First-passage time for superstatistical Fokker-Planck models

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The first-passage-time (FPT) problem is studied for superstatistical models assuming that the mesoscopic system dynamics is described by a Fokker-Planck equation. We show that all moments of the random intensive parameter associated to the superstatistical approach can be put in one-to-one correspondence with the moments of the FPT. For systems subjected to an additional uncorrelated external force, the same statistical information is obtained from the dependence of the FPT moments on the external force. These results provide an alternative technique for checking the validity of superstatistical models. As an example, we characterize the mean FPT for a forced Brownian particle.

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I. INTRODUCTION

Superstatistics is an approximation that allows one to model complex nonequilibrium system dynamics [1]. It applies when temporal or inhomogeneous spatial fluctuations of the environment occur on a large time scale, while the local system dynamics relaxes in a faster time scale [2], staying in equilibrium for some time. Hence, its dynamics follows from a double average (superstatistics) consisting in a local equilibrium probability distribution which in turn depends on a *random intensive parameter* associated to the environment fluctuations. A central feature of this approach is the possibility of modeling non-Gibbsian equilibriumlike system distributions [3].

At the beginning of its formulation, superstatistics was applied in the context of turbulent Lagrangian dynamics [4–6]. Nevertheless, since then, it has been also used for studying diverse kinds of complex systems such as ultracold gases [7], quantum entanglement in Ising-like models [8], bacterial DNA architecture [9], migration of tumor cells [10], plasma physics [11], nanoscale electromechanical systems [12], work fluctuation theorems [13], and market signals [14], just to name a few [15–21].

Many efforts were also devoted to deriving the superstatistical approach from underlying descriptions such as entropic principles [22–25], and dynamical ones based on underlying Fokker-Planck equations that include the intensive parameter as a stochastic dynamical variable [26]. More recently, multiscale and hierarchical systems [27,28] were also proposed, as well as Langevin dynamics with diffusing diffusivity [29]. Theoretical results of these kinds are of relevance because they allow one to explore under which conditions superstatistical dynamics may apply. With a similar motivation, the main goal of this paper is to introduce an extra criterion for checking the validity of superstatistical models. We study the *first-passage-time* (FPT) problem for dynamics where this approach applies.

The FPT measures when a system of interest crosses a given boundary or threshold value for the first time. Its statistics has played an important role in many disciplines [30–33]. Its properties have been scarcely studied in the context of the superstatistical approach [20]. We show that the statistics of the FPT can be put in one-to-one correspondence with the statistics of the random intensive parameter of the superstatistical model. The same information can be obtained by driving the system with an external uncorrelated force. In addition to their theoretical interest, these results provide an alternative technique for measuring the underlying environment fluctuations and consequently checking the validity of a superstatistical approximation.

The paper is outlined as follows. In Sec. II we calculate the FPT statistics for both unforced and forced Fokker-Planck dynamics. In Sec. III we apply the main results to a superstatistical (forced) Brownian particle. Section IV is devoted to the conclusions. Calculus details that support the main results are found in the Appendixes.

II. SUPERSTATISTICAL MODEL AND FIRST-PASSAGE-TIME STATISTICS

The FPT problem is set by a domain $\mathcal{D}$ and its frontier $\partial\mathcal{D}$, which defines the first-passage condition. In an "experimental setup," the domain $\mathcal{D}$ may be fixed in "space" or alternatively, it may be located following the "system position." The system relaxes in a faster time scale. Thus, inside the domain $\mathcal{D}$, and during the measurement of the FPT, we can assume that the environment is characterized by a (random) time-independent "intensive parameter" β, which modifies the system dynamics. We consider a system of arbitrary dimensionality, and assume that its mesoscopic local dynamics (during the measurement time) is described by a probability distribution $P_t(\mathbf{y}|\mathbf{x})$ which obeys a Fokker-Planck equation

$$\frac{\partial}{\partial t}P_t(\mathbf{y}|\mathbf{x}) = \frac{1}{\beta}\mathbb{L}_0 P_t(\mathbf{y}|\mathbf{x}). \quad (1)$$

Here, $\mathbb{L}_0$ is the Fokker-Planck differential operator, where $\mathbf{y}$ labels the relevant system coordinate, while $\mathbf{x} \in \mathcal{D}$ is its value at the initial time $t_0 = 0$.

The previous description is consistent if each measurement of the FPT is performed at separate time intervals larger than the correlation time of the environment fluctuations. Thus, in agreement with the superstatistical approach [29], the value of β in each measurement assumes a different random value. It probability density $p(\beta)$ remains unspecified.

As is well known, for stochastic processes that obey a Fokker-Planck equation, the *mean* FPT obeys a differential equation, named as the Dynkin equation [34]. It is defined by the adjoint Fokker-Planck operator, where the spatial coordinate labels the initial system position [31–33]. Hence, for each "realization" of β, the mean FPT $T^{(1)}(\mathbf{x})$ obeys

$$\beta^{-1}\mathbb{L}_0^{\dagger}T^{(1)}(\mathbf{x}) = -1, \tag{2}$$

where $\mathbb{L}_0^{\dagger}$ is the adjoint Fokker-Planck differential operator. This equation must be solved for $\mathbf{x} \in \mathcal{D}$, where the domain $\mathcal{D}$ and its frontier $\partial\mathcal{D}$ define the first-passage condition. Hence, $T^{(1)}(\partial\mathcal{D}) = 0$. By denoting the average over $p(\beta)$ with $\langle\cdots\rangle$, the previous equation straightforwardly leads to

$$\mathbb{L}_0^{\dagger}\langle T^{(1)}(\mathbf{x})\rangle = -\langle\beta\rangle. \tag{3}$$

Hence, the mean FPT is the same as that in the absence of β fluctuations, but with an effective value $\beta \to \langle\beta\rangle$. Now, we explore if a similar relation is valid for higher moments of the FPT.

In general, higher moments $T^{(n)}(\mathbf{x})$ obey the recurrence relation [31,33]

$$\beta^{-1}\mathbb{L}_0^{\dagger}T^{(n)}(\mathbf{x}) = -nT^{(n-1)}(\mathbf{x}). \tag{4}$$

Given that $T^{(n)}(\mathbf{x})$ and β are correlated, in this case averaging with $p(\beta)$ is not a trivial task. Techniques arising in disordered systems [35,36] may be proposed for performing the average. Nevertheless, a simpler solution is available. First, we notice that Eq. (4) can formally be solved as $(\mathbb{L}_0^{\dagger})^n T^{(n)}(\mathbf{x}) = (-1)^n n!\beta^n$, where the boundary conditions of this differential equation are $(\mathbb{L}_0^{\dagger})^m T^{(n)}(\partial\mathcal{D}) = 0$, $\forall m \leqslant n$. This solution tells us that $T^{(n)}(\mathbf{x})$ only depends on β^n. Hence, averaging over β becomes an easy task:

$$(\mathbb{L}_0^{\dagger})^n\langle T^{(n)}(\mathbf{x})\rangle = (-1)^n n!\langle\beta^n\rangle. \tag{5}$$

This result generalizes Eq. (3). It gives us one of the central results of this paper: as the n moment $\langle T^{(n)}(\mathbf{x})\rangle$ of the FPT *only* depends on the n moment of $\langle\beta^n\rangle$, their measurement allows one to determine the full statistic (moments) of the random intensive parameter β. The dependence on $\mathbf{x}$ only takes into account the geometry of the problem. Notice that the previous equation can be rewritten as

$$\langle T^{(n)}(\mathbf{x})\rangle = T^{(n)}(\mathbf{x})|_{\beta^n\to\langle\beta^n\rangle}. \tag{6}$$

Hence, any moment $\langle T^{(n)}(\mathbf{x})\rangle$ can be obtained from the standard solution $T^{(n)}(\mathbf{x})$ under the replacement $\beta^n \to \langle\beta^n\rangle$. In contrast with Eq. (5), this result, jointly with Eq. (4), provides a simpler technique for calculating $\langle T^{(n)}(\mathbf{x})\rangle$.

