Self-organized emergence of multilayer structure and chimera states in dynamical networks with adaptive couplings

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We report the phenomenon of self-organized emergence of hierarchical multilayered structures and chimera states in dynamical networks with adaptive couplings. This process is characterized by a sequential formation of subnetworks (layers) of densely coupled elements, the size of which is ordered in a hierarchical way, and which are weakly coupled between each other. We show that the hierarchical structure causes the decoupling of the subnetworks. Each layer can exhibit either a two-cluster state, a periodic traveling wave, or an incoherent state, and these states can coexist on different scales of subnetwork sizes.

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I. INTRODUCTION

Synchronous behavior in networks of coupled oscillatory systems is a universal phenomenon observed in many natural and engineering applications. Along with complete synchronization, the formation of more complex synchronization patterns has attracted increasing interest recently. In particular, such patterns include cluster synchronization, when the network splits into two or more groups of synchronized elements [1], and a special type of spatial coexistence of coherent (synchronized) and incoherent (desynchronized) patterns known as chimera states [2–6]. Examples of different types of synchronization patterns may be found in a wide range of systems, including optoelectronic networks [7], networks of chemical oscillators [8,9], neural networks [10–13], and ecological and climate systems [14]. Numerous theoretical works have shown that the underlying connection topology of a network has a crucial effect on the process of synchronization. A diversity of synchronous patterns have been shown to arise in systems with fixed complex topology, including modular organization [15–17] and nonlocal and hierarchical interaction [18,19]. In addition, the formation of clustered states is an important collective effect emerging in adaptive networks [20,21], in which the coupling strength and the topology of connection evolve over time, depending on the states of interacting oscillators.

Some previous works [22,23] have discussed hierarchical organization of the synchronization behavior in complex networks with static couplings. Such a scenario involves the formation of local synchronization clusters in areas with a high density of connections between elements and the gradual evolution of the clusters due to the synchronization of elements from their neighborhood.

In this paper, we present a scenario for the formation of cluster and chimera states, which is realized in networks with adaptive couplings. According to this scenario, in the course of the temporal evolution, subnetworks (or layers [24,25]) emerge, which are ordered in a hierarchical way. Each subnetwork may exhibit different coexisting dynamics, including two-cluster, periodic traveling wave, or incoherent states. The formation of these synchronous groups is accompanied by a corresponding transformation of the properties of the network connection structure.

II. THE MODEL

The most successful approach to study various aspects of synchronization phenomena in complex networks of coupled oscillators is given by the paradigmatic Kuramoto model [26]. Since its original formulation, this model has been extensively studied in many different versions [27,28], taking into account the influence of various factors on the collective behavior of coupled oscillator networks for a wide range of applications. We consider a network of N globally coupled identical phase oscillators described by the following Kuramoto-type equations:

$$\frac{d\phi_i}{dt} = \omega + \frac{1}{N}\sum_{j=1}^{N}\kappa_{ij}\Gamma(\phi_i - \phi_j), \qquad (1)$$

where ϕ_i represents the phase of the ith oscillator ($i = 1,\dots,N$), and ω is the natural frequency, which we set $\omega = 1$. The function $\Gamma(\phi)$ characterizes the interaction between the oscillators and κ_{ij} denotes the coupling strength of the connection from the jth to the ith oscillator. We consider a simple model $\Gamma(\phi) = -\sin(\phi + \alpha)$, where parameter α can be regarded as the phase lag in the coupling.

The evolution of the coupling strength κ_{ij} from the jth to the ith oscillator is controlled by the following equation:

$$\frac{d\kappa_{ij}}{dt} = \varepsilon[\Pi(\phi_i - \phi_j) - \kappa_{ij}], \qquad (2)$$

where $\Pi(\phi)$ is the adaptation function, which determines the dynamics of the coupling strength as a function of the relative phase differences of interacting oscillators. We choose it in the form $\Pi(\phi) = -\sin(\phi + \beta)$, where the parameter β controls the properties of the adaptation function. The dynamics of the coupling is slower than the phase dynamics, and this is reflected by a small parameter $\varepsilon \ll 1$.

