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Controlling percolation with limited resources

Malte Schröder,[1,*] Nuno A. M. Araújo,[2,3,†] Didier Sornette,[4,‡] and Jan Nagler[5,6,§]

[1]*Network Dynamics, Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany*

[2]*Departamento de Física, Faculdade de Ciências, Universidade de Lisboa, P-1749-016 Lisboa, Portugal*

[3]*Centro de Física Teórica e Computacional, Universidade de Lisboa, 1749-016 Lisboa, Portugal*

[4]*Department of Management, Technology and Economics (D-MTEC), ETH Zürich, Scheuchzerstrasse 7, CH-8092 Zurich, Switzerland*

[5]*Computational Physics for Engineering Materials, Institute for Building Materials, ETH Zürich, Wolfgang-Pauli-Strasse 27, HIT, CH-8093 Zurich, Switzerland*

[6]*Computational Social Science, Department of Humanities, Social and Political Sciences, ETH Zürich, Clausiusstrasse 50, CH-8092 Zurich, Switzerland*

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Connectivity, or the lack thereof, is crucial for the function of many man-made systems, from financial and economic networks over epidemic spreading in social networks to technical infrastructure. Often, connections are deliberately established or removed to induce, maintain, or destroy global connectivity. Thus, there has been a great interest in understanding how to control percolation, the transition to large-scale connectivity. Previous work, however, studied control strategies assuming unlimited resources. Here, we depart from this unrealistic assumption and consider the effect of limited resources on the effectiveness of control. We show that, even for scarce resources, percolation can be controlled with an efficient intervention strategy. We derive such an efficient strategy and study its implications, revealing a discontinuous transition as an unintended side effect of optimal control.

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We are living in a globalized world. Large-scale connectivity, in particular, is essential for the proper functioning of many socioeconomic and technical systems. Examples include technical networks like the internet [1–3] or the world aviation network [4] and a wide range of socioeconomic and financial systems [5–7]. In other cases connectivity may be a liability, allowing the spreading of diseases and other contagion processes [8–10]. Ideally, control of connectivity has the goal to prevent widespread failure, for example, by immunizing a subset of the population to prevent an epidemic. Identifying efficient strategies that use minimal resources is an ongoing problem [11–13]. In many cases, however, one cannot completely prevent an undesirable transition, such as a recession or financial crisis, and tries to delay it as long as possible, often resulting in more severe consequences when the transition inevitably occurs [6,14,15]. Thus, it is essential to understand how to control and delay the emergence of connectivity under the constraint of limited resources and what such unintended consequences may be.

Percolation theory describes the emergence or breakdown of global connectivity depending on the structure of the underlying network with stochastic link addition processes [16–20]. A large body of work has studied the impact of an unlimited number of small interventions in modified models of network growth with the goal to delay the percolation transition. Most of these processes are based on a specific link addition rule. Typically, two (or more) possible candidate links are evaluated at each step and the link is added that delays (or enhances) the percolation transition the most [21]. This "competitive" percolation [22] leads to an extremely sudden, but still continuous transition, sometimes referred to as "explosive" [22–24]. Other models introduce explicit control over the largest cluster, which further delays the transition and can result in a genuine discontinuous percolation transition [25–28]. Many more models with similar motivation have been studied, leading to a surprising diversity of phenomena [18,21–24,29–37].

In all these examples control is inherent to the link addition process, implicitly assuming unlimited resources and allowing indefinite control. Control in realistic settings, however, will be restricted by limited resources. Here, we derive an efficient resource limited control strategy to delay percolation and discuss the consequences for the resulting percolation transition. In particular, while the delayed transition remains smooth for suboptimal interventions, optimizing the control parameters to maximize the delay results in a discontinuous transition.

I. RESULTS

A. Model

We develop our framework to efficiently delay the percolation transition based on the prototypical model of classical network formation, percolation of a random graph: new links e_{ij} between nodes i and j are chosen uniformly at random and sequentially added to a set of N initially unconnected nodes [38]. We implement control of link addition by preventing the chosen link from being added (see Fig. 1). This control is costly and preventing a link incurs a cost $c[S(i),S(j)]$, where $S(i)$ and $S(j)$ are the sizes of the respective connected components (clusters) that include the nodes i and j. Once a total budget B is spent, we can no longer control the link addition process. We track the evolution of the relative size of the largest connected component S_1/N as a function of the link density $p = L/N$, where L is the number of links added to the network. For the results presented here, the cost of an intervention is kept constant $c[S(i),S(j)] = 1$ and we assume a budget that scales

