Thin structure of the transit time distributions of open billiards

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It is known that typical open billiards distribution of transit times is an exponentially decaying function, possibly with a power-law tail. In the paper we show that on small scales some of such distributions change their appearance. These distributions contain a quasiperiodic thin structure, which carries a significant amount of information about the system. Origin and properties of this structure are discussed.

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I. INTRODUCTION

Mathematical billiard is a standard model system in the chaos theory. The particle in the billiard moves rectilinearly and reflects absolutely elastically from the billiards boundary. Well known are classical billiards with a strong type of chaos, such as the Sinai [1] and Bunimovich [2] billiards. Also important are billiards with a weaker type of chaos, the phase space of which contains islands of stability.

A billiard is called open if there are sites in its boundary, through which particles can both enter and leave the billiard (see, e.g. [3]). Such billiards are currently intensively studied, due to a large number of applications. A variety of systems may be reduced to an open billiards. In particular, systems are from such fields of physics as acoustics, optics [4–6], hydrodynamics [7,8], plasma physics [9], etc. A billiard type of dynamics is also associated with some theoretical issues—the justification of statistical physics, interconnection between the classical and quantum descriptions of a system, etc. [10–15]. Detailed description of the relationship between open billiards and these fields can be found in the review in [16].

Considering open billiards, typically the distributions of escape times (for particles initially inside) or transit times (for particles entering through a hole then exiting) are studied. In general, exponential distributions are typical for chaotic behavior. In addition to the main exponential decay there may be a power-law tail of the distribution. It is connected with such chaos phenomena as intermittency and stickiness to the boundary between regular and chaotic regions [17,18]. Fine structure of sticky sets in mushroom billiards was considered in the paper in [19]. Power spectrum $1/f$ of cellular automata may also contain a quasiperiodic structure [20].

The sedate law of escape times distribution is typical in the case of a regular behavior (see, e.g. [21]). As shown in the papers in [22–24], the distributions of the same billiard may either possess or not an algebraic tail depending on the system parameters, on initial distribution of particles and position of the hole. There are recent studies of the return time distributions in the case of a hole moving along the border of billiard. In the paper [25] these distributions were examined for the Bunimovich stadium and uniformly moving hole. In the paper in [26] are considered the distributions for the billiard of the oval form, with the islands of stability in the phase space, and the hole moving either periodically or randomly.

Understanding of the rules of formation of return time distributions is important for many reasons. It allows one to restore the phase portrait of the system, or to conduct the nondestructive monitoring of systems, inaccessible to the direct observations [27]. The paper [27] considers the case of the Bunimovich billiard with particles, initially located inside of it. The dependence of the small amendments to a leading order of the escape rate on the size and location of the hole was studied. We also consider the Bunimovich stadium, but in the case of particles entering the system through the hole. The precise value of the escape rate generally depends on the hole parameters, but in this particular case it is precisely proportional to the size of the hole and does not depend on its position [28]. However, as will be shown below, the distribution of transit times for some billiards, including Bunimovich stadium, is not purely exponential, but contains some additional thin structure. This structure is smoothed on large scales, with the result that distribution looks like exponential decay.

II. CONSIDERED BILLIARDS AND THEIR DISTRIBUTIONS

In the paper we consider the distributions of transit times of the two focusing billiards, in the case when particles enter and leave the billiard through the same single hole. One of these billiards is a well-known Bunimovich stadium; another one is a parabolic billiard. The Bunimovich billiard is a classical object of study in the chaos theory. It has two parameters: the length l and the height h; its general view is shown in Fig. 1(a). The properties of the escape and transit time distributions of this billiard were heavily studied previously. However, to build a distribution it is necessary to choose some degree of roughening. The time axis has to be divided on the intervals of duration Δ_t. And all the returns getting into the same time interval are joined together and represented by one point of the histogram. Below we will show that return time distributions may contain subtle details that are visible only when $\Delta_t \ll \bar{t} = \bar{l}/v$, where $\bar{t}$ is an average time between collisions of the particle with billiards boundary, $\bar{l} = \frac{\pi S}{P}$ is an average length of free path [29], S and P are the area