Thus, the adaptive network considered here is described by

$$\frac{d\phi_i}{dt} = 1 - \frac{1}{N}\sum_{j=1}^{N}\kappa_{ij}\sin(\phi_i - \phi_j + \alpha),$$

$$\frac{d\kappa_{ij}}{dt} = -\varepsilon[\sin(\phi_i - \phi_j + \beta) + \kappa_{ij}]. \qquad (3)$$

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The state of the network (3) is studied as a function of control parameters α and β. Initial conditions are chosen randomly, with uniform distributions of the phase ϕ_i in the interval $[0,2\pi]$ and coupling strengths κ_{ij} in $[-1,1]$. The choice of such a range of initial conditions for the coupling strength is determined by the presence of the attracting region G in phase space,

$$G = \{\phi_i,\kappa_{ij} : \phi_i \in S^1, |\kappa_{ij}| \leqslant 1,\ i,j = 1,\ldots,N\}.$$

III. EMERGENCE OF MULTILAYERED STRUCTURES AND CHIMERA STATES

We have found several scenarios of self-organization of the adaptive network, depending on the properties of the adaptation function. Regardless of the specific characteristics of the final states of the network, the mechanisms of their formation have a common feature, namely, a sequential hierarchical formation of new densely connected subnetworks, accompanied by a corresponding emergence of either synchronized, cluster, traveling wave, or incoherent dynamics within each subnetwork. These subnetworks arise on different time scales, and their size decreases at each subsequent stage of the network evolution. Note that the existence of some of these cluster states, but neither the hierarchical sequential mechanism nor the emergence of chimeras, were reported in [29].

To illustrate such a hierarchical synchronization, Fig. 1 displays a series of snapshots of the coupling matrix κ_{ij} calculated at different stages of the formation of the synchronization pattern. In order to identify the synchronization patterns and the coupling structure, the following ordering procedure of the oscillator indices was performed: After skipping a sufficiently long transient time, the average frequencies of each oscillator were calculated, and the oscillators were ordered according to increasing frequencies. However, the oscillators belonging to the same cluster possess also the same average frequency. Hence, in this case, the indices within one cluster are ordered according to increasing value of the oscillators' instantaneous phase.

Starting from a random initial state [Fig. 1(a)], there appears a group of elements which are quickly synchronized (here two cluster). The initially emerging group is characterized by strong interelement couplings within the group [Fig. 1(b)] against the background of relatively weak interaction with the rest of the network elements. Further, in the remaining incoherent part of the network, a second synchronous group [Fig. 1(c)] is formed, which is also characterized by a strong interaction between the elements within the group. At the same time, this process is accompanied by the suppression of couplings between the elements of different groups formed at different stages of the evolution of the network. Similar processes continue in the incoherent part of the network until the network reaches the final state [Figs. 1(e) and 1(f)], which has a multilayered structure; see also the space-time plot [Fig. 1(g)] and the average frequency profile [Fig. 1(h)].

According to a second scenario of the network evolution, the formation of synchronous groups terminates at some stage, and the remaining oscillators stay unsynchronized; see Fig. 2. In particular, Fig. 2(e) indicates that the relative phase differences between oscillators within the coherent groups are unchanged with time, which indicates coherent traveling waves, while those in the incoherent part of the network change irregularly. The formation of such chimera states [Figs. 2(f) and 2(g)] is also accompanied by similar structural changes: the coupling strength between the elements of the different groups is decreased while the coupling within the group remains strong.