*malte.schroeder@ds.mpg.de

†nmaraujo@fc.ul.pt

‡dsornette@ethz.ch

§jnagler@ethz.ch

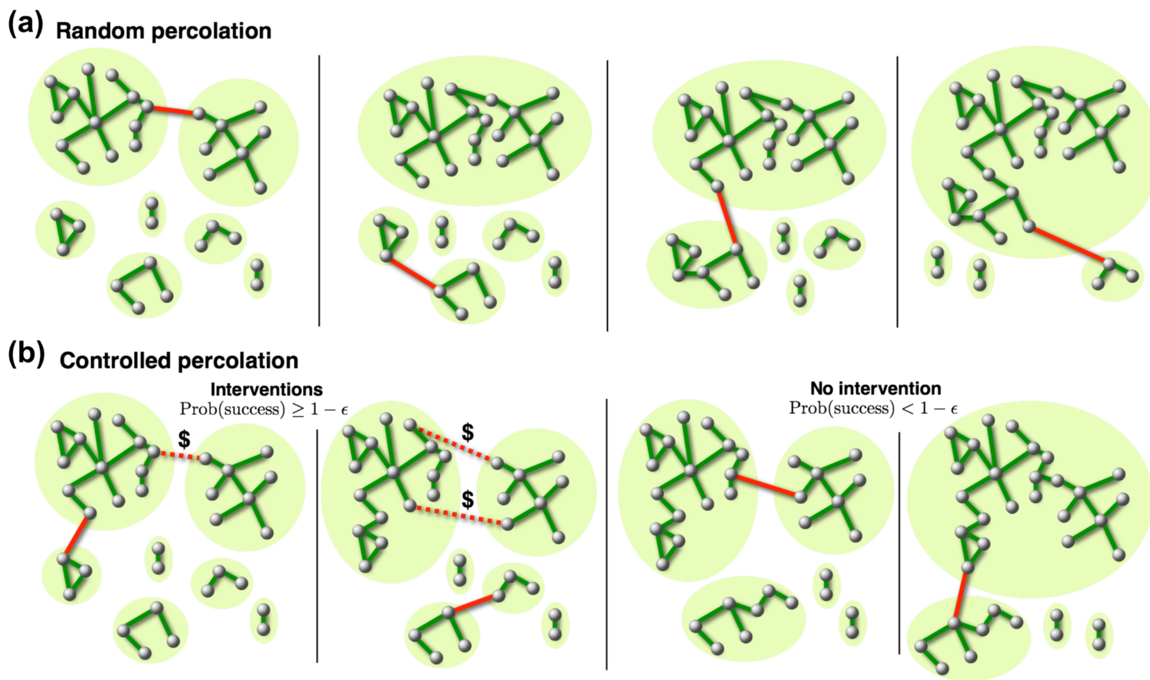

FIG. 1. Controlling the percolation transition. (a) Random percolation: in each step a link is selected uniformly at random and added to the network. (b) Controlled percolation: In each step, we can prevent the chosen link from being added to the network, paying a cost $c[S(i),S(j)]$ from a limited budget B. The constraint of a limited budget requires an efficient control strategy. As described in the text, we only prevent links when the probability that the intervention is successful is sufficiently large, Prob(success) $\geqslant 1-\epsilon$. We consider an intervention unsuccessful if a similarly large cluster is likely to appear again with the next link e_{kl}. Consequently, we intervene when the probability of such a failure Prob(failure) $\approx$Prob$[S(k)+S(l)\geqslant S_{ij}]<\epsilon$ is small (the expected time until a similarly large cluster appears is large). When this failure probability is too large or the budget is exhausted, we do not intervene. As illustrated, this control delays the creation of large clusters and the onset of percolation.

linearly with the number of nodes, $B = bN$, where b is a (finite) constant. Corresponding results are obtained for other cost functions that scale with the size of the clusters, such as $c[S(i),S(j)] = S(i) + S(j)$ (see Supplemental Material [39]). In this case, avoiding the transition completely would clearly require preventing most of the links, which is impossible with limited resources.

In order to efficiently utilize the available resources and decide which links to prevent, we derive a control protocol based on the effect of a single intervention. Consider preventing a link e_{ij} that, when added to the network, would create a cluster of size $S_{ij} = S(i) + S(j)$. If the next link e_{kl} creates a cluster of size $S_{kl} = S(k) + S(l) \geqslant S_{ij}$, we spent some of our budget in vain, since we did not delay the emergence of a large cluster. Conversely, we can consider the intervention effective, when the next links e_{kl} only create smaller clusters $S_{kl} < S_{ij}$ and the emergence of a large cluster was delayed. Based on this idea we propose a control protocol where we prevent a link e_{ij} only if the expected impact is sufficiently large. We measure this impact by the (expected) number of links $\Delta L_{S_{ij}}$ until a cluster of size at least S_{ij} appears again. Clearly, if $\Delta L_{S_{ij}}$ is large, the intervention is more likely to delay the growth of a large cluster. If this delay is larger than some threshold ΔL_{thres}, we consider the intervention effective and prevent the link, otherwise we do not intervene. In practice, we estimate the expected $\Delta L_{S_{ij}}$ from the current cluster-size distribution n_S as the inverse of the probability that a new link e_{kl} creates a cluster $S_{kl} \geqslant S_{ij}$,

$$\frac{1}{\langle \Delta L_{S_{ij}} \rangle} \approx \text{Prob}[S_{kl} = S(k) + S(l) \geqslant S_{ij}]$$
$$= \sum_{\substack{S(l)\neq S(k) \\ S(k)+S(l)\geqslant S_{ij}}} \frac{S(k)n_{S(k)}}{N}\frac{S(l)n_{S(l)}}{N-1}$$
$$+ \sum_{2S(k)\geqslant S_{ij}} \frac{S(k)n_{S(k)}}{N}\frac{S(k)(n_{S(k)}-1)}{N-1}, \qquad (1)$$

where the first sum describes the probability of a merger of clusters of different size resulting in a cluster at least as large as S_{ij} and the second sum describes similar mergers between clusters with equal size. For simplicity, we ignore that a link already present cannot be added again. Hence, we prevent a link from being added if Prob$[S(k) + S(l) \geqslant S_{ij}] < 1/\Delta L_{\text{thres}} := \epsilon$, where ϵ denotes the "intervention intensity," which is the expected link rejection rate. This protocol is equivalent to stopping the ϵ-fraction most extreme events during the percolation process given sufficient budget. Other control strategies based, for example, on constraining the variance of the cluster size distribution are less efficient but give qualitatively similar results (see Supplemental Material [39]).