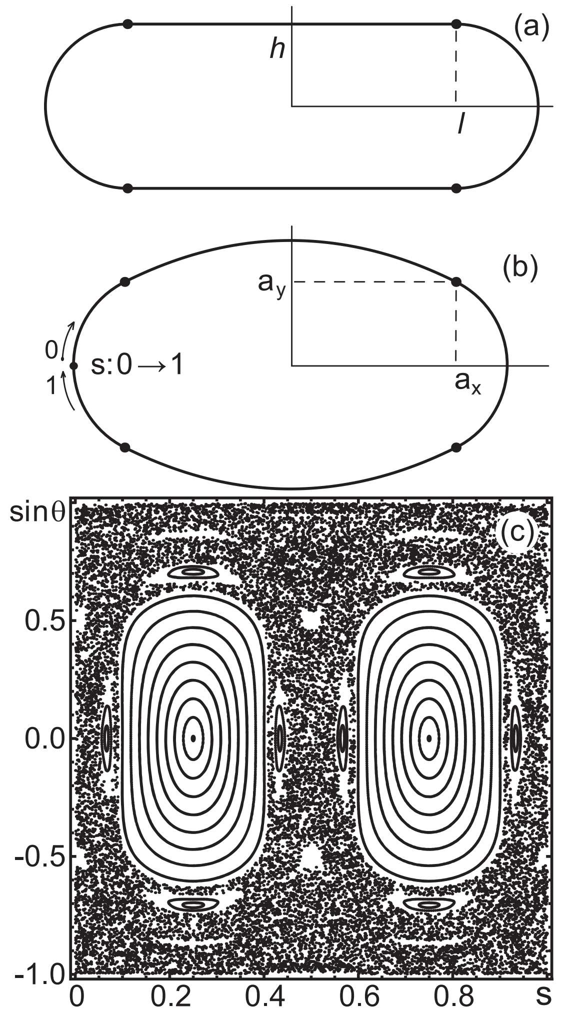

FIG. 1. General view of (a) the Bunimovich stadium with the parameters $l = 20$, $h = 10$ and (b) the parabolic billiard with the parameters $a = 40$, $a_x = 20$, $a_y = 10$. Also shown is (c) the phase portrait of the parabolic billiard with the same parameters.

and perimeter of the billiard, and v is the particle's velocity. Previously, since the main interest was in the behavior of the tail of distribution, the distributions were built for $\Delta_t \gg \bar{t}$. As a result, the thin distributions structure was smoothed out, and contained in the distribution information was partly lost.

In order to determine how general is the presence of this thin structure of the distributions we consider two billiards, not only a Bunimovich stadium. Also it is interesting what features of this structure are typical and common for the different billiards. The Bunimovich stadium has no islands of stability in its phase space. However, a phase space of general billiard may contain them. For this reason we also consider the parabolic billiard that has islands of stability in its phase space. The parabolic billiard can be obtained from the billiard of Bunimovich via the replacement of the flat border sides by the segments of parabolas of the form $y = \pm(a_y + \frac{a_x^2 - x^2}{2a})$, $x \in [-a_x, a_x]$. With the corresponding modification of the lateral circle segments, they become sewn with parabolic segments smoothly. These border sides became the segments of circles of the radius $r = \sqrt{a_y^2 + (\frac{a_x a_y}{a})^2}$ with the centers at the points $\pm(a_x - \frac{a_x a_y}{a}, 0)$. They are joined together with parabolic segments at the points $(\pm a_x, \pm a_y)$. A general view of the thus obtained parabolic billiard is shown in Fig. 1(b). It is easy to notice that in the limit $a \to \infty$ this billiard transfers to the Bunimovich stadium, with the parameters of parabolic billiard a_x and a_y being transferred to the parameters l and h. The parameters a_x and a_y determine the coordinates of the sides' end points, similar to l and h of the Bunimovich billiard. Parameter a determines the curvature of the parabolic sides; $a = \infty$ corresponds to an infinite radius of curvature, i.e., the transformation of parabola to the flat border side.

The typical phase portrait of the parabolic billiard in Birkhoff coordinates $(s_i, \sin\Theta_i)$ is shown in Fig. 1(c). This phase portrait consists of the chaotic sea and the islands of stability, among which the central one may be distinguished. The configuration of the islands of stability depends on the billiards parameters; for their certain choice there is only one central island of stability. The phase space volume, occupied by this island, may vary from zero to almost the entire phase space. In particular, when $a = 100$, $a_x = 80$, and $a_y \in (20, 34)$, there is only one island in the phase space, the size of which is limited by the unstable periodic trajectory of the "bird" type. In the Birkhoff coordinates it has the form of a rhombus, which is surrounded by the chaotic sea [30]. For other parameters, the size of the central island can be limited by the end points of the parabolic segments, as, for example, in the cases shown in Figs. 4(a) and 4(b). The parabolas (of the parabolic border sides) generally are not confocal, but the motion of particles between them largely inherits the properties of motion between confocal parabolas, which is integrable. As a result, in the phase space appears a large region of regular motion. Such simple structure of the phase portrait makes the parabolic billiard a convenient model system to study the effects related to the presence of the islands of stability.

