Social dilemmas in multistrategy evolutionary potential games

György Szabó*
Institute of Technical Physics and Materials Science, Centre for Energy Research, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary

Gergely Bunth†
Budapest University of Technology and Economics, Budafoki út 8, H-1111 Budapest, Hungary

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The nature of social dilemmas is studied in n-strategy evolutionary potential games on a square lattice with nearest-neighbor interactions and the logit rule. For symmetric games with symmetric payoff matrices there are no dilemmas because of the coincidence of individual and common interests. The dilemmas are caused by the antisymmetric parts of the self- and cross-dependent payoff components if it modifies the preferred Nash equilibrium. The contentment of players and the emergence of dilemmas in the preferred Nash equilibria are illustrated on some two-dimensional cross sections of the parameter space.

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I. INTRODUCTION

Social dilemmas were first recognized by Flood and Dresher in 1950 [1,2] when studying two-player two-strategy games within the framework of traditional game theory [3]. The observed situation became a world-wide phenomenon via the story of the prisoner's dilemma suggested by Tucker. The original story of the prisoner's dilemma hid the importance of this phenomenon occurring in many other real-life situations when the selfish participants cannot receive optimum payoffs. The relevance of this phenomenon in the level of interactions raised plenty of questions about the applicability of game theory in different fields of science, including political decisions, economy, biology, and social sciences. At the same time the systematic investigation of these dilemmas was delayed by the "folk theorem" predicting the elimination of the dilemma for the repeated games [4]. The progressive activity in the study of social dilemmas was initiated by the computer tournaments conducted by Axelrod [5] and also by the development of evolutionary game theory providing a general mathematical framework to analyze quantitatively the living systems [6–14].

In the last decades numerous attempts have been developed to find ways to avoid the undesired consequences of social dilemmas. These approaches include the application of different protocols [15,16], the repetition of games together with the introduction of evolutionary processes [5,6,17,18], and the reduction of the number of interacting players to a small quenched [19,20] or evolving neighborhood [21,22]. It is now well known that the maintenance of cooperation can be increased if the evolution is controlled by the imitation of a better neighbor. The efficiency of the latter mechanism depends on the noise level and some topological features of the connectivity structure [20,23–25]. In the level of cooperation the most relevant improvement is achieved for irregular networks [20]. Similar positive mechanisms can be generated when the models are extended with personal features (e.g., reputation [26–29], age [30], or fraternity [31,32]). Many other additional features can also be considered by extending the number of strategies (possible third strategies can illustrate the voluntarism [33] or punishment [34–37]) that reflect the presence of dilemmas for the multistrategy systems, too.

The identification and distinction of social dilemmas require the determination and comparison of Nash equilibria [38] when studying their Pareto inefficiency in the space of strategy profiles [39]. In the literature of game theory a wide range of methods are described which can be used for the classification of games. For example, a taxonomy of two-player two-strategy games (henceforth 2×2 games) has been suggested by Rapoport and Guyer [40] who simplified the problem by considering only the rank of payoffs. In evolutionary game theory the determination of evolutionarily stable strategy (ESS) [6] can be used to distinguish the games or interactions. If replicator dynamics controls the evolution then phase portraits (characterizing fixed points) classify the games [41,42]. Further aspects of the general features of social dilemmas are also discussed in some recent papers [43–45] with further references therein.

The classification of interactions becomes transparent for the symmetric 2×2 games when the possible payoffs are defined by four values of the payoff matrix. The payoff components, however, can be modified by a constant and we can choose a suitable unit, as detailed later. In the corresponding two-dimensional parameter space, four types of games are distinguished [8,9,11,46], for which the different features of social dilemmas are well discussed. All these symmetric 2×2 games are potential games which allow us to determine the preferred Nash equilibrium identified by the maximum value of the potential [47–51]. The preferred Nash equilibrium resembles the ground state of a physical system. Additionally, the multi-agent systems with equivalent players

*szabo@mfa.kfki.hu
†gbunthy@gmail.com

and interactions evolve into a Boltzmann distribution when the so-called logit rule controls the random sequential strategy updates in the system. Now we extend the analysis for the symmetric $n \times n$ potential games. It will be shown that all the relevant features and types of dilemmas are inherited if n is increased.

The present analysis is based on the concept of matrix decomposition surveyed briefly in the next section (for a detailed analysis of this approach we suggest reading our previous papers [51–53]). First, the application of this method will be illustrated for the symmetric 2×2 games. Subsequently, the different effects of the antisymmetric matrix components will be demonstrated and discussed by considering typical examples. Most of the deduced statements can be derived from the analysis of the pair potential matrix. The variation of the preferred Nash equilibrium causes striking consequences in the spatial distribution of strategies for evolutionary games when the players are located on the sites of a square lattice and the noise level of the logit rule is tuned.

II. FORMALISM AND GENERAL FEATURES

In the evolutionary games discussed here equivalent players are located on the sites (x) of a square lattice. Each player can choose one of her n pure strategies $\mathbf{s}_x$ denoted by the traditional n-dimensional Cartesian unit vectors or by an integer i if the player uses her ith strategy. Using their strategies the players play the same game with all their nearest neighbors (located at sites $y = x + \delta$). The accumulated payoff $\tilde{u}_x(\mathbf{s}_x)$ is given by the expression [3]

$$\tilde{u}_x(\mathbf{s}_x) = \sum_{\delta} \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_{x+\delta}, \tag{1}$$

where the A_{ij} component of the $n \times n$ payoff matrix defines the player's income if she chooses her ith strategy while the co-player selects the jth one. The given pair interaction is a potential game if we can introduce a symmetric potential matrix $\mathbf{V}$ ($V_{ij} = V_{ji}$) that satisfies the following conditions:

$$\mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y - \mathbf{s}'_x \cdot \mathbf{A}\mathbf{s}_y = \mathbf{s}_x \cdot \mathbf{V}\mathbf{s}_y - \mathbf{s}'_x \cdot \mathbf{V}\mathbf{s}_y, \tag{2}$$

for all possible pure strategies $\mathbf{s}_x$, $\mathbf{s}'_x$, and $\mathbf{s}_y$. This quantity summarizes the incentive of active players and its meaning is similar to the negative potential energy for an interacting pair of players. For multi-agent lattice systems the total potential $U(\mathbf{s})$ depends on the strategy profile $\mathbf{s} = \{\mathbf{s}_x\}$ and summarizes the contributions of all nearest-neighbor pairs; that is,

$$U(\mathbf{s}) = \frac{1}{2} \sum_{x,\delta} \mathbf{s}_x \cdot \mathbf{V}\mathbf{s}_{x+\delta}. \tag{3}$$

A similar expression can be used to describe the potential energy of a multistate lattice system in physics.