External forcing

Now, we consider that an external driving force is applied over the system. This situation is of interest from an experimental point of view. In fact, the external force may delay the exit time, or it may even "localize" the system in the surroundings of the domain $\mathcal{D}$.

We assume that the external force has no correlation with the environment fluctuations. Hence, the Fokker-Planck equation becomes

$$\frac{\partial}{\partial t}P_t(\mathbf{y}|\mathbf{x}) = (\varepsilon\mathbb{L}_f + \beta^{-1}\mathbb{L}_0)P_t(\mathbf{y}|\mathbf{x}). \tag{7}$$

Here, $\mathbb{L}_f$ is the contribution induced by the external force, while the parameter ε measures its strength. For each value of β, the FPT nth moments $T^{(n)}(\mathbf{x})$ obey the equation

$$(\varepsilon\mathbb{L}_f^{\dagger} + \beta^{-1}\mathbb{L}_0^{\dagger})T^{(n)}(\mathbf{x}) = -nT^{(n-1)}(\mathbf{x}). \tag{8}$$

In this case, averaging over β is also a nontrivial task [36]. Nevertheless, as in the previous case, Eq. (8) can formally be solved as

$$(\varepsilon\mathbb{L}_f^{\dagger} + \beta^{-1}\mathbb{L}_0^{\dagger})^n T^{(n)}(\mathbf{x}) = (-1)^n n!. \tag{9}$$

For solving this equation we propose a series solution in the strength force, $T^{(n)}(\mathbf{x}) = \sum_{j=0}^{\infty}\varepsilon^j\beta^{n+j}\tau_j(\mathbf{x})$, where the set of functions $\{\tau_j(\mathbf{x})\}_{j=0}^{\infty}$ (*defined for each n*) satisfy the same boundary conditions as $T^{(n)}(\mathbf{x})$. This series expansion implies that the set of functions $\{\tau_j(\mathbf{x})\}_{j=0}^{\infty}$ does not depend on β (Appendix A). Thus, we arrive at the second main result:

$$\langle T^{(n)}(\mathbf{x})\rangle = \sum_{j=0}^{\infty}\varepsilon^j\langle\beta^{n+j}\rangle\tau_j(\mathbf{x}). \tag{10}$$

This expression demonstrates that all moments $\langle\beta^n\rangle$ can alternatively be determined by studying the dependence of $\langle T^{(n)}(\mathbf{x})\rangle$ on the force strength ε. On the other hand, notice that the functions $\{\tau_j(\mathbf{x})\}_{j=0}^{\infty}$ are the same as those that arise for a deterministic value of β. They only depend on the geometry and symmetries of the problem.

From Eq. (10), for the mean FPT we get

$$\langle T^{(1)}\rangle(\mathbf{x}) = \langle\beta\rangle\tau_0(\mathbf{x}) + \varepsilon\langle\beta^2\rangle\tau_1(\mathbf{x}) + \varepsilon^2\langle\beta^3\rangle\tau_2(\mathbf{x}) + \cdots, \tag{11}$$

where the functions $\{\tau_j(\mathbf{x})\}_{j=0}^{\infty}$ satisfy (Appendix A)

$$\mathbb{L}_0^{\dagger}\tau_0(\mathbf{x}) = -1, \quad \mathbb{L}_0^{\dagger}\tau_j(\mathbf{x}) = -\mathbb{L}_f^{\dagger}\tau_{j-1}(\mathbf{x}), \tag{12}$$

which in fact do not depend on β.

III. SYMMETRICALLY FORCED BROWNIAN MOTION

The previous results are valid for models defined with arbitrary Fokker-Planck operators $\mathbb{L}_0$ and $\mathbb{L}_f$. In order to exemplify how they apply, we consider a diffusive particle whose *velocity* relaxes in a faster time scale when compared with the environment fluctuations. Thus, in the last time scale, its *position* performs a Brownian motion [31–33]. In addition, we consider that an external uncorrelated *constant force*, $\mathbf{F}(\mathbf{x}) = \varepsilon\mathbf{n}$, is applied over it. The (unit) vector $\mathbf{n}$ gives its direction in space. In consequence, the probability density $P_t(\mathbf{y}|\mathbf{x})$ for its position $\mathbf{y}$ obeys the (Smoluchowski) Fokker-Planck equation

$$\frac{\partial}{\partial t}P_t(\mathbf{y}|\mathbf{x}) = (\varepsilon\nabla\cdot\mathbf{n} + \beta^{-1}\nabla^2)P_t(\mathbf{y}|\mathbf{x}), \tag{13}$$

where $\mathbb{L}_f = \nabla$ and $\mathbb{L}_0 = \nabla^2$ are the gradient and Laplacian operators in dimension $d = 1,2,3$. Notice that for this

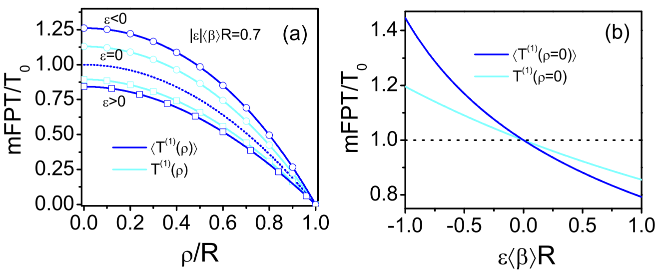

FIG. 1. Mean first-passage time (mFPT) $\langle T^{(1)}(\rho)\rangle$ in the presence of a constant uncorrelated radial force in 3D [Eq. (15)] and gamma distributed β fluctuations [Eq. (17)]. $T^{(1)}(\rho)$ is the solution in the absence of fluctuations, $p(\beta)=\delta(\beta-\langle\beta\rangle)$. Time is measured in units $T_0=\langle\beta\rangle R^2/6$. (a) Dependence on the initial position ρ, for negative (circles), null (dotted line), and positive (squares) forces. The parameters fulfill $|\varepsilon|\langle\beta\rangle R=0.7$. (b) Dependence with the external dimensionless force $\varepsilon\langle\beta\rangle R$ at the initial position $\rho=0$.

dynamics, β^{-1} becomes the *diffusion coefficient*. Therefore, β may be associated with inverse-temperature fluctuations of the environment.

