact can be verified by substituting the solution $f =$ const in Eq. (73). One obtains the condition $\mathfrak{B} = 0$. Beyond diffusion driven by Beltrami operators, the nonvanishing of $\mathfrak{B}$ obstructs, in general, the determination of a suitable metric g where an H theorem can be obtained. A possible way out is the extension method of Eq. (46), which enables the handling of a general antisymmetric operator by extending it to a measure preserving form. However, there are cases that can be solved explicitly even for $\mathfrak{B} \neq 0$, as shown at the end of Sec. III.

	H FUNCTION	SOURCES OF HETEROGENEITY	REFERENCE IN THE TEXT
SYMPLECTIC	$\int_\Omega f \log f d\boldsymbol{p} d\boldsymbol{q}$	ENERGY	
POISSON	$\int_\Omega f \log(f g_c) dV$	ENERGY, FOLIATION BY CASIMIR INVARIANTS	Examples a, b, c sec. IV Equation (71)
MEASURE PRESERVING	$\int_\Omega f \log(f g_{IM}) dV$	ENERGY, FOLIATION	Equation (61)
BELTRAMI	$\int_\Omega f \log(f g_{\mathfrak{B}}) dV$		Theorem III.1 Example d sec. IV Equation (74)
ANTISYMMETRIC ($\hat{\boldsymbol{b}} = \nabla\zeta$)	$\int_\Omega f \log(f g_\zeta) dV$	ENERGY, FOLIATION, FIELD CHARGE	Example e sec. IV Equation (18)
ANTISYMMETRIC			Examples f, g, h sec. IV

FIG. 13. Relation between operator properties and self-organizing behavior. When available, the relevant H function is shown. The red and blue lines indicate the set in of geometric sources of heterogeneity. In a canonical system (symplectic operator) the only source of heterogeneity is energy. In noncanonical systems (Poisson operators), heterogeneity can arise by Casimir invariants. For Poisson operators, H is determined by the Jacobian g_c sending dV to phase space variables as $dV = g_c d\boldsymbol{p}\, d\boldsymbol{q}$. Measure preserving operators are not endowed with phase space and H is built on the invariant measure $dV_{IM} = g_{IM}^{-1} dV$. Beltrami operators do not possess an invariant measure and H is given on the coordinates $dV_{\mathfrak{B}} = g_{\mathfrak{B}}^{-1} dV$ where the field charge $\mathfrak{B}$ vanishes. Antisymmetric operators exhibit a new kind of self-organization caused by the nonvanishing of $\mathfrak{B}$. When $\hat{\boldsymbol{b}} = \nabla\zeta$, $g_\zeta = we^\zeta$. Notice that foliation may arise also in non-Hamiltonian systems if the kernel of the antisymmetric operator admits an integrable part.

X. CONCLUSION

In this study we have investigated the properties of diffusion in systems that lack canonical phase space. Such defect is caused by topological constraints that break the Hamiltonian structure of the dynamics and is mathematically represented by the violation of the Jacobi identity. Under these circumstances, the usual arguments of statistical mechanics relying on the invariant measure provided by Liouville's theorem do not apply, and diffusion causes, in general, the creation of heterogeneous distributions.

The characterization of diffusion processes in non-Hamiltonian systems requires deeper understanding of the notion of homogenization or equilibration. The primitive idea of homogeneity is the constancy of some density distribution. However, remembering the fact that any density is not a scalar function, but is dependent on the metric of the space, we have to enquire about the "proper space" where the appropriate density is defined. Indeed, the recent works on the theory of foliated phase spaces [4,5] elucidated that some structures are the reflections of heterogeneous metric of effective phase spaces. In this work, the standard notion of "flatness" is thus generalized by allowing change of coordinates that restore the entropy law by providing a suitable H function.

As a result, it is found that Beltrami operators are the limit beyond which such relativization of the notion of homogeneity is no longer applicable: the impossibility of annihilating the field charge, possibly by some coordinate change, implies that there is no reference frame where the application of a white noise process flattens the corresponding distribution function. The diagram shown in Fig. 13 summarizes the relationship between the geometry of antisymmetric operators and the H theorem as elucidated in this study.

At the opposite pole of homogeneity, the notion of heterogeneity also acquires a new meaning. While in Hamiltonian systems the sources of heterogeneity are either the special form of the energy or the foliation of the phase space dictated by Casimir invariants [4], we have shown that in non-Hamiltonian systems the determinant is the field charge, which measures the degree at which an antisymmetric operator (field tensor) departs from a Beltrami field. We proved an H theorem for systems characterized by a vanishing field charge, and demonstrated the role of a finite field charge in generating heterogeneous structures.