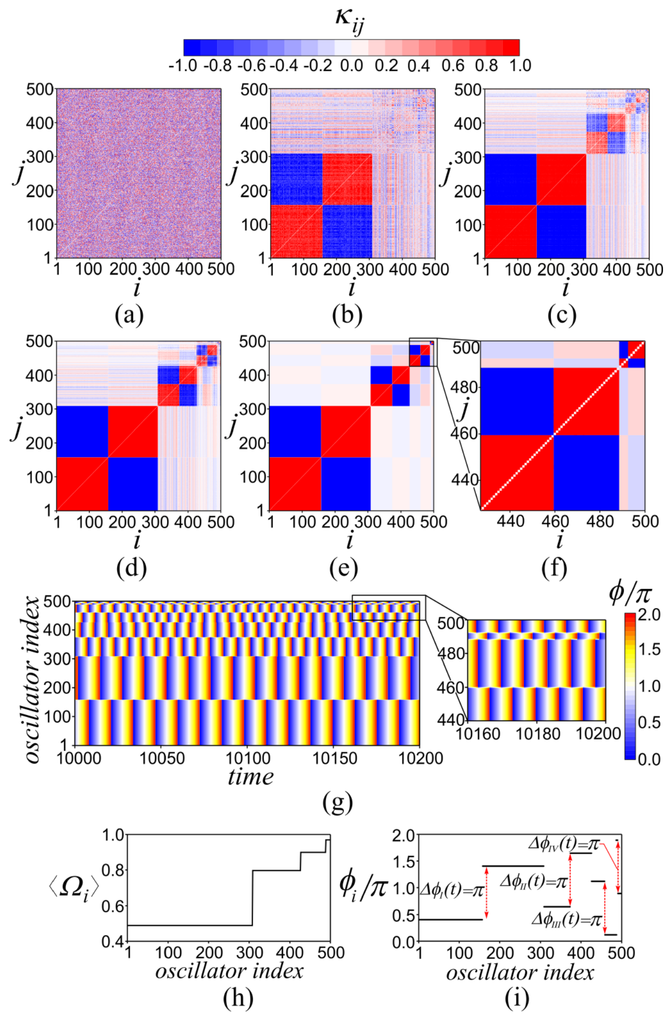

FIG. 1. Hierarchical cluster formation in the evolution of the network structure. (a)–(f) Coupling matrix κ_{ij} at different stages of emergence of synchronization patterns: (a) $t = 0$, (b) $t = 200$, (c) $t = 350$, (d) $t = 500$, (e) and (f) $t = 10\,000$; (g) dynamics of phases $\phi_i(t)$ in the final state; (h) average frequencies of oscillators; (i) phase snapshot of the oscillators at $t = 10\,010$. Parameter values: $\varepsilon = 0.01$, $\alpha = 0.3\pi$, $\beta = -0.53\pi$, $N = 500$.

It should be noted that in adaptive networks, unlike complex networks with static connections, considered in [18], the set of oscillators forming synchronous groups or a chimera state essentially depends on the initial conditions. For fixed parameter values, the network demonstrates a large number of chimera states that have different sizes of the synchronous part of the network, different numbers of synchronous groups, and different sets of elements that form synchronous groups.

The emerging groups may exhibit different properties of synchronous behavior depending on the characteristics of the

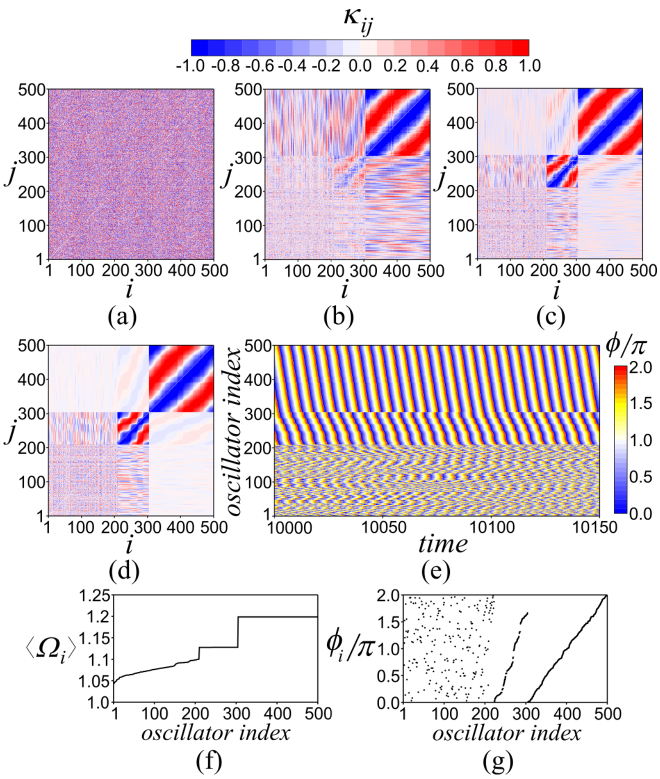

FIG. 2. Hierarchical formation of a chimera state. (a)–(d) Coupling matrix κ_{ij} at different stages of emergence of a chimera state: (a) $t = 0$, (b) $t = 350$, (c) $t = 1500$, (d) $t = 10\,000$; (e) dynamics of phases $\phi_i(t)$ in the final state; (f) average frequencies of oscillators; (g) phase snapshot of the oscillators at $t = 10\,050$, indicating traveling waves within the two coherent clusters. Parameter values: $\varepsilon = 0.01$, $\alpha = 0.3\pi$, $\beta = 0.3\pi$, $N = 500$.

adaptation function. Figure 3 shows a diagram illustrating the properties of synchronous groups as a function of parameters α and β. This diagram is obtained as a result of an ensemble averaging of 20 different sets of initial conditions. For each initial condition, we construct two-parameter diagrams of a number of characteristics, such as time-averaged order