The Dynkin equation for the mean FPT becomes

$$(\varepsilon\mathbf{n}\cdot\nabla+\beta^{-1}\nabla^2)T^{(1)}(\mathbf{x})=-1. \qquad (14)$$

For simplifying and showing the main features of the problem, we consider symmetrical domains and forces that do not break their symmetry. Hence, in $d=2{,}3$ the domain $\mathcal{D}$ is the inner region of a circle and a sphere of radius R, respectively. The force is radial in both cases, which defines the direction $\mathbf{n}$. In $d=1$, for mimicking the higher-dimensional cases, a reflecting boundary is taken at the origin. In all cases, the only relevant (radial) initial coordinate is denoted with ρ. Hence, the frontier $\partial\mathcal{D}$ is taken into account through the absorbing boundary condition $T^{(1)}(\rho=R)=0$.

The solution of Eq. (14) for arbitrary ε and β can be found in an exact way (Appendix B). After performing a series expansion in ε and averaging over β, we get the exact expression

$$\langle T^{(1)}(\rho)\rangle=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{r_d(n)}\varepsilon^{n-1}\langle\beta^n\rangle(R^{n+1}-\rho^{n+1}), \qquad (15)$$

where the function $r_d(n)$, depending on the space dimension ($d=1{,}2{,}3$), reads $r_1(n)=(n+1)n!$, $r_2(n)=(n+1)(n+1)!$, and $r_3(n)=(n+1)(n+2)!/2$, respectively. We notice that Eq. (15) is consistent with the general result (11). In the limit $\varepsilon\ll 1$, it follows that

$$\langle T^{(1)}(\rho)\rangle\simeq\frac{\langle\beta\rangle}{a}(R^2-\rho^2)-\varepsilon\frac{\langle\beta^2\rangle}{b}(R^3-\rho^3), \qquad (16)$$

where the pairs of constants (a,b), depending on the space dimension ($d=1{,}2{,}3$), read (2,6), (4,18), and (6,36), respectively.

In Fig. 1 we plot the mean FPT $\langle T^{(1)}(\rho)\rangle$ [Eq. (15)] for a three-dimensional (3D) Brownian particle, in the case in which the β fluctuations are gamma distributed [2],

$$p(\beta)=\frac{1}{\Gamma(\alpha)}\left(\frac{\alpha}{\langle\beta\rangle}\right)^{\alpha}\beta^{\alpha-1}\exp\left(\frac{-\alpha\beta}{\langle\beta\rangle}\right), \qquad (17)$$

with $\alpha=3/2$. The moments are

$$\langle\beta^n\rangle=2^n\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\left(\frac{\langle\beta\rangle}{2\alpha}\right)^n. \qquad (18)$$

In addition, we plot the solution in absence of fluctuations $p(\beta)=\delta(\beta-\langle\beta\rangle)$, which is denoted as $T^{(1)}(\rho)$. Time is measured in units of $T_0=\langle\beta\rangle R^2/6$, which corresponds to the mean FPT starting at the origin in the undriven 3D case, Eq. (16) with $\rho=0$ and $\varepsilon=0$, respectively.

In Fig. 1(a) we observe that, for any particular initial condition ρ, with respect to the undriven case (dotted line) a decrease (increase) of the mean FPT is observed for outward ($\varepsilon>0$, squares) (inward; $\varepsilon>0$, circles) radial forces. These two behaviors are expected. Nevertheless, we notice that a crossover between $\langle T^{(1)}(\rho)\rangle$ and $T^{(1)}(\rho)$ occurs when the force changes sign. In fact, $\langle T^{(1)}(\rho)\rangle\gtrless T^{(1)}(\rho)$ for $\varepsilon\lessgtr 0$. This effect is independent of the specific value of ε, a property confirmed in Fig. 1(b). In this case, we plot the same objects as a function of the dimensionless force $\varepsilon\langle\beta\rangle R$ at the initial condition $\rho=0$.

The crossover induced by the environment fluctuations (Fig. 1) can be quantified from Eq. (16), which gives

$$\langle T^{(1)}(\rho)\rangle-T^{(1)}(\rho)\simeq-\varepsilon[\langle\beta^2\rangle-\langle\beta\rangle^2]\tau_1(\rho), \qquad (19)$$

where $\tau_1(\rho)=(R^3-\rho^3)/36$. Thus, the crossover around small forces is governed by the cumulant $\langle\beta^2\rangle-\langle\beta\rangle^2$. This result gives us a procedure for checking the environment fluctuations. From the undriven dynamics it is possible to determine $\langle\beta\rangle$. By subjecting the system to a small constant force, the mean FPT allows one to measure the environment quadratic fluctuations. In fact, in the forced case, $\langle T^{(1)}(\rho)\rangle$ cannot be fit with the single parameter $\langle\beta\rangle$. From Eq. (15) we deduce that higher n orders in ε are weighted by $[\langle\beta^n\rangle-\langle\beta\rangle^n]$. Thus, the crossover gives us direct information about all centered moments of the bath fluctuations.

We remark that the previous results do not depend on the specific studied model. In fact, Eq. (19) remains valid in general. The only change is the term $\tau_1(\rho)$ which takes into account the force and geometry of the problem. For example, the same behaviors such as those shown in Fig. 1 also arise for radial harmonic forces $\mathbf{F}(\mathbf{x})=\varepsilon\mathbf{x}$ and Coulomb-like ones $\mathbf{F}(\mathbf{x})=\varepsilon f(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$, where $f(|\mathbf{x}|)=1/|\mathbf{x}|^{d-1}$, and $d=2{,}3$ is the space dimensionality (Appendix B).

Forces that depend on position, as do the previous ones, may be implemented at a fixed position in space. Nevertheless, one may also be interested in tracking the particle position while it diffuses in different spatial regions of the environment. Thus, the successive domains $\mathcal{D}$ should be chosen around the (time-dependent) particle position. In this case, a homogeneous force along space may be the more appropriate one, while $\mathcal{D}$ may be chosen as a "square" surrounding the particle in each FPT measurement. This case can be analytically characterized, and also confirms the proposed approach (Appendix C). In fact, the previous results and conclusions remain valid independently of the nature of the experimental setup chosen for measuring the FPT.

We based our analysis on Eq. (13), which applies when the environment induces *diffusion coefficient* (temperature) fluctuations. Alternatively, the environment may induce fluctuations in the *dissipative coefficient* of the velocity dynamics. For the particle position it becomes a global multiplicative constant [31–33]. After a redefinition of the parameters, its description can be recovered from Eq. (13) under the replacement $\varepsilon \to \varepsilon\beta^{-1}$. Consequently, the Dynkin equation (14) assumes the structure given by Eq. (2). The mean FPT can therefore be written as

$$\langle T^{(1)}(\rho)\rangle = \langle\beta\rangle \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{r_d(n)} \varepsilon^{n-1}(R^{n+1} - \rho^{n+1}). \quad (20)$$

Comparing this expression with Eq. (15), we realize that the measurement of the mean FPT in the forced case allows one, in addition, to discriminate between both situations, that is, diffusion versus dissipative coefficient fluctuations. This property is also valid for arbitrary external forces.

