Exact mean-energy expansion of Ginibre's gas for coupling constants $\Gamma = 2 \times$ (odd integer)

R. Salazar[1,2] and G. Téllez[1]

[1]*Departamento de Física, Universidad de los Andes, Bogotá, Colombia*

[2]*Laboratoire de Physique Théorique, UMR No. 8627, Université Paris–Saclay, Université Paris–Sud, and CNRS, Bâtiment 210, 91405 Orsay Cedex, France*

(Received 11 August 2017; published 26 December 2017)

Using the approach of a Vandermonde determinant to the power $\Gamma = Q^2/k_BT$ expansion on monomial functions, a way to find the excess energy U_{exc} of the two-dimensional one-component plasma (2DOCP) on hard and soft disks (or a Dyson gas) for odd values of $\Gamma/2$ is provided. At $\Gamma = 2$, the present study not only corroborates the result for the particle-particle energy contribution of the Dyson gas found by Shakirov [Shakirov, Phys. Lett. A **375**, 984 (2011)] by using an alternative approach, but also provides the exact N-finite expansion of the excess energy of the 2DOCP on the hard disk. The excess energy is fitted to the ansatz of the form $U_{\text{exc}} = K^1N + K^2\sqrt{N} + K^3 + K^4/N + O(1/N^2)$ to study the finite-size correction, with K^i coefficients and N the number of particles. In particular, the bulk term of the excess energy is in agreement with the well known result of Jancovici for the hard disk in the thermodynamic limit [Jancovici, Phys. Rev. Lett. **46**, 386 (1981)]. Finally, an expression is found for the pair correlation function which still keeps a link with the random matrix theory via the kernel in the Ginibre ensemble [Ginibre, J. Math. Phys. **6**, 440 (1965)] for odd values of $\Gamma/2$. A comparison between the analytical two-body density function and histograms obtained with Monte Carlo simulations for small systems and $\Gamma = 2,6,10,\ldots$ shows that the approach described in this paper may be used to study analytically the crossover behavior from systems in the fluid phase to small crystals.

DOI: 10.1103/PhysRevE.96.062145

I. INTRODUCTION

This paper is devoted to the study of the two-dimensional one-component plasma (2DOCP) in hard- and soft-disk cases. In general, the 2DOCP refers to a system of N identical charges Q existing on a two-dimensional surface S with a neutralizing background with uniform density charge $-\rho_b Q$, where ρ_b is the background number density. For the case of a flat plane, two charges of the 2DOCP located at $\vec{r}_1$ and $\vec{r}_2$ interact with a logarithmic potential of the form

$$v(\vec{r}_1,\vec{r}_2) = -Q^2 \ln\left(\frac{|\vec{r}_1 - \vec{r}_2|}{L}\right),$$

with L an arbitrary length constant. The potential energy U_{inter} of the 2DOCP is given by

$$U_{\text{inter}} = U_{pp} + U_{bp} + U_{bb},$$

where U_{pp} is the particle-particle interaction energy contribution, U_{bp} is the background-particle interaction, and U_{bb} is the background-background interaction. The total average energy E is the usual bidimensional ideal gas energy plus the excess energy U_{exc} contribution $E = Nk_BT + U_{\text{exc}}$, with $U_{\text{exc}} = \langle U_{\text{inter}}\rangle$. Generally, the potential energy U_{inter} depends on the geometry of S. If a 2DOCP on a hard disk of radius R is considered, then the potential energy is [1]

$$U^{\mathcal{H}}_{\text{inter}} = Q^2\left[f^{\mathcal{H}}(N) + \frac{1}{2}\sum_{i=1}^{N}\left(\frac{\sqrt{N}}{R}r_i\right)^2 - \sum_{1\leqslant i<j\leqslant N} \ln\left(\frac{\sqrt{N}}{R}r_{ij}\right)\right], \tag{1}$$

where

$$f^{\mathcal{H}}(N) = -\frac{3}{8}N^2 + \frac{N}{2}\ln\left(\frac{R}{L}\right) + \frac{N^2}{2}\ln\sqrt{N} - \frac{N}{2}\ln N. \tag{2}$$

In this situation the particles repel each other logarithmically while they are bound by an attractive quadratic potential generated by the background and eventually by the circular boundary (see Fig. 1).

The statistical behavior of the system depends only on a coupling parameter $\Gamma = Q^2/k_BT$, where k_B is the Boltzmann constant and T is the temperature. For $\Gamma \to 0$ the system is a two-dimensional ideal gas and fluid for moderately high values of Γ. In contrast, the system becomes a crystal for $\Gamma \to \infty$, where it has an extremely high electric interaction or very low temperature. There are several analytical studies on the 2DOCP in diverse geometries for the special coupling $\Gamma = 2$ [2–6]. In particular, the excess free energy F_{exc} per particle at $\Gamma = 2$ is

$$\frac{F_{\text{exc}}}{N} = -\frac{Q^2}{4}\ln(\rho_b\pi L^2) + \frac{Q^2}{2}[1 - \ln(2\pi)]$$

in the thermodynamic limit, which implies that the particle density of the background should be kept constant $\rho_b = \frac{N}{\pi R^2}$ as N and R tend to infinity. Previously, Jancovici [2] found that the excess parts of the energy and heat capacity C_{exc} per particle in the thermodynamic limit at $\Gamma = 2$ are

$$\frac{U_{\text{exc}}}{N} = -\frac{Q^2}{4}\ln(\rho_b\pi L^2) - \frac{Q^2}{4}\gamma,$$
$$\frac{C_{\text{exc}}}{N} = k_B\left(\ln 2 - \frac{\pi^2}{24}\right),$$

respectively, where $\gamma = 0.577\,215\,664\ldots$ is the Euler-Mascheroni constant. These results are also valid for the

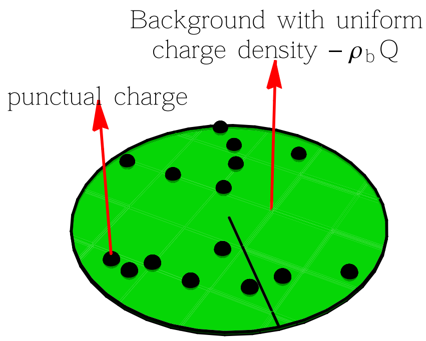


FIG. 1. The 2DOCP on a hard disk.

2DOCP on the soft disk or Dyson gas where the infinite potential barrier at R is removed since in the thermodynamic limit the barrier is moved at the infinite. However, for a finite number of particles both the soft and hard systems have substantial differences. The potential energy of the 2DOCP on the soft disk is

$$U^{\mathcal{S}}_{\text{inter}} = Q^2\left[f^{\mathcal{S}}(N) + \frac{1}{2}\rho_b\pi\sum_{i=1}^{N} r_i^2 - \sum_{1\leqslant i<j\leqslant N}\ln r_{ij}\right], \tag{3}$$

where

$$f^{\mathcal{S}}(N) = f^{\mathcal{H}}(N) - \frac{N(N-1)}{2}\ln\sqrt{\rho_b\pi N}.$$

In Ref. [3] Shakirov computed the average of the last term of Eq. (3) (which differs from the average of the particle-particle energy $\langle U^{\mathcal{S}}_{pp}\rangle$ with some additive constants),

$$\mathcal{U}^{\mathcal{S}}_{pp} = -Q^2\left\langle\sum_{1\leqslant i<j\leqslant N}\ln r_{ij}\right\rangle, \tag{4}$$

by using the replica method, finding the result

$$\begin{aligned}\mathcal{U}^{\mathcal{S}}_{pp} \underset{\rho_b=\frac{1}{\pi},\Gamma=2}{=} \frac{Q^2}{2}\bigg[&\frac{N^2}{2}H_N - \frac{N^2}{4} + \frac{3N}{4} + \frac{1}{4} + \frac{N\gamma}{2}\\ &- \frac{\Gamma(N+3/2)}{(N+1)!\sqrt{\pi}/2}\,{}_3F_2\\ &\times\left(\begin{matrix}1 & N-1 & N+3/2\\ & N+2 & N+1\end{matrix}\middle|1\right)\bigg]\end{aligned} \tag{5}$$

for $\Gamma = 2$ in terms of the hypergeometric function ${}_3F_2$, the harmonic numbers $H_N = \sum_{k=1}^{N}\frac{1}{k}$, and the Gamma function $\Gamma(x)$.[1] Although, analytic solutions for any value of Γ are limited, there are several studies of the 2DOCP in diverse geometries especially for positives integers values of Γ [7–11]. Previously, Jancovici and Téllez [11] described a way to compute the excess energy of the 2DOCP on the sphere based on the expansion the Vandermonde determinant to the power Γ. The main purpose of this work is to obtain the excess energy of the 2DOCP on a hard and a soft disk for even values of Γ.

To this aim, we will show that the approach of [11] applied on the sphere may be used in the flat geometry to obtain analytical results of the excess energy for $\Gamma = 2,6,10,\ldots$, reproducing the result obtained by Shakirov at $\Gamma = 2$ for the Dyson gas as well as the energy of the finite 2DOCP on the hard disk. The monomial expansion method may be used to expand the Vandermonde determinant to the power Γ in terms of monomial functions $\{m_{\mu_i}(z_1,\ldots,z_N)\}_{i=1,\ldots,N}$, where $z_1,\ldots,z_N$ are the particle's complex positions and μ_i are vectors with N integer entries called partitions. The main idea of the method is to write the statistical averages of the quantities as the energy as an average in terms of partitions. We stress the fact that our results for the 2DOCP on a disk are limited to odd values of $\Gamma/2$. This restriction comes from the fact that counting is more easy for odd values of $\Gamma/2$ where partitions do not have repeated elements. However, in the 2DOCP on a sphere, the problem may be solved for $\Gamma = 2,4,6\ldots$, since the expansions do not mix partitions due to the symmetry of the sphere.

In practice, the results for $\Gamma = 4,6,\ldots$ will also be limited to small systems. However, it will be shown that our analytical results are in good agreement with the ones obtained by numerical simulations.

Mermin and Wagner [12] rigorously proved that long-range positional order is not possible in two-dimensional systems with continuous degrees of freedom and sufficiently short-range interactions. Previously, it was demonstrated that systems with long-range interactions such as the 2DOCP do not have a strict long-range positional order at the thermodynamic limit [13,14]. However, this feature does not imply the absence of a solid phase with long-range orientational order and approximate positional long-range order if the 2DOCP is still far from the thermodynamic limit. Moreover, the finite 2DOCP not only crystallizes at vanishing temperature but also exhibits transitions, as it was shown in several studies [15–19]. Most recently, Monte Carlo (MC) simulations in Ref. [20] confirmed that melting of the 2DOCP on the plane with Coulomb interactions $v(r) \sim 1/r$ includes three phases, solid, hexatic, and fluid, according to the two-step scenario proposed by the Kosterlitz-Thouless-Halperin-Nelson-Young theory [21].

The 2DOCP has been considered as an ideally suited model to study strongly coupled matter since it may mimic the phase transitions of real systems, e.g., dusty plasmas [22–28], where the first observations of crystals in the laboratory were realized [22,23]. It is well known that the logarithmic Coulomb interaction between particles comes from the solution of the Poisson equation in two dimensions. However, the typical experimental setup usually confines the particles in a quasibidimensional arrangement. Even when particles may be trapped in a monolayer, they do not have a logarithmic interaction potential because the experimental layer usually has a finite thickness and the electric field does not necessary exist in a plane. Numerical simulations of the 2DOCP with alternative potentials, not necessarily a logarithmic one, may be found in the literature. Examples of these numerical studies on systems with long-range interaction are [20,29,30] for the $1/r$ Coulomb interaction and [31] for the $1/r^3$ dipolar interaction.

[1]Throughout the paper we use bold symbols for the Γ function as well as its incomplete versions in order to avoid any confusion with the coupling parameter Γ. The symbols $\mathcal{H}$ and $\mathcal{S}$ will be used to denote hard and soft-disk cases, respectively.

The paper is organized as follows. The main results of this work for the excess energy and two-body density function will be summarized in the next section. The preliminary material and the basics of the monomial expansion method will be described in Sec. III. Although, the generalities of the method may be also found in [9–11], this section is included in order to introduce the notation used throughout the present paper. The statistical average of the quadratic contribution to the energy [the quadratic sum introduced by the parabolic confining potential in Eq. (1) or (3)] is computed in Sec. IV. This energy contribution may be found without applying a monomial expansion even when $\Gamma > 2$. However, Sec. IV is included because it shows appropriately how the technique works and how several procedures described in the computation of the quadratic contribution may be extended to compute other quantities as the particle-particle interaction energy. The excess energy computation for odd values of $\Gamma/2$ is described in Secs. V–VIII. In particular, the N-finite expansion of excess energy for the 2DOCP on the soft and hard disks at $\Gamma = 2$ is presented in Sec. VII. Section IX is devoted to the analytic determination of the two-point density function for $\Gamma = 2,6,10,\ldots$ and a brief comparison between this function in the strong-coupling regime and the structure of small Wigner crystals.

II. SUMMARY OF RESULTS

The main value of a given observable $g = g(\vec{r}_1 \ldots \vec{r}_N)$ of the Dyson gas in the canonical ensemble is

$$\langle g(\vec{r}_1,\ldots,\vec{r}_N)\rangle = \frac{1}{Z^{\mathcal{S}}_{N,\Gamma}}\frac{1}{N!}\prod_{i=1}^{N}\int_0^{\infty}\int_0^{2\pi} r_i dr_i d\phi_i \times e^{-\beta U^{\mathcal{S}}_{\text{inter}}(r_1,\ldots,r_N)} g(\vec{r}_1,\ldots,\vec{r}_N),$$

where $\vec{r}_1 \ldots \vec{r}_N$ with $Z^{\mathcal{S}}_{N,\Gamma}$ the partition function. The Boltzmann factor is

$$e^{-\beta U^{\mathcal{S}}_{\text{inter}}(r_1,\ldots,r_N)} = |\Delta_N(z_i - z_j)|^{\Gamma} \times \exp\left(-f^{\mathcal{S}}(N) - \rho_b\pi\sum_{i=1}^{N} r_i^2\right),$$

with $\Delta_N(z_i - z_j) = \prod_{1\leqslant i<j\leqslant N}(z_j - z_i)$ the Vandermonde determinant and $z = r\exp(\mathbf{i}\phi)$ the complex positions of the particles. The method described in this paper is based on the expansion of the Vandermonde determinant to even values of Γ in terms of monomial functions $m_\mu(z_1,\ldots,z_N)$ and coefficients $C^{(N)}_\mu$ whose labels μ are called partitions.[2] This enables us to write the usual average $\langle g(\vec{r}_1 \ldots \vec{r}_N)\rangle$ as an average over partitions

$$\langle g(\vec{r}_1,\ldots,\vec{r}_N)\rangle = \langle G(\mu_1,\ldots,\mu_N)\rangle_N,$$

where

$$\langle G(\mu)\rangle_N := \frac{1}{\sum_\mu \frac{1}{(\prod_i m_i!)}\left[C^{(N)}_\mu(\Gamma/2)\right]^2\left(\prod_{j=1}^N \Phi_{\mu_j}\right)} \times \sum_\mu \frac{1}{(\prod_i m_i!)}\left[C^{(N)}_\mu(\Gamma/2)\right]^2\left(\prod_{j=1}^N \Phi_{\mu_j}\right) \times G(\mu_1,\ldots,\mu_N),$$

with $\prod_i m_i!$ the multiplicity of the partition μ and Φ_{μ_j} proportional to $\mu_i!$ or related to the incomplete Γ functions depending on whether the system has a soft or a hard boundary. Using this approach, we compute the excess energy of the Dyson gas $U^{\mathcal{S}}_{\text{exc}} = \langle U^{\mathcal{S}}_{\text{inter}}\rangle$ for odd values of $\Gamma/2$ as

$$U^{\mathcal{S}}_{\text{exc}} = \langle {}_{\mathcal{S}}E_\mu\rangle_N + \langle {}_{\mathcal{S}}\mathbb{U}_\mu\rangle_N. \tag{6}$$

The term $\langle E_\mu\rangle_N$ in Eq. (6) is the partition average of

$${}_{\mathcal{S}}E_\mu = Q^2\left\{-\sum_{1\leqslant i<j\leqslant N}\left[\frac{i^{\mu_i}_{\mu_j}}{\mu_i-\mu_j} + \frac{j(\mu_i,\mu_j)}{2}\right] + \frac{1}{4}N(N-1)\ln(\rho_b\pi\Gamma/2) + \frac{1}{\Gamma}\left[N + N(N-1)\frac{\Gamma}{4}\right] + f^{\mathcal{S}}(N)\right\},$$

where $i^{\mu_i}_{\mu_j}$ and $j(\mu_i,\mu_j)$ are functions of the partitions elements defined by Eqs. (31) and (A6), respectively. The other contribution of Eq. (6) is the partition average of

$${}_{\mathcal{S}}\mathbb{U}_\mu = Q^2\sum_{\nu\in\mathcal{D}_\mu}\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2)(-1)^{p+q+m+n}\frac{1}{2}f\binom{p,q}{m,n},$$

where $\mathcal{D}_\mu := \{\nu|\text{Dim}(\nu\cap\mu) = N-2\}$ corresponds to the set of all partitions ν which share $N-2$ elements with μ; the term $\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2)$, which is equal to $C^{(N)}_\nu(\Gamma/2)/C^{(N)}_\mu(\Gamma/2)$ if $C^{(N)}_\mu(\Gamma/2)\neq 0$ and 0 otherwise, is the ratio between coefficients; and $f\binom{p,q}{m,n}$ is a function of the unshared elements between μ and ν, defined as $(\mu_p,\mu_q)\notin\nu$ and $(\nu_m,\nu_n)\notin\mu$ [see Eq. (C3)]. For $\Gamma = 2$ there is only one partition $\mu = \lambda$, called the root partition, whose elements are $\lambda_i = N - i$ and $\langle {}_{\mathcal{S}}\mathbb{U}_\mu\rangle_N = 0$ since $\mathcal{D}_\mu = 0$. Hence the excess energy is

$$U^{\mathcal{S}}_{\text{exc}} \underset{\Gamma=2}{=} {}_{\mathcal{S}}E_\lambda = Q^2\left\{f^{\mathcal{S}}(N) + \frac{N(N-1)}{4}[\ln(\rho_b\pi)+1] + \frac{N}{2} - \sum_{1\leqslant i<j\leqslant N}\left[\frac{i(\lambda_i,\lambda_j)}{|\lambda_i-\lambda_j|} + \frac{j(\lambda_i,\lambda_j)}{2}\right]\right\}.$$

This result coincides with the one found by Shakirov [3] plus $f^{\mathcal{S}}(N)$ and the quadratic energy contribution $\langle\rho_b\pi\sum_{i=1} r_i^2\rangle$ by using the replica method. In particular, the excess energy per particle at $\Gamma = 2$ is

$$\lim_{N\to\infty}\frac{{}_{\mathcal{S}}E_\lambda}{N} = -0.144\,303\,92\ldots \quad \text{with } \rho_b = \frac{1}{\pi},$$

---

[2] In general, it is possible to use the multinomial theorem to expand $|\Delta_N(z_i - z_j)|^{\Gamma}$ as a polynomial whose terms are of the form $z_1^{n_1}z_2^{n_2}\cdots z_N^{n_N}$ with $(n_1,\ldots,n_N)$ a set of N-integer numbers. In fact, the method described here in some sense is a factorized version of the multinomial theorem where coefficients $C^{(N)}_\mu$ and partitions $\mu = (\mu_1,\ldots,\mu_N)$ are not trivially related to the coefficients of the multinomial theorem and the powers $(n_1,\ldots,n_N)$.

which is in agreement with the result $\frac{U_{\rm exc}}{N} = -\frac{Q^2}{4}\ln(\rho_b\pi L^2) - \frac{Q^2}{4}\gamma$ found by Jancovici [2] in the thermodynamic limit. Similarly, the excess energy of the 2DOCP on the hard disk at $\Gamma = 2$ is

$$\mathcal{U}^{\mathcal{H}}_{\rm exc} \underset{\Gamma=2}{=} {}_{\mathcal{H}}E_\lambda = Q^2\left[f^{\mathcal{H}}(N) + \frac{1}{2}\sum_{j=1}^{N}\frac{\Phi^{\mathcal{H}}_{\lambda_j+1}}{\Phi^{\mathcal{H}}_{\lambda_j}} + \sum_{1\leqslant i<j\leqslant N}\frac{4}{\Phi^{\mathcal{H}}_{\lambda_i}\Phi^{\mathcal{H}}_{\lambda_j}}\left(J^{\lambda_i,\lambda_j}_{\mathcal{H}} + J^{\lambda_j,\lambda_i}_{\mathcal{H}} - \frac{I^{\lambda_j,\lambda_i}_{\mathcal{H}}}{|\lambda_i-\lambda_j|}\right)\right]$$

for any number of particles, where $\Phi^{\mathcal{H}}_{\lambda_j}$ is related to the incomplete Γ function (11). The functions $I^{\lambda_j,\lambda_i}_{\mathcal{H}}$ and $J^{\lambda_i,\lambda_j}_{\mathcal{H}}$ are given by Eqs. (B1) and (B3). The excess energy per particle for the hard disk ${}_{\mathcal{H}}E_\lambda/N$ is also in agreement with the result found in [2] as $N\to\infty$. In this limit the 2DOCP in the hard or soft disk describes practically the same system because the hard boundary goes to the infinity if the background density $\rho_b = N/\pi R^2$ is held constant as N grows.

In this paper we also study the two-body density function

$$\rho^{(2)}_{N,\Gamma}(\vec{r},\vec{r}\,') = \left\langle \sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\delta(\vec{r}-\vec{r}_i)\delta(\vec{r}\,'-\vec{r}_j)\right\rangle,$$

where we find the result for the hard-disk case

$$\mathcal{H}\rho^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) = (\rho_b\pi)^2\big\langle \mathrm{Det}\big[{}_{\mathcal{H}}k^{(N)}_\mu(z_i,z_j)\big]_{i,j=1,2} + {}_{\mathcal{H}}\mathbb{S}_\mu\big\rangle_N,$$

valid for odd values of $\Gamma/2$. The term $\langle \mathrm{Det}[{}_{\mathcal{H}}k^{(N)}_\mu(z_i,z_j)]_{i,j=1,2}\rangle_N$ corresponds to the partition average of the functions

$${}_{\mathcal{H}}k^{(N)}_\mu(z_i,z_j) = \sum_{l=1}^{N}{}_{\mathcal{H}}\psi_{\mu_l}(z_i)\,{}_{\mathcal{H}}\psi^*_{\mu_l}(z_j),$$

depending on the complex particle's positions $z = \frac{\sqrt{N}}{R}r\exp(\mathbf{i}\phi)$ and partitions. It is built with the orthogonal functions

$${}_{\mathcal{H}}\psi_{\mu_l}(z) = \frac{z^{\mu_l}}{\sqrt{\pi\,\Phi^{\mathcal{H}}_{\mu_l}}}\exp(-|z|^2\Gamma/4).$$

When the coupling parameter is $\Gamma = 2$ the function ${}_{\mathcal{H}}k^{(N)}_\mu(z_i,z_j) = {}_{\mathcal{H}}k^{(N)}_\lambda(z_i,z_j)$ coincides with the kernel of the Ginibre ensemble [32,33]. It is remarkable to see that both the excess energy and the two-body density function for $\Gamma > 2$ partially evoke their previous expressions for $\Gamma = 2$ but in terms of partition averages of them. The second contribution ${}_{\mathcal{H}}\mathbb{S}_\mu$ of Eq. (2) is given by

$${}_{\mathcal{H}}\mathbb{S}_\mu = \frac{\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)}{\pi^2}\sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}\mathcal{R}^{(N)}_{\mu,\nu}}{\Phi^{\mathcal{H}}_{\mu_p}\Phi^{\mathcal{H}}_{\mu_q}} \times\left\{h^{\mu_p+\nu_m}_{\mu_q+\nu_n}(\tilde{r}_1,\tilde{r}_2)\cos[(\nu_m-\mu_p)\phi_{12}] - h^{\mu_q+\nu_m}_{\mu_p+\nu_n}(\tilde{r}_1,\tilde{r}_2)\cos[(\nu_m-\mu_q)\phi_{12}]\right\},$$

with $h^b_a(x,y) := x^a y^b + y^a x^b$. A similar result for the two-density function of the soft disk in terms of the rescaled complex positions $u = \sqrt{\frac{\rho_b\pi\Gamma}{2}}r\exp(\mathbf{i}\phi)$ is also found

$${}_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12}) = \left(\frac{\rho_b\pi\Gamma}{2}\right)^2\big\langle \mathrm{Det}\big[{}_{\mathcal{S}}k^{(N)}_\mu(u_i,u_j)\big]_{i,j=1,2} + {}_{\mathcal{S}}\mathbb{S}_\mu\big\rangle_N,$$

where ${}_{\mathcal{S}}k^{(N)}_\mu(u_i,u_j)$ and $\mathbb{S}_\mu$ are given by Eqs. (52) and (53). In general, the two-body density function $\rho^{(2)}_{N,\Gamma}(\vec{r},\vec{r}\,')$ depends on four parameters since $(\vec{r},\vec{r}\,')\in\Re^2$. For a homogeneous system the two-body density function is a function of the relative distance between particles $\rho^{(2)}_{N,\Gamma}(|\vec{r}-\vec{r}\,'|)$. This is not case of the 2DOCP for a finite number of particles where the soft or hard boundary does not allow a translational symmetry. However, in this paper it is shown explicitly that $\rho^{(2)}_{N,\Gamma}$ depends on the radial positions of particles r_1 and r_2 and the angle difference $\phi_{12} = \phi_1 - \phi_2$ between them, as expected, because the finite system has azimuthal symmetry. A mathematical consequence of this dependence on ϕ_{12} is the mixture of partitions contributions in ${}_{\mathcal{S}}\mathbb{S}_\mu$ and ${}_{\mathcal{S}}\mathbb{U}_\mu$ and the hard-disk version of these contributions.[3] Even though the translational should be recovered in the thermodynamic limit, in the following sections it is shown that ${}_{\mathcal{S}}\mathbb{S}_\mu$ plays an important role in the generation of small crystals as the coupling parameter is increased and the two-body density function reveals Gaussian-like functions on the expected lattice positions at vanishing temperature. Finally, it was numerically tested that Wigner crystals on the soft disk are bound by a surface defined by

$$S := \left\{(x,y) : x^2+y^2 = \left(R^{\mathcal{S}}_{N,\Gamma\to\infty}\right)^2\,\forall\, N\in\mathcal{Z}^+\right\},$$

with

$$R^{\mathcal{S}}_{N,\Gamma} = 2\sqrt{\frac{1}{\Gamma\rho_b\pi}\left[(N-1)\frac{\Gamma}{4}+1\right]}.$$

III. PARTITION FUNCTION

Our first objective is to evaluate the configurational partition function. For the hard disk it takes the form

$$Z^{\mathcal{H}}_{N,\Gamma} = \frac{1}{N!}\prod_{j=1}^{N}\int_{\rm disk} dS_j\exp\left(-\beta U^{\mathcal{H}}_{\rm inter}\right),$$

with

$$\int_{\rm disk} dS_i = \int_0^R\int_0^{2\pi} r_i dr_i d\phi_i.$$

[3] Such a mixture of partitions in the energy as well as the two-body density function for the 2DOCP on the sphere never appeared since in the sphere we are always free to put one particle in the north pole because of the symmetry of the system. As a result, the two-body density function depends on only one parameter $\rho^{(2)}_{N,\Gamma}(\theta)$, with θ the usual azimuthal angle of spherical coordinates. Hence, the function $\rho^{(2)}_{N,\Gamma}(\theta)$ describes rings on the sphere as the coupling constant is increased. Such rings are related to the Wigner crystal, which corresponds to the solution of the Thomson problem.