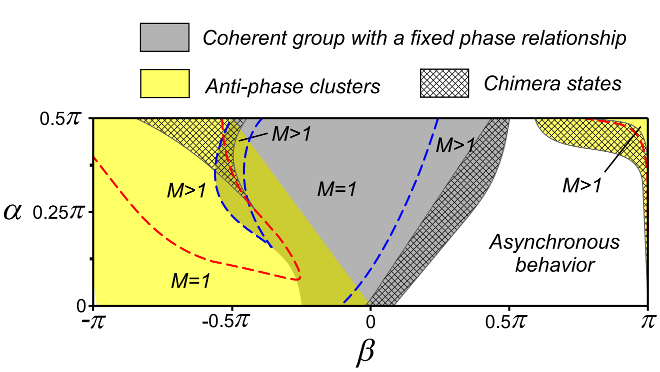

FIG. 3. Diagram of dynamical states in the (β,α) parameter plane. Regions with different properties of synchronous behavior of emerging groups are indicated by color. Dotted lines separate regions with different numbers M of synchronous groups. Chimera states are cross hatched. The ensemble average of 20 sets of initial conditions is used.

parameters,

$$\langle R_k \rangle = \frac{1}{\Delta t}\int_T^{T+\Delta t} \frac{1}{N}\left|\sum_{j=1}^{N} e^{-ik\phi_j}\right| dt, \quad k = 1,2.$$

In addition, we analyzed the domains of synchronized oscillators of the network using information about the degree of synchrony between any two oscillators i and j defined as follows:

$$R_{ij} = \left|\frac{1}{\Delta t}\int_T^{T+\Delta t} e^{i[\phi_i(t)-\phi_j(t)]} dt\right|.$$

The calculation of these characteristics was carried out over a large time window Δt skipping a transient time interval T. Depending on the parameters, each synchronous group can represent two antiphase clusters or demonstrate coherent behavior, maintaining a fixed phase relationship between oscillators within the group, i.e., $\Delta\phi_{ij}(t) \equiv \phi_i(t) - \phi_j(t) =$ const. At the same time, complex network states consisting of M synchronous groups ($M > 1$) can include groups with only one type of synchronous behavior, as well as combinations of these types. We also note that the properties of synchronous part in chimera states are determined by the color of the corresponding region in Fig. 3. The first type of the group synchronous behavior is observed in the region $\beta \in (-\pi,0)$, where in-phase and antiphase synchronization modes exist simultaneously for the model of two coupled oscillators considered in [30]. This behavior is also realized at the level of a large network. Depending on initial conditions, the oscillators within the cluster are in-phase synchronized, and the antiphase synchronization is established between the oscillators of different clusters. Depending on the properties of the adaptation function, the network states are characterized by different number and size of the emerging phase clusters. In particular, multicluster states ($M > 1$) are realized in the vicinity of $\beta = -\pi/2$, when the adaptation function $\Pi(\Delta\phi) = \cos(\Delta\phi)$ can be considered as a Hebbian-like rule. In this case, the interaction between the oscillators within such antiphase clusters is characterized by the maximum values $\kappa_{ij} \approx \pm 1$. Consequently, it becomes possible to form isolated groups of oscillators with strong couplings against the background of a relatively weak interaction of the rest of the network elements.

Figure 1 illustrates the formation of multicluster state, when the network splits into four pairs of antiphase clusters [Figs. 1(g) and 1(i)]. Oscillators belonging to different pairs of synchronous clusters have distinct frequencies, whose magnitudes $\Omega_i(t) = d\phi_i/dt$ oscillate around some mean values $\langle\Omega_i\rangle$ [Fig. 1(h)]. The phase differences of oscillators belonging to the same synchronized groups determine the corresponding connectivity within the groups. The coupling between oscillators within each group takes the values $\kappa_{ij} = -\sin\beta$ for in-phase synchronization and $\kappa_{ij} = \sin\beta$ for oscillators in antiphase. At the same time, the coupling strength between oscillators of different synchronized groups oscillates in time around the zero mean value, and the amplitude of these modulations depends on the frequency difference of the interacting oscillators. It should be noted that the formation of multicluster states is accompanied by the emergence of a modular structure of connections, when the network splits into several weakly coupled groups (layers) of oscillators as shown in Figs. 1(e) and 1(f). The configuration of the layers repeats the structure of the corresponding pairs of antiphase clusters formed in the network. Moving the parameter β to the boundaries of the interval $(-\pi,0)$ results in a decrease of interaction strength between oscillators of antiphase clusters. As a result, either one pair of antiphase clusters ($M = 1$) or a chimera state are formed, as shown in Fig. 3.