As mentioned in the Introduction, a remarkable feature of the multi-agent evolutionary potential games is that, for the application of the logit rule, these systems evolve into a Boltzmann distribution [47,51,54] where in the stationary state the strategy profile $\mathbf{s}$ occurs with a probability

$$p(\mathbf{s}) = \frac{e^{U(\mathbf{s})/K}}{\sum_{\mathbf{s}'} e^{U(\mathbf{s}')/K}}. \tag{4}$$

For the logit rule [55–58], unilateral strategy changes are repeated by randomly selected players who can choose a new strategy $\mathbf{s}'_x$ with a probability

$$w(\mathbf{s}'_x) = \frac{e^{\tilde{u}_x(\mathbf{s}'_x)/K}}{\sum_{\mathbf{s}_x} e^{\tilde{u}_x(\mathbf{s}_x)/K}} \tag{5}$$

that depends on the neighboring strategies and favors exponentially the higher individual income. In evolutionary games K quantifies the noise or errors in the decision processes and its role is similar to temperature in physical systems where the logit rule is a generalized version of the Glauber dynamics [59] introduced for the investigation of the kinetic Ising model. Consequently, in the mathematical analysis of evolutionary potential games one can exploit the concepts, tools, and approaches of equilibrium statistical physics when the number of participants is large. Thus, on the square lattice the ordered strategy arrangement in the low-noise limit ($K \to 0$) is determined by the maximum value of the potential matrix $\mathbf{V}$. For example, if $\max(V_{ij}) = V_{11}$ then all players follow the first strategy. Conversely, both of the chessboard-like arrangements of the first and second strategies are stable when $\max(V_{ij}) = V_{12} = V_{21}$ in the limit $K \to 0$. These ordered states tend to the random strategy distribution if $K \to \infty$.

The existence of potential $\mathbf{V}$ prohibits the presence of rock-paper-scissors–type cyclic components in the payoff matrix [51]. Recently it has turned out [51–53,60] that the payoff matrix $\mathbf{A}$ of a potential game can be built up as a sum of different types of interactions; namely,

$$\mathbf{A} = \mathbf{A}^{(\mathrm{av})} + \mathbf{A}^{(\mathrm{se})} + \mathbf{A}^{(\mathrm{cr})} + \mathbf{A}^{(\mathrm{co})}, \tag{6}$$

where

$$A^{(\mathrm{av})}_{ij} = a^{(\mathrm{av})} = \frac{1}{n^2} \sum_{i,j} A_{ij} \tag{7}$$

represents the contribution of average payoff $a^{(\mathrm{av})}$, and the terms

$$A^{(\mathrm{se})}_{ij} = \varepsilon_i = \frac{1}{n} \sum_{j} A_{ij} - a^{(\mathrm{av})}, \tag{8}$$

$$A^{(\mathrm{cr})}_{ij} = \gamma_j = \frac{1}{n} \sum_{i} A_{ij} - a^{(\mathrm{av})}, \tag{9}$$

define the payoffs for the self- and cross-dependent elementary games. In contrary to the previous notations [51], now both $\mathbf{A}^{(\mathrm{se})}$ and $\mathbf{A}^{(\mathrm{cr})}$ are defined by $(n-1)$ independent parameters because the coefficients satisfy the conditions $\sum_i \gamma_i = \sum_i \varepsilon_i = 0$.

The coordination component summarizes the contributions of coordination between all possible strategy pairs (i,j) $(i < j)$ with a strength of ν_{ij} in a way that $A^{(\mathrm{co})}_{ij} = A^{(\mathrm{co})}_{ji} = -\nu_{ij}$ and the diagonal components ensure that the sums of payoffs become zero in each row and column [60]; that is,

$$\sum_i A^{(\mathrm{co})}_{ij} = \sum_j A^{(\mathrm{co})}_{ij} = 0. \tag{10}$$

Due to the above features the components in Eq. (6) are mutually orthogonal to each other in the sense that $\mathbf{A}^{(\mathrm{av})} \cdot \mathbf{A}^{(\mathrm{se})} = \mathbf{A}^{(\mathrm{av})} \cdot \mathbf{A}^{(\mathrm{cr})} = \cdots = \mathbf{A}^{(\mathrm{cr})} \cdot \mathbf{A}^{(\mathrm{co})} = 0$, where the scalar product of the matrices $\mathbf{A}$ and $\mathbf{A}'$ is defined as

$$\mathbf{A}\cdot\mathbf{A}' = \sum_{i,j} A_{ij}A'_{ij}. \tag{11}$$

An arbitrary $n \times n$ payoff matrix is described by n^2 real values. The potential matrix $\mathbf{V}$ is symmetric and is defined by $[n + n(n-1)/2]$ independent parameters involving an irrelevant or arbitrary constant term proportional to $\mathbf{A}^{(\mathrm{av})}$. The rest of the parameters are determined by the values of ε_i and ν_{ij} because $\mathbf{A}^{(\mathrm{cr})}$ gives zero contribution to $\mathbf{V}$. The existence of potential is prevented by the presence of cyclic components that are given by $(n-1)(n-2)/2$ coefficients measuring the strengths of the independent rock-paper-scissors–type subgames [51,53].

In the friendship or fraternal [31,32,41,61,62] game the payoff matrix is symmetric ($A^{(\mathrm{fr})}_{ij} = A^{(\mathrm{fr})}_{ji}$ or $\mathbf{A}^{(\mathrm{fr})} = \mathbf{A}^{(\mathrm{fr})\mathrm{T}}$). These interactions describe situations when the equivalent players share the income equally for all the possible strategy profiles. For these games the coincidence of the individual and common interest eliminates the source of social dilemmas. Furthermore, these games are potential games and the corresponding potential matrix is equal to the payoff matrix.