It is convenient to use the following change of variables $\tilde{r}_i = \frac{\sqrt{N}}{R} r_i$, keeping $\rho_b = N/\pi R^2$ constant:

$$Z^{\mathcal{H}}_{N,\Gamma} = \frac{e^{-\Gamma f^{\mathcal{H}}(N)}}{N!(\rho_b \pi)^N} \tilde{Z}^{\mathcal{H}}_{N,\Gamma}, \tag{7}$$

with

$$\tilde{Z}^{\mathcal{H}}_{N,\Gamma} = \prod_{j=1}^{N} \int_0^{\sqrt{N}} \int_0^{2\pi} \tilde{r}_i d\tilde{r}_i d\phi_i e^{-\Gamma \tilde{r}_i^2/2} \prod_{1\leqslant i<j\leqslant N} |z_i - z_j|^{\Gamma},$$

where $z_i = \tilde{r}_i \exp(\mathbf{i}\phi_i)$ are related to the particles' positions in the complex xy plane. It is possible to evaluate the partition function for even values of Γ [9,10] by using the expansion

$$\prod_{1\leqslant i<j\leqslant N} (z_j - z_i)^{\Gamma/2} = \sum_{\mu} C^{(N)}_{\mu}(\Gamma/2) m_{\mu}(z_1,\ldots,z_N). \tag{8}$$

The set of indices $\mu := (\mu_1,\ldots,\mu_N)$ is a partition of $\Gamma N(N-1)/4$ with the condition $(N-1)\Gamma/2 \geqslant \mu_1 \geqslant \mu_2 \cdots \geqslant \mu_N \geqslant 0$ for even values of $\Gamma/2$ and a partition of $\Gamma N(N-1)/4$ with the condition $(N-1)\Gamma/2 \geqslant \mu_1 > \mu_2 \cdots > \mu_N \geqslant 0$ for odd values of $\Gamma/2$. The terms $m_{\mu}(z_1,\ldots,z_N)$ are the monomial symmetric or antisymmetric functions, depending on the parity of $\Gamma/2$,

$$m_{\mu}(z_1,\ldots,z_N) = \frac{1}{\prod_i m_i!} \sum_{\sigma\in S_N} \text{sgn}(\sigma)^{b(\Gamma)} \prod_{i=1}^{N} z_i^{\mu_{\sigma(i)}},$$

where $\sum_{\sigma\in S_N}$ denotes the sum over all label permutations of a given partition $\mu_1,\ldots,\mu_N$, the variable m_i is the frequency of the index i in such a partition (one for the odd values of $\Gamma/2$), and $b(\Gamma)$ is defined as

$$b(\Gamma) = \begin{cases} 1 & \text{if } \Gamma/2 \text{ is odd} \\ 0 & \text{if } \Gamma/2 \text{ is even.} \end{cases}$$

Hence, the product $\prod_{1\leqslant i<j\leqslant N} |z_i - z_j|^{\Gamma}$ takes the form

$$\prod_{1\leqslant i<j\leqslant N} |z_i - z_j|^{\Gamma} = \sum_{\mu\nu} \frac{C^{(N)}_{\mu}(\Gamma/2) C^{(N)}_{\nu}(\Gamma/2)}{\left(\prod_i m_i!\right)^2} \times \sum_{\sigma,\omega\in S_N} \text{sgn}_{\Gamma}(\sigma,\omega) \prod_{j=1}^{N} \tilde{r}_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}} \times \exp[\mathbf{i}(\mu_{\sigma(j)} - \nu_{\omega(j)})\phi_j], \tag{9}$$

where we have defined $\text{sgn}_{\Gamma}(\sigma,\omega) := [\text{sgn}(\sigma)\text{sgn}(\omega)]^{b(\Gamma)}$. Substituting Eq. (9) into Eq. (7) and simplifying yields

$$Z^{\mathcal{H}}_{N,\Gamma} = \frac{e^{-\Gamma f^{\mathcal{H}}(N)}}{\rho_b^N} \sum_{\mu} \frac{\left[C^{(N)}_{\mu}(\Gamma/2)\right]^2}{\prod_i m_i!} \prod_{i=1}^{N} \Phi^{\mathcal{H}}_{\mu_i}, \tag{10}$$

where

$$\Phi^{\mathcal{H}}_{\mu_i} = 2\int_0^{\sqrt{N}} \exp(-\tilde{r}^2\Gamma/2) \tilde{r}^{\mu_i+1} d\tilde{r} = \left(\frac{2}{\Gamma}\right)^{2\mu_i+1} [\mu_i! - \Gamma(1+\mu_i, N\Gamma/2)], \tag{11}$$

with $\Gamma(a,x)$ the lower incomplete Gamma function. Similarly, the partition function of the Dyson gas

$$Z^{\mathcal{S}}_{N,\Gamma} = \frac{e^{-\Gamma f^{\mathcal{S}}(N)}}{N!} \tilde{Z}^{\mathcal{S}}_{N,\Gamma}, \tag{12}$$

with

$$\tilde{Z}^{\mathcal{S}}_{N,\Gamma} = \prod_{j=1}^{N} \int_0^{\infty} \int_0^{2\pi} r_i dr_i d\phi_i e^{-\rho_b \pi \Gamma r_i^2/2} \prod_{1\leqslant i<j\leqslant N} |u_i - u_j|^{\Gamma},$$

where $u_i = r_i \exp(\mathbf{i}\phi_i)$ is

$$Z^{\mathcal{S}}_{N,\Gamma} = e^{-\Gamma f^{\mathcal{S}}(N)} \sum_{\mu} \frac{\left[C^{(N)}_{\mu}(\Gamma/2)\right]^2}{\prod_i m_i!} \prod_{i=1}^{N} \Phi^{\mathcal{S}}_{\mu_i}, \tag{13}$$

where

$$\Phi^{\mathcal{S}}_{\mu_i} = 2\int_0^{\infty} \exp(-\rho_b \pi r^2 \Gamma/2) r^{\mu_i+1} dr = \left(\frac{2}{\rho_b \pi \Gamma}\right)^{\mu_i+1} \mu_i!. \tag{14}$$

Finally, the statistical average of any function $g = g(\vec{r}_1,\ldots,\vec{r}_N)$ with explicit dependence on the particles' positions will be computed in the standard form

$$\langle g(\vec{r}_1,\ldots,\vec{r}_N)\rangle = \frac{1}{Z^{\mathcal{H}}_{N,\Gamma}} \frac{1}{N!} \frac{1}{(\rho_b\pi)^N} \prod_{i=1}^{N} \int_0^{\sqrt{N}} \int_0^{2\pi} \tilde{r}_i d\tilde{r}_i d\phi_i \times e^{-\beta U^{\mathcal{H}}_{\text{inter}}(z_1,\ldots,z_N)} g(\vec{r}_1,\ldots,\vec{r}_N)$$

on the hard disk and

$$\langle g(\vec{r}_1,\ldots,\vec{r}_N)\rangle = \frac{1}{Z^{\mathcal{S}}_{N,\Gamma}} \frac{1}{N!} \prod_{i=1}^{N} \int_0^{\infty} \int_0^{2\pi} r_i dr_i d\phi_i \times e^{-\beta U^{\mathcal{S}}_{\text{inter}}(u_1,\ldots,u_N)} g(\vec{r}_1,\ldots,\vec{r}_N)$$

for the Dyson gas.

IV. THE QUADRATIC POTENTIAL CONTRIBUTION

The quadratic contribution to the excess energy of the hard disk is

$$\mathcal{U}^{\mathcal{H}}_{\text{quad}} = \frac{Q^2}{2}\left\langle \sum_{i=1}^{N} \left(\frac{\sqrt{N}}{R} r_i\right)^2 \right\rangle = N\frac{Q^2}{2}\left\langle \left(\frac{\sqrt{N}}{R} r_N\right)^2 \right\rangle. \tag{15}$$

The integrals included in $\mathcal{U}^{\mathcal{H}}_{\text{quad}}$ may be evaluated by using the expansion of Eq. (9),

$$\mathcal{U}^{\mathcal{H}}_{\text{quad}} = \frac{NQ^2}{2Z^{\mathcal{H}}_{N,\Gamma}} \frac{e^{-\Gamma f^{\mathcal{H}}(N)}}{N!(\rho_b\pi)^N} \sum_{\mu\nu} \frac{C^{(N)}_{\mu}(\Gamma/2) C^{(N)}_{\nu}(\Gamma/2)}{\left(\prod_i m_i!\right)^2} \times \sum_{\sigma,\omega\in S_N} \text{sgn}_{\Gamma}(\sigma,\omega) \pi^N \prod_{j=1}^{N-1} \Phi^{\mathcal{H}}_{\mu_{\sigma(j)}} \Phi^{\mathcal{H}}_{\mu_{\sigma(N)+1}} \prod_{l=1}^{N} \delta_{\mu_{\sigma(l)},\nu_{\omega(l)}},$$

where $\Phi^{\mathcal{H}}_{\mu_i}$ is given by Eq. (11). For odd values of $\Gamma/2$ each partition μ will not have repeated elements and the δ product $\prod_{l=1}^{N} \delta_{\mu_{\sigma(l)},\nu_{\omega(l)}}$ may be replaced by $\delta_{\mu,\nu} \prod_{l=1}^{N} \delta_{\sigma(l),\omega(l)}$. This implies that the double sum over partitions and their permutations is zero if $\mu \neq \nu$ or $\mu = \nu$, but their permuted elements are not organized in the same way. Therefore, for nonzero contributions on the sum of permutations the sign function is $\mathrm{sgn}_\Gamma(\sigma,\omega) = 1$ independently of the parity of $\Gamma/2$. In consequence, the sums will collect nonzero terms if $\mu = \nu$ but generating $(N-1)!\prod_i m_i!$ times the same result because partitions may repeat elements for even values of $\Gamma/2$ and the δ product. Hence

$$\mathcal{U}^{\mathcal{H}}_{\text{quad}} = \frac{Q^2}{2}\frac{1}{Z^{\mathcal{H}}_{N,\Gamma}}\frac{e^{-\Gamma f^{\mathcal{H}}(N)}}{\rho_b^N}\sum_\mu \frac{1}{\left(\prod_i m_i!\right)}\left[C^{(N)}_\mu(\Gamma/2)\right]^2 \times\left(\prod_{j=1}^N \Phi^{\mathcal{H}}_{\mu_j}\right)\sum_{j=1}^N\frac{\Phi^{\mathcal{H}}_{\mu_j+1}}{\Phi^{\mathcal{H}}_{\mu_j}}.$$

If the previous result for the partition function (10) of the hard disk is used, then it is possible to simplify $\mathcal{U}^{\mathcal{H}}_{\text{quad}}$ as

$$\mathcal{U}^{\mathcal{H}}_{\text{quad}} = \frac{Q^2}{2}\left\langle\sum_{i=1}^N\left(\frac{\sqrt{N}}{R}r_i\right)^2\right\rangle = \frac{Q^2}{2}\left\langle\sum_{j=1}^N\frac{\Phi^{\mathcal{H}}_{\mu_j+1}}{\Phi^{\mathcal{H}}_{\mu_j}}\right\rangle_N, \quad (16)$$

where we have defined

$$\langle G(\mu)\rangle_N := \frac{1}{\sum_\mu\frac{1}{(\prod_i m_i!)}\left[C^{(N)}_\mu(\Gamma/2)\right]^2\left(\prod_{j=1}^N\Phi_{\mu_j}\right)} \times\sum_\mu\frac{1}{\left(\prod_i m_i!\right)}\left[C^{(N)}_\mu(\Gamma/2)\right]^2\left(\prod_{j=1}^N\Phi_{\mu_j}\right) \times G(\mu_1,\dots,\mu_N) \quad (17)$$

for any function $G = G(\mu_1,\dots,\mu_N)$ with explicit dependence on the partitions elements. Throughout the paper we adopt $\langle g\rangle$ for statistical averages and $\langle G\rangle_N$ (with the subindex N) for averages on the partitions elements. Hereafter, our intention will be to change the average on the phase space of the excess energy U_{exc} for its equivalent version in terms of a permutation average as we have done with $\mathcal{U}^{\mathcal{H}}_{\text{quad}}$ in Eq. (16).

The quadratic potential contribution for the Dyson gas $\mathcal{U}^{\mathcal{S}}_{\text{quad}}$ may be obtained by using an analogous procedure and the result is

$$\mathcal{U}^{\mathcal{S}}_{\text{quad}} = \frac{Q^2}{2}\left\langle\rho_b\pi\sum_{i=1}^N r_i^2\right\rangle = \frac{Q^2}{2}\rho_b\pi\left\langle\sum_{j=1}^N\frac{\Phi^{\mathcal{S}}_{\mu_j+1}}{\Phi^{\mathcal{S}}_{\mu_j}}\right\rangle_N,$$

where $\Phi^{\mathcal{S}}_\mu$ is given by Eq. (14). It is possible to evaluate the average on partitions for the soft case because Φ_μ is proportional to the complete Γ function. Therefore, $\Phi^{\mathcal{S}}_{\mu_j+1}/\Phi^{\mathcal{S}}_{\mu_j}$ is simply $2(\mu_j+1)/\rho_b\pi\Gamma$ and $\langle\sum_{j=1}^N\Phi^{\mathcal{S}}_{\mu_j+1}/\Phi^{\mathcal{S}}_{\mu_j}\rangle_N = 2(N + \langle\sum_{j=1}^N\mu_j\rangle_N)/\rho_b\pi\Gamma$. Since the partitions elements are built holding the sum $\sum_{j=1}^N\mu_j = \sum_{j=1}^N\lambda_j = N(N-1)\Gamma/4$ constant with $\lambda_j = (N-j)\Gamma/2$ the root partition, then

$$\mathcal{U}^{\mathcal{S}}_{\text{quad}} = \frac{Q^2}{2}\left\langle\rho_b\pi\sum_{i=1}^N r_i^2\right\rangle = \frac{Q^2}{\Gamma}\left[N + N(N-1)\frac{\Gamma}{4}\right]. \quad (18)$$

An alternative but more standard way to compute this contribution for the Dyson gas and obtain an identical result is by using [34]

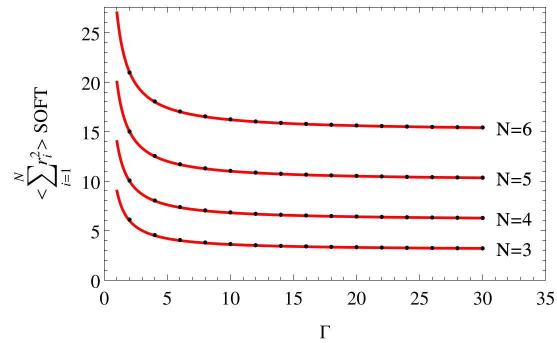


FIG. 2. Quadratic potential contribution. The solid line corresponds to $2\mathcal{U}^{\mathcal{S}}_{\text{quad}}/Q^2$ with $\rho_b = 1/\pi$ of Eq. (18) and black points correspond to the Metropolis method.

$$\mathcal{U}^{\mathcal{S}}_{\text{quad}} = -\frac{Q^2\rho_b}{\Gamma\tilde{Z}^{\mathcal{S}}_{N,\Gamma}}\frac{\partial\tilde{Z}^{\mathcal{S}}_{N,\Gamma}}{\partial\rho_b},$$

with

$$\tilde{Z}^{\mathcal{S}}_{N,\Gamma} = \left(\frac{1}{\rho_b\pi}\right)^{N(N-1)\Gamma/4+N}\prod_{j=1}^N\int_0^\infty\int_0^{2\pi} r'_i dr'_i d\phi_i e^{-\Gamma r'^2_i/2} \times\prod_{1\leqslant i<j\leqslant N}|u'_i - u'_j|^\Gamma.$$

This is Eq. (12) with $r' = \sqrt{\rho_b\pi}r$ and $u' = \sqrt{\rho_b\pi}u$. Unfortunately, it is not easy to use the same trick for the quadratic contribution of the hard disk. However, it is still possible to evaluate $\mathcal{U}^{\mathcal{H}}_{\text{quad}}$ from Eq. (16). A comparison between the quadratic energy contribution of Eq. (18) and numerical simulations with the Metropolis method [35] is shown in Fig. 2.

By definition, the 2DOCP on the hard disk is completely confined in $\mathcal{R}^{\mathcal{H}}_N = \{(x,y)|x^2+y^2\leqslant R^2\}$. In contrast, the Dyson gas is partially bounded by the quadratic potential. In fact, $\mathcal{U}^{\mathcal{S}}_{\text{quad}}$ is more confining as the coupling parameter is increased, but $\mathcal{U}^{\mathcal{S}}_{\text{quad}}$ cannot indefinitely compress the gas because of the repulsion among charges. It is expected that the 2DOCP on the soft disk in its crystal phase occupies on average a finite circular region $\mathcal{R}^{\mathcal{S}}_N$, which depends on the number of particles. Numerically, the region occupied by the crystal will have small variations due to the initial conditions used in the Metropolis simulation as well as the chain of random numbers generated. We remark that the mean square radius

$$r^{\mathcal{S}}_{N,\Gamma} = \sqrt{\frac{2}{\Gamma\rho_b\pi}\left[(N-1)\frac{\Gamma}{4}+1\right]}$$

extracted from Eq. (18) in the strong-coupling regime $r^{\mathcal{S}}_{N,\Gamma\to\infty} = \sqrt{(N-1)/2\rho_b\pi}$ defines a region of area $\pi(r^{\mathcal{S}}_{N,\Gamma\to\infty})^2$ which tends to grow proportionally to the expected area $\mathcal{R}^{\mathcal{S}}_N$ at least for a large number of particles. In order to find the radius $R^{\mathcal{S}}_N = R^{\mathcal{S}}_{N,\Gamma\to\infty}$ of the circular region $\mathcal{R}^{\mathcal{S}}_N = \{(x,y)|x^2+y^2\leqslant R^2\}$ we may begin with an extremely crude approximation of the crystal, considering it as a flat disk of uniformly distributed charge. In this scenario the mass density would be a constant $\sigma = \frac{dm}{dA} = \frac{M}{\pi(R^{\mathcal{S}}_N)^2}$ with M the total

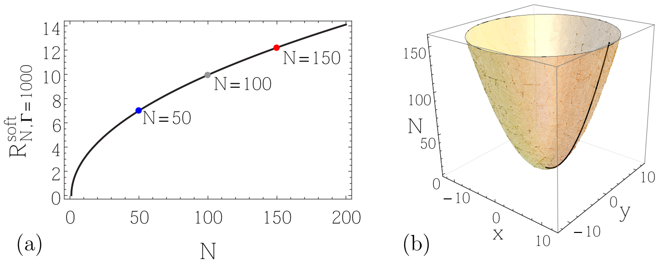


FIG. 3. Bound radius of the Dyson gas at $\Gamma = 1000$ and $\rho_b = 1/\pi$. (a) Bound radius vs the number of particles. (b) The bound radius defines a surface $S := \{(x,y) : x^2 + y^2 = (R^{\mathcal{S}}_{N,\Gamma\to\infty})^2 \, \forall \, N \in \mathcal{Z}^+\}$, which contains the Wigner crystal for the soft disk.

mass; then

$$r^{\mathcal{S}}_{N,\Gamma} \approx \sqrt{\frac{1}{M}\int_{\text{disk}} dm r^2} = \sqrt{\frac{2\pi\sigma}{M}\int_0^{R^{\mathcal{S}}_{N,\Gamma}} r^3 dr}$$

$$\therefore R^{\mathcal{S}}_{N,\Gamma} = 2\sqrt{\frac{1}{\Gamma\rho_b\pi}\left[(N-1)\frac{\Gamma}{4}+1\right]} \qquad (19)$$

and $R^{\mathcal{S}}_{N,\Gamma\to\infty} = \sqrt{(N-1)/\rho_b\pi}$. A plot of $R^{\mathcal{S}}_{N,\Gamma}$ for $\Gamma = 1000$ is shown in Fig. 3. Numerical simulations for $\rho_b = 1/\pi$ and $\Gamma = 1000$ show that the corresponding Wigner crystal of the Dyson gas tends to occupy a well-defined portion of the plane depending only on the number of particles for a fixed value of the background density (see Fig. 4).

If the background density is set as $\rho_b = N/\pi R^2$ then the radius of the 2DOCP on the hard disk would be $R^{\mathcal{H}}_N = \sqrt{N/\rho_b\pi}$. Therefore, in the strong-coupling regime we have $R^{\mathcal{S}}_{N,\Gamma\to\infty} < R^{\mathcal{H}}_N$ and thus the Wigner crystal in the hard case will never touch the hard boundary because it is completely bounded by the quadratic potential. In contrast, particles are effectively bounded by the hard frontier in the fluid phase $\Gamma = 2$ because $R^{\mathcal{S}}_{N,\Gamma=2} = \sqrt{(N+1)/\rho_b\pi} > R^{\mathcal{H}}_N$. In this situation, even the Dyson gas in the fluid phase is not necessary in the region $\mathcal{R}^{\mathcal{S}}_N$ because of the thermal fluctuations.