For the considered open billiards the transit time distributions were built. In all cases, the boundary of the billiard had only one fixed hole. Initially the particles were distributed uniformly along the length of the hole and had Lambert distribution of the initial directions of motion $I(\varphi) = I_0 \cos\varphi$, where $-\frac{\Pi}{2} < \varphi < \frac{\Pi}{2}$. The trajectories were partly constructed with an increased precision, with the use of a library of long numbers. We consider it necessary because of the rounding errors, which inevitably arise due to the limited accuracy of the determination of coordinates and velocities of particles. For chaotic trajectories they grow exponentially. To verify that the discovered distributions structure is not connected with this, all distributions in the paper were built with the first 100 collisions calculated with the guaranteed accuracy. Next collisions were calculated with a usual double precision of 64 bits per number. The velocity of all particles is considered to be a unit by modulus, respectively, for example, for the Bunimovich billiard with parameters $l = 20$ and $h = 10$, the first 100 collisions corresponded to the return times of approximately 2450 units. The thin structure of distributions is already observed at the lesser times, and the structure of distributions before and after this value does not differ. Some of the distributions were also rebuilt with the lesser statistics, but with the precision that guarantees the absence of significant rounding error on all

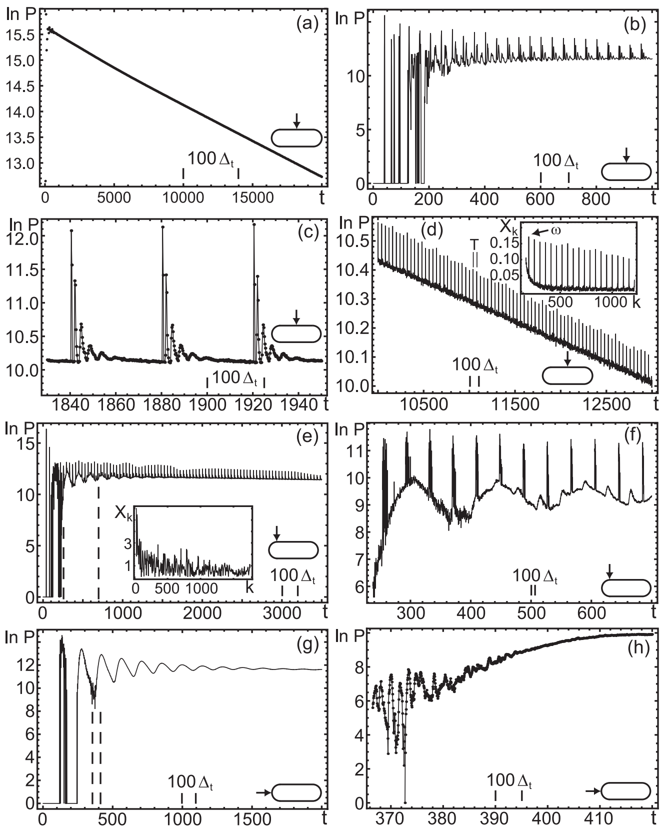

FIG. 2. Distribution of transit times for the Bunimovich stadium with the parameters $l = 20$, $h = 10$, $d = 0.5$ (the mean free path $\bar{l} \approx 24.5$) on different scales. Positions of the hole are shown by the arrows in the insets. The initial data of $N_r = 10^9$ trajectories were distributed evenly along the hole and had the Lambert distribution of the initial directions of motion. Each point presents the sum of all escape times within the same time interval Δ_t. It has the following duration values: (a) $\Delta_t = 40$, (b) $\Delta_t = 1$, (c) $\Delta_t = 0.25$, (d) $\Delta_t = 1$, (e) $\Delta_t = 2$, (f) $\Delta_t = 0.05$, (g) $\Delta_t = 1$, and (h) $\Delta_t = 0.05$. The insets on figures (d) and (e) show the discrete Fourier transform spectrum of the presented distribution part.

considered times. The correctness of obtained trajectories was tested by the double sweep method, which is the return of the particle to its initial position. As it turned out, built with the increased accuracy distributions does not differ from the distributions constructed with the usual double precision. Thus, to study the thin structure of the distributions, it is possible to use the numbers of regular accuracy. The associated rounding error does not affect the statistical properties of the distribution. In some cases the spectrum of the presented distribution part was also shown, i.e., the modulus of complex coefficients of discrete Fourier transformation, chopped left half, with some of the characteristic quasiperiods and corresponding frequencies.

The distributions of transit times were built on different time scales with different Δ_t. The appearance of a rapidly oscillating distribution depends on the choice of this value. The value of the distribution over the time interval $[t, t + \Delta_t]$ was equal to the total number $P(t)$ of returns, getting in this time interval $[t, t + \Delta_t]$.

Distributions built for the Bunimovich stadium are shown at Fig. 2. All distributions are shown on the logarithmic scale and contain at least $N_r = 10^9$ number of returns. The insets show the form of the billiard, for which the distribution was built; an arrow marks the position of the hole. Figure 2(a) shows the roughened distribution of transit times in the case of the hole located in the middle of the flat billiards side, built for $\Delta_t \sim \bar{t}$. For this choice of Δ_t the distribution looks like a pure exponential decay without any structure. Such distribution appearance was observed previously many times. However, when we build the same distribution with a higher resolution $\Delta_t \ll \bar{t}$, the quasiperiodic structure of peaks became visible. It is shown at Fig. 2(b) and in more detail in Fig. 2(c). It is visible that each peak is followed by the several secondary peaks of smaller amplitude and greater width, with the gradually increasing distance between them. These groups of local maximums of the distribution follow recurrently. Figure 2(d) shows the behavior of this distribution on the larger times. The periodically following peaks are also observed. Thus the exponentially decaying distribution of escape times is not always purely exponential, as it was considered previously, but may also contain some thin structure, like the one shown at Fig. 2(d).