IV. SUMMARY AND CONCLUSIONS

We have characterized the FPT problem for superstatistical models where the mesoscopic local system dynamics is described by a Fokker-Planck operator. In the first case, the (slow) environmental fluctuations are taken into account by an intensive parameter that affects the total system dynamics. It was shown that all moments of the random intensive parameter can be put in one-to-one correspondence with the moments of the FPT. In addition, in the second case, it was shown that the same statistical information can be obtained by subjecting the system to an external force that is independent of the environment fluctuations. The approach was exemplified with the paradigmatic case of a superstatistical (forced) Brownian particle. In the independent force case, depending on its action, a crossover between the mean FPT in the presence and absence of environmental fluctuations is observed. Furthermore, we showed that external driving forces allow one to discriminate between diffusion and dissipative coefficient fluctuations. These properties are a fingerprint of the environment fluctuations which are found whenever a superstatistical approach applies.

The obtained results provide two complementary experimental methods for checking the consistence of the hypothesis (spatiotemporal scales) under which a superstatistical modeling may apply. In fact, the superstatistical approach leads to a precise structure of the FPT problem that goes beyond the possible consistence with stationary non-Gibbsian equilibriumlike distributions. Given the present technological advances in single-particle tracking, as well as the possibility of processing measurement signals of diverse complex systems, the developed results provide a theoretical tool that may be of relevance in different realistic experimental setups.

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APPENDIX A: SOLUTION FOR THE FIRST-PASSAGE *n* MOMENT

Here, we demonstrate that the n moment of the FPT $T^{(n)}(\mathbf{x})$, in the forced case, can be expressed as the series expansion

$$T^{(n)}(\mathbf{x}) = \beta^n \tau_0(\mathbf{x}) + \varepsilon\beta^{n+1}\tau_1(\mathbf{x}) + \varepsilon^2\beta^{n+2}\tau_2(\mathbf{x}) + \cdots, \quad (A1)$$

which lead to Eq. (10). $T^{(n)}(\mathbf{x})$ satisfies the recursive relation

$$(\varepsilon\mathbb{L}_f^\dagger + \beta^{-1}\mathbb{L}_0^\dagger)T^{(n)}(\mathbf{x}) = -nT^{(n-1)}(\mathbf{x}), \quad (A2)$$

which is formally solved by

$$(\varepsilon\mathbb{L}_f^\dagger + \beta^{-1}\mathbb{L}_0^\dagger)^n T^{(n)}(\mathbf{x}) = (-1)^n n!. \quad (A3)$$

The differential operator is rewritten as

$$(\varepsilon\mathbb{L}_f^\dagger + \beta^{-1}\mathbb{L}_0^\dagger)^n = \sum\nolimits_{i=0}^{n} \frac{\varepsilon^i}{\beta^{n-i}}\mathbb{S}_i, \quad (A4)$$

where $\{\mathbb{S}_i\}_{i=0}^n$ are operators that can be written in terms of $\mathbb{L}_0^\dagger$ and $\mathbb{L}_f^\dagger$. Applying the previous expression to Eq. (A1), after solving each order in ε, Eq. (A3) leads to a set of consistent equations for the functions $\{\tau_j(\mathbf{x})\}_{j=0}^{\infty}$. We explicitly get

$$\mathbb{S}_0[\tau_0(\mathbf{x})] = (-1)^n n!, \quad (A5)$$

while for $1 \leqslant i < n$

$$\sum\nolimits_{j=0}^{i} \mathbb{S}_j[\tau_{i-j}(\mathbf{x})] = 0, \quad 1 \leqslant i < n. \quad (A6)$$

For $i \geqslant n$ it follows that

$$\sum\nolimits_{j=0}^{n} \mathbb{S}_j[\tau_{i-j}(\mathbf{x})] = 0, \quad n \leqslant i. \quad (A7)$$

This set of equations does not include the parameter β. Therefore, the series Eq. (A1) gives a consistent solution of Eq. (A3). In fact, after solving $\tau_0(\mathbf{x})$ one can obtain $\tau_1(\mathbf{x})$, and so on. The posterior average over β is straightforward.

For $n = 1$, after taking $\mathbb{S}_0 = \mathbb{L}_0^\dagger$ and $\mathbb{S}_1 = \mathbb{L}_f^\dagger$, the previous relations lead to

$$\mathbb{L}_0^\dagger\tau_0(\mathbf{x}) = -1, \quad \mathbb{L}_0^\dagger\tau_j(\mathbf{x}) = -\mathbb{L}_f^\dagger\tau_{j-1}(\mathbf{x}). \quad (A8)$$

For $n = 2$, the equations are

$$\mathbb{S}_0[\tau_0(\mathbf{x})] = 2, \quad (A9)$$

while for $\tau_1(\mathbf{x})$

$$\mathbb{S}_0[\tau_1(\mathbf{x})] + \mathbb{S}_1[\tau_0(\mathbf{x})] = 0. \quad (A10)$$

For $\tau_j(\mathbf{x})$ with $j \geqslant 2$,

$$\mathbb{S}_0[\tau_j(\mathbf{x})] + \mathbb{S}_1[\tau_{j-1}(\mathbf{x})] + \mathbb{S}_2[\tau_{j-2}(\mathbf{x})] = 0. \quad (A11)$$

The operators are

$$\mathbb{S}_0 = (\mathbb{L}_0^\dagger)^2, \quad \mathbb{S}_1 = \mathbb{L}_0^\dagger\mathbb{L}_f^\dagger + \mathbb{L}_f^\dagger\mathbb{L}_0^\dagger, \quad \mathbb{S}_2 = (\mathbb{L}_f^\dagger)^2. \quad (A12)$$

APPENDIX B: EXACT SOLUTIONS FOR THE MEAN FPT FOR RADIAL FORCES

In the presence of radial forces, $\mathbf{F}(\mathbf{x}) = \varepsilon f(|\mathbf{x}|)\mathbf{x}/|\mathbf{x}|$, the Fokker-Planck equation is (uncorrelated force)

$$\frac{\partial}{\partial t}P_t(\mathbf{y}|\mathbf{x}) = \left(\varepsilon\nabla\cdot\frac{\mathbf{y}}{|\mathbf{y}|}f(|\mathbf{y}|) + \beta^{-1}\nabla^2\right)P_t(\mathbf{y}|\mathbf{x}). \quad (B1)$$

The Dynkin equation hence becomes

$$\left(\varepsilon f(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}\cdot\nabla + \beta^{-1}\nabla^2\right)T^{(1)}(\mathbf{x}) = -1. \quad (B2)$$

This equation can be solved for different space dependences $f(|\mathbf{x}|)$ and configurations of interest. In each case, the parameter ε changes its units.