The symmetric part of the payoff matrix can be separated as

$$\mathbf{A}^{(\mathrm{fr})} = \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \mathbf{A}^{(\mathrm{av})} + \mathbf{A}^{(\mathrm{co})} + \mathbf{A}^{(\mathrm{ex})}, \tag{12}$$

where

$$\mathbf{A}^{(\mathrm{ex})} = \tfrac{1}{2}(\mathbf{A}^{(\mathrm{se})} + \mathbf{A}^{(\mathrm{cr})} + \mathbf{A}^{(\mathrm{se})\mathrm{T}} + \mathbf{A}^{(\mathrm{cr})\mathrm{T}}). \tag{13}$$

For later convenience the above symmetric portion of the self- and cross-dependent components is written as a sum of n elementary games as

$$\mathbf{A}^{(\mathrm{ex})} = \sum_i \beta_i \mathbf{F}(i), \tag{14}$$

where $\beta_i = (\varepsilon_i + \gamma_i)/2$ with $\sum_i \beta_i = 0$,

$$\mathbf{F}(1) = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{15}$$

and the other $\mathbf{F}(k)$ matrices can be constructed from $\mathbf{F}(1)$ by exchanging its first and kth rows and columns. These matrices can also be expressed by Kronecker δ symbols as $F_{ij}(k) = \delta_{ik} + \delta_{jk}$. The potential of $\mathbf{A}^{(\mathrm{ex})}$ is equal to itself and this term acts like a multidimensional external field in the n-state Potts model. More precisely, $\mathbf{A}^{(\mathrm{ex})}$ favors the dominance of the jth strategy if $\max(\beta_i) = \beta_j$ and the corresponding payoff is positive in the preferred Nash equilibrium.

The antisymmetric part of the payoff matrix is defined as

$$\mathbf{A}^{(\mathrm{as})} = \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \tfrac{1}{2}(\mathbf{A}^{(\mathrm{se})} + \mathbf{A}^{(\mathrm{cr})} - \mathbf{A}^{(\mathrm{se})\mathrm{T}} - \mathbf{A}^{(\mathrm{cr})\mathrm{T}}), \tag{16}$$

and arises from the self- and cross-dependent components. This term is responsible for the appearance of social dilemmas in the potential games and can be described as a linear combination of the adjacency matrices of directed star graphs [53]. More quantitatively,

$$\mathbf{A}^{(\mathrm{as})} = \sum_i \alpha_i \mathbf{H}(i), \tag{17}$$

with $\alpha_i = (\varepsilon_i - \gamma_i)/2$ ($\sum_i \alpha_i = 0$) and the first matrix is given as

$$\mathbf{H}(1) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ -1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{18}$$

The matrices $\mathbf{H}(i)$ ($i > 1$) can be obtained from $\mathbf{H}(1)$ by exchanging its first and ith rows and columns.

$\mathbf{H}(1)$ defines a two-player zero-sum game that does not modify the players' total income. In this curious game the players have two options: to be a winner (strategy 1) or looser (strategy $i > 1$). A unit payoff transfer from the looser to the winner occurs for the suitable choices, otherwise they receive nothing. The rationality or selfishness compels both players to choose the first strategy without any rewards. The potential matrix of $\mathbf{H}(1)$ is equivalent to $\mathbf{F}(1)$ defined by Eq. (15).

Games with a payoff matrix $\mathbf{H}(i)$ exhibit similar features and favor the choice of the ith strategy. For the linear combinations of these interactions, as defined by Eq. (17), the corresponding potential matrix can be given as

$$\mathbf{V}^{(\mathrm{as})} = \sum_i \alpha_i \mathbf{F}(i). \tag{19}$$

Notice that in the whole parameter space the possible potential matrices of $\mathbf{A}^{(\mathrm{ex})}$ and $\mathbf{A}^{(\mathrm{as})}$ span the same subset. If $\mathbf{A} = \mathbf{A}^{(\mathrm{as})}$ then rationality favors the choice of the jth strategy for both players (as well as for all the players on the square lattice) if $\max(\alpha_i) = \alpha_j$.

The presence of the antisymmetric payoff matrix component ($\mathbf{A}^{(\mathrm{as})} \neq 0$) can help the players and also the whole society to get optimum payoffs if the preferred Nash equilibria of $\mathbf{A}^{(\mathrm{as})}$ and $\mathbf{A}^{(\mathrm{fr})}$ coincide, as will be illustrated later. In these cases $\mathbf{A}^{(\mathrm{as})}$ acts as the "invisible hand" offered by Adam Smith (for a short discussion of the invisible hand see the books [11,13] and papers [39,63]). In most of the cases, however, the preferred Nash equilibria of $\mathbf{A}^{(\mathrm{as})}$ and $\mathbf{A}^{(\mathrm{fr})}$ are different. Social dilemmas occur when $\mathbf{A}^{(\mathrm{as})}$ is sufficiently strong to change the preferred Nash equilibrium dictated by $\mathbf{A}^{(\mathrm{fr})}$. In the latter potential games $\mathbf{A}^{(\mathrm{as})}$ acts as Ate's hand and can be depicted as the root of all evil. In Greek mythology Ate (the eldest daughter of Zeus) is the goddess of delusion, infatuation, and mischief. To preserve the harmony in heaven Zeus threw her down to Earth.

In the next sections we discuss the effects of $\mathbf{A}^{(\mathrm{as})}$ for $n = 2$ and 3 by considering typical examples.

III. TWO-STRATEGY GAMES

First, we remind the reader that all symmetric two-strategy games are potential games. Furthermore, for $n = 2$, the above criteria simplify the decomposition of the 2×2 payoff matrix into the sum of four orthogonal elementary components that reflect the general features mentioned above.

For the traditional notation of the symmetric two-strategy social dilemmas the strategies are denoted by C (cooperation) and D (defection) and the four values of a single payoff matrix are denoted as R (reward), S (sucker's payoff), T (temptation), and P (punishment) [1]. For the given matrix

$$\mathbf{A} = \begin{pmatrix} R & S \\ T & P \end{pmatrix}, \tag{20}$$

it is assumed that $R > P$. Most of the analyses are constrained to the cases when $R = 1$, $P = 0$, and $T + S < 2$. The last condition excludes the parameter region where the players would receive the highest income when choosing alternately the (C,D) and D,C strategy pairs in repeated games.