V. THE $\mathcal{U}_{pp}$ ENERGY CONTRIBUTION

We have written the excess energy contribution as $U_{\text{exc}} = Q^2 f(N) + \mathcal{U}_{\text{quad}} + \mathcal{U}_{pp}$. For the hard-disk $\mathcal{U}_{pp}$ contribution is given by

$$\mathcal{U}^{\mathcal{H}}_{pp} = -Q^2\left\langle \sum_{1\leqslant i<j\leqslant N} \ln\left(\frac{\sqrt{N}}{R} r_{ij}\right)\right\rangle. \qquad (20)$$

If the Vandermonde term is expanded according to Eq. (9), then $\mathcal{U}^{\mathcal{H}}_{pp}$ takes the form

$$\mathcal{U}^{\mathcal{H}}_{pp} = \frac{N(N-1)Q^2 e^{-\Gamma f^{\mathcal{H}}(N)}}{2Z^{\mathcal{H}}_{N,\Gamma} N!\rho_b^N} \sum_{\mu\nu} \frac{C^{(N)}_\mu(\Gamma/2)C^{(N)}_\nu(\Gamma/2)}{(\prod_i m_i!)^2}$$
$$\times \sum_{\sigma,\omega\in S_N} \text{sgn}_\Gamma(\sigma,\omega) 2\,\Xi^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\nu_{\omega(1)},\nu_{\omega(2)}} \prod_{j=3}^{N} \delta_{\mu_{\sigma(j)},\nu_{\omega(j)}} \Phi^{\mathcal{H}}_{\mu_{\sigma(j)}},$$

where we have defined

$$\Xi^{\mu_1,\mu_2\mathcal{H}}_{\nu_1,\nu_2} := \frac{1}{2\pi^2}\prod_{j=1}^{2}\int_0^{2\pi} d\phi_j e^{\mathbf{i}(\nu_j-\mu_j)\phi_j}$$
$$\times \int_0^{\sqrt{N}} \tilde{r}_j^{\mu_j+\nu_j+1} d\tilde{r}_j e^{-\Gamma\tilde{r}_j^2/2}(-\ln|z_1-z_2|).$$

In principle, if $\mathcal{N}(N,\Gamma)$ is the number of partitions for a given value of the coupling parameter then the double sum over partitions and their corresponding permutations would have a large number of terms $\sum_{\mu\nu}\sum_{\sigma,\omega\in S_N} 1 = \mathcal{N}(N,\Gamma)^2[\mathcal{N}(N,\Gamma)]!^2$. Fortunately, many of these terms are zero because of the incomplete δ product $\prod_{j=3}^N \delta_{\mu_{\sigma(j)},\nu_{\omega(j)}}$. In

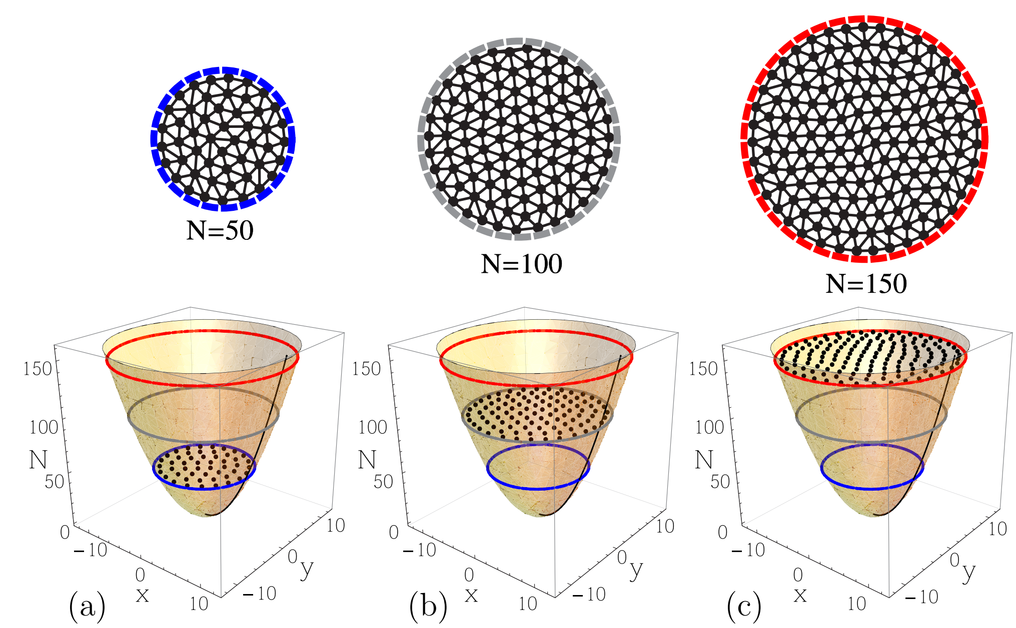


FIG. 4. Dyson gas at $\Gamma = 1000$ and $\rho_b = 1/\pi$ for (a) 50, (b) 100, and (c) 150 particles. The radii of the dashed circles are given by Eq. (19). Each crystal was obtained after 10^6 MC cycles starting at a random initial configuration.

the previous computation of $\mathcal{U}_{\text{quad}}^{\mathcal{H}}$ the complete δ product $\prod_{j=1}^{N}\delta_{\mu_{\sigma(j)},\nu_{\omega(j)}}$ selected only one partition ν for a given μ and it was $\nu=\mu$. A similar situation appears in the computation of the excess energy of the 2DOCP on the sphere [11] because the symmetry of the system enables us to write the correlation function in terms of a single parameter instead of two as happens in the hard- and soft-disk cases. The computation of $\mathcal{U}_{pp}^{\mathcal{H}}$ may be particularly difficult in comparison with the one done for $\mathcal{U}_{\text{quad}}^{\mathcal{H}}$ because in the current procedure the δ product is not complete and the term $\Xi_{\nu_1,\nu_2}^{\mu_1,\mu_2}$ for a given μ tends to select nonzero contributions of partitions ν not necessarily equal to μ. In order to deal with this potential task it is possible to split into two parts the logarithmic term of $\Xi_{\nu_1,\nu_2}^{\mu_1,\mu_2}$ as

$$-\ln|z_1-z_2|=\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\tilde{r}_<}{\tilde{r}_>}\right)^n\cos[n(\phi_2-\phi_1)]-\ln\tilde{r}_>,$$

where $\tilde{r}_<=\min(\tilde{r}_1,\tilde{r}_2)$ and $\tilde{r}_>=\max(\tilde{r}_1,\tilde{r}_2)$. This also enables us to split the whole computation into two parts

$$\mathcal{U}_{pp}^{\mathcal{H}}=\mathcal{U}_{pp\mathbf{L}}^{\mathcal{H}}+\mathcal{U}_{pp\mathbf{R}}^{\mathcal{H}}, \tag{21}$$

where the subindices $\mathbf{L}$ and $\mathbf{R}$ denote left and right, respectively, evoking each contribution of $-\ln|z_1-z_2|$, and we have defined

$$\mathcal{U}_{pp\mathbf{L}}^{\mathcal{H}}=\frac{N(N-1)Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{2Z_{N,\Gamma}^{\mathcal{H}}N!\rho_b^N}\sum_{\mu\nu}\frac{C_\mu^{(N)}(\Gamma/2)C_\nu^{(N)}(\Gamma/2)}{\left(\prod_i m_i!\right)^2}\times\sum_{\sigma,\omega\in S_N}\text{sgn}_\Gamma(\sigma,\omega)2\,\Xi_{\nu_{\omega(1)},\nu_{\omega(2)}\mathbf{L}}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}\Phi_{\mu_{\sigma(j)}}^{\mathcal{H}}, \tag{22}$$

$$\mathcal{U}_{pp\mathbf{R}}^{\mathcal{H}}=\frac{N(N-1)Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{2Z_{N,\Gamma}^{\mathcal{H}}N!\rho_b^N}\sum_{\mu\nu}\frac{C_\mu^{(N)}(\Gamma/2)C_\nu^{(N)}(\Gamma/2)}{\left(\prod_i m_i!\right)^2}\times\sum_{\sigma,\omega\in S_N}\text{sgn}_\Gamma(\sigma,\omega)2\,\Xi_{\nu_{\omega(1)},\nu_{\omega(2)}\mathbf{R}}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}\Phi_{\mu_{\sigma(j)}}^{\mathcal{H}},$$

$$\Xi_{\nu_1,\nu_2\mathbf{L}}^{\mu_1,\mu_2\mathcal{H}}:=\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{1}{n}\prod_{j=1}^{2}\int_0^{2\pi}d\phi_j e^{\mathbf{i}(\nu_j-\mu_j)\phi_j}\cos[n(\phi_2-\phi_1)]\times\int_0^{\sqrt{N}}\tilde{r}_j^{\mu_j+\nu_j+1}d\tilde{r}_j e^{-\Gamma\tilde{r}_j^2/2}\left(\frac{\tilde{r}_<}{\tilde{r}_>}\right)^n, \tag{23}$$

and

$$\Xi_{\nu_1,\nu_2\mathbf{R}}^{\mu_1,\mu_2\mathcal{H}}:=\frac{1}{2\pi^2}\prod_{j=1}^{2}\int_0^{2\pi}d\phi_j e^{\mathbf{i}(\nu_j-\mu_j)\phi_j}\times\int_0^{\sqrt{N}}\tilde{r}_j^{\mu_j+\nu_j+1}d\tilde{r}_j e^{-\Gamma\tilde{r}_j^2/2}(-\ln r_>).$$

The angular integrals of $\Xi_{\nu_1,\nu_2\mathbf{R}}^{\mu_1,\mu_2\mathcal{H}}$ are proportional to the Kronecker δ's which complete the product $\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}$ in Eq. (23). This enables us to simplify $\mathcal{U}_{pp\mathbf{R}}^{\mathcal{H}}$ as (see Appendix A)

$$\mathcal{U}_{pp\mathbf{R}}^{\mathcal{H}}=Q^2\left\langle\sum_{1\leqslant i<j\leqslant N}4\frac{J_{\mathcal{H}}^{\mu_i,\mu_j}+J_{\mathcal{H}}^{\mu_j,\mu_i}}{\Phi_{\mu_i}^{\mathcal{H}}\Phi_{\mu_j}^{\mathcal{H}}}\right\rangle_N, \tag{24}$$

where $J_{\mathcal{H}}^{\mu_j,\mu_i}$ is given by Eq. (B3). The analogous formula for the soft disk is

$$\mathcal{U}_{pp\mathbf{R}}^{\mathcal{S}}=-\frac{Q^2}{2}\left\langle\sum_{1\leqslant i<j\leqslant N}j(\mu_i,\mu_j)\right\rangle_N+\frac{Q^2}{4}N(N-1)\ln(\rho_b\pi\Gamma/2), \tag{25}$$

with $j(\mu_i,\mu_j)$ given by Eq. (A6). Although, the reduction of $\mathcal{U}_{pp\mathbf{R}}^{\mathcal{S}}$ in terms of the partition average is possible, the procedure for $\mathcal{U}_{pp\mathbf{L}}^{\mathcal{S}}$ is less evident because for a given partition μ it is possible to find another partition ν which may contribute in the expansion, as will be pointed out in the next section.

VI. COMMENTS ABOUT $\mathcal{U}_{pp\mathbf{L}}$

If the angular part of the integral $\Xi_{\nu_1,\nu_2\mathbf{L}}^{\mu_1,\mu_2\mathcal{H}}$ is evaluated, then

$$\Xi_{\nu_1,\nu_2\mathbf{L}}^{\mu_1,\mu_2\mathcal{H}}=\frac{\delta_{\mu_1+\mu_2,\nu_1+\nu_2}}{|\mu_1-\nu_1|_{\mu_1\neq\nu_1}}\tilde{\Xi}_{\nu_1,\nu_2\mathbf{L}}^{\mu_1,\mu_2\mathcal{H}}$$

where

$$\tilde{\Xi}_{\nu_1,\nu_2\mathbf{L}}^{\mu_1,\mu_2\mathcal{H}}=\prod_{j=1}^{2}\int_0^{\sqrt{N}}d\tilde{r}_j\tilde{r}_j^{\mu_j+\nu_j+1}\exp\left(-\Gamma\tilde{r}_j^2/2\right)\left(\frac{r_<}{r_>}\right)^{|\mu_1-\nu_1|}$$

and the $\mathcal{U}_{pp\mathbf{L}}$ contribution for the hard disk may be written as

$$\mathcal{U}_{pp\mathbf{L}}^{\mathcal{H}}=\frac{N(N-1)Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{Z_{N,\Gamma}^{\mathcal{H}}N!\rho_b^N}\sum_{\mu\nu}\frac{C_\mu^{(N)}(\Gamma/2)C_\nu^{(N)}(\Gamma/2)}{\left(\prod_i m_i!\right)^2}\mathcal{B}_{\mu\nu}^{\mathcal{H}}, \tag{26}$$

with

$$\mathcal{B}_{\mu\nu}^{\mathcal{H}}=\sum_{\sigma,\omega\in S_N}\text{sgn}_\Gamma(\sigma,\omega)\frac{\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\nu_{\omega(1)}+\nu_{\omega(2)}}}{|\mu_{\sigma(1)}-\nu_{\omega(1)}|_{\mu_{\sigma(1)}\neq\nu_{\omega(1)}}}\tilde{\Xi}_{\nu_{\omega(1)},\nu_{\omega(2)}\mathbf{L}}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}\times\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}\Phi_{\mu_{\sigma(j)}}^{\mathcal{H}}. \tag{27}$$

Our first task is to identify which elements of $\mathcal{B}_{\mu\nu}^{\mathcal{H}}$ are not zero. It is expected that many of the matrix elements in $\mathcal{B}_{\mu\nu}^{\mathcal{H}}$ should be zero because of the product $\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\nu_{\sigma(1)}+\nu_{\sigma(2)}}\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}$. For simplicity, we study the case $\Gamma/2=$ (odd value) where each partition μ does not have repeated elements. Defining

$$n_{\mu,\nu}:=\text{Dim}[(\mu_1,\ldots,\mu_N)\cap(\nu_1,\ldots,\nu_N)]$$

as the number of common elements between μ and ν, then for a given partition μ only a partition ν with $n_{\mu,\nu}\geqslant N-2$ or $n_{\mu,\nu}=N$ will generate a nonzero value of $\mathcal{B}_{\mu\nu}^{\mathcal{H}}$. Note that $\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}$ may be replaced by $\prod_{j=3}^{N}\delta_{\omega(j)}^{\sigma(j)}$ for odd values of $\Gamma/2$ where a change of subindex in a given partition element means strictly a change of partition value, that is, $\mu_i\neq\mu_j$ if $i\neq j$. As a result, $\prod_{j=3}^{N}\delta_{\nu_{\omega(j)}}^{\mu_{\sigma(j)}}$ is not zero only if μ and ν share $N-2$ or more elements placed in correct order after permutations. The possibilities for $\mathcal{B}_{\mu\nu}^{\mathcal{H}}\neq 0$ are reduced by noting that the case $n_{\mu,\nu}=N-1$ is forbidden because partitions are obtained by applying squeezing operations on the root partition and these types of operations do not allow $n_{\mu,\nu}=N-1$. Finally, the case $n_{\mu,\nu}=N$ must be taken into account because it is possible to permute labels to give a nonzero value of $\mathcal{B}^{\mathcal{H}}_{\mu\mu}$. Summarizing, we know that

$$\begin{aligned}\mathcal{B}^{\mathcal{H}}_{\mu\nu} &\neq 0 \quad \text{if } n_{\mu\nu} = N-2 \text{ or } n_{\mu\nu} = N\\ &= 0 \quad \text{otherwise}\end{aligned} \tag{28}$$

for odd values of $\Gamma/2$. The analysis is far from trivial when the term $\Gamma/2$ adopts even values because partitions may repeat elements and a simple condition on $n_{\mu\nu}$ is not enough to identify the nonzero contributions because the multiplicity of each partition plays an important role.

It is instructive to obtain explicitly $\mathcal{U}_{pp\mathbf{L}}$ for the simplest case $\Gamma = 2$ before continuing with $\Gamma =$ (even value). So the plan for the next section is to compute $\mathcal{U}_{pp\mathbf{L}}$ and excess energy for $\Gamma = 2$ on the hard disk and the Dyson gas, comparing with the previous results of other authors and then jumping to the most general case.

VII. EXCESS ENERGY FOR $\Gamma = 2$

The easiest case is $\Gamma = 2$ because there is only one partition $\mu = \nu = \lambda$ and therefore we only have to find $\mathcal{B}_{\lambda\lambda}$ with λ the root partition. In this section $\mathcal{B}_{\mu\mu}$ (where μ is any partition) for the case $\Gamma/2 =$ (odd value) will be computed because it contains $\mathcal{B}_{\lambda\lambda}$. Since the sign term is $\text{sgn}_\Gamma(\sigma,\omega) = (\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)})^{b(\Gamma)}$ with $\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}$ the Levi-Cività symbol, then

$$\mathcal{B}^{\mathcal{H}}_{\mu\mu} = \sum_{\sigma,\omega\in S_N} (\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)})^{b(\Gamma)} \times \frac{\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\mu_{\omega(1)}+\mu_{\omega(2)}}}{|\mu_{\sigma(1)}-\mu_{\omega(1)}|_{\mu_{\sigma(1)}\neq\mu_{\omega(1)}}}\tilde{\Xi}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}_{\mu_{\omega(1)},\mu_{\omega(2)}\mathbf{L}}\prod_{j=3}^{N}\delta^{\mu_{\sigma(j)}}_{\mu_{\omega(j)}}\Phi^{\mathcal{H}}_{\mu_{\sigma(j)}}.$$

We may use the property $\mu_i \neq \mu_j$ if $i \neq j$ for odd values of $\Gamma/2$ to obtain

$$\mathcal{B}^{\mathcal{H}}_{\mu\mu} = -\sum_{\sigma\in S_N}\frac{\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\mu_{\sigma(1)}+\mu_{\sigma(2)}}}{|\mu_{\sigma(1)}-\mu_{\sigma(1)}|_{\sigma(1)\neq\sigma(1)}}\tilde{\Xi}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}_{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathbf{L}} \times \frac{1}{\Phi^{\mathcal{H}}_{\mu_{\sigma(1)}}\Phi^{\mathcal{H}}_{\mu_{\sigma(2)}}}\prod_{j=1}^{N}\Phi^{\mathcal{H}}_{\mu_j}.$$

Here the sum over permutations will generate $(N-2)!$ times the same result for a given value of $(\sigma(1),\sigma(2))$. At the same time $\sigma(1)$ and $\sigma(2)$ will take integer values from 1 to N, therefore

$$\mathcal{B}^{\mathcal{H}}_{\mu\mu} = -(N-2)!\left(\prod_{j=1}^{N}\Phi^{\mathcal{H}}_{\mu_j}\right) \times \sum_{1\leqslant i<j\leqslant N}\frac{1}{|\mu_i-\mu_j|}\frac{\tilde{\Xi}^{\mu_i,\mu_j\mathcal{H}}_{\mu_j,\mu_i\mathbf{L}}}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}}\Bigg|_{\mu_i\neq\mu_j} \tag{29}$$

for odd values of $\Gamma/2$. The version of $\mathcal{B}_{\mu\mu}$ for the Dyson gas is obtained by changing $\Phi^{\mathcal{H}}_{\mu_i}$ and $\tilde{\Xi}^{\mu_i,\mu_j\mathcal{H}}_{\mu_j,\mu_i\mathbf{L}}$ with $\Phi^{\mathcal{S}}_{\mu_i}$ and $\tilde{\Xi}^{\mu_i,\mu_j\mathcal{S}}_{\mu_j,\mu_i\mathbf{L}}$,

$$\mathcal{B}^{\mathcal{S}}_{\mu\mu} = -(N-2)!\left(\prod_{j=1}^{N}\Phi^{\mathcal{S}}_{\mu_j}\right) \times \sum_{1\leqslant i<j\leqslant N}\left(\frac{1}{|\mu_i-\mu_j|}\right)_{\mu_i\neq\mu_j}\frac{\tilde{\Xi}^{\mu_i,\mu_j\mathcal{S}}_{\mu_j,\mu_i\mathbf{L}}}{\Phi^{\mathcal{S}}_{\mu_i}\Phi^{\mathcal{S}}_{\mu_j}}$$

for odd values of $\Gamma/2$, where

$$\begin{aligned}\tilde{\Xi}^{\mu_i,\mu_j\mathcal{S}}_{\mu_j,\mu_i\mathbf{L}} &\underset{i<j}{=} 2I^{\mu_i\mu_j}_{\mathcal{S}},\\ I^{m,n}_{\mathcal{S}} &:= \int_0^\infty dy\int_0^y dx\, x^{2m+1}y^{2n+1}e^{-(x^2+y^2)\rho_b\pi\Gamma/2}\\ &= \frac{1}{4}\left(\frac{2}{\rho_b\pi\Gamma}\right)^{m+n+2}\mathcal{I}^{m,n}.\end{aligned}$$

Hence, the $\mathcal{B}^{\mathcal{S}}_{\mu\mu}$ term takes the form

$$\mathcal{B}^{\mathcal{S}}_{\mu\mu} = -(N-2)!\left(\prod_{j=1}^{N}\Phi^{\mathcal{S}}_{\mu_j}\right)\sum_{1\leqslant i<j\leqslant N}\frac{i(\mu_i,\mu_j)}{|\mu_i-\mu_j|}\Bigg|_{\mu_i\neq\mu_j} \tag{30}$$

for odd values of $\Gamma/2$, where

$$i(k_1,k_2) = i^{k_1}_{k_2} = \frac{\mathcal{I}^{k_1,k_2}}{k_1!k_2!} = \frac{1}{2^{k_1+1}}\sum_{l=0}^{k_2}\frac{(k_1+1)_l}{l!}\left(\frac{1}{2}\right)^l. \tag{31}$$

For $\Gamma = 2$ the sum of Eq. (26) has only one term with the coefficient $C^{(N)}_\lambda(\Gamma/2) = 1$ and multiplicity $\prod_i m_i! = 1$ corresponding to the root partition λ; then

$$\begin{aligned}\mathcal{U}^{\mathcal{H}}_{pp\mathbf{L}} &\underset{\Gamma=2}{=} \frac{N(N-1)Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{Z^{\mathcal{H}}_{N,\Gamma=2}N!\rho_b^N}\mathcal{B}^{\mathcal{H}}_{\lambda\lambda}\\ &= -Q^2\sum_{1\leqslant i<j\leqslant N}\frac{1}{|\lambda_i-\lambda_j|}\frac{\tilde{\Xi}^{\lambda_i,\lambda_j\mathcal{H}}_{\lambda_j,\lambda_i\mathbf{L}}}{\Phi^{\mathcal{H}}_{\lambda_i}\Phi^{\mathcal{H}}_{\lambda_j}},\end{aligned}$$

where the partition function of the hard disk and the result of Eq. (29) were replaced. The $\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}}$ contribution and the quadratic energy contribution $\mathcal{U}^{\mathcal{H}}_{\text{quad}}$ for $\Gamma = 2$ are obtained from Eqs. (16) and (A3). Therefore,

$$\begin{aligned}\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}} &\underset{\Gamma=2}{=} Q^2\sum_{1\leqslant i<j\leqslant N}4\frac{J^{\lambda_i,\lambda_j}_{\mathcal{H}}+J^{\lambda_j,\lambda_i}_{\mathcal{H}}}{\Phi^{\mathcal{H}}_{\lambda_i}\Phi^{\mathcal{H}}_{\lambda_j}},\\ \mathcal{U}^{\mathcal{H}}_{\text{quad}} &\underset{\Gamma=2}{=} \frac{Q^2}{2}\sum_{j=1}^{N}\frac{\Phi^{\mathcal{H}}_{\lambda_j+1}}{\Phi^{\mathcal{H}}_{\lambda_j}}.\end{aligned}$$

As a result, the excess energy $\mathcal{U}^{\mathcal{H}}_{\text{exc}} = Q^2f^{\mathcal{S}}(N) + \mathcal{U}^{\mathcal{H}}_{\text{quad}} + \mathcal{U}^{\mathcal{H}}_{pp\mathbf{L}} + \mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}}$ of the hard disk for

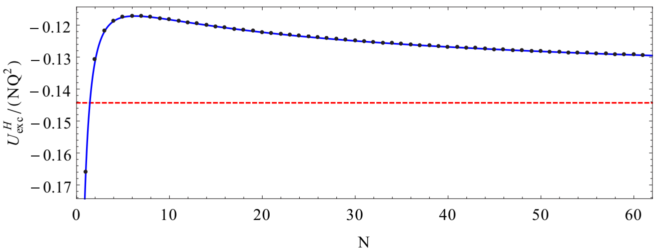


FIG. 5. Excess energy per particle and squared charge of the hard disk for $\Gamma = 2$ and setting $\rho_b = 1/\pi$. The black points correspond to Eq. (32). The red dashed line is the value of $\mathcal{U}^{\mathcal{H}}_{\rm exc}/NQ^2$ in the thermodynamic limit obtained by Jancovici [2] and the blue solid curve is the interpolation according to the ansatz of Eq. (33).

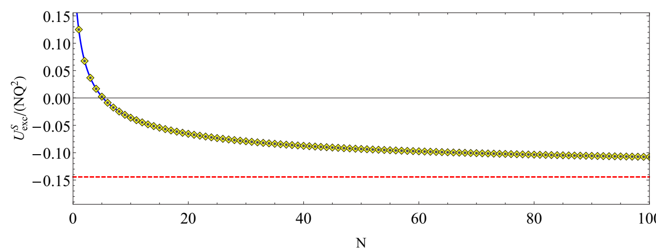


FIG. 6. Excess energy per particle and squared charge of the Dyson gas for $\Gamma = 2$ and $\rho_b = 1/\pi$. The black points correspond to Eq. (34) and the green diamond symbols are the quadratic potential contribution given by Eq. (18) plus Shakirov's result (5) and the term $Q^2 f^{\mathcal{S}}(N)$. The red dashed line is the result obtained by Jancovici for the 2DOCP on the disk in the thermodynamic limit and the blue solid curve is the interpolation with the ansatz of Eq. (35).