1. Constant radial forces

For constant radial forces, it follows that $f(|\mathbf{x}|) = 1$. Solutions of the Dynkin equation depend on the space dimensionality and the chosen domain $\mathcal{D}$.

a. Mean FPT in 1D

In the unidimensional case the Brownian particle moves on a line. Its position is denoted by ρ. Equation (B2) becomes

$$\varepsilon \frac{\partial T^{(1)}(\rho)}{\partial \rho} + \beta^{-1} \frac{\partial^2 T^{(1)}(\rho)}{\partial \rho^2} = -1. \tag{B3}$$

We consider the passage problem in the domain $\mathcal{D} = [0, R]$. Hence, $T^{(1)}(R) = 0$. For simplicity, we consider that the origin is a reflecting boundary $\partial T^{(1)}(\rho)/\partial \rho|_{\rho=0} = 0$. In addition, this election mimics the boundary conditions in higher dimensions.

By a direct integration, and after imposing the boundary conditions, the solution of (B3) can be written as

$$T^{(1)}(\rho) = \frac{1}{\varepsilon}(R - \rho) - \frac{1}{\varepsilon} \int_\rho^R \exp[-\varepsilon\beta\rho'] d\rho'. \tag{B4}$$

By performing a series expansion of the integral contribution, terms proportional to $(1/\varepsilon)$ cancel out, while the remaining ones recover Eq. (15).

b. Mean FPT in 2D

In the plane, the domain $\mathcal{D}$ is taken as a circle of radius R. The external force is directed in the radial direction, whose coordinate is denoted by ρ. Given the symmetry of the problem, Eq. (B2) is

$$\varepsilon \frac{\partial T^{(1)}(\rho)}{\partial \rho} + \frac{\beta^{-1}}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial T^{(1)}(\rho)}{\partial \rho}\right) = -1. \tag{B5}$$

This equation can be solved by finding two homogeneous solutions, $Y_1(\rho)$ and $Y_2(\rho)$, and writing a particular one $Y_p(\rho)$ as a function of them [37], which give rise to $T^{(1)}(\rho) = c_1 Y_1(\rho) + c_2 Y_2(\rho) + Y_p(\rho)$. We get $Y_1(\rho) = 1$, $Y_2(\rho) = \int^\rho d\rho' \exp(-\varepsilon\beta\rho')/\rho'$, and $Y_p(\rho) = -\varepsilon^{-1}[\rho - (\varepsilon\beta)^{-1}\ln(\rho)]$. The two indeterminate constants, c_1 and c_2, are chosen for satisfying the boundary condition $T^{(1)}(R) = 0$ and for avoiding a divergence at the origin. The solution then reads

$$T^{(1)}(\rho) = \frac{1}{\varepsilon}(R - \rho) - \frac{1}{\varepsilon^2\beta} \int_\rho^R \frac{1 - \exp[-\varepsilon\beta\rho']}{\rho'} d\rho'. \tag{B6}$$

By performing a series expansion of the integral contribution, Eq. (15) is recovered.

c. Mean FPT in 3D

In the three-dimensional case, the domain $\mathcal{D}$ is taken as a sphere of radius R. Given the symmetry of the problem, the solution only depends on the radial coordinate ρ. The Dynkin differential equation becomes [Eq. (B2)]

$$\varepsilon \frac{\partial T^{(1)}(\rho)}{\partial \rho} + \frac{\beta^{-1}}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T^{(1)}(\rho)}{\partial \rho}\right) = -1, \tag{B7}$$

with boundary condition $T^{(1)}(R) = 0$. The solution can be found as in the two-dimensional case, $T^{(1)}(\rho) = c_1 Y_1(\rho) + c_2 Y_2(\rho) + Y_p(\rho)$, with $Y_1(\rho) = 1$, $Y_2(\rho) = \int^\rho d\rho' \exp(-\varepsilon\beta\rho')/(\rho')^2$, and $Y_p(\rho) = \varepsilon^{-3}\beta^{-2}[2\rho^{-1} + 2(\varepsilon\beta)\ln(\rho) - (\varepsilon\beta)^2\rho]$. The exact solution is

$$T^{(1)}(\rho) = \frac{1}{\varepsilon}(R-\rho) - \frac{2\beta^2}{\varepsilon^3} \int_\rho^R \frac{\exp[-\varepsilon\beta\rho'] - (1 - \varepsilon\beta\rho')}{\rho'^2} d\rho'. \tag{B8}$$

By performing a series expansion of the integral contribution, Eq. (15) is recovered.

2. Harmonic radial forces

Here, we consider harmonic radial forces, $f(|\mathbf{x}|) = |\mathbf{x}|$.

a. Mean FPT in 2D

In this case, the domain $\mathcal{D}$ is also taken as a circle of radius R. Given the symmetry of the problem, Eq. (B2) is

$$\varepsilon\rho \frac{\partial T^{(1)}(\rho)}{\partial \rho} + \frac{\beta^{-1}}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial T^{(1)}(\rho)}{\partial \rho}\right) = -1. \tag{B9}$$

Notice that this expression follows from Eq. (B5) under the replacement $\varepsilon \to \varepsilon\rho$. The boundary condition is $T^{(1)}(R) = 0$.

The homogeneous solutions of Eq. (B9) are $Y_1(\rho) = 1$ and $Y_2(\rho) = \int^\rho d\rho'(1/\rho')\exp(-\varepsilon\beta\rho'^2/2)$. The particular solution is $Y_p(\rho) = -\ln(\rho)/\varepsilon$. After imposing the boundary condition, we get the exact solution

$$T^{(1)}(\rho) = \frac{1}{\varepsilon} \int_\rho^R \frac{1 - \exp[-\varepsilon\beta\rho'^2/2]}{\rho'} d\rho'. \tag{B10}$$

By performing a series expansion of the integral contribution, and after performing the average over β, it follows the series

$$\langle T^{(1)}(\rho)\rangle = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{r_2(n)} \varepsilon^{n-1} \langle\beta^n\rangle (R^{2n} - \rho^{2n}). \tag{B11}$$

Here, $r_2(n) = 2^{n+1} n! n$. In the limit $\varepsilon \ll 1$, we get

$$\langle T^{(1)}(\rho)\rangle \simeq \frac{\langle\beta\rangle}{4}(R^2 - \rho^2) - \varepsilon\frac{\langle\beta^2\rangle}{32}(R^4 - \rho^4). \tag{B12}$$

b. Mean FPT in 3D

The domain $\mathcal{D}$ is a sphere of radius R. Given the symmetry of the problem, Dynkin equation (B2) becomes

$$\varepsilon\rho \frac{\partial T^{(1)}(\rho)}{\partial \rho} + \frac{\beta^{-1}}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T^{(1)}(\rho)}{\partial \rho}\right) = -1, \tag{B13}$$

with boundary condition $T^{(1)}(R) = 0$. This equation can be read from Eq. (B7) under the replacement $\varepsilon \to \varepsilon\rho$.