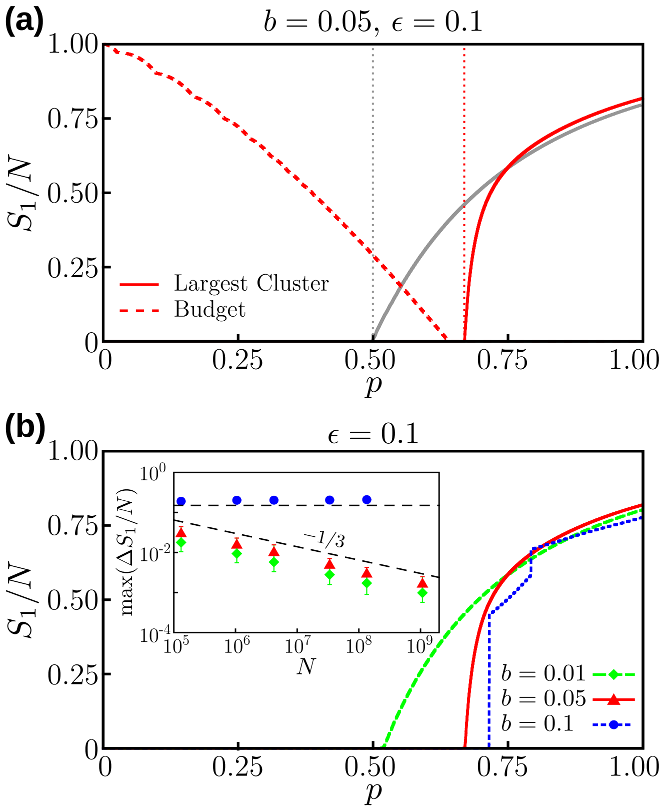

FIG. 2. Effects of resource limited control of percolation. (a) Single realization of the evolution of the relative size of the largest cluster for $N = 2^{25}$ (red solid line) and remaining fraction of the budget (red dashed line) for budget parameter $b = 0.05$ and intervention intensity $\epsilon = 0.1$. Compared to zero budget, the percolation threshold is shifted from $p_c = 0.5$ (gray line, showing random percolation without control) to $p_c \approx 0.67$. Interestingly, the transition remains continuous and in the same universality class. (b) Single realizations of the evolution of the relative size of the largest cluster for $N = 2^{25}$, $\epsilon = 0.1$, and different values of b. Surprisingly, when b becomes large enough, the transition becomes discontinuous. Inset: the largest gap $\max(\Delta S_1/N)$, averaged over 2^{10} to 2^6 realizations; error bars indicate the standard deviation. For small b, the scaling is the same as expected for random percolation, $\max(\Delta S_1/N) \sim N^{-1/3}$. However, for a sufficiently large budget, the largest gap is independent of the network size and the transition is discontinuous.

B. Efficient control of percolation

How much and how efficiently can the percolation transition be delayed with limited resources? As shown in Fig. 2, even with a small budget $B = bN = 0.05N$, meaning less than one intervention in ten link additions until $p_c \geqslant 1/2$, we can significantly delay the percolation transition compared to random percolation. Compared to the sudden transitions in the models of explosive percolation [21–24,31,37], our control protocol is more effective in delaying the transition (see Appendix A). Interestingly, the transition remains smooth and still belongs to the same universality class as random percolation when the budget is exhausted before the transition (see Table I and Appendix B for results of a finite-size scaling analysis).

Note that in Fig. 2(a) the budget runs out at $p =: p_{\rm last} < p_c$, before the percolation threshold p_c, and the transition itself is uncontrolled. We can estimate how long the budget lasts: With a constant intervention rate ϵ we would expect $\Delta L_{\rm int} = \epsilon \Delta L_{\rm total}$ interventions to occur during the sampling of $\Delta L_{\rm total}$ links. During this period, we add only $N\Delta p = \Delta L = (1-\epsilon)\Delta L_{\rm total}$ links. Taking $\Delta L_{\rm int} = \Delta B = N\Delta b$, we find the budget used in this interval $\Delta b = \frac{\epsilon}{1-\epsilon}\Delta p$.

However, the budget decays nonlinearly, as seen in Fig. 2(a), which means the true intervention rate also varies with p. This nonlinear dependency results from the behavior of the intervention rate oscillating around an effective linear increase $\epsilon_{\rm eff}(p) = \epsilon(1 + p/p_c^{\rm max})/2$ for $p \leqslant p_c$, where $p_c^{\rm max}$ is the position of the critical point of controlled percolation with intervention intensity ϵ and infinite budget (see Appendix C for details). This observation, together with integration over p, then yields the closed expression defining $p_{\rm last}$

$$b = \int_0^{p_{\rm last}} \frac{\epsilon_{\rm eff}(p)}{1-\epsilon_{\rm eff}(p)} dp. \qquad (2)$$

As expected, a larger (effective) intervention rate requires a larger budget. Consequently, for a small budget, (i) the budget runs out before the onset of percolation at $p_{\rm last} < p_c$ (interventions stop), (ii) the process is uncontrolled in a short but extensive window prior to the transition point, and (iii) one observes a continuous transition in the same universality class as random percolation. In contrast to previous percolation rules where delaying the transition changes its universality class, the limited resources in our model are exhausted before the transition. At this point the largest cluster has a fixed finite size and uncontrolled random percolation takes over, resulting in a continuous transition similar to random percolation for different initial cluster-size distributions [40].

TABLE I. Finite-size scaling. Exponents $-\beta/\nu$ (top) and γ/ν (bottom) found by finite-size scaling analysis. The corresponding fits are shown in Fig. 7 in Appendix B. The values agree with the exponents expected for random percolation $-\beta/\nu = -1/3$ and $\gamma/\nu = 1/3$ when the interventions end before the transition. For $\epsilon = 0.1$ and $b = 0.1$ the result is consistent with the expected $\beta = 0$ of a discontinuous transition.

	$-\beta/\nu$		
$b\backslash\epsilon$	0.1	0.2	0.5
0.01	−0.325(3)	−0.338(8)	−0.337(9)
0.05	−0.35(1)	−0.336(3)	−0.333(8)
0.10	−0.03(5)	−0.338(6)	−0.337(5)
	γ/ν		
0.01	0.331(3)	0.338(5)	0.331(2)
0.05	0.347(5)	0.343(7)	0.333(7)
0.10	0.40(5)	0.334(7)	0.339(5)

C. Optimal control leads to discontinuity

Increasing the budget also increases the delay of the transition. Interestingly, too large a budget also leads to a discontinuous transition [see Fig. 2(b)]. At the same time, increasing the budget further no longer increases the delay of the transition and p_c becomes constant. Clearly, when the budget survives the percolation threshold, additional

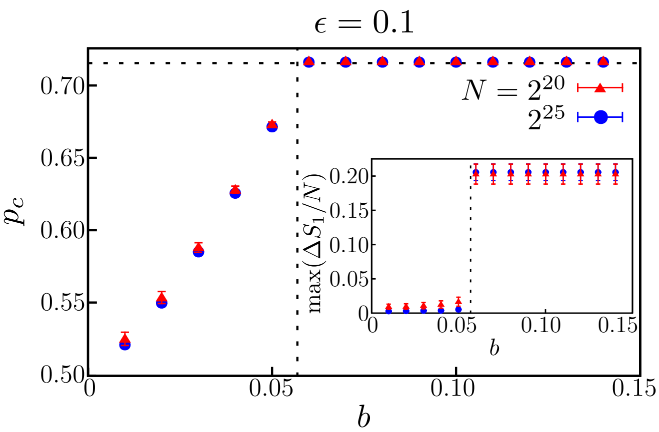

FIG. 3. Discontinuous transition above a critical budget. Percolation threshold p_c measured by the position of the largest gap of S_1 for different values of b. Results are averaged over 1024 and 256 realizations for networks of size $N = 2^{20}$ and 2^{25}, respectively. Error bars indicate the standard deviation. The delay increases with an increasing budget until it becomes constant above a critical budget $b_c \approx 0.058$. At the same time, the transition changes from continuous to discontinuous at $b = b_c$. Inset: The size of the largest gap max$(\Delta S_1/N)$ for different b.