$\Gamma = 2$ is

$$\mathcal{U}^{\mathcal{H}}_{\rm exc} \underset{\Gamma=2}{=} Q^2\Bigg[f^{\mathcal{H}}(N) + \frac{1}{2}\sum_{j=1}^{N}\frac{\Phi^{\mathcal{H}}_{\lambda_j+1}}{\Phi^{\mathcal{H}}_{\lambda_j}} + \sum_{1\leqslant i<j\leqslant N}\frac{4}{\Phi^{\mathcal{H}}_{\lambda_i}\Phi^{\mathcal{H}}_{\lambda_j}}\left(J^{\lambda_i,\lambda_j}_{\mathcal{H}} + J^{\lambda_j,\lambda_i}_{\mathcal{H}} - \frac{I^{\lambda_j,\lambda_i}_{\mathcal{H}}}{|\lambda_i-\lambda_j|}\right)\Bigg]. \tag{32}$$

A plot of the excess energy for the disk at $\Gamma = 2$ is shown in Fig. 5. It is possible to propose the expansion

$$\mathcal{U}^{\mathcal{H}}_{\rm exc} = {}_{\mathcal{H}}K^1_\Gamma N + {}_{\mathcal{H}}K^2_\Gamma\sqrt{N} + {}_{\mathcal{H}}K^3_\Gamma + {}_{\mathcal{H}}K^4_\Gamma/N + O(1/N^2) \tag{33}$$

for large values of N. A fitting of Eq. (32) with the ansatz of Eq. (33) gives us the result $\mathcal{U}^{\mathcal{H}}_{\rm exc}/Q^2 \underset{\Gamma=2}{=} -0.144\,103N + 0.137\,482\sqrt{N} - 0.178\,439 + 0.019\,528\,8/N$, where ${}_{\mathcal{H}}K^1_{\Gamma=2}/Q^2 = -0.144\,103$ is in agreement with the expected value in the thermodynamic limit $\lim_{N\to\infty}\mathcal{U}^{\mathcal{H}}_{\rm exc}/NQ^2 = -0.144\,304\ldots$ with $\rho_b = 1/\pi$ computed in [2]. Similarly, for the Dyson gas at $\Gamma = 2$ we have, from Eqs. (A7) and (18), the results

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{R}} = -\frac{Q^2}{2}\sum_{1\leqslant i<j\leqslant N} j(\mu_i,\mu_j) + \frac{Q^2}{4}N(N-1)\ln(\rho_b\pi),$$

$$\mathcal{U}^{\mathcal{S}}_{\rm quad} = \frac{Q^2}{2}\left[N + N(N-1)\frac{1}{2}\right].$$

Hence, the excess energy of the Dyson gas is

$$\mathcal{U}^{\mathcal{S}}_{\rm exc} \underset{\Gamma=2}{=} Q^2\Bigg\{ f^{\mathcal{S}}(N) + \frac{N(N-1)}{4}[\ln(\rho_b\pi)+1] + \frac{N}{2} - \sum_{1\leqslant i<j\leqslant N}\left[\frac{i(\lambda_i,\lambda_j)}{|\lambda_i-\lambda_j|} + \frac{j(\lambda_i,\lambda_j)}{2}\right]\Bigg\}, \tag{34}$$

where $i(\lambda_i,\lambda_j)$ and $j(\lambda_i,\lambda_j)$ are given by Eqs. (31) and (A6), respectively. A plot of the excess energy according to Eq. (34) is shown in Fig. 6. This result is consistent with the one found in [3] by using the replica method. In fact, Eq. (34) provides the same result of the sum of the energy contributions of $Q^2 f^{\mathcal{S}}(N)$ [Eqs. (18) and (5)] by setting the background density as $\rho = 1/\pi$. The expansion

$$\mathcal{U}^{\mathcal{S}}_{\rm exc} = {}_{\mathcal{S}}K^1_\Gamma N + {}_{\mathcal{S}}K^2_\Gamma\sqrt{N} + {}_{\mathcal{S}}K^3_\Gamma + {}_{\mathcal{S}}K^4_\Gamma/N + O(1/N^2) \tag{35}$$

has been proposed also for the soft disk, obtaining $\mathcal{U}^{\mathcal{S}}_{\rm exc}/(Q^2) \underset{\Gamma=2}{=} -0.144\,358N + 0.377\,118\sqrt{N} - 0.109\,725 + 0.001\,571\,09/N + O(1/N^2)$. Here the bulk coefficient ${}_{\mathcal{S}}K^1_{\Gamma=2}/Q^2 = -0.144\,358$ is in agreement with the expected value in the thermodynamic limit [2]. In principle, the coefficients ${}_{\mathcal{H}}K^1_{\Gamma=2}$ and ${}_{\mathcal{S}}K^1_\Gamma$ should be equal since they correspond to the bulk energy of the 2DOCP. On the other hand, the coefficients ${}_{\mathcal{H}}K^1_\Gamma$ and ${}_{\mathcal{S}}K^1_\Gamma$ are different because they are associated with the surface tensions of the 2DOCP on hard and soft disks.

Previously, Téllez and Forrester [9] studied the N-finite expansion of the form

$$\beta F^{\rm exc}_{N,\Gamma} = \beta f_\Gamma N + B_\Gamma\sqrt{N} + k_\Gamma \ln N + C_\Gamma + D_\Gamma/N$$

for the excess free energy $\beta F^{\rm exc}_{N,\Gamma}$, where $f_\Gamma N$, B_Γ, k_Γ, C_Γ, and D_Γ are coefficients depending on Γ. These coefficients were computed exactly by Jacovici *et al.* [36] at $\Gamma = 2$ (see Table I). Note that in the ansatz of Eqs. (33) and (35) for the internal energy there is not a $\ln N$ term as in the free energy, because the study of [9] suggests that this term is a universal finite-size correction for the free energy, independent of the temperature. Since $U^{\rm exc}_{N,\Gamma} = Q^2\partial_\Gamma(\beta F^{\rm exc}_{N,\Gamma})$, this $\ln N$ correction is not present in $U^{\rm exc}_{N,\Gamma}$. We must also remark that the coefficient associated with the $\sqrt{N}$ dependence of $\beta F^{\rm exc}_{N,\Gamma=2}$ is zero at $\Gamma = 2$ only for the soft disk; this is ${}_{\mathcal{S}}B_{\Gamma=2} = 0$. However, the

TABLE I. Coefficients of $\beta F^{\rm exc}_{N,\Gamma}$ at $\Gamma = 2$. Here ${}_{\mathcal{H}}B_{\Gamma=2} = -0.477\,535\,3\ldots$ and $\zeta(x)$ is the Riemann zeta function.

Coefficient	Hard disk	Soft disk
$\beta f_{\Gamma=2}$	$\ln(\rho_b/2\pi^2)/2$	βf_2
$B_{\Gamma=2}$	$\sqrt{2}\int_0^\infty \ln[(1+\mathrm{erf}y)/2]dy$	0
$k_{\Gamma=2}$	1/12	1/12
$C_{\Gamma=2}$	0	$-\zeta'(-1)$

coefficient is activated for ${}_{\mathcal{S}}B_\Gamma > 0$ at $\Gamma > 2$ since ${}_{\mathcal{S}}K_\Gamma^2 = \partial_\Gamma B_\Gamma = 0.377\,118\ldots$ at $\Gamma = 2$. Similarly, ${}_{\mathcal{H}}K_\Gamma^2 > 0$ ensures that ${}_{\mathcal{H}}B_\Gamma$ will tend to grow around $\Gamma = 2$ as the coupling parameter is increased, which is consistent with the results of [9].

VIII. EXCESS ENERGY OF THE SOFT DISK FOR ODD VALUES OF Γ/2

The $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}}$ energy contribution for the soft disk may be found by following an analogous procedure to get Eq. (26). Then the expansion may be split into two sums

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}} = \frac{N(N-1)Q^2 e^{-\Gamma f^{\mathcal{H}}(N)}}{Z^{\mathcal{S}}_{N,\Gamma} N!\rho_b^N}\left\{\sum_\mu \left[C^{(N)}_\mu(\Gamma/2)\right]^2 \mathcal{B}^{\mathcal{S}}_{\mu\mu} + \sum_\mu \sum_{\nu\in\mathcal{D}_\mu} C^{(N)}_\mu(\Gamma/2) C^{(N)}_\nu(\Gamma/2)\mathcal{B}^{\mathcal{S}}_{\mu\nu}\right\},$$

one for $\mu = \nu$, where diagonal terms of the $\mathcal{B}^{\mathcal{S}}_{\mu\nu}$ matrix are given by Eq. (30), and the other sum for the nonzero diagonal terms of $\mathcal{B}^{\mathcal{S}}_{\mu\nu}$, where

$$\mathcal{D}_\mu := \{\nu|\dim(\mu\cap\nu) = N-2\}$$

implies that μ and ν necessarily differ in two elements, say, $(\mu_p,\mu_q) \notin \nu$ and $(\nu_m,\nu_n) \notin \mu$, with (p,q,m,n) the index positions of the unshared elements. The nonzero diagonal terms of $\mathcal{B}^{\mathcal{S}}_{\mu\nu}$ are given by (see Appendix C)

$$\mathcal{B}^{\mathcal{S}}_{\mu\nu}\begin{pmatrix} p,q \\ m,n\end{pmatrix} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{j=1}^N \Phi^{\mathcal{S}}_{\mu_j}\right)\frac{1}{2} f\begin{pmatrix} p,q \\ m,n\end{pmatrix} \quad (36)$$

for odd values of $\frac{\Gamma}{2}$ and $\dim(\mu\cap\nu) = N-2$, where $f\binom{p,q}{m,n}$ is a function defined in Eq. (C3) depending on the position of the indices of the unshared elements between partitions μ and ν. The sign $(-1)^{\tau_{\mu\nu}}$ is related to the number of transpositions required to accommodate the unshared elements of ν in the same index positions of the unshared elements of μ or vice versa (see Appendix D). Therefore, the $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}}$ energy takes the form

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}} = \frac{Q^2}{\sum_\mu \left[C^{(N)}_\mu(\Gamma/2)\right]^2}\sum_\mu \left[C^{(N)}_\mu(\Gamma/2)\right]^2\left\{-\sum_{1\leqslant i<j\leqslant N}\frac{i^{\mu_i}_{\mu_j}}{\mu_i-\mu_j} + \sum_{\nu\in\mathcal{D}_\mu}\frac{C^{(N)}_\nu(\Gamma/2)}{C^{(N)}_\mu(\Gamma/2)}(-1)^{\tau_{\mu\nu}}\frac{1}{2}f\begin{pmatrix} p,q \\ m,n\end{pmatrix}\right\}.$$

If the notation defined in Eq. (17) and the result of Eq. (D1) are used, then

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}} = Q^2\left\langle -\sum_{1\leqslant i<j\leqslant N}\frac{i^{\mu_i}_{\mu_j}}{\mu_i-\mu_j} + \sum_{\nu\in\mathcal{D}_\mu}\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2)(-1)^{p(\mu)+q(\mu)+m(\nu)+n(\nu)}\frac{1}{2}f\begin{pmatrix} p,q \\ m,n\end{pmatrix}\right\rangle_N, \quad (37)$$

where

$$\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2) = \begin{cases} \frac{C^{(N)}_\nu(\Gamma/2)}{C^{(N)}_\mu(\Gamma/2)} & \text{if } C^{(N)}_\mu(\Gamma/2) \neq 0 \\ 0 & \text{otherwise.}\end{cases}$$

The result of Eq. (37) is identical to the one obtained from

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}} = Q^2\left\langle \sum_{1\leqslant i<j\leqslant N}\sum_{n=1}^\infty \frac{1}{n}\left(\frac{\min(r_i,r_j)}{\max(r_i,r_j)}\right)^n \cos[n(\phi_j-\phi_i)]\right\rangle, \quad (38)$$

but using an average on partitions instead of computing it on the phase space. In theory, it is possible to use the Monte Carlo method to evaluate the term in angular brackets of Eq. (38) to find its thermodynamic average. However, it is more practical to evaluate the excess energy and subtract from it the contribution $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}}$. A comparison between analytical and numerical results for the $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}}$ energy is shown in Fig. 7. Now the particle-particle energy is $\mathcal{U}^{\mathcal{S}}_{pp} = \mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}} + \mathcal{U}^{\mathcal{S}}_{pp\mathbf{R}}$, with $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{R}}$ given by Eq. (A7). Hence, the $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{R}}$ contribution takes the form

$$\mathcal{U}^{\mathcal{S}}_{pp} = Q^2\left\langle -\sum_{1\leqslant i<j\leqslant N}\left[\frac{i^{\mu_i}_{\mu_j}}{\mu_i-\mu_j} + \frac{j(\mu_i,\mu_j)}{2}\right] + \sum_{\nu\in\mathcal{D}_\mu}\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2)(-1)^{p+q+m+n}\frac{1}{2}f\begin{pmatrix} p,q \\ m,n\end{pmatrix}\right\rangle_N + \frac{Q^2}{4}N(N-1)\ln(\rho_b\pi\Gamma/2). \quad (39)$$

Finally, the excess energy $U^{\mathcal{S}}_{\text{exc}} = Q^2 f^{\mathcal{S}}(N) + \mathcal{U}^{\mathcal{S}}_{\text{quad}} + \mathcal{U}^{\mathcal{S}}_{pp}$ is

$$U^{\mathcal{S}}_{\text{exc}} = Q^2\left\langle -\sum_{1\leqslant i<j\leqslant N}\left[\frac{i^{\mu_i}_{\mu_j}}{\mu_i-\mu_j} + \frac{j(\mu_i,\mu_j)}{2}\right] + \sum_{\nu\in\mathcal{D}_\mu}\mathcal{R}^{(N)}_{\mu,\nu}(\Gamma/2)(-1)^{p+q+m+n}\frac{1}{2}f\begin{pmatrix} p,q \\ m,n\end{pmatrix}\right\rangle_N + \frac{Q^2}{4}N(N-1)\ln(\rho_b\pi\Gamma/2) + \frac{Q^2}{\Gamma}\left[N + N(N-1)\frac{\Gamma}{4}\right] + Q^2 f^{\mathcal{S}}(N), \quad (40)$$

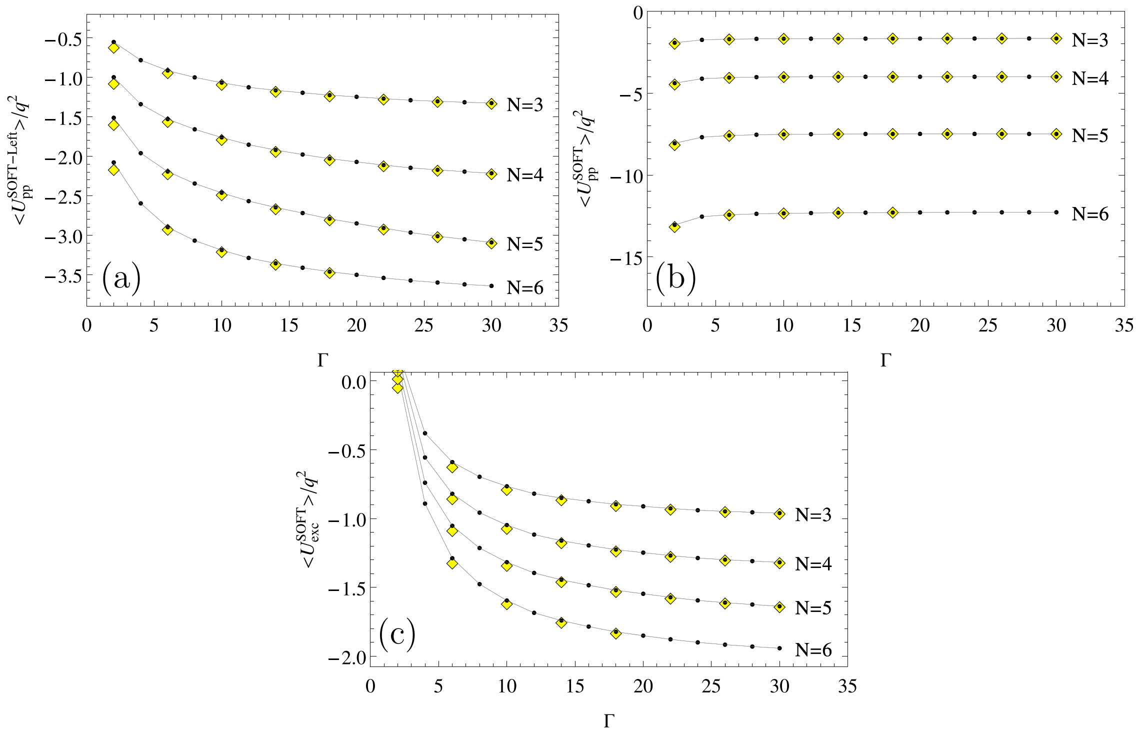


FIG. 7. Three different energy contributions for the 2DOCP on a soft disk. Comparison between numerical and analytical results for the (a) $\mathcal{U}^{\mathcal{S}}_{pp\mathbf{L}}$, (b) $\mathcal{U}^{\mathcal{S}}_{pp}$, and (c) $U^{\mathcal{S}}_{\text{exc}}$ energy contributions on the soft disk given by Eqs. (37), (39), and (40), respectively. Yellow symbols correspond to analytical results and black points to the Metropolis method. The solid lines are drawn to guide the eye.

with $\mathcal{U}^{\mathcal{S}}_{\text{quad}}$ given by Eq. (18).

IX. PAIR CORRELATION FUNCTION FOR ODD VALUES OF $\Gamma/2$

A. Hard disk

The density function associated with the probability of finding n particles in the differential area $\prod_{j=1}^{n} dS_j$ is given as

$$\rho^{(n)}_{N,\Gamma}(\vec{r}_1,\ldots,\vec{r}_n) = \frac{1}{(N-n)!}\frac{1}{Z_{N,\Gamma}}\prod_{j=n+1}^{N}\int_{\mathcal{R}}\exp(-\beta U_{\text{exc}})dS_j,$$

where $\mathcal{R}=\{(x,y)|x^2+y^2\leqslant R^2\}$ for the hard disk or the real plane $\mathcal{R}=\Re^2$ for the soft disk. This function is also known as the n-body density function and it takes the form

$$_{\mathcal{H}}\rho^{(n)}_{N,\Gamma}(\vec{z}_1,\ldots,\vec{z}_n) = \frac{N!}{(N-n)!}\frac{(\rho_b\pi)^n}{Z^{\mathcal{H}}_{N,\Gamma}}\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)\prod_{j=n+1}^{N}\int_0^{2\pi}d\phi_j\int_0^{\sqrt{N}}\tilde{r}_j d\tilde{r}_j\exp\left(-\Gamma\tilde{r}_j^2/2\right)\prod_{1\leqslant i<j\leqslant N}|z_i-z_j|^{\Gamma}$$

for the hard-disk case, where

$$\tilde{Z}^{\mathcal{H}}_{N,\Gamma} = \prod_{j=1}^{N}\int_0^{\infty}\int_0^{2\pi}\tilde{r}_i d\tilde{r}_i d\phi_i e^{-\rho_b\pi\Gamma r_i^2/2}\prod_{1\leqslant i<j\leqslant N}|z_i-z_j|^{\Gamma}$$

is the rescaled partition function, $z_i=\tilde{r}_i\exp(\mathbf{i}\phi_i)$ is the complex position on the plane of the ith particle, and $\tilde{r}_i=\sqrt{N}r_i/R$. In order to evaluate the integrals on the n-body density function, it is possible to expand the Vandermonde determinant term according to Eq. (8), as it was done with the particle-particle interaction energy in previous sections. The result is

$$_{\mathcal{H}}\rho^{(n)}_{N,\Gamma}(z_1,\ldots,z_n) = \frac{\rho_b^n\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)}{_{\mathcal{H}}\underset{\sim}{Z}_{N,\Gamma}(N-n)!}\sum_{\mu,\nu}\frac{C^{(N)}_{\mu}(\Gamma/2)C^{(N)}_{\nu}(\Gamma/2)}{\left(\prod_i m_i!\right)^2}$$

$$\times \sum_{\sigma,\omega\in S_N} \mathrm{sgn}_\Gamma(\sigma,\omega) \prod_{j=1}^{n} \tilde{r}_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}} e^{\mathbf{i}(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j} \prod_{j=n+1}^{N} \delta_{\mu_{\sigma(j)}\nu_{\omega(j)}} \Phi^{\mathcal{H}}_{\mu_{\sigma(j)}},$$

with

$$_{\mathcal{H}}\underset{\sim}{Z}_{N,\Gamma} := \sum_{\mu} \frac{\left[C_\mu^{(N)}(\Gamma/2)\right]^2}{\prod_i m_i!} \prod_{i=1}^{N} \Phi^{\mathcal{H}}_{\mu_i} = \frac{1}{N!\pi} \tilde{Z}^{\mathcal{H}}_{N,\Gamma}. \tag{41}$$

In particular, for $n = 2$ we may write

$$_{\mathcal{H}}\rho^{(2)}_{N,\Gamma}(z_1,z_2) = \frac{\rho_b^n \exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)}{_{\mathcal{H}}\underset{\sim}{Z}_{N,\Gamma}\,(N-n)!} \sum_{\mu,\nu} \frac{C_\mu^{(N)}(\Gamma/2) C_\nu^{(N)}(\Gamma/2)}{(\prod_i m_i!)^2} \underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}, \tag{42}$$

where

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu} := \sum_{\sigma,\omega\in S_N} \mathrm{sgn}_\Gamma(\sigma,\omega) \prod_{j=1}^{2} \tilde{r}_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}} e^{\mathbf{i}(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j} \prod_{j=3}^{N} \delta_{\mu_{\sigma(j)}\nu_{\omega(j)}} \Phi^{\mathcal{H}}_{\mu_{\sigma(j)}}, \tag{43}$$

which is practically a nonintegrated version of the matrix $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$ given by Eq. (27). The underdotted symbol on $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ is used only to avoid confusion with the previous notation $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$, which together share the following properties.

(i) $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$ and $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ have a nonzero contribution if $n_{\mu\nu} = \dim(\mu\cap\nu) = N$. In general, $\mathcal{B}^{\mathcal{H}}_{\mu\mu}$ and $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\mu}$ are real for any partition μ.

(ii) The case $n_{\mu\nu} = N-1$ is impossible due to the restrictions on the partitions construction. Hence, for this case we may set $\mathcal{B}^{\mathcal{H}}_{\mu\nu} = 0$ and $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu} = 0$.

(iii) $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$ and $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ may have a nonzero contribution if $n_{\mu\nu} = \dim(\mu\cap\nu) = N-2$. However, $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ is in general a complex function.

(iv) As it will be shown later, another property of $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ is its dependence only on the angle difference $\delta\phi_{12} = \phi_1 - \phi_2$ instead of both angles ϕ_1 and ϕ_2 separately because of the rotational symmetry of the system. On the other hand, $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$ is by definition independent of the angle.

This matrix may be written as (see Appendix E)

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu} = \begin{cases} (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{\substack{i=1\\ i\neq p,q}}^{N} \Phi^{\mathcal{H}}_{\mu_i}\right)\left(z_1^{\mu_p}z_2^{\mu_q} - z_1^{\mu_q}z_2^{\mu_p}\right)^*\left(z_1^{\nu_m}z_2^{\nu_n} - z_1^{\nu_n}z_2^{\nu_m}\right) & \text{if } \mu\neq\nu\in\mathcal{D}_\mu \text{ otherwise } 0\\ (N-2)!\left(\prod_{i=1}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right) \quad \mathrm{Det}\,[_{\mathcal{H}}K_\mu^{(N)}(z_i z_j^*)]_{i,j=1,2} & \text{if } \mu=\nu, \end{cases}$$

where we defined

$$_{\mathcal{H}}K_\mu^{(N)}(z) := \sum_{l=1}^{N} \frac{z^{\mu_l}}{\Phi^{\mathcal{H}}_{\mu_l}}\ .$$

Now splitting the two-body density function of Eq. (42) in two parts corresponding to $\mu=\nu$ and $\mu\neq\nu$, we obtain

$$\begin{aligned} _{\mathcal{H}}\rho^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) = \rho_b^2 \exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)&\Bigg\{\left\langle \mathrm{Det}\left[_{\mathcal{H}}K_\mu^{(N)}(z_i z_j^*)\right]_{i,j=1,2}\right\rangle_N \\ &+ \left\langle \sum_{\nu\in\mathcal{D}_\mu} (-1)^{\tau_{\mu\nu}} \mathcal{R}^{(N)}_{\mu,\nu}\left(z_1^{\mu_p}z_2^{\mu_q} - z_2^{\mu_p}z_1^{\mu_q}\right)^*\left(z_1^{\nu_m}z_2^{\nu_n} - z_2^{\nu_m}z_1^{\nu_n}\right)\right\rangle_N \Bigg\}, \end{aligned} \tag{44}$$

valid for odd values of $\Gamma/2$. The hard-disk one-body density function $_{\mathcal{H}}\rho^{(1)}_{N,\Gamma}$ (or simply the density function) may be found by applying the same technique

$$_{\mathcal{H}}\rho^{(1)}_{N,\Gamma}(\tilde{r}_1) = \rho_b e^{-(\Gamma/2)\tilde{r}_i^2} \left\langle \sum_{i=1}^{N} \frac{\tilde{r}_1^{2\mu_i}}{\Phi^{\mathcal{H}}_{\mu_i}} \right\rangle_N .$$

Therefore, the hard-disk pair correlation function would be, according to its definition,

$$_{\mathcal{H}}g^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) = \frac{_{\mathcal{H}}\rho^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12})}{_{\mathcal{H}}\rho^{(1)}_{N,\Gamma}(\tilde{r}_1)\,_{\mathcal{H}}\rho^{(1)}_{N,\Gamma}(\tilde{r}_2)}.$$

For $\Gamma = 2$ there is only one partition $\mu = \lambda$ with $\lambda_i = (N-i)\Gamma/2$. It implies that $\mathcal{D}_\lambda = 0$ and the average on partitions has only one term corresponding to the root partition. Hence

$$ {}_{\mathcal{H}}\rho^{(2)}_{N,\Gamma=2}(\tilde{r}_1,\tilde{r}_1,\phi_{12}) = \rho_b^2 \exp\left(-\sum_{i=1}^{2}\tilde{r}_i^2\right)\mathrm{Det}\big[{}_{\mathcal{H}}K^{(N)}_{\lambda}(z_i z_j^*)\big]_{i,j=1,2}, $$

where

$$ {}_{\mathcal{H}}K^{(N)}_{\lambda}(z_i z_j^*) = \sum_{l=1}^{N}\frac{(z_i z_j^*)^{\lambda_l}}{\Phi^{\mathcal{H}}_{\lambda_l}} $$

is related to the usual kernel ${}_{\mathcal{H}}k^{(N)}_{\lambda}(z_i,z_j)$ of the Ginibre ensemble but in terms of the partition $\mu = \mu(\Gamma/2)$ as

$$ {}_{\mathcal{H}}k^{(N)}_{\mu}(z_i,z_j) = \frac{1}{\pi}\exp\left(-\sum_{i=1}^{2}\tilde{r}_i^2\Gamma/4\right){}_{\mathcal{H}}K^{(N)}_{\mu}(z_i z_j^*), $$

where

$$ {}_{\mathcal{H}}k^{(N)}_{\mu}(z_i,z_j) = \sum_{l=1}^{N}{}_{\mathcal{H}}\psi_{\mu_l}(z_i)\,{}_{\mathcal{H}}\psi^*_{\mu_l}(z_j), $$

with

$$ {}_{\mathcal{H}}\psi_{\mu_l}(z) = \frac{z^{\mu_l}}{\sqrt{\pi\Phi^{\mathcal{H}}_{\mu_l}}}\exp(-|z|^2\Gamma/4) $$

orthogonal functions since they satisfy

$$ \int_{|z|<\sqrt{N}}{}_{\mathcal{H}}\psi_{\mu_l}(z)\,{}_{\mathcal{H}}\psi_{\mu_m}(z)d^2z = \delta\mu_l,\mu_m. $$