The homogeneous solutions of Eq. (B13) are $Y_1(\rho) = 1$ and $Y_2(\rho) = \int^\rho d\rho'(1/\rho'^2)\exp(-\varepsilon\beta\rho'^2/2)$. The particular solution is $Y_p(\rho) = -\beta\int^\rho d\rho'[F(\rho')/\rho'^2]\exp(-\varepsilon\beta\rho'^2/2)$, where the function $F(\rho)$ is defined by the relation

$$\frac{d}{d\rho}F(\rho) = \rho^2 \exp\left(\frac{\varepsilon\beta\rho^2}{2}\right). \tag{B14}$$

The solution $Y_2(\rho)$ is divergent at the origin. Thus, the final exact solution of Eq. (B13) can be written as

$$T^{(1)}(\rho) = \beta \int_\rho^R \frac{\exp[-\varepsilon\beta\rho'^2/2]}{\rho'^2} F(\rho') d\rho'. \tag{B15}$$

For performing a series expansion in ε, first we note that $F(\rho)$, from Eq. (B14), can be written as

$$F(\rho) = \sum_{n=0}^{\infty} \frac{\varepsilon^n \beta^n \rho^{2n+3}}{2^n n!(2n+3)}. \tag{B16}$$

By inserting this expression into Eq. (B15), developing in series the exponential factor, and after performing the average over β, we get

$$\langle T^{(1)}(\rho)\rangle = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{r_3(n)} \varepsilon^{n-1} \langle \beta^n \rangle (R^{2n} - \rho^{2n}). \tag{B17}$$

Here, $r_3(n) = (2n)!(1+2n)/[2^{n-1}(n-1)!]$. For $\varepsilon \ll 1$,

$$\langle T^{(1)}(\rho)\rangle \simeq \frac{\langle\beta\rangle}{6}(R^2-\rho^2) - \varepsilon \frac{\langle\beta^2\rangle}{60}(R^4-\rho^4). \tag{B18}$$

In deriving Eq. (B17) we have used the product series result: $\sum_{i=0}^{\infty} a_i x^i \sum_{j=0}^{\infty} b_j x^j = \sum_{k=0}^{\infty} c_k x^k$, with coefficient $c_k = \sum_{s=0}^{k} a_s b_{k-s}$. Furthermore, the addition $\sum_{s=0}^{n} (-1)^s [s!(2s+3)(n-s)]^{-1} = \sqrt{\pi}/[4\Gamma(n+\frac{5}{2})]$, jointly with $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}+n) = \sqrt{\pi}(2n)!/(4^n n!)$, where $\Gamma(x)$ is the gamma function [38].

3. Coulomb-like forces

Here, we consider radial forces whose strengths change in space as Coulomb electrical forces do.

a. Mean FPT in 2D

As an underlying physical model, we consider an electric charge uniformly distributed on a (effectively infinite) cylinder of radius R_a. The charge generates an electrical field (force) which, over the plane perpendicular to the cylinder, is directed in the radial direction. Its amplitude decays as $f(\rho) = 1/\rho$. It is assumed that diffusion in the direction parallel to the cylinder axes can be disregarded. Thus, the problem is a bidimensional one. The domain $\mathcal{D}$ is taken as a circle of radius $R_b > R_a$. Given the symmetry of the problem, Eq. (B2) becomes

$$\frac{\varepsilon}{\rho}\frac{\partial T^{(1)}(\rho)}{\partial\rho} + \frac{\beta^{-1}}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial T^{(1)}(\rho)}{\partial\rho}\right) = -1. \tag{B19}$$

This expression follows from Eq. (B5) under the replacement $\varepsilon \to \varepsilon/\rho$. We consider that the Brownian particle cannot diffuse inside the cylinder. Hence, $(\partial/\partial\rho)T^{(1)}(\rho)|_{\rho=R_a} = 0$. Furthermore, $T^{(1)}(R_b) = 0$.

The homogeneous solutions of Eq. (B19) are $Y_1(\rho) = 1$ and $Y_2(\rho) = \rho^{-\varepsilon\beta}$, while the particular one reads $Y_p(\rho) = -\rho^2/[2(2\beta^{-1}+\varepsilon)]$. After imposing the boundary conditions it follows ($R_a \leqslant \rho \leqslant R_b$),

$$T^{(1)}(\rho) = \frac{(R_b^2-\rho^2)}{2(2\beta^{-1}+\varepsilon)} + \frac{\beta^{-1}R_a^{2+\varepsilon\beta}}{\varepsilon(2\beta^{-1}+\varepsilon)}\left(R_b^{-\varepsilon\beta} - \rho^{-\varepsilon\beta}\right). \tag{B20}$$

In the limit $R_a \to 0$, it reduces to ($R_b \to R$),

$$T^{(1)}(\rho) = \frac{(R^2-\rho^2)}{2(2\beta^{-1}+\varepsilon)}. \tag{B21}$$

This last solution is only valid for $\varepsilon > -\beta/2$, while for $\varepsilon \leqslant -\beta/2$ a divergence is obtained, $T^{(1)}(\rho) = \infty$. This singular property follows because for $\varepsilon \leqslant -\beta/2$ the origin leads to a nonregular boundary condition [32,33].

By performing a series expansion in ε, both Eqs. (B20) and (B21) confirm the series expansion found in the paper. For the former one, we get

$$\langle T^{(1)}(\rho)\rangle \simeq \left[\frac{R_b^2-\rho^2}{4} + \frac{R_a^2 \ln\left(\frac{\rho}{R_b}\right)}{2}\right]\langle\beta\rangle - \varepsilon\langle\beta^2\rangle \times\left\{\frac{R_b^2-\rho^2}{8} + \frac{R_a^2\ln\left(\frac{\rho}{R_b}\right)\left[1+\ln\left(\frac{R_b\rho}{R_a^2}\right)\right]}{4}\right\}, \tag{B22}$$

while for the last one it follows that

$$\langle T^{(1)}(\rho)\rangle \simeq \frac{(R^2-\rho^2)}{4}\left(\langle\beta\rangle - \varepsilon\frac{\langle\beta^2\rangle}{2} + \varepsilon^2\frac{\langle\beta^3\rangle}{4}\cdots\right). \tag{B23}$$

b. Mean FPT in 3D

In this example, the electric charge is uniformly distributed on a sphere of radius R_a. This distribution generates a radial force with intensity $f(\rho) = 1/\rho^2$. The domain $\mathcal{D}$ is taken as a sphere of radius $R_b > R_a$. Given the symmetry of the problem, Eq. (B2) becomes

$$\frac{\varepsilon}{\rho^2}\frac{\partial T^{(1)}(\rho)}{\partial\rho} + \frac{\beta^{-1}}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial T^{(1)}(\rho)}{\partial\rho}\right) = -1, \tag{B24}$$

which can be read from Eq. (B7) under the replacement $\varepsilon \to \varepsilon/\rho^2$. Given that the particle cannot diffuse inside the sphere, it follows that $(\partial/\partial\rho)T^{(1)}(\rho)|_{\rho=R_a} = 0$, while $T^{(1)}(R_b) = 0$.