The second type of synchronous behavior within the group is observed in the region around $\beta = 0$. In this case, oscillators within each group demonstrate coherent behavior, maintaining a fixed phase relationship between the oscillators. For $\beta = 0$ the adaptive function has the form $\Pi(\Delta\phi) = -\sin(\Delta\phi)$, which was referred to as spiking-time-dependent-plasticity (STDP) or causal rule in a number of previous papers [31,32]. The sign of the adaptive function depends on the temporal order of the oscillators. It means that the coupling strength has opposite sign for the oscillators with phase differences $\Delta\phi = \pm\Delta\phi_0$. The number of emerging coherent groups with a fixed phase relationship also depends on the properties of the adaptation function. Thus the number of such coherent groups increases as the form of the adaptive function becomes similar to a Hebbian-like ($\beta = -\pi/2$) or anti-Hebbian-like ($\beta = \pi/2$) rule. Along with the increase of the number of coherent groups, such a change of the adaptation function can also lead to the formation of chimera states (Fig. 3). In Fig. 2 we present an example of a chimera state, when two coherent groups are organized in the synchronous part of the network. Each group is characterized by its own frequency [Fig. 2(f)], which takes some fixed value [$\Omega_i(t) = \text{const}$] in the final state. At the same time, the frequencies of the oscillators of the incoherent part of the network vary over time, and their average values are distributed in some interval. The emergence of such states is also accompanied by a corresponding modification of the network connectivities. In particular, the coupling coefficients between oscillators within coherent groups take a fixed values [Fig. 2(d)] depending on the phase difference between corresponding oscillators $\kappa_{ij} = -\sin(\Delta\phi_{ij} + \beta)$, where $\Delta\phi_{ij} = \phi_i(t) - \phi_j(t) = \text{const}$. Assuming that the frequencies of the oscillators are fixed, the evolution of the coupling coefficient between two oscillators i and j belonging to different coherent groups follows from the second equation of Eqs. (3):

$$\kappa_{ij}(t) = \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta\Omega_{ij}^2}} \sin(\Delta\Omega_{ij} t - \chi_{ij} + \xi), \qquad (4)$$

where $\Delta\Omega_{ij} = \Omega_i - \Omega_j$, $\cos\chi_{ij} = \varepsilon/\sqrt{\varepsilon^2 + \Delta\Omega_{ij}^2}$, and $\xi = \phi_i(0) - \phi_j(0) + \beta$. According to Eq. (4), the amplitude of the oscillations of the coupling coefficients κ_{ij} characterizing the interaction between different groups decreases with increasing frequency difference $\Delta\Omega_{ij}$ of the interacting oscillators. This dependence explains the process of formation of the network states characterized by the presence of several groups of synchronized elements. The possibility of coexistence of these groups is also determined by the weak interaction between them due to a relatively large frequency difference between the oscillators of the emerging groups. The frequency of group M can be found using the ansatz $\phi_i(t) - \phi_j(t) = \text{const}$ within the group and assuming that the coupling between different groups is negligible:

$$\Omega_i(t) = 1 + \frac{N_M}{2N}\cos(\alpha - \beta) - \frac{1}{2N}\sum_{j\in\mathcal{N}_M}\cos\left(\frac{4\pi}{N_M}(i-j) + \alpha + \beta\right), \qquad (5)$$

where $i \in \mathcal{N}_M$, $\{\mathcal{N}_M\}$ is a set of oscillators forming the group M and N_M is the number of oscillators in group M. In particular, for the case shown in Fig. 2, this expression gives the numbers $\Omega_{M1} \approx 1.197\,658$, $\Omega_{M2} \approx 1.126\,705$ in good agreement with the numerical values. The emergence of synchronous groups with sizes ordered in a hierarchical way provides a significant difference in the frequencies of the oscillators of different groups and, in accordance with Eq. (4) leads to the decoupling of the groups.

IV. CONCLUSION

In conclusion, we have investigated the organization of synchronous behavior in networks of phase oscillators with adaptive coupling. We have found the emergence of hierarchical synchronization patterns including multicluster and chimera states. This process represents a sequential formation of a few groups of synchronized oscillators, ordered in a hierarchy. Depending on the properties of the adaptation function the self-organized groups may exhibit different local synchronous behavior. In the final state, the network can consist of a finite number of groups of synchronized oscillators (multicluster states), split into several coherent groups with a fixed phase relationship within the groups, and also demonstrate a variety of chimera states. The emergence of these states is accompanied by a transformation of the network connectivities, when the globally coupled network splits into a number of weakly interacting subnetworks. The subnetworks with fixed phase relation exhibit, in fact, a generalized ring coupling structure, which is symmetric with respect to index shift of the oscillators within the subnetwork. Such a self-organized emergence of the symmetry is another interesting phenomenon, which should be studied in the future.

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