In the generalization of the theory to arbitrary dimensions, we developed a geometrical classification of antisymmetric operators. Each of the operators (measure preserving and Beltrami) introduced in this study exhibits peculiar dynamical and statistical properties. We found that all antisymmetric operators can be extended to a measure preserving form, and that the standard results of statistical mechanics can be generalized to the class of measure preserving operators. This latter fact is remarkable because such operators do not possess a Hamiltonian structure.

Finally, the normal Laplacian is a novel object of mathematical interest: this operator shows a clear interplay between integrability in the context of differential geometry and the study of nonelliptic PDEs.

ACKNOWLEDGMENT

The work of N.S. was supported by JSPS KAKENHI Grant No. 16J01486, and that of Z.Y. was supported by JSPS KAKENHI Grant No. 17H01177.

[1] P. J. Morrison, in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, edited by M. Tabor and Y. Treve, AIP Conference Proceedings No. 88 (AIP, New York, 1982), pp. 13–46.
[2] R. Littlejohn, in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, edited by M. Tabor and Y. Treve, AIP Conference Proceedings No. 88 (AIP, New York, 1982), pp. 47–66.
[3] P. J. Morrison, Rev. Mod. Phys. **70**, 467 (1998).
[4] Z. Yoshida and S. M. Mahajan, Prog. Theor. Exp. Phys. **2014**, 073J01 (2014).
[5] Z. Yoshida, Adv. Phys. X **1**, 2 (2016).
[6] Z. Yoshida, H. Saitoh, J. Morikawa, Y. Yano, S. Watanabe, and Y. Ogawa, Phys. Rev. Lett. **104**, 235004 (2010).
[7] N. Sato and Z. Yoshida, J. Phys. A: Math. Theor. **48**, 205501 (2015).
[8] N. Sato, Z. Yoshida, and Y. Kawazura, Plasma Fus. Res. **11**, 2401009 (2016).
[9] N. Sato and Z. Yoshida, Phys. Rev. E **93**, 062140 (2016).
[10] T. Frankel, *The Geometry of Physics, An Introduction*, 3rd ed. (Cambridge University Press, Cambridge, 2012), pp. 165–178.
[11] I. Agricola and T. Friedrich, in *Global Analysis, Differential Forms in Analysis, Geometry and Physics* (American Mathematical Society, Providence, 2002), pp. 111–120.
[12] S. Kobayashi and K. Nomizu, *Foundations of Differential Geomtry* (Wiley, New York, 1969), pp. 321–324.
[13] A. M. Bloch, J. E. Marsden, and D. V. Zenkov, Not. AMS **52**, 320 (2005).
[14] A. J. Van Der Shaft and B. M. Maschke, Rep. Math. Phys. **34**, 225 (1994).
[15] L. Bates and J. Sniatycki, Rep. Math. Phys. **32**, 99 (1993).
[16] C. E. Caligan and C. Chandre, Chaos **26**, 053101 (2016).
[17] A. M. Bloch, *Nonholonomic Mechanics and Control*, 2nd ed. (Springer, New York, 2015), pp. 11–15.
[18] M. de León, RACSAM **106**, 191 (2012).
[19] S. A. Chaplygin, Regul. Chaot. Dyn. **13**, 369 (2008).
[20] A. Sergi and M. Ferrario, Phys. Rev. E **64**, 056125 (2001).
[21] A. Sergi, Phys. Rev. E **67**, 021101 (2003).
[22] M. E. Tuckerman *et al.*, J. Chem. Phys. **115**, 1678 (2001).
[23] G. S. Erza, J. Math. Chem. **35**, 1 (2004).
[24] J. R. Cary and A. J. Brizard, Rev. Mod. Phys. **81**, 693 (2009).
[25] R. G. Littlejohn, Phys. Fluids **24**, 1730 (1981).
[26] L. D. Landau and E. M. Lifshitz, Ukr. J. Phys. **53**, 14 (2008).
[27] T. L. Gilbert, IEEE Trans. Magnetics **40**, 3443 (2004).
[28] Z. Yoshida, T. Tokieda, and P. J. Morrison, Phys. Lett. A **381**, 2772 (2017).
[29] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer, Berlin, 2001), p. 31.
[30] L. C. Evans, *Partial Differential Equations*, 2nd ed. (American Mathematical Society, Providence, 2010), p. 314.
[31] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations* (Springer, New York, 2011), p. 294.
[32] M. de León, *Methods of Differential Geometry in Analytical Mechanics* (Elsevier, New York, 1989), pp. 250–253.
[33] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd ed. (Springer, New York, 1989), pp. 230–232.
[34] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, 2nd ed. (Springer, Berlin, 1985).
[35] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications*, 2nd ed. (Springer, Berlin, 1989).