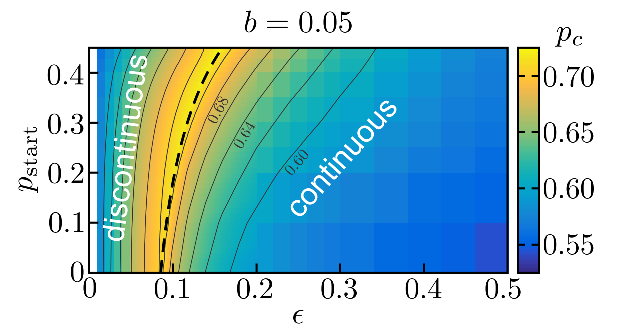

FIG. 4. "Phase diagram" and discontinuous transition as a result of optimal control. Position of the transition p_c for budget parameter $b = 0.05$ as a function of intervention intensity ϵ and intervention start $p_{\rm start}$. Results for system size $N = 2^{20}$, averaged over $R = 128$ realizations. The largest delay with $p_c \approx 0.72$ is achieved for a set of optimal intervention parameters (bright yellow) that separate the continuous from the discontinuous transition regime. The transition becomes discontinuous as a result of the optimal resource limited control. The black dashed line represents our estimate for this optimal parameter set in $(\epsilon, p_{\rm start})$ space (see text). The thin lines indicate lines of constant p_c.

interventions have no effect on the transition. This suggests that the optimal delay is achieved for an optimal budget lasting exactly until the percolation threshold, $p_{\rm last} = p_c$. At this point no uncontrolled window exists before the transition and the transition becomes discontinuous.

A similar logic defines the optimal parameters for speeding up the percolation transition (see Appendix D): interventions taken after the transition have no effect, while intervention-free uncontrolled link addition will reduce the effect of previous interventions. Optimal interventions necessarily end exactly at the percolation threshold, regardless of the intended result of the control.

Substituting $p_{\rm last} = p_c^{\max} \approx 0.72$ in Eq. (2) as the largest observed value of the critical point, we predict the critical budget required for a discontinuous transition for $\epsilon = 0.1$ to be $b_c^{\rm est} \approx 0.058$. Indeed, this is confirmed by the numerical results shown in Fig. 3: the transition is continuous for $b \leqslant 0.05$, while the transition for $b \geqslant 0.06$ is already discontinuous.

But how can the transition become discontinuous for $b > b_c$? Stopping the ϵ-fraction most extreme events prevents any cluster above a certain size $C_{\rm thresh}$ to appear in the network. As more links are added, this threshold slowly increases. This is similar to the dynamics of the Bohman-Frieze-Wormald (BFW) model [41]. In fact, we observe comparable behavior in the subcritical regime: there is a hierarchy of thresholds $p_k > 0$, $k = 3,4,\ldots$ where a new largest cluster of size $S_1 = k$ first appears. As in the BFW model, these p_k converge to constant, finite values $0 < p_k < p_c$ for large systems and announce the critical transition as $p_k \to p_c$ for $k \to \infty$ (see Appendix C). Thus, the same mechanism that leads to a discontinuous transition in the BFW model causes a discontinuous transition for optimal resource limited control of percolation [27,41,42].

We have studied other control strategies and cost functions, for example, cost proportional to the size of the clusters involved in the link, $c[S(i),S(j)] = S(i) + S(j)$ (see Supplemental Material [39]). We find for all of the studied cost functions that a small budget leads to a continuous transition, whereas a larger budget further delays the transition and eventually leads to a discontinuous transition. However, when the cost scales with the size of the clusters, the transition only becomes discontinuous when the budget scales superlinearly $B \sim O(N^a)$ with $a > 1$.

D. Limited observability

One realistic limitation to the control of connectivity is observability. In particular, we might not be aware of problems, such as emerging large clusters, early on in the process and only begin interventions after some time $p_{\rm start}$. Under these conditions, how do we best utilize a limited budget? Adapting Eq. (2) to include $p_{\rm start}$ leads to the relation $b = \int_{p_{\rm start}}^{p_c^{\max}} \frac{\epsilon_{\rm eff}(p)}{1-\epsilon_{\rm eff}(p)} dp$ describing the optimal intervention parameters (see also Appendix C). Calculating the optimal start and intensity of the interventions with $b = 0.05$ and the observed $p_c^{\max} = 0.72$, we obtain good agreement with the numerical results in Fig. 4. As explained above, the line of optimal control parameters separates the regimes of continuous and discontinuous transitions. As required by the constraint of limited resources, our control scheme is much more efficient than explosive percolation models at controlling percolation: We achieve $p_c = 0.72$ with only about one intervention per 15 added links; much less than comparable competitive percolation models, which reject one link for each link added (see Appendix A).

Interestingly, we find that for fixed intervention cost interventions close to the percolation threshold are slightly more effective than early interventions (p_c slowly increases as a function of $p_{\rm start}$ along the critical line). This result, however, is specific to constant intervention costs as other cost functions can lead to a different behavior: interventions as early as possible, $p_{\rm start} = 0$, are optimal for intervention costs that grow with the size of the connected clusters (see Supplemental Material [39]).

II. DISCUSSION

We have derived a control strategy to efficiently delay percolation with limited resources. In contrast to previous models constructed to delay the percolation transition [18,21–24,29–37], we find that the transition remains smooth and in the same universality class as random percolation for nonoptimal control when the resources are exhausted before the transition. Given a fixed budget, maximal delay of the percolation transition is achieved by optimizing the control protocol such that the budget is exhausted exactly at the percolation threshold. While the percolation transition can be delayed by control interventions, this resource optimal delay inevitably results in a discontinuous percolation transition that becomes effectively *un*controllable, since the addition of a single link induces a macroscopic change in the connectivity.

It is commonly believed that interventions taken as early as possible can have the biggest impact to avoid large-scale connectivity [6]. We have shown that this is not always the case: a strong effort to intervene right at the beginning can diminish the budget to such an extent that more timely interventions become impossible in crucial stages.