Figures 2(e) and 2(f) show a similar distribution, constructed for the hole located at the end of the flat side of the Bunimovich billiard. This distribution also contains quasiperiodic peaks. In addition to the peaks, there is also a quasiperiodical damped oscillation at the beginning of the distribution. Also visible is a sequence of local maximums of smaller amplitude, with the period close to the peak's quasiperiod. Thus a structure shown at Fig. 2(f) contains at least three characteristic periods. Figures 2(g) and 2(h) show the distribution for a hole located at the middle of the circular side of the billiard. It is visible that this distribution contains only a decaying quasiperiodic component; the peaks are completely absent. This distribution looks very noisy between the second and third maximums. It is not due to the lack of statistics, but the distribution itself. More precisely this region is shown in Fig. 2(h). It is visible that left and right distribution parts are very different.

Distributions for the parabolic billiard were built in an analogous way. Two hole positions were considered, at the center of the circular arc and at the center of the parabolic arc. In the first case, all entering the billiard particles get into the chaotic sea, like it was in case of the Bunimovich stadium. In the second case, when the hole is in the parabolic border side, the particles mostly get on the central island of stability. Here it is necessary to remember that the creation of a hole in the billiards boundary destroys those islands, whose trajectories partly fall on the removed part of the border. Thus, in the second case, a part of entering particles would move regularly all the time until their escape from billiard. Even a roughened distribution of return times in this case may be unusually arranged. In particular, there may be power-law sites at the beginning of the distribution, chopped on some depending on the islands value, and then an exponential tail.

Built for the parabolic billiard distributions are shown in Fig. 3. The number of particles, their initial distributions, and all the other parameters were taken the same as for the Bunimovich billiard. Figure 3(a) shows the general view of the roughened distribution. It is visible, that, except for the initial distribution part, it looks like a usual exponential

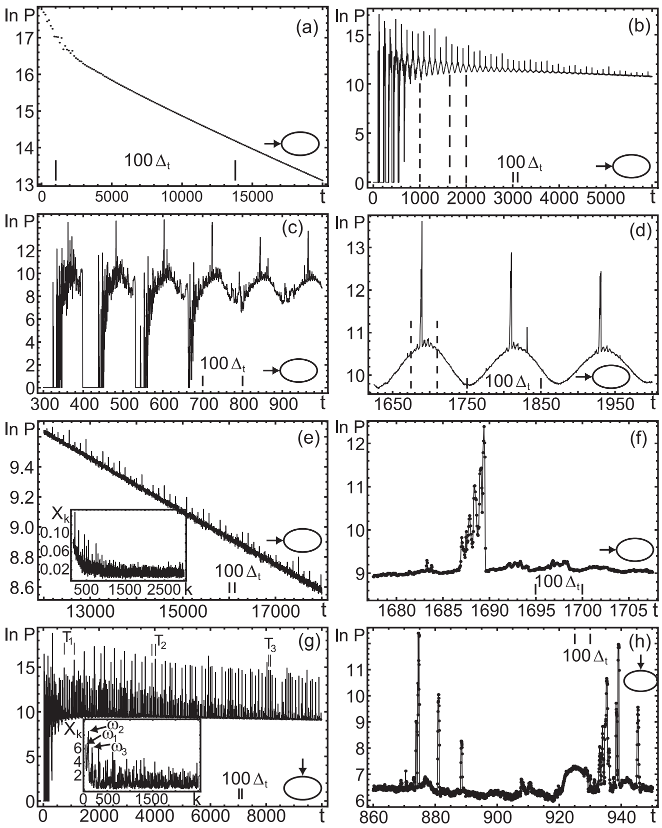

FIG. 3. Distributions of transit times for the parabolic billiard with parameters $a = 40$, $a_x = 20$, $a_y = 10$, $d = 0.5$, $\bar{t} \approx 29.4$ on different scales. The initial data of $N_r = 10^9$ trajectories were distributed evenly along the length of the hole and had Lambert distribution of the initial directions of motion. Each point on the graphs shows the sum of all escape times within the same time interval Δ_t. It has the following duration values: (a) $\Delta_t = 128$, (b) $\Delta_t = 1$, (c) $\Delta_t = 1$, (d) $\Delta_t = 1$, (e) $\Delta_t = 1$, (f) $\Delta_t = 0.05$, (g) $\Delta_t = 1$, and (h) $\Delta_t = 0.05$. The insets on figures (e) and (g) show the discrete Fourier transform spectrum of the presented distribution part.