Usually, four types of games are distinguished for rescaled payoffs on the T-S parameter plane [8,9,11,46]. In the range of harmony game ($T < 1$ and $S > 0$) the system has one Nash equilibrium (C,C) when both players receive $R = 1$. For the prisoner's dilemma the game also has a single Nash equilibrium (D,D) that provides zero income for both, which is smaller then what they would receive for the opposite choices (and hence the dilemma). The stag-hunt game ($T < 1$ and $S < 0$) has two Nash equilibria: (C,C) and (D,D). The maximum value of the potential matrix

$$\mathbf{V} = \begin{pmatrix} R & T \\ T & T - S + P \end{pmatrix} \tag{21}$$

favors the choice of (C,C) if $R > T - S + P$ (or $S > T - 1$). For the opposite case ($S < T - 1$) the dilemma occurs, too. In the region of the hawk-dove game ($T > 1$ and $S > 0$) the system has three Nash equilibria: (C,D), (D,C), and a symmetric mixed strategy profile that is an evolutionarily stable strategy [6]. The latter one dominates the system behavior in well-mixed populations and also on lattices if imitation of a better neighboring strategy controls the dynamics. For the application of the logit rule, however, the neighboring players favor the choice of one of the two equivalent preferred pure strategies that results in a sublattice ordered strategy arrangement on the square lattice in the zero-noise limit [64].

Figure 1 illustrates the possible preferred Nash equilibria and the contentment of players on the T-S parameter plane. In this map the preferred Nash equilibria are denoted by a pair of white (C) and/or black (D) symbols. The pairs of symbols are located horizontally if the players receive equivalent payoffs. Smiling faces indicate satisfied players getting optimum payoffs. On the contrary, the frowning faces of D players refer to their disappointment where the selfish players fall into the trap of the tragedy of the commons. Notice that the C players can also be unsatisfied for $T + S > 2$ and $T < 1$ when the choices (D,C) or (C,D) would result in higher total income for them. For the latter parameters, however, one of the players should make a sacrifice for increasing the total payoff. In other words, here the profit of the C player exceeds the loss of his or her co-player changing from C to D.

For the vertical arrangements of symbols, the upper player gets higher income and smiles. All these pairs refer to twofold degenerate preferred Nash equilibria. In these cases the lower player is smiling (frowning) if the given choice provides higher (lower) total income for them.

The above analyses become more convenient and adaptable for a larger number of strategies if the payoff matrix is built up from the four orthogonal elementary games. In this frame the present matrix can be written as

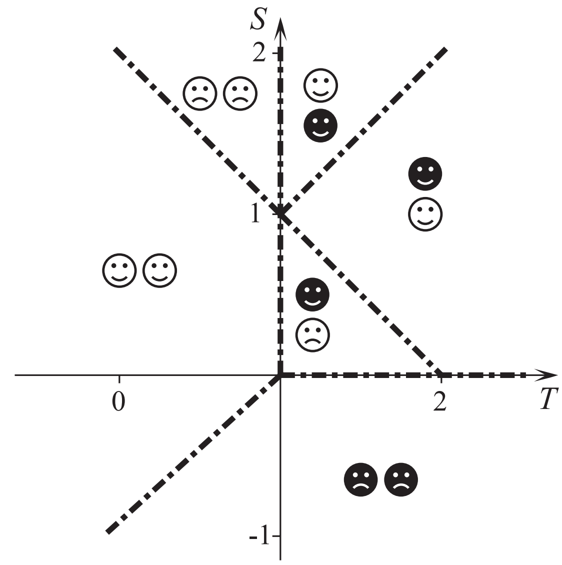

FIG. 1. Contentment of players for the preferred Nash equilibria as a function of T and S for $R = 1$ and $P = 0$. The axes T and S divide the plane into four segments characterizing the harmony, hawk-dove, stag-hunt, and prisoner's dilemma games. Thick dash-dotted lines separate regions possessing similar behavior. The explanation of the symbols is given in the text.

$$\mathbf{A} = a^{(\mathrm{av})}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \nu_{12}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \beta_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \alpha_1\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{22}$$

with

$$a^{(\mathrm{av})} = \frac{R + S + T + P}{4}, \quad \nu_{12} = \frac{R - S - T + P}{4}, \quad \beta_1 = \frac{R - P}{2}, \quad \alpha_1 = \frac{S - T}{2}. \tag{23}$$

Here the first coefficient denotes the average payoff, and the second one defines the strength of the coordination between the two strategies. Additionally, we have exploited that β_is can be expressed by one independent parameter ($\beta_2 = -\beta_1$) and $\mathbf{F}(1) - \mathbf{F}(2)$ is given by the third matrix in Eq. (22). Similarly, for the fourth term $\alpha_1 = -\alpha_2$ and $\mathbf{H}(2) = -\mathbf{H}(1)$. It is worth mentioning that the above matrices are orthogonal to each other and this fact can be exploited when determining the corresponding coefficients.

In this frame, if $\nu_{12} = 0$, $\mathbf{A}$ is defined by two parameters (α_1, β_1) and it is equivalent to the donation game [11,65], representing the simplest version of the social dilemma. In that case the maximum value of the potential matrix prefers one of the homogeneous pure strategies; namely, (1,1) if $\alpha_1 + \beta_1 > 0$ or (2,2) for $\alpha_1 + \beta_1 < 0$, meanwhile the payoffs are β_1 or $-\beta_1$, respectively. Thus, the tragedy of the commons emerges if $\alpha_1 + \beta_1 < 0$ and $\beta_1 > 0$ or $\alpha_1 + \beta_1 > 0$ and $\beta_1 < 0$ (see Fig. 2). This situation remains unchanged in the presence of coordination ($\nu_{12} > 0$) because it increases equally the relevant elements of $\mathbf{A}$ and $\mathbf{V}$.

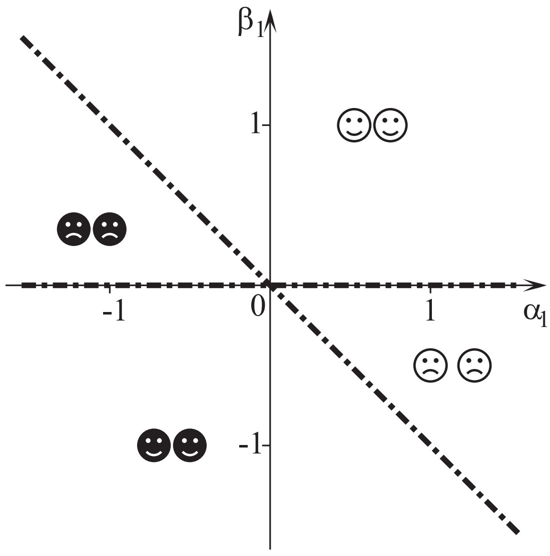

FIG. 2. Contentment map on the α_1-β_1 plane for $\nu_{12} \geqslant 0$.