The determinant of the kernel

$$ \mathrm{Det}\big[{}_{\mathcal{H}}k^{(N)}_{\mu}(z_i,z_j)\big]_{i,j=1,2} = \frac{1}{\pi^2}\exp\left(-\sum_{i=1}^{2}\tilde{r}_i^2\Gamma/2\right)\mathrm{Det}\big[{}_{\mathcal{H}}K^{(N)}_{\mu}(z_i z_j^*)\big]_{i,j=1,2} $$

depends only on the radial positions $\tilde{r}_1$ and $\tilde{r}_2$ of the particles on the disk and the difference of their angular positions $\phi_{12} = \phi_1 - \phi_2$ since

$$ \mathrm{Det}\big[{}_{\mathcal{H}}K^{(N)}_{\mu}(z_i z_j^*)\big]_{i,j=1,2} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}}\left\{\tilde{r}_1^{2\mu_i}\tilde{r}_2^{2\mu_j} - (\tilde{r}_1\tilde{r}_2)^{\mu_i+\mu_j}\cos[(\mu_j-\mu_i)(\phi_1-\phi_2)]\right\} $$

and it is real. Therefore,

$$ \begin{aligned} {}_{\mathcal{H}}\rho^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) = \Bigg\{&(\rho_b\pi)^2\big\langle\mathrm{Det}\big[{}_{\mathcal{H}}k^{(N)}_{\mu}(z_i,z_j)\big]_{i,j=1,2}\big\rangle_N + \rho_b^2\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)\\ &\times\left\langle\sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}}{\Phi^{\mathcal{H}}_{\mu_p}\Phi^{\mathcal{H}}_{\mu_q}}\mathcal{R}^{(N)}_{\mu,\nu}\big(z_1^{\mu_p}z_2^{\mu_q} - z_2^{\mu_p}z_1^{\mu_q}\big)^*\big(z_1^{\nu_m}z_2^{\nu_n} - z_2^{\nu_m}z_1^{\nu_n}\big)\right\rangle_N\Bigg\}. \end{aligned} $$

It is important to remark that crystals in the hard or soft disk do not have translational symmetry except in the thermodynamic limit where the crystal is filling the whole plane. This feature appears in the two-body density function as an explicit dependence on the angle difference ${}_{\mathcal{H}}\rho^{(2)}_{N,\Gamma} = {}_{\mathcal{H}}\rho^{(2)}_{N,\Gamma}(\tilde{r}_1,\tilde{r}_2,\phi_{12})$ and a mixture of partitions on the term

$$ {}_{\mathcal{H}}S_\mu(\tilde{r}_1,\tilde{r}_2,\phi_{12}) = \frac{1}{\pi^2}\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)\sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}}{\Phi^{\mathcal{H}}_{\mu_p}\Phi^{\mathcal{H}}_{\mu_q}}\mathcal{R}^{(N)}_{\mu,\nu}\big(z_1^{\mu_p}z_2^{\mu_q} - z_2^{\mu_p}z_1^{\mu_q}\big)^*\big(z_1^{\nu_m}z_2^{\nu_n} - z_2^{\nu_m}z_1^{\nu_n}\big). $$

Although, the function ${}_{\mathcal{H}}S_\mu(\tilde{r}_1,\tilde{r}_2,\phi_{12})$ is complex, its average over partitions is real

$$ \langle{}_{\mathcal{H}}S_\mu\rangle_N = \frac{1}{\pi^2}\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde{r}_i^2\right)\left\langle\sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}}{\Phi^{\mathcal{H}}_{\mu_p}\Phi^{\mathcal{H}}_{\mu_q}}\mathcal{R}^{(N)}_{\mu,\nu}\mathrm{Re}\big[\big(z_1^{\mu_p}z_2^{\mu_q} - z_2^{\mu_p}z_1^{\mu_q}\big)^*\big(z_1^{\nu_m}z_2^{\nu_n} - z_2^{\nu_m}z_1^{\nu_n}\big)\big]\right\rangle_N. $$

Now it may be simplified by using $(z_1^{\mu_p}z_2^{\mu_q}-z_2^{\mu_p}z_1^{\mu_q})^*(z_1^{\nu_m}z_2^{\nu_n}-z_2^{\nu_m}z_1^{\nu_n})=f_{m,n}^{p,q}(\tilde r_1,\tilde r_2,\phi_{12})+f_{m,n}^{p,q}(\tilde r_2,\tilde r_1,\phi_{21})$. As a result, the two-body density function of the hard disk for odd values of $\Gamma/2$ takes the form

$$ {}_{\mathcal H}\rho^{(2)}_{N,\Gamma}(\tilde r_1,\tilde r_2,\phi_{12})=(\rho_b\pi)^2\big\langle \mathrm{Det}\big[{}_{\mathcal H}k^{(N)}_{\mu}(z_i,z_j)\big]_{i,j=1,2}+{}_{\mathcal H}\mathbb S_\mu\big\rangle_N, \tag{45}$$

where

$$ {}_{\mathcal H}k^{(N)}_{\mu}(z_i,z_j)=\sum_{l=1}^{N}{}_{\mathcal H}\psi_{\mu_l}(z_i)\,{}_{\mathcal H}\psi^*_{\mu_l}(z_j)$$

is built with the orthogonal functions

$$ {}_{\mathcal H}\psi_{\mu_l}(z)=\frac{z^{\mu_l}}{\sqrt{\pi\,\Phi^{\mathcal H}_{\mu_l}}}\exp(-|z|^2\Gamma/4),$$

the term ${}_{\mathcal H}\mathbb S_\mu:=\mathrm{Re}[{}_{\mathcal H}S_\mu]$ is given by

$$ {}_{\mathcal H}\mathbb S_\mu=\frac{\exp\left(-\frac{\Gamma}{2}\sum_{i=1}^{2}\tilde r_i^2\right)}{\pi}\sum_{\nu\in\mathcal D_\mu}\frac{(-1)^{\tau_{\mu\nu}}\mathcal R^{(N)}_{\mu,\nu}}{\Phi^{\mathcal H}_{\mu_p}\Phi^{\mathcal H}_{\mu_q}}\left\{h^{\mu_p+\nu_m}_{\mu_q+\nu_n}(\tilde r_1,\tilde r_2)\cos[(\nu_m-\mu_p)\phi_{12}]-h^{\mu_q+\nu_m}_{\mu_p+\nu_n}(\tilde r_1,\tilde r_2)\cos[(\nu_m-\mu_q)\phi_{12}]\right\}$$

with $h_a^b(x,y):=x^ay^b+y^ax^b$, and the average over partitions is defined according to Eq. (17) with Φ_{μ_j} replaced by $\Phi^{\mathcal H}_{\mu_j}$.

B. Soft disk

The n-body density function of the soft disk is

$$ {}_{\mathcal S}\rho^{(n)}_{N,\Gamma}(z_1,\dots,z_n)=\frac{\exp\left(-\rho_b\pi\frac{\Gamma}{2}\sum_{i=1}^{n}r_i^2\right)}{{}_{\mathcal S}\tilde Z_{N,\Gamma}(N-n)!}\sum_{\mu,\nu}\frac{C^{(N)}_\mu(\Gamma/2)C^{(N)}_\nu(\Gamma/2)}{(\prod_i m_i!)^2}\times\sum_{\sigma,\omega\in S_N}\mathrm{sgn}_\Gamma(\sigma,\omega)\prod_{j=1}^{n}r_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}}e^{\mathbf i(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j}\prod_{j=n+1}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\Phi^{\mathcal S}_{\mu_{\sigma(j)}},$$

where ${}_{\mathcal S}\tilde Z_{N,\Gamma}$ is defined in Eq. (12). It is convenient to write $\Phi^{\mathcal S}_{\mu_i}=(\frac{2}{\rho_b\pi\Gamma})^{\mu_i+1}\mu_i!$ explicitly in terms of the partitions' elements factorial [see Eq. (14)] so the product $\prod_{j=n+1}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\Phi^{\mathcal S}_{\mu_{\sigma(j)}}$ may be written as $(\rho_b\pi\Gamma/2)^n(2/\rho_b\pi\Gamma)^{N(N+1)\Gamma/4+1}\prod_{j=1}^{n}(\rho_b\pi\Gamma/2)^{\mu_{\sigma(j)}}\prod_{j=n+1}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\mu_{\sigma(j)}!$ and the n-body density function takes the form

$$ {}_{\mathcal S}\rho^{(n)}_{N,\Gamma}(z_1,\dots,z_n)=\frac{(\rho_b\Gamma/2)^n\exp\left(-\rho_b\pi\frac{\Gamma}{2}\sum_{i=1}^{n}r_i^2\right)}{{}_{\mathcal S}\underset{\sim}{Z}_{N,\Gamma}(N-n)!}\sum_{\mu,\nu}\frac{C^{(N)}_\mu(\Gamma/2)C^{(N)}_\nu(\Gamma/2)}{\left(\prod_i m_i!\right)^2}\times\sum_{\sigma,\omega\in S_N}\mathrm{sgn}_\Gamma(\sigma,\omega)\prod_{j=1}^{n}\left(\frac{\rho_b\pi\Gamma}{2}\right)^{\mu_{\sigma(j)}}r_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}}e^{\mathbf i(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j}\prod_{j=n+1}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\mu_{\sigma(j)}!,$$

with

$$\tilde Z_{N,\Gamma}=\left(\frac{2}{\rho_b\pi\Gamma}\right)^{N(N+1)\Gamma/4+1}\underset{\sim}{Z}_{N,\Gamma},\quad {}_{\mathcal S}\underset{\sim}{Z}_{N,\Gamma}:=\sum_{\mu}\frac{[C^{(N)}_\mu(\Gamma/2)]^2}{\prod_i m_i!}\prod_{i=1}^{N}\mu_i!. \tag{46}$$

The two-body density function may be written as

$$ {}_{\mathcal S}\rho^{(2)}_{N,\Gamma}(z_1,z_2)=\frac{\rho_b^2\exp\left(-\rho_b\pi\frac{\Gamma}{2}\sum_{i=1}^{2}r_i^2\right)}{{}_{\mathcal S}\underset{\sim}{Z}_{N,\Gamma}(N-n)!}\sum_{\mu,\nu}\frac{C^{(N)}_\mu(\Gamma/2)C^{(N)}_\nu(\Gamma/2)}{(\prod_i m_i!)^2}\mathcal B^{\mathcal S}_{\bullet\,\mu\nu}, \tag{47}$$

where

$$\mathcal B^{\mathcal S}_{\bullet\,\mu\nu}:=\left(\frac{\Gamma}{2}\right)^2\sum_{\sigma,\omega\in S_N}\mathrm{sgn}_\Gamma(\sigma,\omega)\prod_{j=1}^{2}\left(\frac{\rho_b\pi\Gamma}{2}\right)^{\mu_{\sigma(j)}}r_j^{\mu_{\sigma(j)}+\nu_{\omega(j)}}e^{\mathbf i(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j}\prod_{j=3}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\mu_{\sigma(j)}!, \tag{48}$$

as we have done with the hard case [see Eqs. (42) and (43)]. The δ product $\prod_{j=3}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}$ in Eq. (48) ensures that $\mu_{\sigma(1)}+\mu_{\sigma(2)}=\nu_{\sigma(1)}+\nu_{\sigma(2)}$ and several of the permutations $(\nu_{\sigma(1)},\nu_{\sigma(2)})$ are just squeezing operations on the partition μ. Hence,

$$\prod_{j=1}^{2}\left(\frac{\rho_b\pi\Gamma}{2}\right)^{\mu_{\sigma(j)}}=\prod_{j=1}^{2}\left(\frac{\rho_b\pi\Gamma}{2}\right)^{\nu_{\sigma(j)}}=\prod_{j=1}^{2}\left(\sqrt{\frac{\rho_b\pi\Gamma}{2}}\right)^{\mu_{\sigma(j)}+\nu_{\sigma(j)}}$$

enables us to write the two-body density function in terms of the dimensionless complex variable $u=\sqrt{\frac{\rho_b\pi\Gamma}{2}}r\exp(\mathbf{i}\phi)$ as

$$_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(u_1,u_2)=\frac{\rho_b^2\exp\left(-\sum_{i=1}^{2}|u_i|^2\right)}{_{\mathcal{S}}\underset{\sim}{Z}_{N,\Gamma}\,(N-n)!}\sum_{\mu,\nu}\frac{C^{(N)}_{\mu}(\Gamma/2)C^{(N)}_{\nu}(\Gamma/2)}{\left(\prod_i m_i!\right)^2}\underset{\bullet}{\mathcal{B}}^{\mathcal{S}}_{\mu\nu}(u_1,u_2), \tag{49}$$

where

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{S}}_{\mu\nu}:=\left(\frac{\Gamma}{2}\right)^2\sum_{\sigma,\omega\in S_N}\mathrm{sgn}_{\Gamma}(\sigma,\omega)\prod_{j=1}^{2}|u_j|^{\mu_{\sigma(j)}+\nu_{\omega(j)}}e^{\mathbf{i}(\nu_{\omega(j)}-\mu_{\sigma(j)})\phi_j}\prod_{j=3}^{N}\delta_{\mu_{\sigma(j)}\nu_{\omega(j)}}\mu_{\sigma(j)}!. \tag{50}$$

A similar argument may be used to write the n-body density function of the soft disk in terms of $u_1,\dots,u_n$. We may simplify $\underset{\bullet}{\mathcal{B}}^{\mathcal{S}}_{\mu\nu}$ as it was done for $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ to obtain the result

$$\frac{\underset{\bullet}{\mathcal{B}}^{\mathcal{S}}_{\mu\nu}}{(\Gamma/2)^2}=\begin{cases}(-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{\substack{i=1\\ i\neq p,q}}^{N}\mu_i!\right)\left(u_1^{\mu_p}u_2^{\mu_q}-u_1^{\mu_q}u_2^{\mu_p}\right)^*\left(u_1^{\nu_m}u_2^{\nu_n}-u_1^{\nu_n}u_2^{\nu_m}\right) & \text{if } \mu\neq\nu\in\mathcal{D}_\mu \text{ otherwise } 0\\ (N-2)!\left(\prod_{i=1}^{N}\mu_i!\right)\mathrm{Det}\left[_{\mathcal{S}}K^{(N)}_{\mu}(u_iu_j^*)\right]_{i,j=1,2} & \text{if } \mu=\nu,\end{cases}$$

where

$$_{\mathcal{S}}K^{(N)}_{\mu}(u_iu_j^*):=\sum_{l=0}^{N}\frac{(u_iu_j^*)^{\mu_l}}{\mu_l!}.$$

Therefore,

$$_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})=\left(\frac{\rho_b\pi\Gamma}{2}\right)^2\left\langle\mathrm{Det}\left[_{\mathcal{S}}k^{(N)}_{\mu}(u_i,u_j)\right]_{i,j=1,2}+{_{\mathcal{S}}\mathbb{S}_\mu}\right\rangle_N, \tag{51}$$

where the average over partitions is defined according to Eq. (17) with Φ_{μ_j} replaced by $\mu_j!$,

$$_{\mathcal{S}}k^{(N)}_{\mu}(u_i,u_j)=\sum_{l=1}^{N}{_{\mathcal{S}}\psi_{\mu_l}(u_i)}{_{\mathcal{S}}\psi^*_{\mu_l}(u_j)},\quad {_{\mathcal{S}}\psi_{\mu_l}(u)}=\frac{u^{\mu_l}}{\sqrt{\pi\mu_l!}}\exp\left(-\frac{1}{2}|u|^2\right), \tag{52}$$

with $_{\mathcal{S}}\psi_{\mu_m}(u)$ orthogonal functions

$$\int_{\Re^2}{_{\mathcal{S}}\psi_{\mu_l}(u)}{_{\mathcal{S}}\psi_{\mu_m}(u)}d^2z=\frac{\delta\mu_l,\mu_m}{2^{\mu+1}}$$

and

$$_{\mathcal{S}}\mathbb{S}_\mu=\frac{e^{-(|u_1|^2+|u_2|^2)}}{\pi^2}\sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}\mathcal{R}^{(N)}_{\mu,\nu}}{\mu_p!\mu_q!}\left(h^{\mu_p+\nu_m}_{\mu_q+\nu_n}(|u_1|,|u_2|)\cos[(\nu_m-\mu_p)\phi_{12}]-h^{\mu_q+\nu_m}_{\mu_p+\nu_n}(|u_1|,|u_2|)\cos[(\nu_m-\mu_q)\phi_{12}]\right). \tag{53}$$

We have checked that Eq. (51) fulfills the normalization condition

$$\prod_{j=1}^{2}\int_{\mathcal{R}}dS_j\,{_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})}=N(N-1),$$

which ensures that, for any measurement, it is possible to find $N(N-1)$ pairs of particles in the total area (the real xy plane). On the other hand, the limit

$$_{\mathcal{S}}\rho^{(2),II}_{N,\Gamma}(r,\phi_{12}):=\lim_{r_1\to r_2=r}{_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})}$$

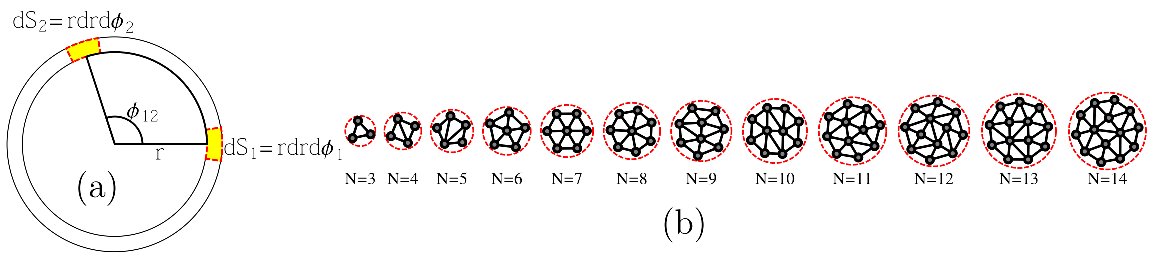


FIG. 8. (a) Probability density ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$. (b) Small crystals. The first 14 configurations are at vanishing temperature of the 2DOCP on the soft disk obtained with the Metropolis method with their corresponding Delaunay triangulation. The radius of the red dashed circles are given by Eq. (19).

is the density function related to the probability to find a particle in the differential element $dS_1 = rdrd\phi_1$ at (r,ϕ_1) and another particle in $dS_2 = rdrd\phi_2$ located at (r,ϕ_2) [see Fig. 8(a)]. Explicitly, the function ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$ is

$$
{}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12}) = \left(\frac{\rho_b\Gamma}{2}\right)^2 e^{-2|u|^2}\left\{\left\langle 2\sum_{m=1}^{N}\sum_{l=m+1}^{N}\frac{|u|^{2(\mu_l+\mu_m)}}{\mu_l!\mu_m!}\{1-\cos[(\mu_l-\mu_m)\phi_{12}]\}\right\rangle_N \right.
$$

$$
\left. + \left\langle \sum_{\nu\in\mathcal{D}_\mu}\frac{(-1)^{\tau_{\mu\nu}}\mathcal{R}^{(N)}_{\mu,\nu}}{\mu_p!\mu_q!}2|u|^{\mu_p+\mu_q}\{\cos[(\nu_m-\mu_p)\phi_{12}]-\cos[(\nu_m-\mu_q)\phi_{12}]\}\right\rangle_N\right\}. \tag{54}
$$

Note that, independently of N and Γ, the probability density ${}_S\rho^{(2),II}_{N,\Gamma}(r,0)$ for $\phi_{12}=0$ is zero because it is not possible to find in the equilibrium state two charged particles located at the same position. Similarly, the limit

$$
\lim_{r\to\infty} {}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12}) = 0
$$

because the partition average terms generate a polynomial $\mathcal{P}_{N,\Gamma}$ for finite values of N and Γ. The number of terms of the polynomial $\mathcal{P}_{N,\Gamma}$ becomes large but finite as the number of particles or the coupling parameter is increased. Therefore, the product $e^{-2|u|^2}\mathcal{P}_{N,\Gamma}$ goes to zero as $|u|\to\infty$. It obeys the fact that the radial parabolic potential generated by the background tends to confine the charges and the probability to find a pair of particles far from the origin becomes negligible.

A plot of Eq. (54) for three particles and several values of the coupling parameter is shown in Figs. 9 and 10. When the coupling parameter is $\Gamma = 2$ the function ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$ is reduced to

$$
{}_S\rho^{(2),II}_{N,\Gamma=2}(r,\phi_{12}) = \rho_b^2 e^{-2|u|^2} 2\sum_{m=1}^{N}\sum_{l=m+1}^{N}\frac{|u|^{2(\lambda_l+\lambda_m)}}{\lambda_l!\lambda_m!} \times\{1-\cos[(m-l)\phi_{12}]\}.
$$

In theory, if we would perform a measurement finding a particle in the position (r,ϕ_1) and another at (r',ϕ_2), then it is possible to rotate the system $-\phi_1$ due to the rotational invariance. The result of several measurements yields that the first particle should be somewhere in the line $\{(r,\phi=0)|r\geqslant 0\}$ and the second particle at $(r',\phi_2-\phi_1)$. Hence, it is not a surprise that plots of ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$ vanish around the line $\{(r,\phi=0)|r\geqslant 0\}$. As the coupling constant is increased the plot of ${}_S\rho^{(2),II}_{N=3,\Gamma}(r,\phi_{12})$ for three particles splits into two Gaussian-like functions and the locations of the peaks of these Gaussian-like functions are related to the minimal energy configuration: three point charges located at the vertices of an equilateral triangle (see Fig. 8). In fact, if we set $\phi=0$ for one of the three particles of the Wigner crystal by a rotation, then the corresponding positions of the other two particles coincide with maximum locations of ${}_S\rho^{(2),II}_{N=3,\Gamma}(r,\phi_{12})$ as Γ increases.

Even when small systems of $N\leqslant 5$ particles do not exhibit phase transitions, they show crystallization features (localization of the $\rho^{(2),II}_{N,\Gamma}$ around the particle's positions at vanishing temperature) for $\Gamma\simeq 14$. Numerical simulations performed on the 2DOCP with logarithmic interactions [16,17] expected a liquid-solid phase transition at $\Gamma\simeq 130$–140 in the thermodynamic limit, which is far from $\Gamma\simeq 14$. This difference may be attributed to the shifting of the critical coupling constants $\Gamma_c=\Gamma_c(N)$ for finite systems where $\Gamma_c(N_1)<\Gamma_c(N_2)$ if $N_1<N_2$, as it is shown numerically in 2DOCP with Coulomb interactions [20].

Roughly speaking, the 2DOCP mimics some properties of the dusty plasmas realized in the laboratory. Commonly, there is more interest in the generation of dusty plasmas with a large number of particles, which enables measurements in the thermodynamic limit. However, monolayer plasma systems with a low number of particles have also been obtained experimentally. In particular, Goree *et al.* [28] reported small plasma crystals with $N\in(1,19)$. The experiment and the 2DOCP plasma have in common a radial parabolic potential which confines the microspheres. In the laboratory the charged particles are microspheres of diameter 9 μm with charge $Q=-12.3e$ which tend to arrange essentially in the same configurations as in Fig. 8 up to a scale factor because the interparticle repulsion for the experiment practically comes from a Yukawa potential instead of a logarithmic one. In fact, Goree *et al.* [28] expected a Yukawa interaction potential since the positions of particles for small crystals are accurately

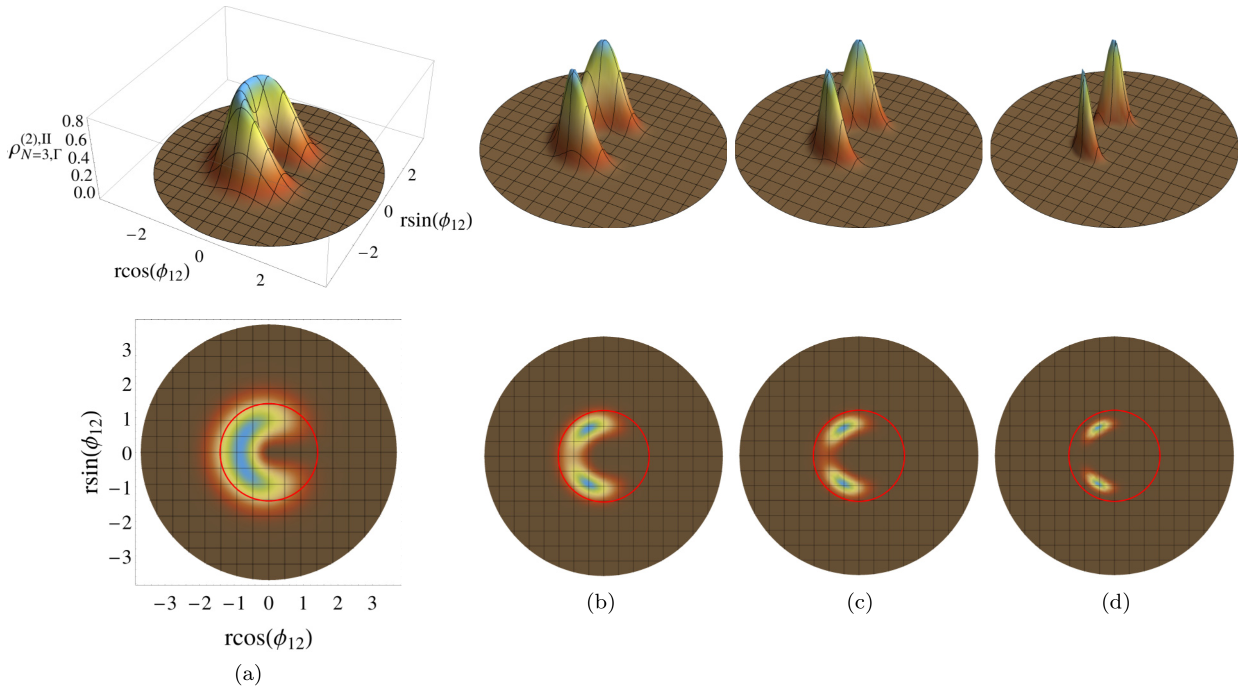


FIG. 9. Analytical probability density ${}_S\rho^{(2),II}_{N=3,\Gamma}(r,\phi_{12})$ for three particles computed from Eq. (54). Shown, from left to right, are plots of ${}_S\rho^{(2)}_{N=3,\Gamma}$ setting the coupling parameter as (a) $\Gamma = 2$, (b) $\Gamma = 6$, (c) $\Gamma = 10$, and (d) $\Gamma = 26$. The radius of the red circle on the density plots is given by Eq. (19).