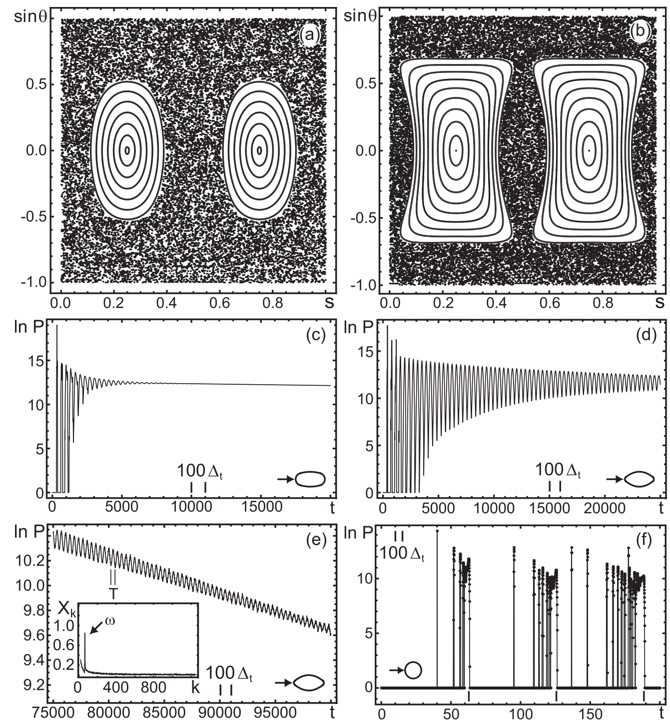

FIG. 4. Phase portraits and escape time distributions for the parabolic billiard (a),(c) with the parameters $a = 180$, $a_x = 50$, $a_y = 30$, $d = 0.5$, $\Delta_t = 10$ and (b),(d),(e) with the parameters $a = 100$, $a_x = 80$, $a_y = 12$, $d = 0.5$, $\Delta_t = 10$. (f) Initial part of the distribution of transit times for the Bunimovich stadium with the parameters $l = 0.01$, $h = 10$, $d = 0.5$, $\Delta_t = 0.05$. The marks below show the multiples of the billiards perimeter. The inset on figure (e) shows the discrete Fourier transform spectrum of this part of distribution.

decay. On the smaller scales, as shown in Figs. 3(b)–3(e), a quasiperiodic thin structure became visible, like it was in the case of the Bunimovich stadium. Some difference is that for the Bunimovich billiard this structure of periodic peaks was observed for a hole at the center of the flat side, while for the parabolic billiard it is observed for a hole in the circular side of the billiard. It is interesting to notice that the peaks quasiperiod is very close to the quasiperiod of the damped oscillation. As can be seen in Fig. 3(e), quasiperiodically following peaks are observed not only at the beginning, but also on further sites of the distribution. Their variation in amplitude is substantially larger than it was in the case of the Bunimovich billiard. Figure 3(f) shows in details the construction of the first of the peaks that are shown in Fig. 3(d). It is visible that, similar to the case of the Bunimovich stadium, the peaks have their own internal structure. They consist of a sequence of closely located subpeaks of rising amplitude.

Figure 3(g) and in more detail Fig. 3(h) show the distribution of escape times for the hole at the center of the parabolic border segment. In this case the distribution also contains multiple peaks that form a complicated structure. This structure contains a superposition of the sequences of periodically following peaks of different periods. It is due to the fact that entering the billiard particles not only get in the chaotic sea, but also on a number of destroyed islands of stability. Each of these islands generates its own characteristic distribution that depends on the parameters of this island. As a result of their superposition, there occurs a complex distribution that contains a large amount of information about the structure of the phase space.

III. STRUCTURE COMPONENTS AND DEPENDENCE ON BILLIARD PARAMETERS

The observed structure can be divided on two main components: the smooth quasiperiodic decaying component and the high sharp peaks of the distribution. The initial part of the first component was found to be associated with the trajectories that move along the border of the billiard all the time until the escape from it. These trajectories are similar to the whispering gallery trajectories, existing in all convex and smooth enough billiards. In order to understand how the first maximums of this component arise, let us consider in more detail the arrangement of the initial part of a typical return times distribution. For example, Fig. 4(f) shows the initial part of the distribution for the Bunimovich billiard with the parameters, when it is almost a circle. The first maximum corresponds to the trajectory in the form of a horizontal line, on which the particle reflects one time from the opposite to the hole border site. This is the fastest of all possible ways to leave the billiard. The next peak corresponds to the trajectory in the form of an approximately equilateral triangle, on which the particle leaves the billiard after two reflections from the border. The trajectories of the next pikes have a form of regular rectangle, pentagon, etc. It is easy to see that the length of these trajectories with the increase of their order tends to the perimeter of the billiard, but does not exceed it on more than one size of the hole. The peaks caused by these trajectories after some point merge together in a continuous distribution, the exact form of which depends on the initial distributions of entering the billiard particles. However, the position of this maximum of the distribution generally depends only on the billiards perimeter. Next, maximums of the distribution are formed in the same way; only merged together are the peaks from the trajectories that passed along the perimeter twice, then three times, etc. Generally, different trajectories of a whispering gallery type are one of the fastest ways out of the billiard (by a trajectory length, not a number of collisions). On smallest return times their contribution is significant. The length of each of these trajectories is approximately a multiple of the perimeter of the billiard. As a result, periodically following distribution maximums are formed, with a period equal to the billiard's perimeter. With the increase of a return time value, the share of such trajectories reduces and associated maximums gradually fade.