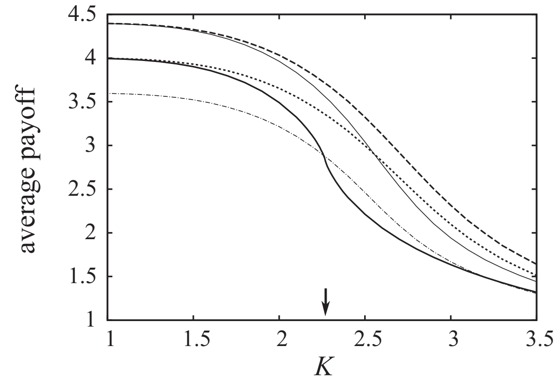

FIG. 4. Average payoff vs noise if $\nu_{12} = 1$ for several values of α_1 and β_1: $\alpha_1 = \beta_1 = 0$ (thick solid line); $\alpha_1 = 0, \beta_1 = 0.1$ (dashed line); $\alpha_1 = 0.1, \beta_1 = 0$ (dotted line); $\alpha_1 = -0.15, \beta_1 = 0.1$ (dashed-dotted line); and $\alpha_1 = -0.05, \beta_1 = 0.1$ (thin line). The arrow shows the value of K_c for $\alpha_1 = -\beta_1$.

On the contrary, the presence of an anticoordination component ($\nu_{12} < 0$) causes relevant changes on the map of contentment as is summarized in Fig. 3. Comparison of Figs. 2 and 3 illustrates well that, for large values of $|\alpha_1|$ and $|\beta_1|$, the contribution of coordination becomes negligible except in the close vicinity of the line $\alpha_1 + \beta_1 = 0$ where the opposite effects of $\mathbf{A}^{(ex)}$ and $\mathbf{A}^{(as)}$ are balanced. Along this line ($\alpha_1 + \beta_1 = 0$) the (1,2) or (2,1) strategy pairs are preferred equally which yields a sublattice ordered strategy arrangement on a square lattice.

Before detailing the noise dependence we remind the reader that, in the terminology of Ising model [66,67], the strategies are replaced by spin-up and -down states, $\mathbf{A}$ summarizes the interactions between the neighboring spins, which includes an irrelevant constant in the form of the ferro- ($\nu_{12} > 0$) or antiferromagnetic ($\nu_{12} < 0$) interactions, and the effect of a homogeneous magnetic field. The fourth (antisymmetric or non-Hermitian) term is missing in physical systems.

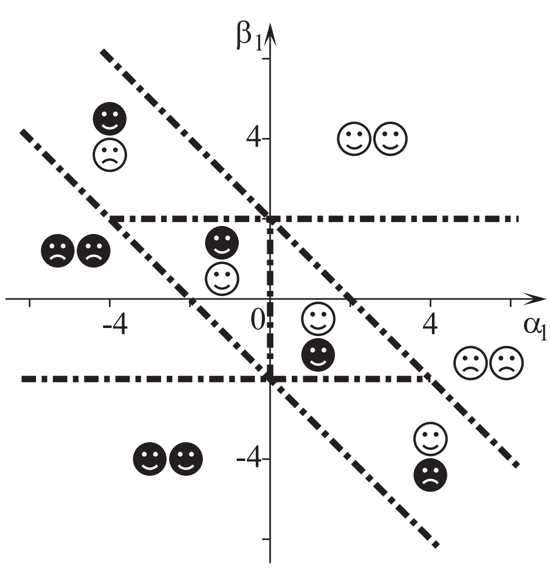

FIG. 3. Contentment map on the α_1-β_1 plane for $\nu_{12} = -1$.

Monte Carlo simulations are performed on a square lattice to quantify the noise dependence of the average strategy frequencies and payoffs for several values of parameters. In these numerical analyses the statistical error is comparable to the line thickness because of the sufficiently large system size a sampling time. First we discuss the effects of α_1 and β_1 for fixed coordination ($\nu_{12} = 1$). This system undergoes an Ising-type critical phase transition at $K = K_c = 2/\ln(\sqrt{2}+1)$ [68,69] from one of the ordered strategy arrangements to a disordered one if K is increased for $\alpha + \beta = 0$. The two ordered phases are equivalent if $\alpha_1 = \beta_1 = 0$. On the contrary, if $\alpha_1 = -\beta_1 \neq 0$ then the opposite effects of $\mathbf{A}^{(ex)}$ and $\mathbf{A}^{(as)}$ are balanced, therefore the strategy frequencies exhibit similar K dependence; meanwhile the average payoffs are different in the two ordered phases. This phenomenon will be demonstrated later for $n = 3$.

In Fig. 4 the thick solid line illustrates the continuous decrease of the average payoff if K is increased in the absence of the self- and cross-dependent components. In this coordination game at low noise the high average payoff is ensured by the dominance of (1,1) or (2,2) strategy pairs. Among the plotted examples the highest average incomes are received by the players for $\alpha_1 = 0$ and $\beta_1 = 0.1$ when most of the players choose strategy 1 at low noise and the critical phase transition in the strategy frequencies is smoothed out. Simultaneously, the average income is increased by the term $\mathbf{A}^{(ex)}$. The situation resembles the application of a homogeneous magnetic field in the Ising model.