The solution of Eq. (B24) can be written as

$$T^{(1)}(\rho) = c_1 + c_2\exp(\varepsilon\beta/\rho) + \frac{(\varepsilon\rho - \beta^{-1}\rho^2)}{6\beta^{-2}} + \frac{\varepsilon^2\exp(\varepsilon\beta/\rho)}{6\beta^{-3}}E_i(-\beta/\rho), \tag{B25}$$

where $E_i(z) \equiv -\int_{-z}^{\infty} dt\, \exp[-t]/t$ is the exponential integral function. The undetermined constants c_1 and c_2 can be obtained after imposing the boundary conditions. We notice that in the case $R_a = 0$, a divergence at the origin is avoided by taking $c_2 = 0$. In this limit case, only solutions for $\varepsilon > 0$ are admissible, while $T^{(1)}(\rho) = \infty$ for $\varepsilon < 0$.

A series expansion in the parameter ε of Eq. (B25) leads to

$$\langle T^{(1)}(\rho)\rangle \simeq \left[\frac{R_b^2-\rho^2}{6} - \frac{R_a^3(R_b-\rho)}{3R_b\rho}\right]\langle\beta\rangle - \varepsilon\langle\beta^2\rangle \times\left\{\frac{(R_b-\rho)\left[R_b^2\rho^2 + R_a^3(R_b+\rho) - 3R_a^2R_b\rho\right]}{6R_b^2\rho^2}\right\}. \tag{B26}$$

For $R_a = 0$, $R_b \to R$, the previous result reduces to

$$\langle T^{(1)}(\rho)\rangle \simeq \langle\beta\rangle\frac{(R^2-\rho^2)}{6} - \varepsilon\frac{\langle\beta^2\rangle}{6}(R-\rho), \tag{B27}$$

which is valid for $\varepsilon > 0$.

APPENDIX C: EXACT SOLUTIONS FOR THE MEAN FPT FOR SPATIALLY HOMOGENEOUS FORCES

In this example the force is spatially homogeneous. We consider a 2D problem, where the coordinates are (x,y). The domain $\mathcal{D}$ is taken as a square of length L, one of its vertices being located at the origin of coordinates. The force is directed in the x direction. Hence, the Dynkin equation becomes

$$\left[\beta^{-1}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\varepsilon\frac{\partial}{\partial x}\right]T^{(1)}(x,y)=-1, \tag{C1}$$

with $T^{(1)}(0,y)=T^{(1)}(L,y)=T^{(1)}(x,0)=T^{(1)}(x,L)=0$. This equation can be solved by proposing the solution

$$T^{(1)}(x,y)=\sum_{m=1}^{\infty}a_m\sin(k_m y)f_m(x), \tag{C2}$$

where $k_m=m\pi/L$, while the coefficients $\{a_m\}$ are defined by the relation

$$\sum_{m=1}^{\infty}a_m\sin(k_m y)=1,\quad y\in(0,L), \tag{C3}$$

leading to $a_m=4/(m\pi)$ for odd m, while $a_m=0$ for even m. Inserting Eq. (C2) into Dynkin equation (C1) it follows that

$$\left[\beta^{-1}\left(\frac{\partial^2}{\partial x^2}-k_m^2\right)+\varepsilon\frac{\partial}{\partial x}\right]f_m(x)=-1. \tag{C4}$$

The solution of this differential equation is

$$f_m(x)=\frac{\beta}{k_m^2}+b_+e^{w_+x}+b_-e^{w_-x}, \tag{C5}$$

where $w_\pm$ are the solutions of $[\beta^{-1}(w^2-k_m^2)+\varepsilon w]=0$. The coefficients $b_\pm$ are chosen such that the boundary conditions $f_m(0)=f_m(L)=0$ are fulfilled. Extension of the solution (C2) to 3D is straightforward from the previous calculation steps, $k_m^2\to k_m^2+k_n^2$, where $k_n=n\pi/L$ introduces the discretization in the z direction.

For small forces, after performing the average over β, from Eq. (C2) we get

$$\langle T^{(1)}\rangle(x,y)\simeq\langle\beta\rangle\tau_0(x,y)+\varepsilon\langle\beta^2\rangle\tau_1(x,y)+\varepsilon^2\langle\beta^3\rangle\tau_2(x,y), \tag{C6}$$

where the auxiliary functions can be written as

$$\tau_i(x,y)=\sum_{m=1}^{\infty}\sin(k_m y)g_m^{(i)}(x). \tag{C7}$$

The addition is over odd m, while

$$g_m^{(0)}(x)=\frac{4L^2}{(m\pi)^3}\left\{1-\frac{\cosh\left[\frac{m\pi}{2}\left(1-2\frac{x}{L}\right)\right]}{\cosh\left[\frac{m\pi}{2}\right]}\right\}, \tag{C8}$$

jointly with

$$g_m^{(1)}(x)=\frac{L^2}{(m\pi)^3}[\coth(m\pi)-1]e^{-m\pi x/L}\big[(L-x)\times(e^{m\pi}-e^{m\pi(L+2x)/L})+x(e^{2m\pi}-e^{2m\pi x/L})\big]. \tag{C9}$$

This function vanishes at $x=L/2$, $g_m^{(1)}(L/2)=0$. This fact follows from the symmetries of the problem. Only for this single segment, $x=L/2$, $0<y<L$, the first correction to Eq. (C6) is of order ε^2. The geometrical factor is

$$g_m^{(2)}(L/2)=L^4\frac{\left[2\tanh\left(\frac{m\pi}{2}\right)-m\pi\right]}{8m^4\pi^4\cosh\left(\frac{m\pi}{2}\right)}. \tag{C10}$$

[1] C. Beck and E. G. D. Cohen, Superstatistics, Physica A **322**, 267 (2003); C. Beck, Superstatistics: Theoretical concepts and physical applications, arXiv:0705.3832.

[2] C. Beck, E. G. D. Cohen, and H. L. Swinney, From time series to superstatistics, Phys. Rev. E **72**, 056133 (2005).

[3] H. Touchette and C. Beck, Asymptotics of superstatistics, Phys. Rev. E **71**, 016131 (2005).

[4] C. Beck, Dynamical Foundations of Nonextensive Statistical Mechanics, Phys. Rev. Lett. **87**, 180601 (2001); A. M. Reynolds, Superstatistical Mechanics of Tracer-Particle Motions in Turbulence, *ibid.* **91**, 084503 (2003).

[5] C. Beck, Statistics of Three-Dimensional Lagrangian Turbulence, Phys. Rev. Lett. **98**, 064502 (2007); E. Van der Straeten and C. Beck, Superstatistical fluctuations in time series: Applications to share-price dynamics and turbulence, Phys. Rev. E **80**, 036108 (2009).

[6] M. Mehrafarin, Superstatistics as the statistics of quasiequilibrium states: Application to fully developed turbulence, Phys. Rev. E **84**, 022102 (2011).

[7] I. Rouse and S. Willitsch, Superstatistical Energy Distributions of an Ion in an Ultracold Buffer Gas, Phys. Rev. Lett. **118**, 143401 (2017).