The framework we developed on the basis of random network growth highlights the unintended consequences of trying to control the percolation transition by delaying it [6,14,15]. Likely, similar effects will occur for other control schemes as well. This work may thus help to design control schemes in other networks, specific to the underlying network dynamics and its constraints, in particular when resources are scarce.

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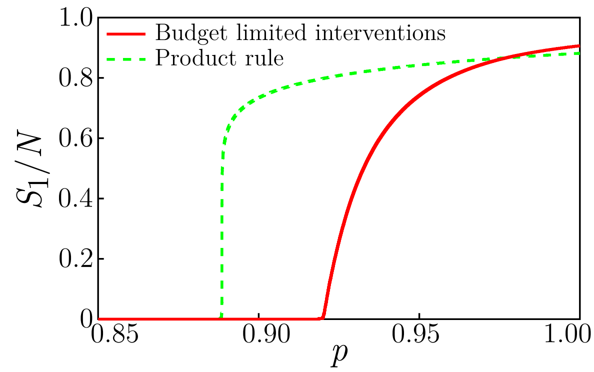

FIG. 5. Effectiveness of resource limited control. Single realizations of the largest cluster size for the budget limited control (red solid line) and the product rule (green dashed line) [21] resulting in explosive percolation ($N = 2^{25}$). The parameters are $b = 0.88$ and $\epsilon = 0.62$. In both models $L_{\mathrm{rej}} \approx 0.88N$ links are rejected until the phase transition occurs. This illustrates that the intervention rule defined in the main text is more effective in delaying the transition and at the same time keeps the transition smoother.

APPENDIX A: COMPARISON TO THE PRODUCT RULE

To illustrate the effectiveness of the proposed control protocol, we explicitly compare it to the product rule of explosive percolation [21]. The product rule is defined as follows: in each step choose two links uniformly at random

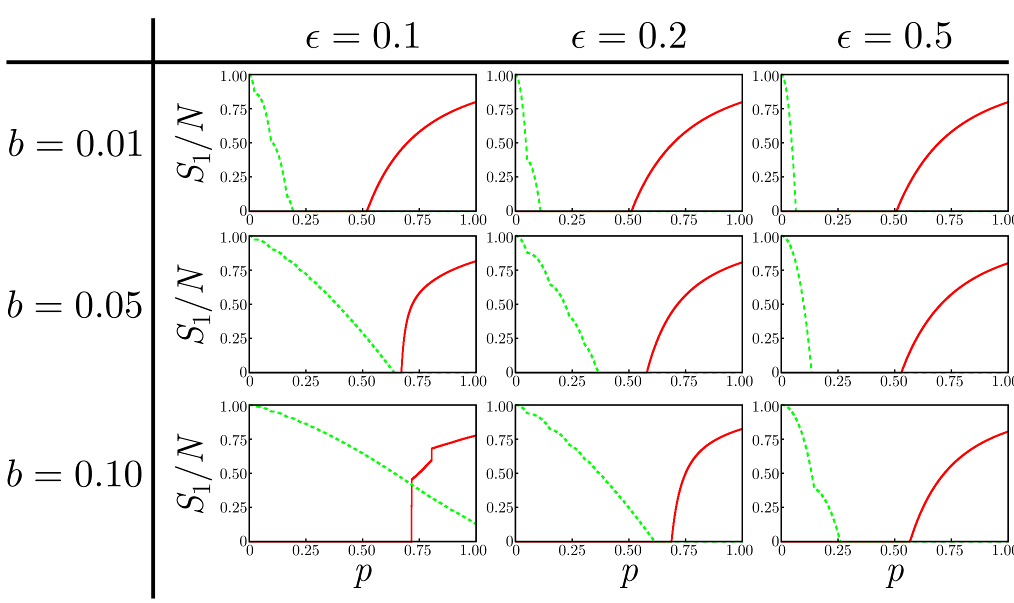

FIG. 6. Controlled percolation. Single realizations of the largest cluster size (red solid lines) and the remaining *fraction* of the budget (green dashed lines) for various parameter combinations and $N = 2^{25}$. Depending on the parameters the delay between the last interventions (budget reaching 0) and the percolation transition changes. The transition is smoothest when this gap is large. When the budget lasts until after the percolation transition, the transition becomes discontinuous.