Exactly the same variation in the strategy frequencies can be observed for $\alpha_1 = 0.1$ and $\beta_1 = 0$. In that case, however, the antisymmetric component does not modify the average payoff arising exclusively from the coordination in the ordered phase. At the same time, the numerical results (dotted line in Fig. 4) indicate the increase of the average payoff in the presence of noise ($K > 0$). The resultant extra payoff comes from the preference of (1,1) strategy pairs at the expense of other constellations. Evidently, the K dependence of the

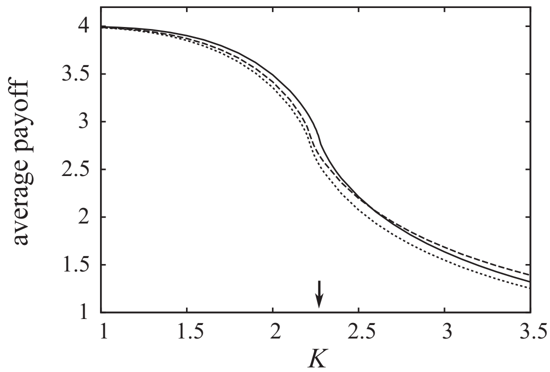

FIG. 5. Average payoff as a function of K for $\nu_{12} = -1$ if $\alpha_1 = \beta_1 = 0$ (thick solid line); $\alpha_1 = 0$, $\beta_1 = 0.4$ (dashed line); and $\alpha_1 = 0.4$, $\beta_1 = 0$ (dotted line). The arrow shows the critical point for $\alpha_1 + \beta_1 = 0$.

average payoff will be similar for $\alpha_1 = -0.1$ and $\beta_1 = 0$ when the strategy pairs (2,2) are preferred by the actual potential.

Two additional curves in Fig. 4 illustrate what happens when the symmetric and antisymmetric terms support different homogeneous strategy pairs. The thin solid line represents a situation when the symmetric term dominates the behavior and increases the average payoff in the preferred ordered state. In the opposite case (see the thin dashed-dotted line) the stronger antisymmetric component will determine the homogeneous preferred strategy pair while the corresponding average payoff is decreased by the symmetric one.

Basically different effects are caused by α_1 and β_1 for $\nu_{12} = -1$ when the anticoordination favors one of the two equivalent checkerboard-like strategy distributions in the low-noise limit. Due to the equivalence between the ferro- and antiferromagnetic Ising models these sublattice ordered states are transformed into a disordered phase at the same critical noise $[K_c = 2/\ln(\sqrt{2}+1)]$ when K is increased. It is emphasized that the K dependence of strategy frequencies (in the sublattices) and average payoff are equivalent to those predicted for $\nu_{12} = 1$. Figure 5 compares the variation of average payoffs when the anticoordination is extended by a symmetric ($\beta_1 = 0.4$) or an antisymmetric term ($\alpha_1 = 0.4$). In the latter cases the variations of the potential matrices are identical, therefore the system exhibits similar (Ising type) phase transitions at a critical point dependent on $|\alpha_1 + \beta_1|$. In agreement with the phase diagram (see Fig. 3), the lattice system evolves into one of the homogeneous states in the low-noise limit if $|\alpha_1 + \beta_1| > 2$. The most striking message of Fig. 5 is that the average payoff remains unchanged at low noise. Furthermore, the changes in the average payoff are significantly smaller than those we observed for $\nu_{12} = 1$. In fact, this is the reason why we used higher values of α_1 and β_1 when demonstrating the effects.

Finally, it is worth emphasizing that, in the sublattice ordered phases ($K < K_c$) of the examples in Fig. 5, the average payoff is always decreased by the appearance of additional (1,1) and (2,2) strategy pairs. The mentioned trend is also recognizable for $\beta_1 = 0$, and $\alpha_1 \neq 0$ in the disordered states because the additional symmetric (1,1) or (2,2) pairs are not rewarded by extra payoffs. On the other hand, the variation in the ratio of the pairs (1,1) and (2,2) is accompanied with a suitable payoff increase at sufficiently high noise level if it caused by $\mathbf{A}^{(ex)}$ (instead of $\mathbf{A}^{(as)}$).

IV. THREE-STRATEGY GAMES

Despite the high degree of freedom, most of the general features (discussed above) are preserved in the multistrategy games. An exhaustive analysis of all the possible phenomena goes beyond the scope of a paper. Instead of it, the most relevant features of social dilemmas will be illustrated via several examples for $n = 3$.

First we detail the effects of the antisymmetric components in the absence of all other terms when the concept of the tragedy of the commons is meaningless. Then the payoff matrix is defined by Eqs. (17) and (18). Accordingly, for $n = 3$ the payoff matrix is described by two independent parameters (α_1 and α_2) as

$$\mathbf{A}^{(as)} = \begin{pmatrix} 0 & \alpha_1 - \alpha_2 & 2\alpha_1 + \alpha_2 \\ -\alpha_1 + \alpha_2 & 0 & \alpha_1 + 2\alpha_2 \\ -2\alpha_1 - \alpha_2 & -\alpha_1 - 2\alpha_2 & 0 \end{pmatrix}, \tag{24}$$

and the corresponding potential matrix obeys the form

$$\mathbf{V}^{(as)} = \begin{pmatrix} 2\alpha_1 & \alpha_1 + \alpha_2 & -\alpha_2 \\ \alpha_1 + \alpha_2 & 2\alpha_2 & -\alpha_1 \\ -\alpha_2 & -\alpha_1 & -2(\alpha_1 + \alpha_2) \end{pmatrix}. \tag{25}$$

According to the straightforward determination of the largest component of the potential matrix the strategy pair (1,1) is the preferred Nash equilibrium if $\alpha_1 > \alpha_2$ and $\alpha_1 > -\alpha_2/2$. Similarly, the strategy pair (2,2) is recommended for the players if $\alpha_1 > -2\alpha_2$ and $\alpha_1 < \alpha_2$. In the rest of the α_1-α_2 parameter plane ($\alpha_1 < -2\alpha_2$ and $\alpha_1 < -\alpha_2/2$) the preferred Nash equilibrium is (3,3).

Now we consider the combination of $\mathbf{A}^{(as)}$ given by Eq. (24) and a simple symmetric component $\mathbf{A}^{(ex)}$ favoring the first strategy. More precisely, $\mathbf{A}^{(ex)}$ is defined by Eqs. (14) and (18) with values of β_1 and $\beta_2 = \beta_3 = -\beta_1/2$ (assuming $\beta_1 > 0$). The results are illustrated in Fig. 6 by a contentment map for $\beta_1 = 1$. In this phase diagram the three straight and thick dashed-dotted lines divide the parameter plane into three territories where the preferred Nash equilibria are (1,1), (2,2), or (3,3). The slopes of the phase boundaries are 1, $-\frac{1}{2}$ and -2. Similar phase diagrams are obtained for any positive values of β_1. More quantitatively, the phase boundaries are shifted parallel and meet at the point ($\alpha_1 = -\beta_1, \alpha_2 = \beta_1/2$). Evidently, the above-discussed case is reproduced for $\beta_1 = 0$. For $\beta_1 > 0$ the players are satisfied when the preferred Nash equilibrium is (1,1). For other preferred Nash equilibria the players are not contented because the strategy profile (1,1) always provides higher income for both.