The quasiperiodic maximums may still be observed after this associated with the whispering gallery trajectories part of the smooth quasiperiodic component. A quasiperiod of these maximums well correlates with the perimeter of the billiard, but does not coincide with it exactly. Whether this continuation of the first component of a thin structure will be observed depends on the parameters of the billiard. Figure 4 shows the phase portraits of the parabolic billiard and the distributions of escape times for the two possible cases. Figures 4(a) and 4(b) show the case of the distribution, whose quasiperiodic component is associated only with the whispering gallery trajectories. In this case, this component totally fades after about ten oscillations, both for the parabolic billiard and for the Bunimovich stadium [see Fig. 2(g)]. Although this component fades fast, it may play some role, since in many physically important applications the billiards type of dynamics defines the system behavior specifically on small times.

Figures 4(d) and 4(e) show the distribution of transit times for a parabolic billiard with the parameters $a = 100$, $a_x = 80$, $a_y = 12$, the perimeter of which is equal to $l_p \approx 406$. It is visible that a quasiperiodic component of the distribution persists on the times that are two or three orders of magnitude greater than the length of the billiards perimeter (the billiards particle velocity is a unit). The initial part of this component is still due to the trajectories of the whispering gallery. The further maximums follow with the quasiperiod $t_{\mathrm{qp}} \approx 385$. The origin of this part of the quasiperiodic component is currently unclear.

The second component of the observed distributions thin structure is the system of sharp narrow distribution peaks. This system of peaks may be present or absent in the distribution, depending on the billiard parameters and the position of the hole. For both considered in the paper billiards there are parameters when this structure is present. It allows us to suggest that its presence in the escape times distribution is a general case. This peak's structure is connected with the existence of the families of billiard trajectories with the lengths that are the multiples of some basic length. For the case of the Bunimovich stadium shown in Figs. 2(b)–2(d), the peak's structure may be simply explained. This structure is associated with the family of trajectories on which the billiards particle moves between the flat sides until the first collision with the circular border side, after which it moves in the opposite direction between the flat sides until the return to the hole. A typical flattening of such trajectory is shown on the right in Fig. 5. All but one of the collisions of such trajectories occur with the flat border sides, so the path length can be easily calculated as

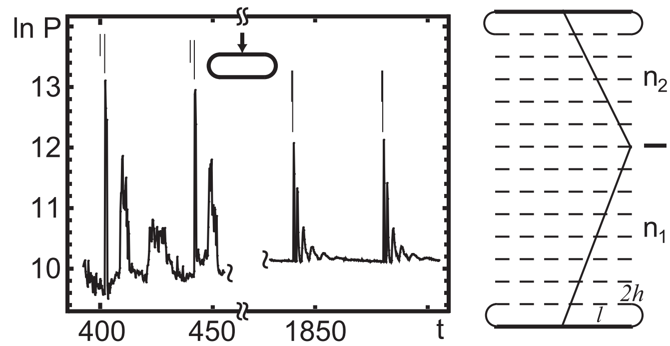

FIG. 5. Two parts of the same distribution of escape times for the Bunimovich billiard with the parameters $l = 20$, $h = 10$, $d = 0.5$. Each of these two parts of the distribution contains two peaks. The marks at the top show the peaks positions calculated according to the formula (1). Short mark—without the second term; long mark—according to the complete formula. On the right is shown the typical flattening of the trajectory belonging to the distribution peak.

$$t = \sqrt{l^2 + (2hn_1)^2} + \sqrt{l^2 + (2hn_2)^2}$$
$$\approx (n_1 + n_2)\left(2h + \frac{1}{n_1 n_2}\frac{l^2}{4h}\right), \quad (1)$$

where it is considered that $l \ll 2hn_1$ and $l \ll 2hn_2$. The sum of n_1 and n_2 must be even for the trajectory to return to the hole, not to the opposite border site. A simple analysis of this formula shows that, starting from some values of n_1 and n_2, the second term will be much lesser than the first one. For this reason, all the lengths of such trajectories are approximately the multiples of $4h$. For the different trajectories with different n_1 and n_2, but with their sum being equal, the lengths of these trajectories will be almost identical. A consequence of the existence of such family of trajectories is the appearance of the periodic peaks in the distribution of transit times. Subpeaks in this distribution are coursed by similar bouncing trajectories, but with more than one reflection forming the circular border part.