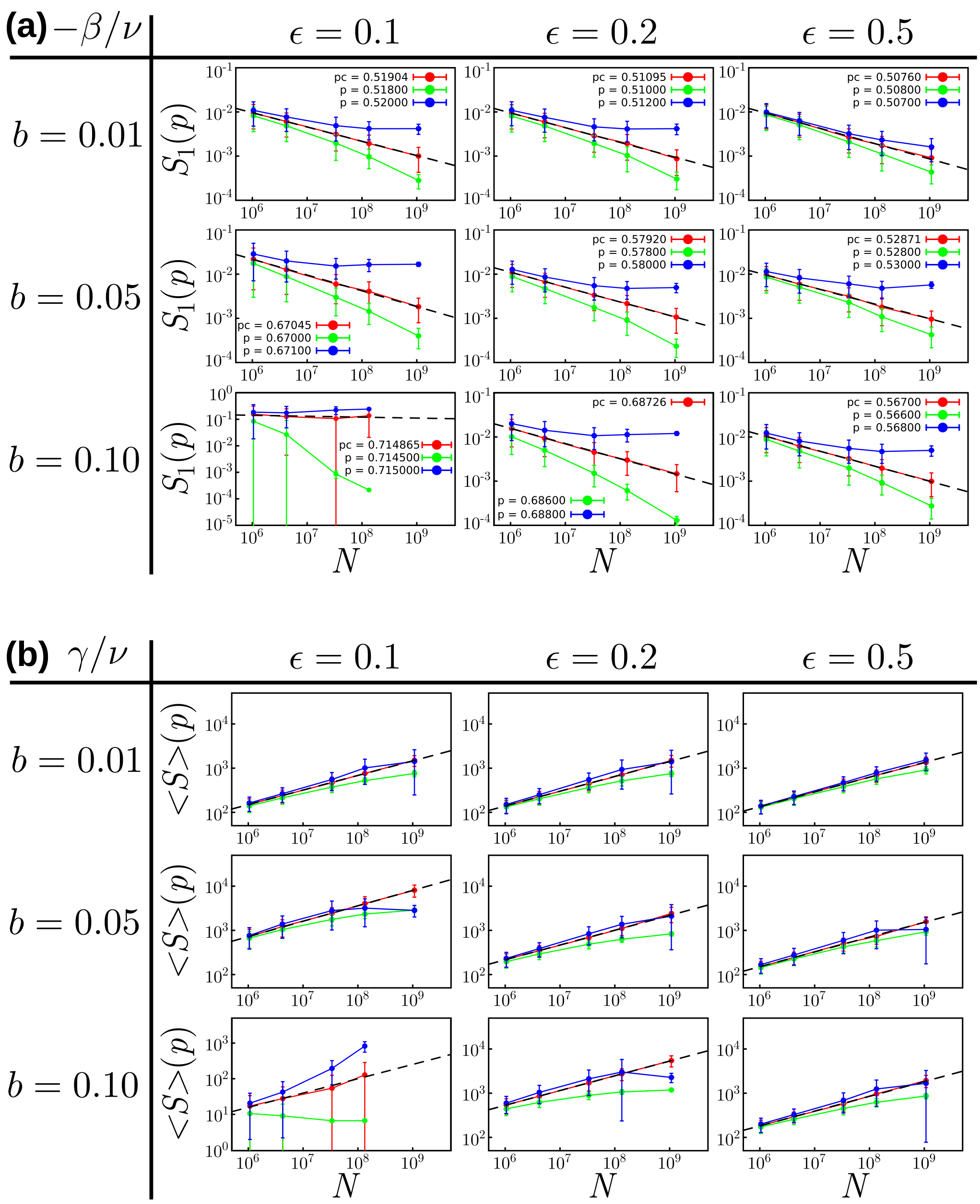

FIG. 7. Finite-size scaling for controlled percolation. Results for finite-size scaling for the estimated critical point and two values of p slightly below and above. The error bars indicate the standard deviation; lines are guides to the eye. Averages are taken over 1024 to 64 realizations for system sizes from $N = 2^{20}$ to $N = 2^{30}$. The black dashed lines indicate the best fits; the resulting exponents are listed in Table I above. (a) Results for the exponent $-\beta/\nu \approx -1/3$, showing the same behavior as expected for random percolation for all continuous transitions. For $b = 0.1$, $\epsilon = 0.1$ we find $\beta \approx 0$, corresponding to a discontinuous transition. (b) Results for the exponent $\gamma/\nu \approx 1/3$, showing the same behavior as expected for random percolation for all continuous transitions.

and add the link that minimizes the product $S(i)S(j)$. This significantly delays the percolation transition, but results in a very abrupt, explosive transition that is continuous but almost indistinguishable from a discontinuous transition even in very large systems. To compare the models, consider the "budget" required for the product rule: in each step one link is rejected, thus for constant cost, $c[S(i),S(j)] = 1$, the product rule requires a budget $B(p) = pN$ up until p. Therefore, until the phase transition at $p_c^{\mathrm{PR}} \approx 0.889$ it uses a budget $B = 0.88N$. We use the same budget, $b = 0.88$, for our intervention rule and choose a good (although not optimal) intervention intensity $\epsilon = 0.62$. As shown in Fig. 5 our intervention rule delays the percolation transition more efficiently while also keeping the transition in the same universality class as random percolation.

APPENDIX B: FINITE-SIZE SCALING ANALYSIS

Compared to explosive percolation our control scheme does not change the universality class of the transition when the

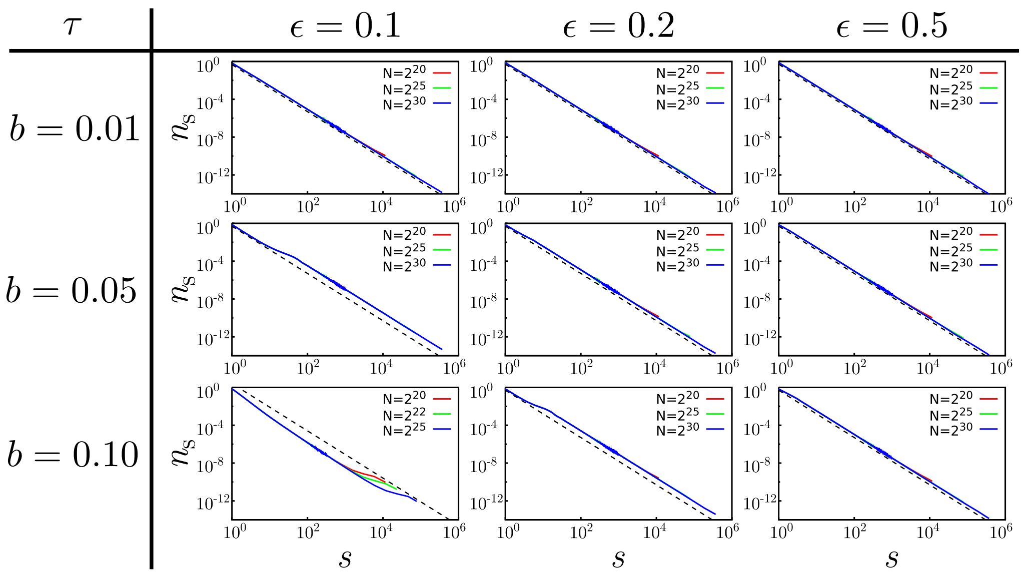

FIG. 8. Critical cluster size distribution. Cluster size distribution n_S for various system sizes N [$N = 2^{20}$, 2^{25}, and 2^{30} averaged over 1024, 256, and 64 realizations, respectively ($N = 2^{20}$, 2^{22}, and 2^{25} for $b = 0.1, \epsilon = 0.1$)]. Here, n_S describes the *relative* frequency of clusters of size S. The results are aggregated into logarithmic bins for clusters with size $S > N^{1/3}$ up to $S = 16N^{1/2}$. The scaling is expected to follow a power law $n_S \sim S^{-\tau}$ for large S; the dashed black lines show the scaling expected for random percolation with exponent $\tau = 5/2$ (not normalized). The peak in the cluster size distribution for small S is a signature of the finite size of clusters in the system when the interventions stop. Larger budgets allow for more interventions shifting the peak to larger S and making it more pronounced. Higher intervention intensities use the budget earlier, shifting the peak to lower S.

budget is exhausted before the transition. To demonstrate this, we conduct a finite-size scaling analysis for various values of the parameters b and ϵ. Figure 6 shows single realizations for these parameters. The transition is continuous in all cases where the budget is exhausted early. Only for $b = 0.1$ and $\epsilon = 0.1$ does the budget last until after the transition and the transition becomes discontinuous.