Figure 6 shows clearly that the players are not content in two of three domains of parameters that represent about two-thirds of the α_2-α_1 plane. The portion of the region of contented players decreases if n is increased because $\mathbf{A}^{(as)}$ supports the preferred Nash equilibrium of $\mathbf{A}^{(ex)}$ only if it acts in the same direction. Disregarding the opposite effect, there are, however, additional ($n - 2$) orthogonal directions which can prevent contentment if the strength of $\mathbf{A}^{(as)}$ exceeds a threshold dependent on $\mathbf{A}^{(ex)}$.

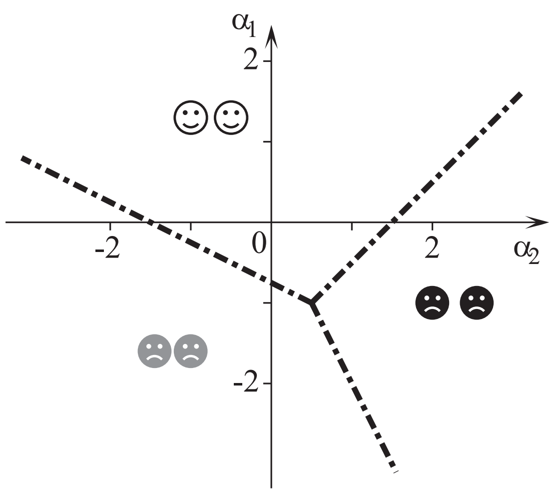

FIG. 6. Contentment map for the preferred Nash equilibria of a three-strategy game on the α_2-α_1 plane when $\mathbf{A} = \mathbf{A}^{(\mathbf{ex})} + \mathbf{A}^{(\mathbf{as})}$ for $\beta_1 = 1$ and $\beta_2 = \beta_3 = -1/2$. The preferred Nash equilibria (1,1), (2,2), and (3,3) are denoted by white, black, and gray pairs of symbols with faces indicating the contentment of players.

A similar situation can be observed when the highest income for the strategy pair (1,1) is ensured by the coordination components. For example, if we consider the interplay between $\mathbf{A}^{(as)}$ and $\mathbf{A}^{(co)}$ for $\nu_{12} = 1$, $\nu_{13} = 1/2$, and $\nu_{23} = 0$ then we get a contentment map similar to those plotted in Fig. 6. The only difference is the location of the point ($\alpha_1 = -1/4, \alpha_2 = 0$) where the three phase boundaries meet.

In the last example we discuss a model where anticoordination dominates the symmetric part, i.e., when $\nu_{12} = -1$ and $\nu_{13} = \nu_{23} = \beta_1 = \beta_2 = 0$. The straightforward analysis of the contentment of players in the preferred Nash equilibria is summarized in Fig. 7. For two players in this peculiar case the average payoff is 1 for the strategy profiles (1,2) or (2,1) and 0 or -1 for all other choices. In the α_1-α_2 plane the trap of social dilemmas is avoided along the line $\alpha_1 = \alpha_2 > -\frac{1}{6}$. In the corresponding preferred Nash equilibria the effect of $\mathbf{A}^{(as)}$ can only cause some difference in the players' income, as indicated in the contentment map. It is remarkable that all the other strategy profiles can be a preferred Nash equilibrium with unsatisfied players.

Some of the above features are preserved for the n-strategy potential games ($n > 3$) with straightforward adaptation of the above results. For example, the deluding component $\mathbf{A}^{(as)}$ is defined by $(n-1)$ parameters; it forces the players to choose one of the symmetric strategy profiles for nothing; the potential matrices of $\mathbf{A}^{(as)}$ and $\mathbf{A}^{(ex)}$ span the same subspace of parameters.

In the spatial evolutionary games the coordination components determine the noise dependence of strategy frequencies if $\mathbf{A}^{(se)} = 0$ because $\mathbf{A}^{(cr)}$ does not modify the value of the potential matrix while the payoffs are changed. This feature results in a peculiar consequence in the noise dependence of average payoffs if the system has equivalent preferred Nash equilibria. Figure 8 illustrates the possible variations of average payoffs in a system where the coordination component is equivalent to a three-state Potts (or clock) model (quantitatively, $\nu_{12} = \nu_{13} = \nu_{23} = \frac{1}{2}$), and this term is extended by the following cross-dependent components: $\gamma_1 = -\gamma_2 = 0.1$ and $\gamma_3 = 0$. For the logit rule, the stationary strategy frequencies are equivalent to those described by the Potts model exhibiting an order-disorder phase transition at $K_c = 1.5/\ln(\sqrt{3}+1)$ [70]. In this plot the solid line represents the average payoff for all the three ordered phases in the absence of cross-dependent terms. In each ordered phases the latter value is modified separately by the cross-dependent components if $K < K_c$. If the simulations are started from a random initial state for $K < K_c$ then after a domain-growing process the system will evolve into one of the ordered phases with the same probability and the society receives the corresponding average payoff. Finally, we emphasize that the

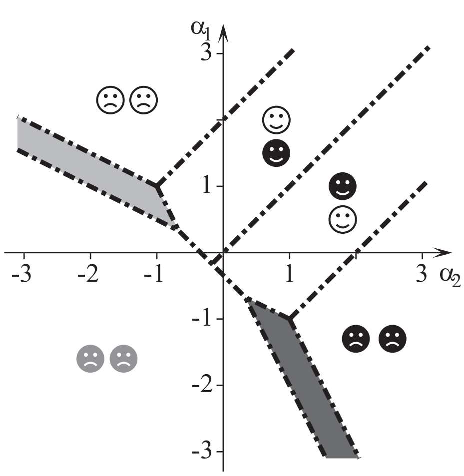

FIG. 7. Contentment of players in the preferred Nash equilibria as a function of α_1 and α_2 if the symmetric part of $\mathbf{A}$ describes anticoordination between strategies 1 (white symbols) and 2 (black symbols) for a neutral third (gray symbols) strategy. In the light- and dark-gray territories both players are discontent in the preferred Nash equilibria (1,3) or (3,1) and (2,3) or (3,2).