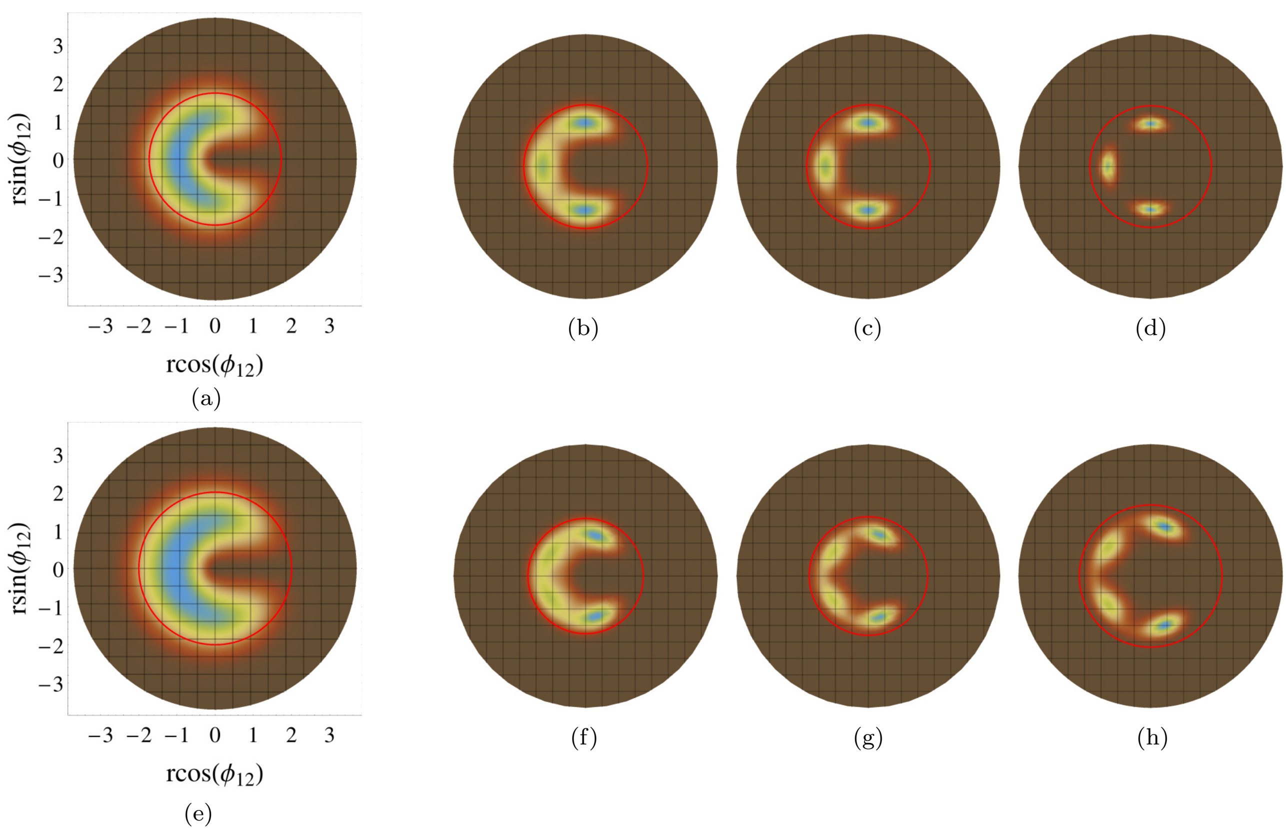


FIG. 10. Analytical probability density ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$ for (a)–(d) $N = 4$ particles and (e)–(h) $N = 5$ particles computed from Eq. (54). The probability density ${}_S\rho^{(2)}_{N=4,\Gamma}$ has been plotted by setting the coupling parameter as (a) $\Gamma = 2$, (b) $\Gamma = 6$, (c) $\Gamma = 10$, and (d) $\Gamma = 26$. The plots of ${}_S\rho^{(2)}_{N=5,\Gamma}$ correspond to the coupling parameters (e) $\Gamma = 2$, (f) $\Gamma = 6$ (g) $\Gamma = 10$, and (h) $\Gamma = 14$.

modeled by simulations performed with Yukawa molecular dynamics. Previously, Šamaj and Percus [37] performed numerically exact expansions of the free-energy and kinetic pressure for the 2DOCP on the hard disk with a small number of particles with Γ ranging from 2 to 14 which agree reasonably well with MC simulations.

It is possible to continue an exploration of the radial dependence of the two-body density function by looking for the density function related to the probability to find a particle in the origin and another particle at (r_2,ϕ_2). Hence, the following limit must be considered:

$$_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}(r_2,\phi_{12}) := \lim_{r_1\to 0} {_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}}(r_1,r_2,\phi_{12}).$$

This limit is simplified because

$$\lim_{r_1\to 0} {_{\mathcal{S}}\mathbb{S}_\mu} = 0$$

and the only contribution on $_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}(r_2,\phi_{12})$ comes from the kernel's determinant

$$\begin{aligned}\mathrm{Det}\big[{_{\mathcal{S}}k^{(N)}_\mu}(u_i u_j^*)\big]_{i,j=1,2} &= \frac{e^{-|u_1|^2-|u_2|^2}}{\pi^2}\sum_{m=1}^{N}\sum_{\substack{n=1\\ n\neq m}}^{N}\frac{1}{\mu_m!\mu_n!}\{|u_1|^{2\mu_m}|u_2|^{2\mu_n}\\ &\quad -(|u_1||u_2|)^{\mu_m+\mu_n}\cos[(\mu_n-\mu_m)\phi_{12}]\}. \qquad (55)\end{aligned}$$

Now the term $(|u_1||u_2|)^{\mu_m+\mu_n}$ in the limit $r_1\to 0$ is always zero because $\mu_m\neq\mu_n$ once a partition is selected. However, one term of the kernel's determinant may contribute

$$\lim_{r_1\to 0}|u_1|^{2\mu_m} = \delta_{\mu_m,0}$$

since the partitions restriction $\mu_1>\mu_2>\cdots>\mu_N$ implies that $\delta_{\mu_m,0}=\delta_{\mu_m,\mu_N=0}$ where only partitions whose last element is zero would contribute. Therefore,

$$\lim_{r_1\to 0}\mathrm{Det}\big[{_{\mathcal{S}}k^{(N)}_\mu}(u_i u_j^*)\big]_{i,j=1,2} = \delta_{\mu_N,0}\sum_{\substack{n=1\\ n\neq N}}^{N}\frac{1}{\mu_N!\mu_n!}|u_2|^{2\mu_n}.$$

As a result, the limit $\lim_{r_1\to 0}{_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}}(r_1,r_2,\phi_{12})$ does not depend on ϕ_{12} and

$$\begin{aligned}{_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(r_2) &= \left(\rho_b\pi\frac{\Gamma}{2}\right)^2\exp\left(-\frac{\rho_b\pi\Gamma}{2}r_2^2\right)\\ &\quad\times\left\langle\sum_{n=1}^{N-1}\frac{1}{\mu_n!}\left(\frac{\rho_b\pi\Gamma}{2}r_2^2\right)^{\mu_n}\right\rangle_{N-1}. \qquad (56)\end{aligned}$$

Here the subscript $N-1$ on the average means that only partitions with $\mu_N=0$ must be considered. Since the contribution of $_{\mathcal{S}}\mathbb{S}_\mu$ for this case vanishes, then there is not a mixture of partitions on the average computations and the result of Eq. (56) remains valid for $\Gamma=2,4,6,\ldots$, not only odd values of $\Gamma/2$. A plot of this function for several values of N and Γ is shown in Fig. 11. The behavior of the radial two-body density function vanishes as r_2^Γ for short distances because of the direct repulsion between particles. Šamaj and Percus [38] found the property

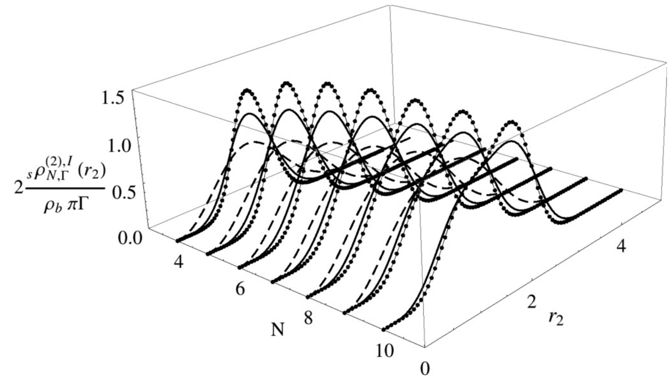


FIG. 11. Exact probability function $_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}(r_2)$. The solid line, dashed line and black points correspond to $\Gamma=2$, 4, and 6, respectively.

$$\rho^{(2)}_{N,\Gamma}(\chi) = (-1)^{\Gamma/2}e^{-\chi^2}\rho^{(2)}_{N,\Gamma}(\mathbf{i}\chi), \qquad (57)$$

with $\chi=\sqrt{\rho_b\pi\Gamma/2}r$, valid for an arbitrary value of the coupling parameter in the thermodynamic limit where the correlations functions are translationally invariant. In order to check the property of Eq. (57) it is necessary to evaluate Eq. (56) and

$${_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(\mathbf{i}\chi) = \left(\rho_b\pi\frac{\Gamma}{2}\right)^2\exp(\chi^2)\left\langle\sum_{n=1}^{N-1}\frac{(-1)^{\mu_n}}{\mu_n!}(\chi^2)^{\mu_n}\right\rangle_{N-1} \qquad (58)$$

at $N\to\infty$. At $\Gamma=2$ we have

$$\begin{aligned}\lim_{N\to\infty}\frac{{_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma=2}}(\mathbf{i}\chi)}{{_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma=2}}(\chi)} &= \frac{e^{\chi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\chi^{2n}}{e^{-\chi^2}\sum_{n=1}^{\infty}\frac{1}{n!}\chi^{2n}}\\ &= \frac{e^{\chi^2}(1-e^{-\chi^2})}{e^{-\chi^2}(1-e^{\chi^2})} = -e^{-\chi^2},\end{aligned}$$

where the partition averages of Eqs. (56) and (58) only have a term corresponding to the root partition $\mu_n=N-n$. A plot of the ratio ${_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(\mathbf{i}\chi)/{_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(\chi)$ for $N=4$ and 6 at $\Gamma=8$ and 10 is shown in Fig. 12. Surprisingly, we observed that the ratio ${_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(\mathbf{i}\chi)/{_{\mathcal{S}}\rho^{(2),I}_{N,\Gamma}}(\chi)$ converges quickly to the exponentially decaying $e^{-\chi^2}$ with the appropriate branch of $(-1)^{\Gamma/2}$ for small values of N too far from the thermodynamic limit.

C. Numerical computation of $_{\mathcal{S}}\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$

It is possible to use the data from the MC simulation to build $_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})$ as it is typically done for the radial distribution function for systems in the fluid phase or with translational symmetry. We start by defining a circular region $\mathscr{A}$ of radius $\mathscr{R}$ where $_{\mathcal{S}}\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})$ will be numerically computed. Since the pair correlation function is small outside the bound radius (19), then we may choose $\mathscr{R}\approx 1.5R^{\mathcal{S}}_{N,\Gamma}$.

Once the system is equilibrated, M configurations $c^{(n)}=\{\vec{r}^{\,(n)}_i|i=1,\ldots,N\}$ are selected from the simulation for each MC cycle

$$\Omega(M)=\{c^{(n)}|n=1,\ldots,M\}.$$

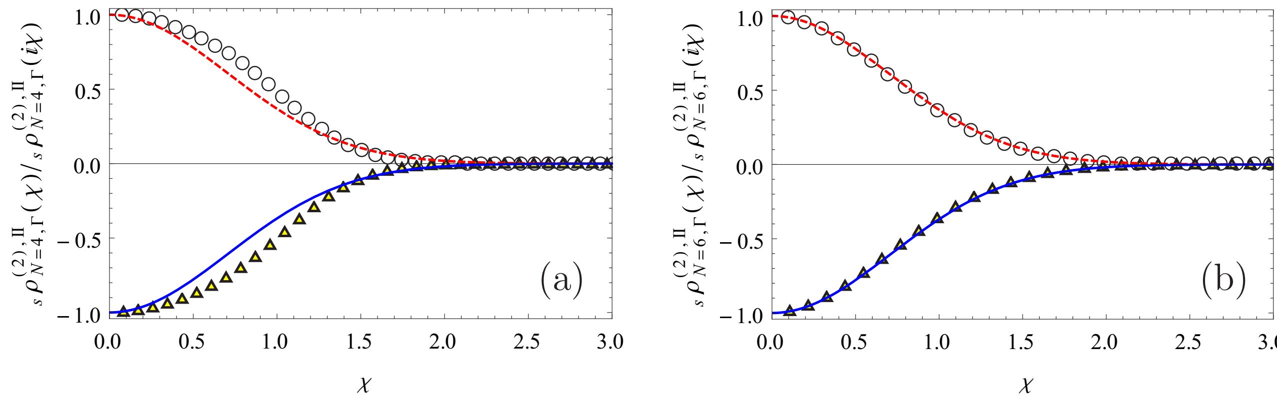


FIG. 12. Ratio ${}_S\rho^{(2),I}_{N,\Gamma}(\mathbf{i}\chi)/{}_S\rho^{(2),I}_{N,\Gamma}(\chi)$. The red dashed and blue solid curves are the plots of $(-1)^{\Gamma/2}e^{-\chi^2}$ for even and odd values of $\Gamma/2$, respectively. Circles and triangles represent the values of ${}_S\rho^{(2),I}_{N,\Gamma}(\mathbf{i}\chi)/{}_S\rho^{(2),I}_{N,\Gamma}(\chi)$ for $\Gamma = 8$ and 10 computed from Eqs. (56) and (58), respectively. (a) $N = 4$ and (b) $N = 6$.

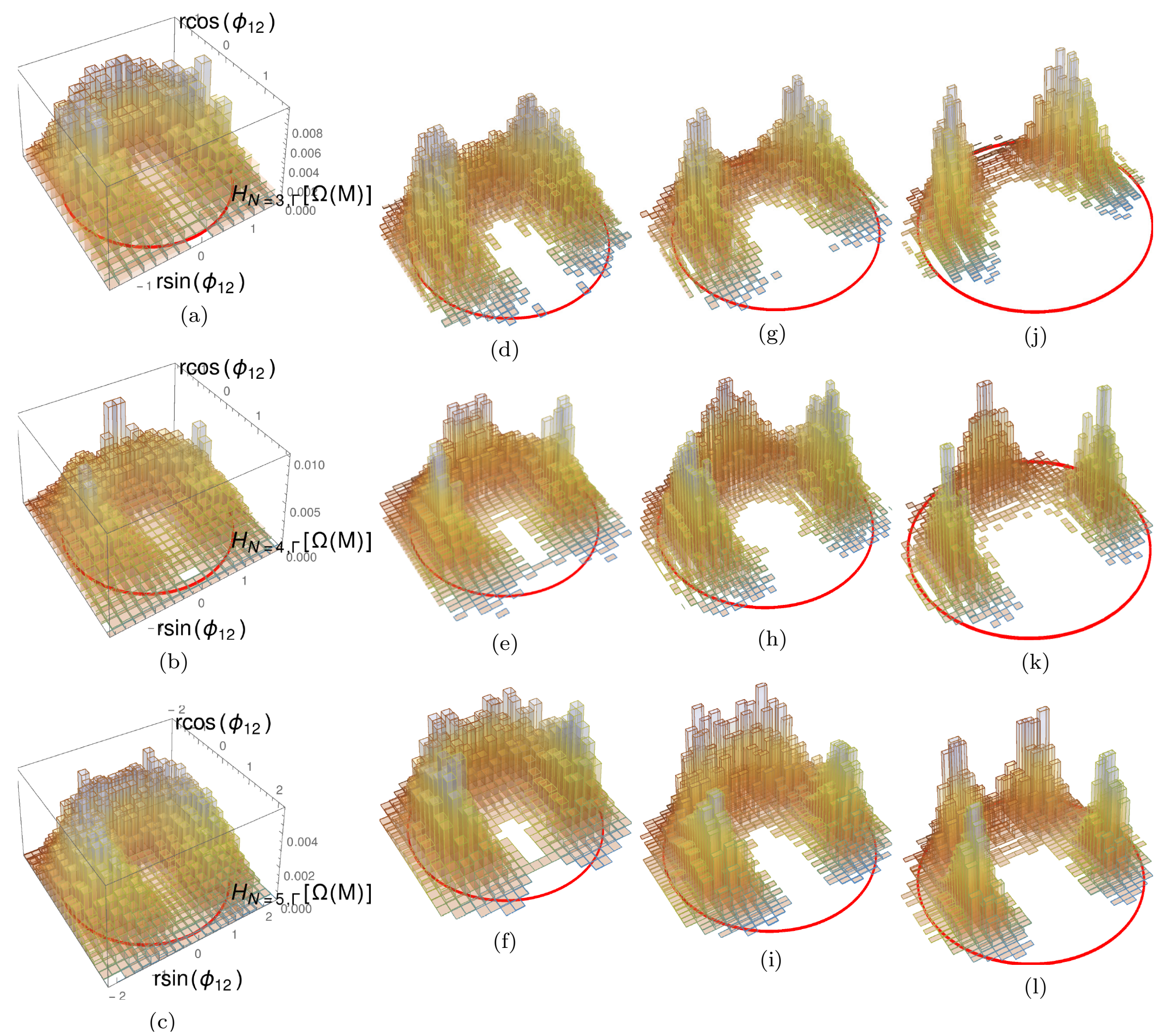
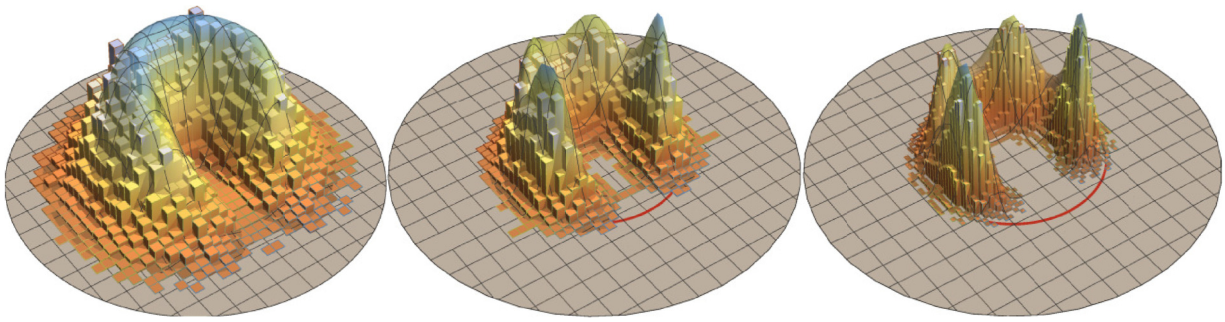


FIG. 13. Numerical computation of ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$ for (a), (d), (g), and (j) $N = 3$, (b), (e), (h), and (k) $N = 4$, and (c), (f), (i), and (l) $N = 5$ and (a)–(c) $\Gamma = 2$, (d)–(f) $\Gamma = 6$, (g)–(i) $\Gamma = 10$, and (j)–(l) $\Gamma = 22$. The radius of the red circle is given by Eq. (19) in the strong-coupling regime $\Gamma \to \infty$.



(a) (b) (c)

FIG. 14. Comparison of $H_{N=5,\Gamma}(\Omega(M))$ for five particles with (a) $\Gamma=2$, (b) $\Gamma=6$, and (c) $\Gamma=14$. Histograms correspond to MC simulations and the surface corresponds to Eq. (54). The volume below each plot has been set to one.

Then $\mathscr{A}$ is divided in $\underset{\sim}{N}$ circular regions of spatial step $\delta\underset{\sim}{r}=\mathscr{R}/\underset{\sim}{N}$ in order to count the particles whose radial distance is between $\underset{\sim}{r}_s-\delta\underset{\sim}{r}$ and $\underset{\sim}{r}_s+\delta\underset{\sim}{r}$, with $\underset{\sim}{r}_s=s\delta\underset{\sim}{r}$ and $s=1,\dots,\underset{\sim}{N}$, to build

$$\omega_s := \left\{\vec{r}_i^{\,(n)}\in\Omega(M) : |\underset{\sim}{r}_s-\vec{r}_i^{\,(n)}|<2\delta r\ \forall n=1,\dots,M\wedge i = 1,\dots,N\right\}.$$

The next step is to compute the sets from ω_s,

$$W_s := \{(r_s\cos(\phi_{ij}),r_s\sin(\phi_{ij})) : \vec{r}_i,\vec{r}_j\in\omega_s\ \forall i\neq j\},$$

which keeps the angle difference $\phi_{ij}:=\phi_i-\phi_j$ between pair of particles. Finally, a two-dimensional histogram is computed from $\{W_s : s=1,\dots,\underset{\sim}{N}\}$.

Figure 13 shows the numerical computation of ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$, where $H_{N,\Gamma}(\Omega(M);r_1,r_2,\phi_{12})$ is the bin height of the histogram obtained from $\{W_s : s=1,\dots,\underset{\sim}{N}\}$. Each histogram in Fig. 13 is normalized to one and it is expected that

$${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12}) = \mathrm{B}_{N,\Gamma}\lim_{M\to\infty}H_{N,\Gamma}(\Omega(M);r,\phi_{12}),$$

with $\mathrm{B}_{N,\Gamma}$ a proper normalization parameter depending on N and Γ rather than $N(N-1)$ as occurs with ${}_S\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})$. The normalization parameter $\mathrm{B}_{N,\Gamma}$ is given by

$$\mathrm{B}_{N,\Gamma} = \int_0^\infty r\,dr\int_0^{2\pi}d\phi_{12}\,{}_S\rho^{(2)}_{N,\Gamma}(r_1,r_2,\phi_{12})$$

and it corresponds to the volume of below the surface of ${}_S\rho^{(2),II}_{N,\Gamma}(r,\phi_{12})$. The result in terms of partition averages

$$\mathrm{B}_{N,\Gamma} = \left(\frac{\rho_b\Gamma}{2}\right)\left\langle\sum_{m=1}^{N}\sum_{n=m+1}^{N}\frac{(\mu_m+\mu_n)!}{\mu_m!\mu_n!}\frac{1}{2^{\mu_m+\mu_n}}\right\rangle_N$$

is obtained by solving the corresponding integrals and it reduces to

$$\mathrm{B}_{N,\Gamma=2} = \rho_b\left[2N-\frac{6}{\sqrt{\pi}(N-1)!}\Gamma\left(N+\frac{1}{2}\right)\right]$$

for the particular case $\Gamma=2$. A comparison between the numerical histograms and the exact probability density given by Eq. (54) is shown in Fig. 14.

X. CONCLUSION

In this paper a finite-N expression for the excess energy of the 2DOCP on the hard and soft disks (32) and (34) for $\Gamma=2$ were obtained. Finite expansions of the excess energy of the soft disk are essentially the same as those found in [3] with the replica method. We have also computed the finite expansion of the excess energy at $\Gamma=2$ for the hard-disk case (32), testing that the result of the excess energy per particle would be in agreement with that found in [2].

The excess energy and the two-body density functions of the 2DOCP on the soft and hard disks for odd values of $\Gamma/2$ in terms of the expansions (40), (45), and (51) were also provided. The formulas found for the excess energy throughout the paper for $\Gamma=2,6,8\dots$ are in good agreement with the results obtained with Monte Carlo simulations. In particular, we have studied the analytical density function $\rho^{(2),II}_{N,\Gamma}(r_1,r_2,\phi_{12})$ associated with the probability to find a pair of particles located at two differential area elements dS_1 and dS_2 located at the same radius but different polar angle (54). The density function $\rho^{(2),II}_{N,\Gamma}(r_1,r_2,\phi_{12})$ was used to explore analytically the generation of small crystals and a comparison of the analytical results of Eq. (54) with histograms obtained via MC simulations was performed, finding good agreement between them.