The value of the second term in formula 1 practically does not depend on the exact values of n_1 and n_2, if they are of the same order, only on their sum. Therefore, even for those n_1 and n_2, for which the second term makes a significant contribution to the length of the trajectory, this contribution is almost identical for the different trajectories. As a result, there also are peaks at the beginning of the distribution, shifted compared to the values of $4nh$. Figure 5 shows two parts of

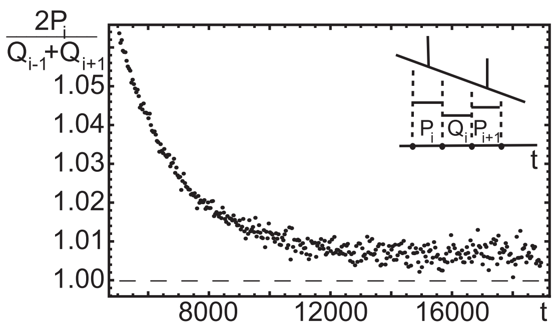

FIG. 6. Ratio of the average number of returns, contained in the distribution sites with a peak, to such for the sites without peak. The distribution, for which this value was calculated, is shown in Fig. 2(d). It is visible that the contribution of peaks in the total number of returns tends to a constant nonzero value.

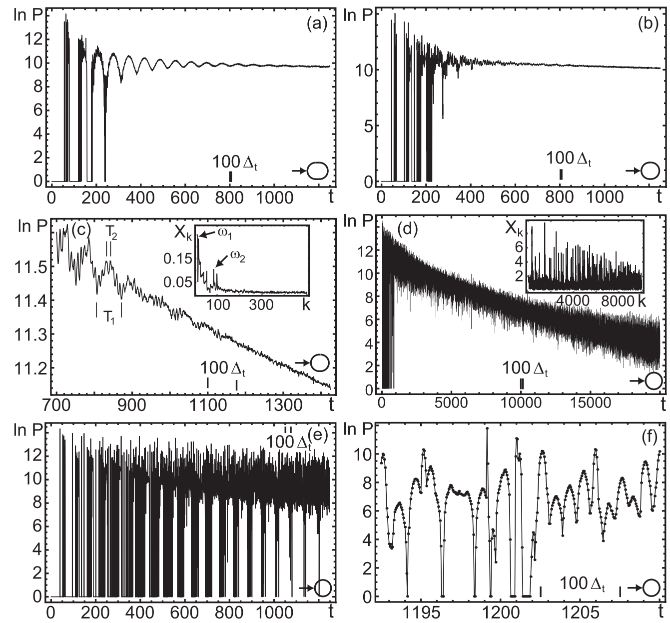

FIG. 7. Modification of the thin structure of distribution for Bunimovich stadium with the decrease of parameter l to zero value, which corresponds to the integrable circle billiard. The distributions were built for the parameters $h = 10$, $d = 0.5$ and (a) $l = 4$, $\Delta_t = 0.25$, (b) $l = 1.2$, $\Delta_t = 0.25$, (c) $l = 1.2$, $\Delta_t = 0.75$, (d) $l = 0.01$, $\Delta_t = 1$, (e) $l = 0.01$, $\Delta_t = 0.25$, and (f) $l = 0.01$, $\Delta_t = 0.05$. The insets on figures (d) and (e) show the discrete Fourier transform spectrum of the presented distribution part.

the same transit times distribution for the Bunimovich stadium with the parameters $l = 20$, $h = 10$. The short marks show the values of $4nh$ and the long marks the positions of the peaks, calculated according to the full formula (1). It is visible that the position of the beginning of each peak coincides well with the calculated one.

It is interesting to note that knowing the positions of, for example, the first two peaks shown in Fig. 5, it is possible to restore the parameters l and h of the billiard, for which the distribution was built. The coordinates of the beginning of these peaks lie within the range of $p_1 \in (402.0,402.25)$ and $p_2 \in (441.75,442.0)$, from where we can get $h \approx 9.94$. From this approximate value of h it can be determined that for the first peak $n_1 + n_2 = 20$ and, respectively, for the second $n_1 + n_2 = 22$. Then it is possible to write and solve a system of two equations for the locations of these two peaks, and to obtain the values of $h \approx 9.99$ and $l \approx 21.5$. The error of determination of the value l is mainly caused by the insufficient accuracy of the determination of the peaks' positions. It can be reduced by consideration of more than two peaks. Thus the structure of peaks does carry a significant amount of information about the system, unlike the value of the exponent of the roughened distribution. The knowledge of the positions of two consecutive peaks of this structure already allows the restoration of both parameters of the billiard, for which the distribution was built.

Another important question about the quasiperiodically following distribution peaks is the dependence of their amplitude on the peak number. The trajectories that give rise to these peaks can be arbitrarily long, allowing in principle the existence of peaks at the arbitrarily large return times. So it is of interest whether the structure of peaks persist on such times or the peaks amplitude gradually decrease to zero. In order to find out, we took the previously built distribution of transit times for the Bunimovich billiard, shown in Figs. 2(b)–2(d). The time axis was divided into the intervals of duration, equal to half of the quasiperiod, so that all the peaks were within one-half of these intervals, and the second half contained a distribution without peaks. Then the ratio was computed that shows the excess of returns to the interval containing a peak compared to the neighboring intervals without peaks. The dependence of this ratio value on the return time is shown in Fig. 6. It is visible that the contribution of the peaks to the overall number of returns decreases gradually, tending to some constant nonzero value, which for a given distribution is approximately equal to 0.68%. This suggests that the distribution peaks will occur at all the return times. In other words, some share of returns will always belong to a quasiperiodic component of the distribution.