With the standard assumption of the critical scaling $S_1(p) \sim |p - p_c|^{\beta}$ and $\langle S\rangle(p) \sim |p - p_c|^{-\gamma}$ for the size of the largest cluster and the mean cluster size, respectively, as well as for the correlation length $\xi(p) \sim |p - p_c|^{-\nu}$, we find the exponents $-\beta/\nu$ and γ/ν from the finite-size scaling fits shown in Fig. 7. The results for the exponents are shown in Table I above. All exponents of the continuous transitions agree well with those expected for random percolation. For $b = 0.1$ and $\epsilon = 0.1$ we obtain exponents expected for a discontinuous transition, $-\beta/\nu = 0$. In particular, the transition never becomes weakly discontinuous or explosive. Similarly, we find the exponent $\tau = 5/2$ for the cluster size distribution at the critical point in all cases, the same as for random percolation. The corresponding results are shown in Fig. 8.

APPENDIX C: OPTIMAL CONTROL PARAMETERS

In order to estimate optimal intervention parameters, we need to predict the point p_{last} when the budget is exhausted. We first illustrate that the intervention rate is not constant as one might have expected from the definition of the intervention rule. Instead it fluctuates, dropping to small values immediately after the largest cluster size in the system grew (see Fig. 9). This is easiest to understand by considering the first link: we never prevent the first link since the probability to create a cluster of size 2 is $\text{Prob}[S(k) + S(l) \geqslant 2] = 1 > \epsilon$. Thus the probability of an intervention $\epsilon(p = 0) = 0$. Similarly, the first few links are unlikely to be prevented, since a link creating a cluster of size 3 or larger is chosen with vanishing probability.

We can think about the intervention rule in the following way: We always prevent the most extreme links. This is equivalent to preventing all clusters above a certain size (until these links become too likely). This means, when the size of the largest cluster just changed to S_1, the probability to create a larger cluster is usually smaller than ϵ. However, the links creating a cluster of size S_1 are not prevented as the probability to create a cluster larger *or equal* to S_1 is larger than ϵ. Thus, after these microtransitions of the largest cluster size, the intervention probability drops. In fact, we find that these transitions to a new largest cluster size happen at well defined times, constant across different system sizes (see Fig. 9). This behavior is similar to the subcritical evolution of the BFW model [41,42]. This observation also supports the discontinuity of the transition.

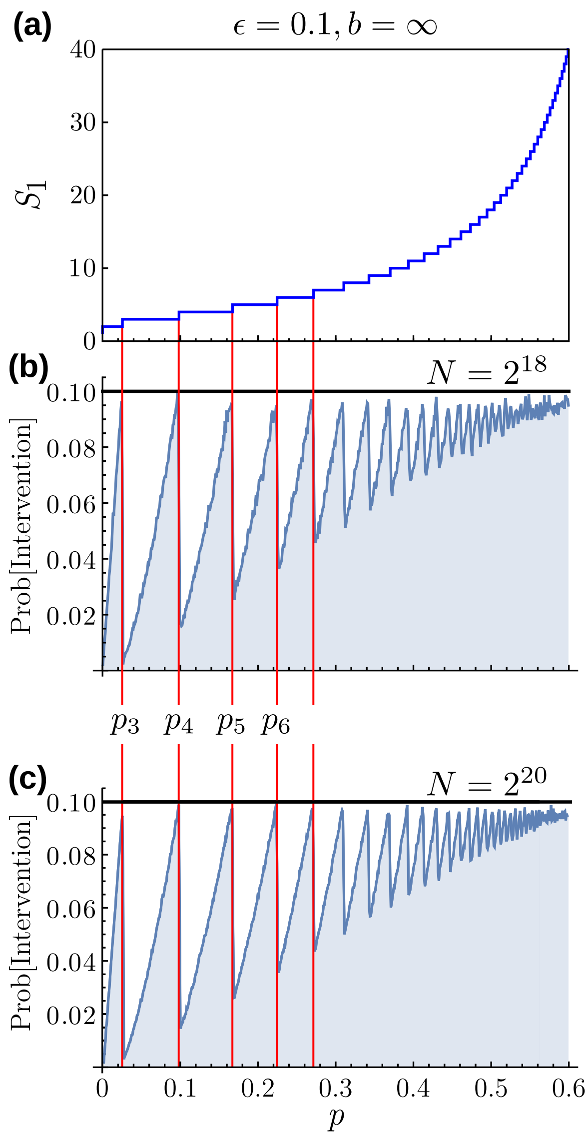

FIG. 9. Distribution of interventions. (a) Single realization of the largest cluster size in the subcritical regime for $N = 2^{25}$ with unlimited budget and $\epsilon = 0.1$. (b), (c) Probability of an intervention for a single link chosen at p for two different system sizes averaged over 100 realizations each. When a new cluster size appears in the network the intervention probability "resets." This causes the transitions to $S_1 = 3$ at p_3, $S_1 = 4$ at p_4, and so on to occur at fixed positions. This behavior is similar to the subcritical evolution of the largest cluster in the BFW model, leading to a discontinuous transition at $p_k \to p_c$ for $k \to \infty$.

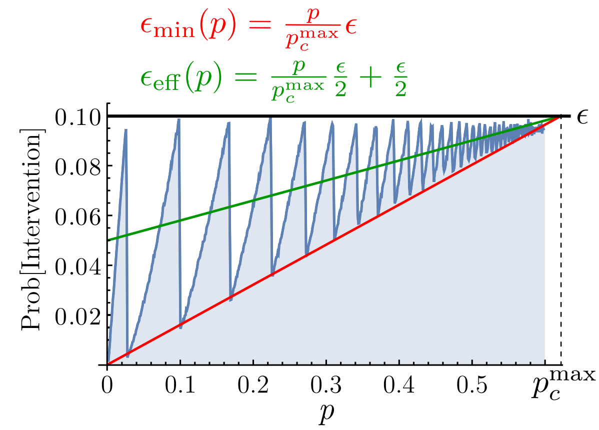

FIG. 10. Estimating the intervention rate. Probability of an intervention for a single link chosen at p for system size $N = 2^{20}$ averaged over 100 realizations. The red and green lines illustrate the approximation used to define the effective intervention rate ϵ_{eff} (green), describing a local average of the true intervention rate (here for $p_{\text{start}} = 0$).