[8] K. Ourabah and M. Tribeche, Quantum entanglement and temperature fluctuations, Phys. Rev. E **95**, 042111 (2017).

[9] M. I. Bogachev, O. A. Markelov, A. R. Kayumov, and A. Bunde, Superstatistical model of bacterial DNA architecture, Sci. Rep. **7**, 43034 (2017).

[10] C. Metzner, C. Mark, J. Steinwachs, L. Lautscham, F. Stadler, and B. Fabry, Superstatistical analysis and modeling of heterogeneous random walks, Nat. Commun. **6**, 7516 (2015).

[11] K. Ourabah, L. A. Gougam, and M. Tribeche, Nonthermal and suprathermal distributions as a consequence of superstatistics, Phys. Rev. E **91**, 012133 (2015).

[12] V. García-Morales and K. Krischer, Superstatistics in nanoscale electrochemical systems, Proc. Natl. Acad. Sci. USA **108**, 19535 (2011).

[13] C. Beck and E. G. D. Cohen, Superstatistical generalization of the work fluctuation theorem, Physica A **344**, 393 (2004); A. A. Budini, Generalized fluctuation relation for power-law distributions, Phys. Rev. E **86**, 011109 (2012).

[14] K. Kiyono and H. Konno, Log-amplitude statistics for Beck-Cohen superstatistics, Phys. Rev. E **87**, 052104 (2013).

[15] D. N. Sob'yanin, Hierarchical maximum entropy principle for generalized superstatistical systems and Bose-Einstein condensation of light, Phys. Rev. E **85**, 061120 (2012); Generalization of the Beck-Cohen superstatistics, **84**, 051128 (2011).

[16] P. Jizba and H. Kleinert, Superstatistics approach to path integral for a relativistic particle, Phys. Rev. D **82**, 085016 (2010).

[17] A. Y. Abul-Magd, Modeling highway-traffic headway distributions using superstatistics, Phys. Rev. E **76**, 057101 (2007); A. Y. Abul-Magd, B. Dietz, T. Friedrich, and A. Richter, Spectral fluctuations of billiards with mixed dynamics: From time series to superstatistics, *ibid.* **77**, 046202 (2008).

[18] A. Y. Abul-Magd, Random matrix theory within superstatistics, Phys. Rev. E **72**, 066114 (2005); Superstatistical random-matrix-theory approach to transition intensities in mixed systems, **73**, 056119 (2006).

[19] C. Vignat and A. Plastino, Power-law random walks, Phys. Rev. E **74**, 051124 (2006); A. A. Budini, Extended q-Gaussian and q-exponential distributions from gamma random variables, *ibid.* **91**, 052113 (2015).

[20] C. Beck, Superstatistical Brownian motion, Prog. Theor. Phys. Suppl. **162**, 29 (2006).

[21] C. Beck, Application of generalized thermostatistics to fully developed turbulence, Physica A **277**, 115 (2000); Stretched exponentials from superstatistics, **365**, 96 (2006); K. Briggsa and C. Beck, Modelling train delays with q-exponential functions, *ibid.* **378**, 498 (2007); L. L. Chena and C. Beck, A superstatistical model of metastasis and cancer survival, *ibid.* **387**, 3162 (2008); G. C. Yalcin and C. Beck, Environmental superstatistics, *ibid.* **392**, 5431 (2013); P. Rabassa and C. Beck, Superstatistical analysis of sea-level fluctuations, *ibid.* **417**, 18 (2015); D. Xu and C. Beck, Transition from lognormal to χ^2-superstatistics for financial time series, *ibid.* **453**, 173 (2016); C. Kosun and S. Ozdemir, A superstatistical model of vehicular flow, *ibid.* **444**, 466 (2016); G. C. Yalcin, P. Rabassa, and C. Beck, Extreme event statistics of daily rainfall: dynamical systems approach, J. Phys. A **49**, 154001 (2016); I. Lubashevsky, R. Friedrich, A. Heuer, and A. Ushakovd, Generalized superstatistics of nonequilibrium Markovian systems, Physica A **388**, 4535 (2009).

[22] E. Van der Straeten and C. Beck, Superstatistical distributions from a maximum entropy principle, Phys. Rev. E **78**, 051101 (2008).

[23] S. Abe, C. Beck, and E. G. D. Cohen, Superstatistics, thermodynamics, and fluctuations, Phys. Rev. E **76**, 031102 (2007); S. Abe, Fluctuations of entropy and log-normal superstatistics, *ibid.* **82**, 011131 (2010).

[24] C. Tsallis and A. M. C. Souza, Constructing a statistical mechanics for Beck-Cohen superstatistics, Phys. Rev. E **67**, 026106 (2003).

[25] C. Beck, Generalized statistical mechanics for superstatistical systems, Philos. Trans. R. Soc. London A **369**, 453 (2011).

[26] S. Abe, Fokker-Planck theory of nonequilibrium systems governed by hierarchical dynamics, Found. Phys. **44**, 175 (2014).

[27] D. S. P. Salazar and G. L. Vasconcelos, Multicanonical distribution: Statistical equilibrium of multiscale systems, Phys. Rev. E **86**, 050103(R) (2012).

[28] A. M. S. Macedo, I. R. Roa González, D. S. P. Salazar, and G. L. Vasconcelos, Universality classes of fluctuation dynamics in hierarchical complex systems, Phys. Rev. E **95**, 032315 (2017).

[29] A. V. Checkin, F. Seno, R. Metzler, and I. M. Sokolov, Brownian yet Non-Gaussian Diffusion: From Superstatistics to Subordination of Diffusing Diffusivities, Phys. Rev. X **7**, 021002 (2017).

[30] *First-Passage Phenomena and their Applications*, edited by S. Redner, R. Metzler, and G. Oshanin (World Scientific, Singapore, 2014).

[31] C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1985).

[32] N. G. van Kampen, *Stochastic Process in Physics and Chemistry*, 2nd ed. (North-Holland, Amsterdam, 1992).

[33] M. O. Cáceres, *Non-equilibrium Statistical Physics with Application to Disordered Systems* (Springer, New York, 2017).

[34] E. B. Dynkin and A. A. Juschkewitz, *Satze und Aufgaben uber Markoffsche Prozesse* (Springer, Berlin, 1969).

[35] E. Hernández-García and M. O. Cáceres, First-passage-time statistics in disordered media, Phys. Rev. A **42**, 4503 (1990); A. Valle, M. A. Rodriguez, and L. Pesquera, Diffusion in a continuous medium with space-correlated disorder, *ibid.* **43**, 948 (1991).

[36] P. A. Pury, M. O. Cáceres, and E. Hernández, First-passage time and the fluctuation of the quenched disorder in biased media, Phys. Rev. E **49**, R967 (1994); P. A. Pury and M. O. Cáceres, Survival and residence times in disordered chains with bias, *ibid.* **66**, 021112 (2002); Mean first-passage and residence times of random walks on asymmetric disordered chains, J. Phys. A **36**, 2695 (2003).

[37] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).

[38] K. Oldham, J. Myland, and J. Spanier, *An Atlas of Functions* (Springer, New York, 2009).

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