IV. TRANSITION TO THE INTEGRABLE CASE

In the limit case $l \to 0$ the Bunimovich billiard transfers to a circle billiard, whose dynamics is integrable. Also integrable is the motion of the billiard particle between confocal (for a certain choice of billiard parameters) parabolic border sides. With the change of parameters of the billiard with a smooth border, each trajectory modifies in a continuous manner, and therefore the distribution of their return times modifies continuously. In particular, for the small values of the parameter l of the Bunimovich billiard, the significant difference with regular dynamics occurs only at large enough times. It is interesting to find out how the thin structure of the distribution modifies during such transition from the chaotic to the regular motion type.

The distributions of transit times for the Bunimovich billiard with progressively decreasing values of the parameter l and fixed h are shown in Fig. 7. Until some value of l the distributions do not change substantially. Shown in Fig. 7(a) distribution for $l = 4$ has the same form as shown in Fig. 2(g) for $l = 20$. The only difference is the change of the value of the quasiperiod, according to the change of the billiards perimeter. With a further decrease of l, a smooth quasiperiodic structure gradually starts to disappear. And starting from some value of l, which for $h = 10$ is approximately equal to

$l \approx 1.2$, this structure almost completely disappears and the distribution takes the form shown in Figs. 7(b) and 7(c). Such distribution appearance is a consequence of the intermediate dynamics, already fairly regular over short periods of time, although strictly speaking still chaotic. The initial part of this distribution, as can be seen in Fig. 7(c), contains some structure with at least two characteristic frequencies. At the larger times, after some intermediate part, the distribution becomes purely exponential.

In the case of the integrable billiard in a circle of radius r, the residence time of the particle in the billiard is determined by the equation $t = 2nr \sin \frac{\Delta\varphi}{2}$, where n is the first natural number that satisfies the inequality: $\varphi_{bgn} < \mathrm{mod}(\varphi_0 + n\Delta\varphi, 2\pi) < \varphi_{\mathrm{end}}$. Here φ_{bgn} and φ_{end} are the angles that determine the start and end points of the hole in the circle border. The distributions of values φ_0 and $\Delta\varphi$ are determined by the distributions of initial conditions of entering particles. The initial part of the distribution of return times for the Bunimovich billiard with a sufficiently small value of the parameter l does not differ from the similar part of the same distribution for the circle billiard. The typical view of this distribution is shown in Figs. 7(d)–7(f). Despite the simplicity of the underlying dynamics, this distribution is extremely complicated. It is interesting to note that, even at the times the order of magnitude is greater than the billiards perimeter, there are the time intervals entirely without returns.

Thus the change of the underlying dynamics from the chaotic to the regular type affects explicitly both the roughened distribution and the thin distribution structure. In the case above, the approach to the regular dynamics first leads to the disappearance of this structure, with the result that the distribution becomes maximally close to the pure exponent. Then there appears another, extremely complicated component of the distribution, which looks like a noise of the high frequency and significant amplitude.

V. CONCLUSIONS

The paper considers the distributions of transit times of particles for the two billiards, the parabolic billiard and the Bunimovich stadium. These distributions refer to the case when there is one hole in the billiards boundary, through which the particles enter the billiard. Distributions of both considered billiards were found to contain a thin structure, typically quasiperiodic with one or more characteristic periods. This structure can be divided on the two main components that can occur separately or superimposed one on another.

The first component is a smooth quasiperiodical decaying oscillation of the distribution of transit times. It was shown that the initial part of this component is associated with the whispering gallery type of trajectories, which is present in all convex billiards. The origin of the possible further oscillations is currently unclear. The quasiperiod of this component well correlates with the billiards perimeter. The decay rate of this component depends significantly on the type of billiard and its parameters. For the parabolic billiard this component may be clearly visible at the times two or three orders of magnitude greater than the billiards perimeter.

The second component is the system of high narrow distribution peaks that follow quasiperiodically or in a more complex manner. It may be present or absent depending on the billiards parameters. These peaks, in their turn, have an internal structure and are related to the families of different trajectories of the same length. There is some evidence provided in the paper that such structure of peaks will be present in the distribution at arbitrarily large times. This structure of peaks was found for both Bunimovich and parabolic billiards. This allows us to suggest that the possible presence of such structure is a general property of the billiards distributions. These issues currently require further investigations.

The characteristic distance between quasiperiodically following peaks generally depends only on the billiards parameters. It is not sensitive to the initial distribution of incoming particles, or to the hole's size, unlike the exponent of the roughened distribution. For the Bunimovich stadium it was shown that, knowing the positions of two consequent peaks of the distribution, it is possible to recover both parameters l and h of the billiard, for which this distribution was built. Thus the thin structure carries a significant amount of additional information about the system.

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