It may be concluded that the monomial expansion approach enables us to perform exact numerical computations of some thermodynamic quantities of the 2DOCP. Unfortunately, the number of terms of this expansions grows quickly as the number of particles or the coupling parameter is increased. This feature limits drastically the practical application of the method, e.g., in the analytical study with phase transitions where the system is large as well as the typical critical values of the coupling constant. Nevertheless, for systems far from the thermodynamic limit, it is possible to use the monomial expansion approach to study analytically the generation of small crystals, as we did throughout the paper with the Dyson gas. It was found that the two-point density function for $\Gamma\geqslant 2$ inherited not only the well-known kernel determinant of the Ginibre ensemble averaged under partitions, but also an additional contribution which appears only for $\Gamma>2$ since this term is responsible for mixing partitions. Both contributions contain the structural information of the system, especially in the strong-coupling regime.

ACKNOWLEDGMENTS

The authors would like to thank to Nicolas Regnault for kindly facilitating computational tools [39] used in some of our computations. This work was supported by ECOS NORD/COLCIENCIAS-MEN-ICETEX, the Programe de Movilidad Doctoral (COLFUTURO-2014), and Fondo de Investigaciones, Facultad de Ciencias, Universidad de los Andes, project "Estudio numérico y analítico del plasma bidimensional en cercanías de su estado de mínima energía," No. 2017-1.

APPENDIX A: COMPUTATION OF $\mathcal{U}_{pp\mathbf{R}}$

If the angular integral of $\Xi^{\mu_1,\mu_2\mathcal{H}}_{\nu_1,\nu_2\mathbf{R}}$ is evaluated, then

$$\Xi^{\mu_1,\mu_2\mathcal{H}}_{\nu_1,\nu_2\mathbf{R}} = 2\left(\prod_{j=1}^{2}\delta^{\mu_j}_{\nu_j}\right)\tilde{\Xi}^{\mathcal{H}}_{\mu_1,\mu_2\mathbf{R}}, \tag{A1}$$

with

$$\tilde{\Xi}^{\mathcal{H}}_{\mu_1,\mu_2\mathbf{R}} := \int_0^{\sqrt{N}} d\tilde{r}_1 \int_0^{\sqrt{N}} d\tilde{r}_2 \tilde{r}_1^{2\mu_j+1}\tilde{r}_1^{2\mu_2+1} e^{-(\tilde{r}_1+\tilde{r}_2)\Gamma/2}(-\ln r_>).$$

Substituting Eq. (A1) into Eq. (23), then it is possible to complete again the δ product in order to simplify the expansion of $\mathcal{U}_{pp\mathbf{R}}$ as

$$\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}} = \frac{N(N-1)Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{2Z^{\mathcal{H}}_{N,\Gamma}N!\rho_b^N}\sum_{\mu\nu}\frac{C^{(N)}_{\mu}(\Gamma/2)C^{(N)}_{\nu}(\Gamma/2)}{(\prod_i m_i!)^2}\sum_{\sigma,\omega\in S_N}\mathrm{sgn}_\Gamma(\sigma,\omega)4\frac{\tilde{\Xi}^{\mathcal{H}}_{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathbf{R}}}{\Phi_{\mu_{\sigma(1)}}\Phi_{\mu_{\sigma(2)}}}\prod_{j=1}^{N}\delta^{\mu_{\sigma(j)}}_{\nu_{\omega(j)}}\Phi^{\mathcal{H}}_{\mu_{\sigma(j)}}.$$

As a result, for a given partition μ only partitions $\nu=\mu$ with elements strictly organized in the same order [which implies $\mathrm{sgn}_\Gamma(\sigma,\omega)=1$] will give a nonzero contribution reducing the sum $\sum_{\mu\nu}\sum_{\sigma,\omega\in S_N}$ to $\sum_{\mu}\sum_{\sigma\in S_N}(\prod_i m_i!)$. Taking into account that

$$\sum_{\sigma\in S_N}4\frac{\tilde{\Xi}^{\mathcal{H}}_{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathbf{R}}}{\Phi_{\mu_{\sigma(1)}}\Phi_{\mu_{\sigma(2)}}} = (N-2)!\sum_{1\leqslant i<j\leqslant N}8\frac{\tilde{\Xi}^{\mathcal{H}}_{\mu_i,\mu_j\mathbf{R}}}{\Phi_{\mu_i}\Phi_{\mu_j}},$$

then

$$\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}} = \frac{Q^2e^{-\Gamma f^{\mathcal{H}}(N)}}{2Z^{\mathcal{H}}_{N,\Gamma}\rho_b^N}\sum_{\mu}\frac{\left[C^{(N)}_{\mu}(\Gamma/2)\right]^2}{\left(\prod_i m_i!\right)}\left(\prod_{j=1}^{N}\Phi^{\mathcal{H}}_{\mu_j}\right)\sum_{1\leqslant i<j\leqslant N}8\frac{\tilde{\Xi}^{\mathcal{H}}_{\mu_i,\mu_j\mathbf{R}}}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}},$$

or most compactly

$$\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}} = \frac{Q^2}{2}\left\langle\sum_{1\leqslant i<j\leqslant N}8\frac{\tilde{\Xi}^{\mathcal{H}}_{\mu_i,\mu_j\mathbf{R}}}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}}\right\rangle_N.$$

It is useful to write $\tilde{\Xi}^{\mathcal{H}}_{\mu_i,\mu_j\mathbf{R}}$ as

$$\tilde{\Xi}^{\mathcal{H}}_{\mu_i,\mu_j\mathbf{R}} = J^{\mu_i,\mu_j}_{\mathcal{H}} + J^{\mu_j,\mu_i}_{\mathcal{H}},$$

where we defined

$$J^{\mu_i,\mu_j}_{\mathcal{H}} := \int_0^{\sqrt{N}} dy \int_0^{y} dx\, x^{2\mu_i+1}y^{2\mu_j+1}(-\ln y)e^{-(x^2+y^2)\Gamma/2}, \tag{A2}$$

and hence

$$\mathcal{U}^{\mathcal{H}}_{pp\mathbf{R}} = Q^2\left\langle\sum_{1\leqslant i<j\leqslant N}4\frac{J^{\mu_i,\mu_j}_{\mathcal{H}}+J^{\mu_j,\mu_i}_{\mathcal{H}}}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}}\right\rangle_N. \tag{A3}$$

An analogous computation for the Dyson gas gives the result

$$\mathcal{U}^{\mathcal{S}}_{pp\mathbf{R}} = Q^2\left\langle\sum_{1\leqslant i<j\leqslant N}4\frac{J^{\mu_i,\mu_j}_{\mathcal{S}}+J^{\mu_j,\mu_i}_{\mathcal{S}}}{\Phi^{\mathcal{S}}_{\mu_i}\Phi^{\mathcal{S}}_{\mu_j}}\right\rangle_N,$$

with

$$J^{\mu_i,\mu_j}_{\mathcal{S}} := \int_0^{\infty} dy \int_0^{y} dx\, x^{2\mu_i+1}x^{2\mu_j+1}(-\ln y)e^{-(x^2+y^2)\rho_b\pi\Gamma/2}. \tag{A4}$$

This integral may also be written as

$$J_{\mathcal{S}}^{m,n} = -\frac{1}{8}\left(\frac{2}{\rho_b\pi\Gamma}\right)^{m+n+2}\left[\mathcal{J}^{m,n} - \ln\left(\frac{\rho_b\pi\Gamma}{2}\right)\mathcal{I}^{m,n}\right],$$

with

$$\mathcal{J}^{m,n} = \underset{t_1<t_2}{\int_0^\infty\int_0^\infty} t_1^m t_2^n \ln(t_2)\exp(-t_1-t_2), \quad \mathcal{I}^{m,n} = \underset{t_1<t_2}{\int_0^\infty\int_0^\infty} t_1^m t_2^n \exp(-t_1-t_2). \tag{A5}$$

For the Dyson gas it is advantageous to write $J_{\mathcal{S}}^{\mu_i,\mu_j}$ in terms of the factor $(\frac{2}{\rho_b\pi\Gamma})^{\mu_i+\mu_j+2}$ because the same factor is in $\Phi_{\mu_i}^{\mathcal{S}}\Phi_{\mu_j}^{\mathcal{S}} = (\frac{2}{\rho_b\pi\Gamma})^{\mu_i+\mu_j+2}\mu_i!\mu_j!$. Additionally, the integral $\mathcal{I}^{\mu_i,\mu_j}$ has the property

$$\mathcal{I}^{\mu_i,\mu_j} + \mathcal{I}^{\mu_j,\mu_i} = \mu_i!\mu_j! \quad \therefore\ \mathcal{U}_{pp\mathbf{R}}^{\mathcal{S}} = -\frac{Q^2}{2}\left\langle\sum_{1\leqslant i<j\leqslant N}\frac{\mathcal{J}^{\mu_i,\mu_j}+\mathcal{J}^{\mu_j,\mu_i}}{\mu_i!\mu_j!} - \ln(\rho_b\pi\Gamma/2)\sum_{1\leqslant i<j\leqslant N}1\right\rangle_N.$$

Now the sum $\mathcal{J}^{\mu_i,\mu_j}+\mathcal{J}^{\mu_j,\mu_i}$ may be written in terms of the factorial product $\mu_i!\mu_j!$ as $\mathcal{J}^{\mu_i,\mu_j}+\mathcal{J}^{\mu_j,\mu_i} = k_1!k_2!j(k_1,k_2)$, where

$$j(k_1,k_2) = \ln 2 - \gamma + H_{k_1} + H_{k_2} - \frac{1}{2^{k_1+1}}\sum_{l=0}^{k_2}\frac{(k_1+1)_l}{l!}\left(\frac{1}{2}\right)^l H_{k_1+l} - \frac{1}{2^{k_2+1}}\sum_{l=0}^{k_1}\frac{(k_2+1)_l}{l!}\left(\frac{1}{2}\right)^l H_{k_2+l}. \tag{A6}$$

Here $H_n = \sum_{i=1}^n \frac{1}{i}$ are the harmonic numbers and $(n)_l = \prod_{i=1}^l (n+i-1)$ is the Pochhammer symbol $(n)_l = (n+l-1)!/(n-1)!$. As a result, the $\mathcal{U}_{pp\mathbf{R}}$ contribution on the Dyson gas is

$$\mathcal{U}_{pp\mathbf{R}}^{\mathcal{S}} = -\frac{Q^2}{2}\left\langle\sum_{1\leqslant i<j\leqslant N} j(\mu_i,\mu_j)\right\rangle_N + \frac{Q^2}{4}N(N-1)\ln(\rho_b\pi\Gamma/2). \tag{A7}$$

APPENDIX B: HARD-DISK INTEGRALS

We are interested in the evaluation of the integral

$$I_{\mathcal{H}}^{m,n} = {}_{\mathcal{H}}I_n^m = \int_0^{\sqrt{N}} y^{2n+1}\exp(-y^2\Gamma/2)\mathcal{F}(m,\Gamma/2,y)dy,$$

where

$$\mathcal{F}(m,\Gamma/2,y) = \int_0^y x^{2m+1}\exp(-x^2\Gamma/2)dx = \frac{2^m}{\Gamma^{m+1}}[\mathbf{\Gamma}(m+1) - \mathbf{\Gamma}(m+1,y^2\Gamma/2)].$$

If m is an integer then $\mathbf{\Gamma}(m+1) = m!$. On the other hand, the incomplete Γ function may be expanded as

$$\mathbf{\Gamma}(m+1,y^2\Gamma/2) = m!\exp(-y^2\Gamma/2)\sum_{k=0}^m\frac{1}{k!}\left(y^2\frac{\Gamma}{2}\right)^k.$$

As a result,

$$I_{\mathcal{H}}^{m,n} = \frac{2^m}{\Gamma^{m+1}}m!\left[\mathcal{F}(m,\Gamma/2,\sqrt{N}) - \sum_{k=0}^m\frac{1}{k!}(\Gamma/2)^k\mathcal{F}(n+k,\Gamma/2,\sqrt{N})\right], \tag{B1}$$

with

$$\mathcal{F}(a,b,c) = \frac{2^a}{(2b)^{a+1}}a!\left[1 - \exp(-c^2b)\sum_{k=0}^a\frac{1}{k!}(c^2b)^k\right], \tag{B2}$$

where a is a positive integer. The second integral included in the energy computations is

$$J_{\mathcal{H}}^{\mu_1,\mu_2} := \int_0^{\sqrt{N}} dy\int_0^y dx\, x^{2\mu_1+1}y^{2\mu_2+1}(-\ln y)e^{-(x^2+y^2)\Gamma/2}.$$

It may be written as

$$J_{\mathcal{H}}^{\mu_1,\mu_2} = \int_0^{\sqrt{N}} dy\, y^{2\mu_2+1}(-\ln y)e^{-y^2\Gamma/2}\mathcal{F}(\mu_1,\Gamma/2,y)$$

and replacing Eq. (B2) we obtain

$$J_{\mathcal{H}}^{\mu_1,\mu_2} = \left(\frac{2}{\Gamma}\right)^{\mu_1}\frac{\mu_1!}{\Gamma}\left[G(\mu_2,\Gamma/2,\sqrt{N}) - \sum_{k=0}^{\mu_1}\frac{1}{k!}\left(\frac{\Gamma}{2}\right)^k G(\mu_2+k,\Gamma,\sqrt{N})\right], \tag{B3}$$

with

$$G(a,b,c) := \int_0^c dy[-\ln(y)]y^{2a+1}e^{-by^2} = \frac{c^{2(a+1)}}{4(a+1)^2}{}_2\mathcal{F}_2(a+1,a+1;a+2,a+2,-bc^2) - \frac{\ln(c)}{2b^{a+1}}[a! - \Gamma(a+1,bc^2)].$$

APPENDIX C: THE $\mathcal{B}_{\mu\nu}$ MATRIX FOR ODD VALUES OF $\Gamma/2$

1. The $\mathcal{B}_{\mu\nu}$ matrix for the hard disk

In this section we will exclude the zero terms of Eq. (27) for odd values of $\Gamma/2$. If the partitions included in $\mathcal{B}_{\mu\nu}$ are not equal, then $\mathcal{B}_{\mu\nu} \neq 0$ only if they share $N-2$ elements $n_{\mu,\nu} = N-2$. Let us suppose that (μ_p,μ_q) are the elements of μ which are not in ν, that is, $(\mu_p,\mu_q)\cap\nu = 0$ and (ν_m,ν_n), the elements of ν such that $(\nu_m,\nu_n)\cap\mu = 0$ with $(\nu_m,\nu_n)\neq(\mu_p,\mu_q)$. Then for a given partition $\mu\neq\nu$ the matrix $\mathcal{B}_{\mu\nu}$ would be different from zero only if ν is of the form

$$\nu = (\mu_2,\mu_N,\ldots,\nu_m,\ldots,\nu_n,\ldots,\mu_1).$$

We define

$$\tilde{\nu} = (\mu_1,\mu_2,\ldots,\mu_{p-1},\nu_m,\mu_{p+1},\ldots,\mu_{q-1},\nu_n,\mu_{q+1},\ldots,\mu_N),$$

that is,

$$\tilde{\nu}_i = \begin{cases}\nu_m & \text{if } i = p\\ \nu_n & \text{if } i = q\\ \mu_i & \text{otherwise.}\end{cases}$$

Since $\tilde{\nu}$ is a permutated version of the original ν, then a permutation of the same indices of ν and $\tilde{\nu}$ will not necessarily have the same sign. However, it is always possible to go from ν to $\tilde{\nu}$ by permutating, for instance, $\tau_{\mu\nu}$ times the labels of ν. Therefore,

$$\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}F(\mu,\nu) = (-1)^{\tau_{\mu\nu}}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}F(\mu,\tilde{\nu})$$

for a given function $F = F(\mu,\nu)$. Hence, the $\mathcal{B}_{\mu\nu}^{\mathcal{H}}$ may be written as

$$\mathcal{B}_{\mu\nu}^{\mathcal{H}} = \sum_{\sigma,\omega\in S_N}(-1)^{\tau_{\mu\nu}}\mathrm{sgn}_\Gamma(\sigma,\omega)\frac{\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\tilde{\nu}_{\omega(1)}+\tilde{\nu}_{\omega(2)}}}{|\mu_{\sigma(1)}-\tilde{\nu}_{\omega(1)}|_{\mu_{\sigma(1)}\neq\tilde{\nu}_{\omega(1)}}}\tilde{\Xi}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}\mathbf{L}}\prod_{j=3}^N\delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}}\Phi^{\mathcal{H}}_{\mu_{\sigma(j)}}$$

or

$$\mathcal{B}_{\mu\nu}^{\mathcal{H}} = (-1)^{\tau_{\mu\nu}}\left(\prod_{i=1}^N\Phi^{\mathcal{H}}_{\mu_i}\right)\sum_{\sigma,\omega\in S_N}[\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}]^{b(\Gamma)}L^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}}\prod_{j=3}^N\delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}}$$

with

$$L^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}} = \frac{\delta_{\mu_{\sigma(1)}+\mu_{\sigma(2)},\tilde{\nu}_{\omega(1)}+\tilde{\nu}_{\omega(2)}}}{|\mu_{\sigma(1)}-\tilde{\nu}_{\omega(1)}|_{\mu_{\sigma(1)}\neq\tilde{\nu}_{\omega(1)}}}\frac{\tilde{\Xi}^{\mu_{\sigma(1)},\mu_{\sigma(2)}\mathcal{H}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}\mathbf{L}}}{\Phi^{\mathcal{H}}_{\mu_{\sigma(1)}}\Phi^{\mathcal{H}}_{\mu_{\sigma(2)}}},\quad \tilde{\nu}_{\omega(j)} = \begin{cases}\tilde{\nu}_p & \text{if } \omega(j) = p\\ \tilde{\nu}_q & \text{if } \omega(j) = q\\ \mu_{\omega(j)} & \text{if } \omega(j)\neq p,\ \omega(j)\neq q.\end{cases}$$

The double sum of permutations will add a zero contribution for $\omega(j) = p$ or $\omega(j) = q$ for $j\geqslant 3$ because of the δ product $\prod_{j=3}^N\delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}}$. Therefore, for nonzero contributions it is only possible to locate the indices (p,q) at $\omega(1),\omega(2)$ and $\sigma(1),\sigma(2)$. Consequently, the term $[\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}]^{b(\Gamma)}L^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}}\prod_{j=3}^N\delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}}$ may be written as $[\epsilon_{\sigma(1)\sigma(2)}\epsilon_{\omega(1)\omega(2)}]^{b(\Gamma)}L^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}}\prod_{j=3}^N\delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}}$. As a result, the sum $\sum_{\sigma,\omega\in S_N}$ will generate $(N-2)!$ times the same nonzero contribution term built from the different permutations of p and q on $(\epsilon_{\sigma(1)=p,\sigma(2)=q}\epsilon_{\omega(1)=p,\omega(2)=q})^{b(\Gamma)}L^{\mu_p,\mu_q}_{\tilde{\nu}_p,\tilde{\nu}_q}$,

$$\mathcal{B}_{\mu\nu}^{\mathcal{H}}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1}^N\Phi^{\mathcal{H}}_{\mu_i}\right)\left[(\epsilon_{pq}\epsilon_{pq})^{b(\Gamma)}L^{\mu_p,\mu_q}_{\tilde{\nu}_p,\tilde{\nu}_q} + (\epsilon_{pq}\epsilon_{qp})^{b(\Gamma)}L^{\mu_q,\mu_p}_{\tilde{\nu}_p,\tilde{\nu}_q} + (\epsilon_{qp}\epsilon_{qp})^{b(\Gamma)}L^{\mu_q,\mu_p}_{\tilde{\nu}_q,\tilde{\nu}_p} + (\epsilon_{qp}\epsilon_{pq})^{b(\Gamma)}L^{\mu_p,\mu_q}_{\tilde{\nu}_q,\tilde{\nu}_p}\right].$$

Now $\epsilon_{pq}\epsilon_{pq} = \epsilon_{qp}\epsilon_{qp} = -\epsilon_{qp}\epsilon_{pq} = -\epsilon_{pq}\epsilon_{qp} = 1$ and $\delta_{\mu_{\sigma(1)=p}+\mu_{\sigma(2)=q},\tilde{\nu}_{\omega(1)=p}+\tilde{\nu}_{\omega(2)=q}} = 1$, as well as the other δ terms, since $\mu_p+\mu_q = \tilde{\nu}_p+\tilde{\nu}_q$; therefore

$$\mathcal{B}_{\mu\nu}^{\mathcal{H}}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1,i\neq p\neq q}^N\Phi^{\mathcal{H}}_{\mu_i}\right)\left(\frac{\tilde{\Xi}^{\mu_p\mu_q\mathcal{H}}_{\nu_m\nu_n\mathbf{L}}}{|\mu_p-\nu_m|} + \frac{\tilde{\Xi}^{\mu_q\mu_p\mathcal{H}}_{\nu_n\nu_m\mathbf{L}}}{|\mu_q-\nu_n|} - \frac{\tilde{\Xi}^{\mu_p\mu_q\mathcal{H}}_{\nu_n\nu_m\mathbf{L}}}{|\mu_p-\nu_n|} - \frac{\tilde{\Xi}^{\mu_q\mu_p\mathcal{H}}_{\nu_m\nu_n\mathbf{L}}}{|\mu_q-\nu_m|}\right),$$

where we have used $\tilde{\nu}_p = \nu_m$ and $\tilde{\nu}_q = \nu_n$. Now the terms

$$\tilde{\Xi}^{\mu_p\mu_q\mathcal{H}}_{\nu_m\nu_n\mathbf{L}} = {}_{\mathcal{H}}I^{(\mu_p+\nu_m+|\mu_p-\nu_m|)/2}_{(\mu_q+\nu_n-|\mu_p-\nu_m|)/2} + {}_{\mathcal{H}}I^{(\mu_q+\nu_n+|\mu_p-\nu_m|)/2}_{(\mu_p+\nu_m-|\mu_p-\nu_m|)/2}$$

may be simplified by using the constraint $\mu_p + \mu_q = \nu_m + \nu_n$ on the partition elements; then $|\mu_p - \mu_n| = \mu_p - \nu_m$ if $\mu_p > \nu_m$ and $|\mu_p - \mu_n| = -\mu_p + \mu_n$ if $\mu_p < \nu_m$. Hence $\tilde{\Xi}^{\mu_p\mu_q\mathcal{H}}_{\nu_m\nu_n\mathbf{L}}/|\mu_p-\nu_m| + \tilde{\Xi}^{\mu_q\mu_p\mathcal{H}}_{\nu_n\nu_m\mathbf{L}}/|\mu_q-\nu_n|$ is

$$2\frac{\tilde{\Xi}^{\mu_p\mu_q\mathcal{H}}_{\nu_m\nu_n\mathbf{L}}}{|\mu_p-\nu_m|} = \frac{2}{|\mu_p-\nu_m|} \times \begin{cases} [{}_{\mathcal{H}}I^{\mu_p}_{\mu_q} + {}_{\mathcal{H}}I^{\nu_n}_{\nu_m}] & \text{if } \mu_p > \nu_m \\ [{}_{\mathcal{H}}I^{\mu_q}_{\mu_p} + {}_{\mathcal{H}}I^{\nu_m}_{\nu_n}] & \text{if } \mu_p < \nu_m \end{cases}$$

and

$$\mathcal{B}^{\mathcal{H}}_{\mu\nu}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1,i\neq p\neq q}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right)2\times\left[\left(\frac{{}_{\mathcal{H}}I^{\mu_p}_{\mu_q}+{}_{\mathcal{H}}I^{\nu_n}_{\nu_m}}{\mu_p-\nu_m}\quad\text{if } \mu_p>\mu_m \text{ or } \frac{{}_{\mathcal{H}}I^{\mu_q}_{\mu_p}+{}_{\mathcal{H}}I^{\nu_m}_{\nu_n}}{\nu_m-\mu_p}\text{ otherwise}\right)\right.$$
$$\left.-\left(\frac{{}_{\mathcal{H}}I^{\mu_p}_{\mu_q}+{}_{\mathcal{H}}I^{\nu_m}_{\nu_n}}{\mu_p-\nu_n}\quad\text{if } \mu_p>\mu_n \text{ or } \frac{{}_{\mathcal{H}}I^{\mu_q}_{\mu_p}+{}_{\mathcal{H}}I^{\nu_n}_{\nu_m}}{\nu_n-\mu_p}\text{ otherwise}\right)\right].$$

An alternative way to write the matrix $\mathcal{B}^{\mathcal{H}}_{\mu\nu}$ is by defining $(\mu,\nu)_\pm := (\mu+\nu\pm|\mu-\nu|)/2$ and taking into account that $|\mu_p-\nu_m| = |\mu_q-\nu_n|$; hence

$$\mathcal{B}^{\mathcal{H}}_{\mu\nu}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1,i\neq p\neq q}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right)2\left[\frac{{}_{\mathcal{H}}I^{(\mu_p,\nu_m)_+}_{(\mu_q,\nu_n)_-}+{}_{\mathcal{H}}I^{(\mu_q,\nu_n)_+}_{(\mu_q,\nu_m)_-}}{|\mu_p-\nu_m|} - \frac{{}_{\mathcal{H}}I^{(\mu_p,\nu_n)_+}_{(\mu_q,\nu_m)_-}+{}_{\mathcal{H}}I^{(\mu_q,\nu_m)_+}_{(\mu_p,\nu_n)_-}}{|\mu_p-\nu_n|}\right], \tag{C1}$$

valid for odd values of $\Gamma/2$ and $\mu\cap\nu = N-2$.