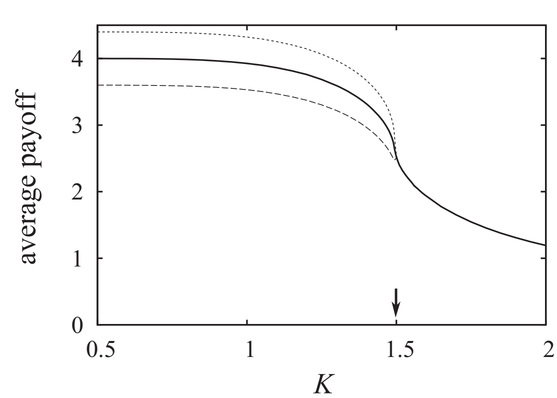

FIG. 8. Possible average payoffs as a function of noise K for degenerate preferred Nash equilibria in a three-strategy potential game where coordination components are equivalent and the cross-dependent components increase (decrease) the payoff for the first (second) strategy while the payoff remains unchanged for the third strategy. The dotted, dashed, and solid lines show the results if the first, second, or third strategy respectively dominates the system behavior below the critical noise level K_c denoted by the arrow.

equivalence of the three ordered phases is destroyed if a payoff-dependent imitation rule controls the evolution. In the knowledge of the above results one can easily find or develop other n-strategy models exhibiting a wider range of social dilemmas.

V. SUMMARY

Applying the frame of matrix decomposition we have studied the emergence of social dilemmas in n-strategy symmetric potential games defined by a suitable payoff matrix $\mathbf{A}$. This approach is based on a suitable rotation of the Cartesian coordinate system of the n^2-dimensional parameter space defining the payoffs. In other words, the payoff matrix is considered as a linear combination of three classes of elementary games representing the coordination ($\mathbf{A}^{(co)}$) between all possible strategy pairs, the symmetric ($\mathbf{A}^{(ex)}$) and antisymmetric ($\mathbf{A}^{(as)}$) combinations of the self- and cross-dependent components. In this subset of games the analyses are simplified by the existence of a pure preferred Nash equilibrium that is easily identified by the largest V_{ij} value of the potential matrix. Disregarding the occasional degeneracy we can distinguish two typical behaviors. In the first case $\max(V_{ij})$ selects one of the diagonal components (e.g., V_{kk}) when the rational players choose the corresponding symmetric strategy profile (k,k). If these games define the interaction between the neighboring players on a lattice then the system evolves into a homogeneous strategy distribution in the low-noise limit of the logit rule and it exhibits a continuous transition towards the random strategy selection when the noise level goes to infinity. In the opposite cases the system behavior is dominated by an anticoordination component that enforces a sublattice ordered (checkerboard) strategy arrangement on a square lattice, which undergoes an Ising-type order-disorder phase transition when the noise level of the logit rule is increased.

In the lattice systems, the above universal behaviors (e.g., order-disorder transition) can be observed for both the presence and absence of social dilemmas. There is no social dilemmas for the friendship or fraternal potential games when the payoff matrix is symmetric, and it defines the potential matrix, too. That happens, for example, when the payoff matrix is composed of coordination-type interactions ($\mathbf{A} = \mathbf{A}^{(co)}$) existing between any different strategy pairs. The social dilemmas; that is, the conflict between the individual and common interest, are caused by the antisymmetric part of the payoff matrix, which can be considered as Ate's hand, originating exclusively from the antisymmetric parts of the self- and cross-dependent components. It is found that the symmetric $\mathbf{A}^{(ex)}$ and antisymmetric $\mathbf{A}^{(as)}$ parts of the latter components span the same subset of potential matrices determined by the possible self-dependent components. Both terms favor the choice of a symmetric (i,i) Nash equilibrium. In itself ($\mathbf{A} = \mathbf{A}^{(as)}$), the antisymmetric term is innocuous. One can find different versions of social dilemmas when studying the interplay between $\mathbf{A}^{(as)}$ and the other components.

The simplest versions of social dilemmas can occur in the absence of coordination components when the deluding strength of $\mathbf{A}^{(as)}$ overcomes the driving force of $\mathbf{A}^{(ex)}$ by suggesting another preferred Nash equilibrium. The possibility of the latter events increases with the number of strategies because the corresponding parameter space has a dimension of $n-1$. Exceptions are represented by games where $\mathbf{A}^{(ex)}$ and $\mathbf{A}^{(as)}$ support the choice of the same strategy. In the latter case $\mathbf{A}^{(as)}$ helps the maintenance of optimum choice in evolutionary games when the stochastic effects (noise) are increased. A similar positive effect of the additional $\mathbf{A}^{(as)}$ is found for games with equivalent strategy-pair coordinations (as it is realized in the n-state Potts model) favoring equally the formation of one of the homogeneous strategy distributions in the low-noise limit. For finite noises $\mathbf{A}^{(as)}$ acts like an external field that drives the system towards the distinguished homogeneous state and suppresses all the other inefficient constellations.

For the typical cases, however, the interplay between $\mathbf{A}^{(as)}$ and a coordination-dominated $\mathbf{A}^{(co)}$ [with $\max(V_{ij}) = V_{ii}$] can be conflicting as is illustrated by the contentment and discontentment of players in the preferred Nash equilibrium dependent on the payoff parameters. The contentment maps elucidated the emergence of different social dilemmas in a large portion of some two-dimensional cross sections of the parameter space. A wider scale of the preferred Nash equilibria (and social dilemmas) is found for games where $\mathbf{A}^{(co)}$ is dominated by an anticoordination component.

The most relevant message of the above-mentioned contentment maps is that the probability of finding discontented players increases with the number of strategies if the payoffs are selected at random in this subset of games. The high frequency of frowning faces is illustrated clearly in the contentment maps. This feature is related to the fact that $\mathbf{A}^{(as)}$ supports the community only if it proposes the selection of the same (pure and symmetric) strategy profile (preferred Nash equilibrium) that is recommended by the rest of payoff components. For opposite or orthogonal $\mathbf{A}^{(as)}$ values, social conflict occurs if its strength is large enough.

We emphasize once again that the present conclusions are valid for the symmetric two-player potential games and multi-agent systems where the payoffs come from pair interactions between the neighbors for suitable connectivity structures [51]. The presence of cyclic components, however, can cause more complex phenomena. By contrast, for low noise the mentioned complexity can be reduced in the so-called ordinal potential games [49] in which the cyclic components are weak and not capable of altering the preferred Nash equilibrium.

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