We can use the observed intervention rate to derive an estimate for the budget used for interventions up to p. Since we do not know the exact form of $\epsilon(p)$, we use an empirically determined "effective intervention rate" $\epsilon_{\text{eff}}(p)$, describing a local average of $\epsilon(p)$ (illustrated in Fig. 10 for $p_{\text{start}} = 0$). This intervention rate depends on the intervention parameter ϵ and the position $p_c^{\max}$ of the critical point of the process with unlimited budget. When $p_{\text{start}} > 0$, the uncontrolled evolution before the control starts will cause the intervention rate to be larger than in the fully controlled process. We assume that the effective intervention rate at p_{start} is $\epsilon_{\text{eff}}(p_{\text{start}}) = p_{\text{start}}\epsilon + \epsilon/2$ (the value obtained by setting $p_c^{\max} = 1/2$). Directly at and after p_c the effective intervention rate is $\epsilon_{\text{eff}}(p \geqslant p_c) = \epsilon$. Together this gives

$$\epsilon_{\text{eff}}(p) = \frac{p - p_{\text{start}}}{p_c^{\max} - p_{\text{start}}}(\epsilon/2 - p_{\text{start}}\epsilon) + \epsilon/2 + p_{\text{start}}\epsilon, \quad \text{(C1)}$$

for $p_{\text{start}} \leqslant p \leqslant p_c$. Obviously, before p_{start} the intervention rate is $\epsilon_{\text{eff}} = 0$ and above p_c the intervention rate is $\epsilon_{\text{eff}} = \epsilon$.

We can now use the argument we gave above and integrate Eq. (C1) over all interventions to find the total budget used. We arrive at the approximate relation

$$b = \int_{p_{\text{start}}}^{p_{\text{last}}} \frac{\epsilon_{\text{eff}}(p)}{1 - \epsilon_{\text{eff}}(p)} dp$$
$$= p_{\text{start}} - p_{\text{last}} - \frac{2(p_c^{\max} - p_{\text{start}})\log\left(\frac{-2(p_c^{\max} - p_{\text{start}}) + \epsilon[p_{\text{last}} + p_c^{\max} - 2p_{\text{start}}(1 + p_{\text{last}} - p_c^{\max})]}{(p_c^{\max} - p_{\text{start}})(\epsilon + 2p_{\text{start}}\epsilon - 2)}\right)}{(1 - 2p_{\text{start}})\epsilon}, \quad \text{(C2)}$$

assuming again $p_c^{\max} \geqslant p_{\text{last}}$ is the critical point of the process with parameters ϵ and p_{start} given unlimited budget.

Substituting $p_{\text{last}} = p_c^{\max}$ gives the condition for optimal intervention parameters, which can be solved numerically to

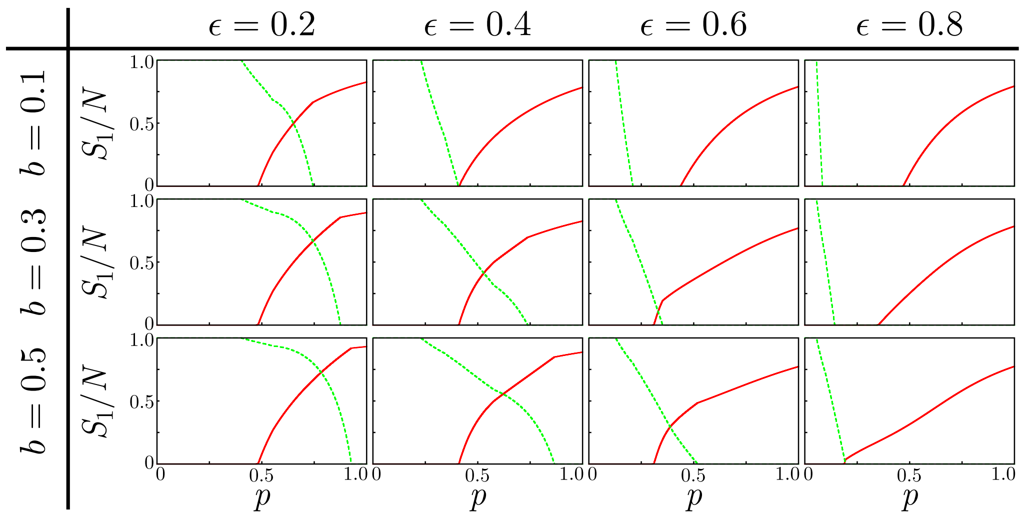

FIG. 11. Enhancing percolation. Single realizations of the largest cluster size (red solid lines) and the remaining *fraction* of the total budget (green dashed lines) for various parameter combinations for interventions enhancing percolation ($N = 2^{25}$). Depending on the parameters the transition is enhanced more or less strongly. As for delaying the transition, interventions are most efficient when the interventions last exactly until the transition.

find the optimal budget or intervention rate (see Fig. 4). The estimate becomes worse for large values of ϵ and p_{start} and very small values of b, where interventions occur only in a small interval and averaging to ϵ_{eff} becomes inaccurate. For the same reason, the effective intervention rate is a good approximation when estimating the optimal intervention parameters, where interventions last until p_c and the error from averaging is small.

APPENDIX D: ENHANCING PERCOLATION

We have illustrated our results for interventions that are designed to delay the percolation transition. Interestingly, the same logic describing the optimal intervention strategy also applies to enhance percolation. Instead of stopping the ϵ-fraction most extreme events, we simply reverse the protocol, $\text{Prob}[S(k) + S(l) \leqslant S_{ij}] < 1/\Delta L_{\text{thres}} := \epsilon$, and stop the ϵ-fraction least extreme events, where we specifically include links connecting nodes in the same cluster as creating a new cluster of size 0. Additionally, we always prevent such intracluster links as there are no less extreme events. However, this is only relevant for nonoptimal interventions after the transition.

Also in this case optimal interventions necessarily end at the percolation threshold. Interventions lasting longer have no additional effect on the threshold and interventions ending earlier create an extensive interval of uncontrolled percolation before the transition, partially negating the effect of the interventions. In Fig. 11 we show examples for single realizations of percolation enhancing interventions. The results confirm that the effect is largest (p_c is smallest) when the budget runs out exactly at $p_{\text{last}} = p_c$. The budget used also shows that it is much more difficult to enhance the percolation than to delay it.

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