2. The $\mathcal{B}_{\mu\nu}$ matrix for the soft disk

The corresponding $\mathcal{B}_{\mu\nu}$ matrix for the soft disk may be found by following the same procedure described in the preceding section for the hard disk. The result is

$$\mathcal{B}^{\mathcal{S}}_{\mu\nu}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1,i\neq p\neq q}^{N}\Phi^{\mathcal{S}}_{\mu_i}\right)2\left[\frac{{}_{\mathcal{S}}I^{(\mu_p,\nu_m)_+}_{(\mu_q,\nu_n)_-}+{}_{\mathcal{S}}I^{(\mu_q,\nu_n)_+}_{(\mu_q,\nu_m)_-}}{|\mu_p-\nu_m|} - \frac{{}_{\mathcal{S}}I^{(\mu_p,\nu_n)_+}_{(\mu_q,\nu_m)_-}+{}_{\mathcal{S}}I^{(\mu_q,\nu_m)_+}_{(\mu_p,\nu_n)_-}}{|\mu_p-\nu_n|}\right],$$

which may be simplified by expressing the integral and product terms as

$${}_{\mathcal{S}}I^{(\mu_p,\nu_m)_+}_{(\mu_q,\nu_n)_-} = \frac{1}{4}\left(\frac{2}{\rho_b\pi\Gamma}\right)^{\mu_p+\mu_q+2}\mathcal{I}^{(\mu_p,\nu_m)_+}_{(\mu_q,\nu_n)_-}, \qquad \prod_{i=1,i\neq p\neq q}^{N}\Phi^{\mathcal{S}}_{\mu_i} = \frac{1}{\mu_p!\mu_q!\left(\frac{2}{\rho_b\pi\Gamma}\right)^{\mu_p+\mu_q+2}}\prod_{i=1}^{N}\Phi^{\mathcal{S}}_{\mu_i},$$

where we have used the property

$$(\mu_p,\nu_m)_+ + (\mu_q,\nu_n)_- = (\mu_q,\nu_n)_+ + (\mu_p,\nu_m)_- = (\mu_p,\nu_n)_+ + (\mu_q,\nu_m)_- = (\mu_q,\nu_m)_+ + (\mu_p,\nu_n)_- = \mu_p+\mu_q.$$

As a result, the $\mathcal{B}^{\mathcal{S}}_{\mu\nu}$ matrix takes the form

$$\mathcal{B}^{\mathcal{S}}_{\mu\nu}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\frac{1}{\mu_p!\mu_q!}\prod_{i=1}^{N}\Phi^{\mathcal{S}}_{\mu_i}\right)\frac{2}{4}\left[\frac{\mathcal{I}^{(\mu_p,\nu_m)_+}_{(\mu_q,\nu_n)_-}+\mathcal{I}^{(\mu_q,\nu_n)_+}_{(\mu_q,\nu_m)_-}}{|\mu_p-\nu_m|} - \frac{\mathcal{I}^{(\mu_p,\nu_n)_+}_{(\mu_q,\nu_m)_-}+\mathcal{I}^{(\mu_q,\nu_m)_+}_{(\mu_p,\nu_n)_-}}{|\mu_p-\nu_n|}\right]$$

or

$$\mathcal{B}^{\mathcal{S}}_{\mu\nu}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\frac{1}{\mu_p!\mu_q!}\prod_{i=1}^{N}\Phi^{\mathcal{S}}_{\mu_i}\right)\frac{1}{2}\left[\left(\frac{\mathcal{I}^{\mu_p}_{\mu_q}+\mathcal{I}^{\nu_n}_{\nu_m}}{\mu_p-\nu_m}\quad\text{if } \mu_p>\nu_m \text{ or } \frac{\mathcal{I}^{\mu_p}_{\mu_q}+\mathcal{I}^{\nu_m}_{\nu_n}}{\nu_m-\mu_p}\text{ otherwise}\right)\right.$$
$$\left.\times\left(\frac{\mathcal{I}^{\mu_p}_{\mu_q}+\mathcal{I}^{\nu_m}_{\nu_n}}{\mu_p-\nu_n}\quad\text{if } \mu_p>\nu_n \text{ or } \frac{\mathcal{I}^{\mu_q}_{\mu_p}+\mathcal{I}^{\nu_n}_{\nu_m}}{\nu_n-\mu_p}\text{ otherwise}\right)\right].$$

The soft disk offers an additional advantage since it is possible to factorize the product $\mu_p!\mu_q!$ from $\mathcal{I}^{\mu_q}_{\mu_p}$ as we did before for $\frac{1}{\mu_p!\mu_q!}\mathcal{I}^{\mu_q}_{\mu_p} = i^{\mu_p}_{\mu_q}$. We may also try to do the same with $\frac{1}{\mu_p!\mu_q!}\mathcal{I}^{\nu_n}_{\nu_m} = \frac{\nu_m!\nu_n!}{\mu_p!\mu_q!}i^{\mu_p}_{\mu_q}$ by writing $\mu_p! = \nu_m!\prod_{i=\nu_m+1}^{\mu_p} i$ for the case $\mu_p > \nu_m$. Since $\mu_p+\mu_q = \nu_m+\nu_n$, then $\mu_p > \nu_m$ implies that $\nu_n > \mu_q$ and hence $\nu_p! = \nu_n!\prod_{i=\mu_q+1}^{\nu_n} i$; therefore

$$\frac{1}{\mu_p!\mu_q!}\mathcal{I}^{\nu_n}_{\nu_m} = \frac{\pi(\mu_q,\nu_n)}{\pi(\nu_m,\mu_q)}i^{\nu_n}_{\nu_m} \quad \text{for} \quad \mu_p>\nu_m \Leftrightarrow \nu_n>\mu_q,$$

with

$$\pi(a,b) := \prod_{i=a+1}^{b} i \quad \text{for } a < b,(a,b) \in Z^+.$$

Proceeding in a similar way, the other terms are then

$$\frac{1}{\mu_p!\mu_q!}\mathcal{I}_{\nu_n}^{\nu_m} = \frac{\pi(\mu_p,\nu_m)}{\pi(\nu_n,\mu_q)} i_{\nu_n}^{\nu_m} \quad \text{for } \mu_p < \nu_m \Leftrightarrow \mu_q > \nu_n,$$

$$\frac{1}{\mu_p!\mu_q!}\mathcal{I}_{\nu_n}^{\nu_m} = \frac{\pi(\mu_q,\nu_m)}{\pi(\nu_n,\mu_p)} i_{\nu_n}^{\nu_m} \quad \text{for } \mu_p > \nu_n \Leftrightarrow \nu_m > \mu_q,$$

$$\frac{1}{\mu_p!\mu_q!}\mathcal{I}_{\nu_m}^{\nu_n} = \frac{\pi(\mu_p,\nu_n)}{\pi(\nu_m,\mu_q)} i_{\nu_m}^{\nu_n} \quad \text{for } \nu_n > \mu_p \Leftrightarrow \mu_q > \nu_m.$$

As a result, the $\mathcal{B}_{\mu\nu}^{\mathcal{S}}$ matrix takes the form

$$\mathcal{B}_{\mu\nu}^{\mathcal{S}}\binom{p,q}{m,n} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{j=1}^{N}\Phi_{\mu_j}^{\mathcal{S}}\right)\frac{1}{2}f\binom{p,q}{m,n} \quad \text{for odd values of } \frac{\Gamma}{2},\ \dim(\mu\cap\nu) = N-2, \tag{C2}$$

where we defined

$$f\binom{p,q}{m,n} := \left[\frac{i_{\mu_q}^{\mu_p} + \frac{\pi(\mu_q,\nu_n)}{\pi(\nu_m,\mu_p)} i_{\nu_m}^{\nu_n}}{\mu_p - \nu_m} \quad \text{if } \mu_p > \nu_m \text{ or } \frac{i_{\mu_p}^{\mu_q} + \frac{\pi(\mu_p,\nu_m)}{\pi(\nu_n,\mu_q)} i_{\nu_n}^{\nu_m}}{\nu_m - \mu_p} \text{ otherwise}\right]$$
$$-\left[\frac{i_{\mu_q}^{\mu_p} + \frac{\pi(\mu_q,\nu_m)}{\pi(\nu_n,\mu_p)} i_{\nu_m}^{\nu_n}}{\mu_p - \nu_n} \quad \text{if } \mu_p > \nu_n \text{ or } \frac{i_{\mu_p}^{\mu_q} + \frac{\pi(\mu_p,\nu_n)}{\pi(\nu_m,\mu_q)} i_{\nu_m}^{\nu_n}}{\nu_n - \mu_p} \text{ otherwise}\right]. \tag{C3}$$

APPENDIX D: THE SIGN $(-1)^{\tau_{\mu\nu}}$

The sign $(-1)^{\tau_{\mu\nu}}$ is required to compute Eq. (C1) or (C2). According to Eq. (28), for nonzero and nondiagonal values of the $\mathcal{B}_{\mu\nu}$ matrix the partitions μ and ν must share all the elements except two, located at the positions (p,q) and (m,n). In general, the sign depends on how these positions are indexed, that is, $(-1)^{\tau_{\mu\nu}} = (-1)^{\tau_{\mu\nu}\binom{p,q}{m,n}}$. If we suppose that $\nu \in \mathcal{D}_\mu$, then these partitions may be, for instance, of the form

$$\mu = (\mu_1,\mu_2,\mu_p,\ldots,\mu_q,\mu_N), \quad \nu = (\mu_1,\nu_m,\mu_3,\ldots,\nu_n,\ldots,\mu_N).$$

We will also define the partition μ' defined as $\mu' = (\mu_p,\mu_q,\mu_3,\mu_4,\ldots,\mu_N)$ by permuting the elements of the original partition $\mu = (\mu_1,\mu_2,\mu_p,\ldots,\mu_q,\mu_N)$. To this aim it is necessary to move the element μ_p from the pth place to the first place by applying $p-1$ movements to the left. Similarly, the element μ_q needs $q-2$ movements to occupy the second place once μ_p is in the first place. So the total number of required transpositions from μ to μ' is $T_\mu = p+q-3$. Similarly, if we would like to build $\nu' = (\nu_m,\nu_n,\mu_3,\mu_4,\ldots,\mu_N)$ it is necessary to apply $T_\nu = m+n-3$ transpositions. Therefore, the total number of transpositions to change the order of the original partitions μ and ν to μ' and ν' is $T_{\nu,\mu} = p+q+m+n-6$. This would introduce the change of sign $(-1)^{T_{\nu,\mu}} = (-1)^{p+q+m+n}$ on the computation of $\mathcal{B}_{\mu\nu}^{\mathcal{H}}\binom{p,q}{m,n}$. Starting from μ' we may return to μ by reversing the $T_\mu = p+q-3$ transpositions used in the other direction

$$\mu' = (\mu_p,\mu_q,\mu_3,\mu_4,\ldots,\mu_N) \underset{T_\mu}{\longmapsto} \mu = (\mu_1,\mu_2,\ldots,\mu_p,\ldots,\mu_q,\ldots\mu_N).$$

Since ν' only differs from μ' in the first two elements, it is also necessarily $T_\mu = p+q-3$ transpositions to build $\tilde{\nu}$,

$$\nu' = (\nu_m,\nu_n,\mu_3,\mu_4,\ldots,\mu_N) \underset{T_\mu}{\longmapsto} \tilde{\nu} = (\nu_1 = \mu_1,\nu_2 = \mu_2,\ldots,\nu_p = \nu_m,\ldots,\nu_q = \nu_n,\ldots,\nu_N = \mu_N).$$

Fortunately, $T_{\mu'\nu'} = 2T_\mu$ is always an even number. Hence, if we go simultaneously from μ' to μ and ν' to $\tilde{\nu}$, then this procedure does not affect the sign. As a result,

$$(-1)^{\tau_{\mu\nu}\binom{p,q}{m,n}} = (-1)^{T_{\nu,\mu}+2T_\mu} = (-1)^{p+q+m+n}. \tag{D1}$$

APPENDIX E: THE UNDERDOTTED $B_{\mu\nu}$ MATRIX FOR ODD VALUES OF $\Gamma/2$

1. Computation of $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ for $\mu \neq \nu$

Nonzero contributions of $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ for $\mu \neq \nu$ are restricted specifically to $\dim(\mu \cap \nu) = N - 2$. Hence, we must focus on a pair of partitions which shares $N-2$ elements. If μ_p,μ_q,ν_m,ν_n are the unshared elements of μ and ν [that is, $(\mu_p,\mu_q)\cap \nu = 0$ and $(\nu_m,\nu_n)\cap\mu = 0$], then the condition $\mu_p + \mu_q = \nu_m + \nu_n$ is expected as a consequence of the squeezing operations behind the construction of any partition μ or ν from the root partition. This condition may be found by recalling that $\sum_{i=1}^N \mu_i = \sum_{i=1}^N \nu_i$ or

$$\mu_p + \mu_q + \sum_{\substack{i=1\\ i\neq p,q}}^{N} \mu_i = \tilde{\nu}_p + \tilde{\nu}_q + \sum_{\substack{i=1\\ i\neq p,q}}^{N} \tilde{\nu}_i,$$

where we have used the fact that it is always possible to go from $\nu = (\dots\nu_m,\dots,\nu_n\dots)$ to a partition $\tilde{\nu} = (\dots\nu_p,\dots,\nu_q\dots)$ whose unshared elements are placed in the same positions of $\mu = (\dots\mu_p,\dots,\mu_q\dots)$ by applying $\tau_{\mu\nu}$ transpositions on ν without changing the value of $\sum_{i=1}^N \nu_i$. Since $(\tilde{\nu}_p,\tilde{\nu}_q) = (\nu_m,\nu_n)$ and $\tilde{\nu}_i = \mu_i$ if $i \neq p,q$, then $\mu_p + \mu_q = \nu_m + \nu_n$. It is convenient to write $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ in terms of $\tilde{\nu}$ by applying $\tau_{\mu\nu}$ transpositions on the partition ν. Thus Eq. (43) takes the form

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu} := (-1)^{\tau_{\mu\nu}}\left(\prod_{i=1}^N \Phi^{\mathcal{H}}_{\mu_i}\right)\sum_{\sigma,\omega\in S_N}[\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}]^{b(\Gamma)}\underset{\bullet}{L}^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\nu}_{\omega(1)},\tilde{\nu}_{\omega(2)}}\prod_{j=3}^N \delta^{\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)}},$$

where we defined

$$\underset{\bullet}{L}^{\mu_{\sigma(j)},\mu_{\sigma(j)}}_{\tilde{\nu}_{\omega(j)},\tilde{\nu}_{\omega(j)}} = \prod_{j=1}^2 \frac{\tilde{r}_j^{\mu_{\sigma(j)}+\tilde{\nu}_{\omega(j)}}}{\Phi^{\mathcal{H}}_{\mu_{\sigma(j)}}}\exp[\mathbf{i}(\mu_{\sigma(j)} - \tilde{\nu}_{\omega(j)})\phi_j].$$

This may be simplified as

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}\begin{pmatrix}p,q\\ m,n\end{pmatrix} = (-1)^{\tau_{\mu\nu}}(N-2)!\left(\prod_{i=1}^N \Phi^{\mathcal{H}}_{\mu_i}\right)\left(\underset{\bullet}{L}^{\mu_p,\mu_q}_{\nu_m,\nu_n} + \underset{\bullet}{L}^{\mu_q,\mu_p}_{\nu_n,\nu_m} - \underset{\bullet}{L}^{\mu_q,\mu_p}_{\nu_m,\nu_n} - \underset{\bullet}{L}^{\mu_p,\mu_q}_{\nu_n,\nu_m}\right). \tag{E1}$$

All the $\underset{\bullet}{L}$ matrices have a dependence on $\phi_{12} = \phi_1 - \phi_2$ and ϕ_2 because of the restriction $\mu_p + \mu_q = \nu_m + \nu_n$ on the unshared elements of μ and ν:

$$\underset{\bullet}{L}^{\mu_p,\mu_q}_{\nu_m,\nu_n} = \frac{1}{\Phi_{\mu_p}\Phi_{\mu_q}}\left[\tilde{r}_1^{\mu_p+\nu_m}\tilde{r}_2^{\mu_q+\nu_n}e^{\mathbf{i}(\nu_m-\mu_p)\phi_{12}}\right].$$

This situation also persists on the other terms $\underset{\bullet}{L}^{\mu_q,\mu_p}_{\nu_n,\nu_m}$, $\underset{\bullet}{L}^{\mu_q,\mu_p}_{\nu_m,\nu_n}$, and $\underset{\bullet}{L}^{\mu_p,\mu_q}_{\nu_n,\nu_m}$. As a result, $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$ may be written as a function of the radial positions and the angular difference as

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}\begin{pmatrix}p,q\\ m,n\end{pmatrix} = (-1)^{p+q+m+n}(N-2)!\left(\prod_{\substack{i=1\\ i\neq p,q}}^{N} \Phi^{\mathcal{H}}_{\mu_i}\right)\left[\underset{\bullet}{f}^{p,q}_{m,n}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) + \underset{\bullet}{f}^{p,q}_{m,n}(\tilde{r}_2,\tilde{r}_1,\phi_{21})\right],$$

where

$$\underset{\bullet}{f}^{p,q}_{m,n}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) := \tilde{r}_1^{\mu_p+\nu_m}\tilde{r}_2^{\mu_q+\nu_n}e^{\mathbf{i}(\nu_m-\mu_p)\phi_{12}} - \tilde{r}_1^{\mu_q+\nu_m}\tilde{r}_2^{\mu_p+\nu_n}e^{\mathbf{i}(\nu_m-\mu_q)\phi_{12}}.$$

Note that Eq. (E1) for $b(\Gamma) = 1$ may be written more compactly as

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu} := (-1)^{p+q+m+n}(N-2)!\left(\prod_{\substack{i=1\\ i\neq p,q}}^{N} \Phi^{\mathcal{H}}_{\mu_i}\right)\left(z_1^{\mu_p}z_2^{\mu_q} - z_1^{\mu_q}z_2^{\mu_p}\right)^*\left(z_1^{\nu_m}z_2^{\nu_n} - z_1^{\nu_n}z_2^{\nu_m}\right), \tag{E2}$$

which also implies that the product

$$\left(z_1^{\mu_p}z_2^{\mu_q} - z_1^{\mu_q}z_2^{\mu_p}\right)^*\left(z_1^{\nu_m}z_2^{\nu_n} - z_1^{\nu_n}z_2^{\nu_m}\right) = \underset{\bullet}{f}^{p,q}_{m,n}(\tilde{r}_1,\tilde{r}_2,\phi_{12}) + \underset{\bullet}{f}^{p,q}_{m,n}(\tilde{r}_2,\tilde{r}_1,\phi_{21})$$

depends on the angle difference ϕ_{12}.

2. Computation of $\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\nu}$

Equation (43), in terms of the matrix $\underset{\bullet}{L}$, is

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\mu} := \left(\prod_{i=1}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right)\sum_{\sigma,\omega\in S_N}\left[\epsilon_{\sigma(1)\sigma(2)\cdots\sigma(N)}\epsilon_{\omega(1)\omega(2)\cdots\omega(N)}\right]^{b(\Gamma)}\underset{\bullet}{L}^{\mu_{\sigma(1)},\mu_{\sigma(2)}}_{\tilde{\mu}_{\omega(1)},\tilde{\mu}_{\omega(2)}}\prod_{j=3}^{N}\delta^{\mu_{\sigma(j)}}_{\tilde{\mu}_{\omega(j)}}.$$

Replacing explicitly $\underset{\bullet}{L}$, we obtain

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\mu} := (N-2)!\left(\prod_{i=1}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right)\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{\Phi^{\mathcal{H}}_{\mu_i}\Phi^{\mathcal{H}}_{\mu_j}}\left\{\tilde{r}_1^{2\mu_i}\tilde{r}_2^{2\mu_j}-(\tilde{r}_1\tilde{r}_2)^{\mu_i+\mu_j}\exp[\mathbf{i}(\mu_i-\mu_j)\phi_{12}]\right\},$$

which may be written most concisely as

$$\underset{\bullet}{\mathcal{B}}^{\mathcal{H}}_{\mu\mu} := (N-2)!\left(\prod_{i=1}^{N}\Phi^{\mathcal{H}}_{\mu_i}\right)\mathrm{Det}\left[{}_{\mathcal{H}}K^{(N)}_{\mu}(z_i z_j^*)\right]_{i,j=1,2}, \tag{E3}$$

with

$${}_{\mathcal{H}}K^{(N)}_{\mu}(z) := \sum_{l=1}^{N}\frac{z^{\mu_l}}{\Phi^{\mathcal{H}}_{\mu_l}}.$$

---

[1] R. R. Sari, D. Merlini, and R. Calinon, On the ground state of the one-component classical plasma, J. Phys. A: Gen. Phys. **9**, 1539 (1976).
[2] B. Jancovici, Exact Results for the Two-Dimensional One-Component Plasma, Phys. Rev. Lett. **46**, 386 (1981).
[3] S. Shakirov, Exact solution for mean energy of 2d Dyson gas at $\beta = 1$, Phys. Lett. A **375**, 984 (2011).
[4] A. Alastuey and J. L. Lebowitz, The two-dimensional one-component plasma in an inhomogeneous background: Exact results, J. Phys. (Paris) **45**, 1859 (1984).
[5] P. Choquard, P. J. Forrester, and E. R. Smith, The two-dimensional one-component plasma at $\Gamma = 2$: The semiperiodic strip, J. Stat. Phys. **33**, 13 (1983).
[6] B. Jancovici and G. Téllez, Two-dimensional Coulomb systems on a surface of constant negative curvature, J. Stat. Phys. **91**, 953 (1998).
[7] L. Šamaj, J. K. Percus, and M. Kolesík, Two-dimensional one-component plasma at coupling $\Gamma = 4$: Numerical study of pair correlations, Phys. Rev. E **49**, 5623 (1994).
[8] L. Šamaj, Is the two-dimensional one-component plasma exactly solvable? J. Stat. Phys. **117**, 131 (2004).
[9] G. Téllez and P. J. Forrester, Exact finite-size study of the 2D OCP at $\Gamma = 4$ and $\Gamma = 6$, J. Stat. Phys. **97**, 489 (1999).
[10] G. Téllez and P. J. Forrester, Expanded Vandermonde powers and sum rules for the two-dimensional one-component plasma, J. Stat. Phys. **148**, 824 (2012).
[11] R. Salazar and G. Téllez, Exact energy computation of the one component plasma on a sphere for even values of the coupling parameter, J. Stat. Phys. **164**, 969 (2016).
[12] N. D. Mermin and H. Wagner, Absence of Ferromagnetism or Antiferromagnetism in One- or Two-Dimensional Isotropic Heisenberg Models, Phys. Rev. Lett. **17**, 1133 (1966).
[13] S. Chakravarty and C. Dasgupta, Absence of crystalline order in two dimensions, Phys. Rev. B **22**, 369 (1980).
[14] A. Alastuey and B. Jancovici, Absence of strict crystalline order in a two-dimensional electron system, J. Stat. Phys. **24**, 443 (1981).
[15] P. Choquard and J. Clerouin, Cooperative Phenomena below Melting of the One-Component Two-Dimensional Plasma, Phys. Rev. Lett. **50**, 2086 (1983).
[16] J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen, A Monte Carlo study of the classical two-dimensional one-component plasma, J. Stat. Phys. **28**, 325 (1982).
[17] S. W. de Leeuw and J. W. Perram, Statistical mechanics of two-dimensional Coulomb systems, Physica A **113**, 546 (1982).
[18] A. Alastuey and B. Jancovici, On the classical two-dimensional one-component Coulomb plasma, J. Phys. France **42**, 1 (1981).
[19] A. Alastuey, Propriétés d'équilibre du plasma classique à une composante en trois et deux dimensions, Ann. Phys. Fr. **11**, 653 (1986).
[20] M. Mazars, The melting of the classical two-dimensional Wigner crystal, Europhys. Lett. **110**, 26003 (2015).
[21] B. I. Halperin and D. R. Nelson, Theory of Two-Dimensional Melting, Phys. Rev. Lett. **41**, 121 (1978).
[22] J. H. Chu and I. Lin, Direct Observation of Coulomb Crystals and Liquids in Strongly Coupled rf Dusty Plasmas, Phys. Rev. Lett. **72**, 25 (1994).
[23] H. Thomas, G. E. Morfill, V. Demmel, J. Goree, B. Feuerbacher, and D. Möhlmann, Plasma Crystal: Coulomb Crystallization in a Dusty Plasma, Phys. Rev. Lett. **73**, 652 (1994).
[24] V. Nosenko and J. Goree, Shear Flows and Shear Viscosity in a Two-Dimensional Yukawa System (Dusty Plasma), Phys. Rev. Lett. **93**, 155004 (2004).
[25] M. Wolter and A. Melzer, Laser heating of particles in dusty plasmas, Phys. Rev. E **71**, 036414 (2005).

[26] Y. Feng, J. Goree, and B. Liu, Solid Superheating Observed in Two-Dimensional Strongly Coupled Dusty Plasma, Phys. Rev. Lett. **100**, 205007 (2008).
[27] B. Liu and J. Goree, Shear Viscosity of Two-Dimensional Yukawa Systems in the Liquid State, Phys. Rev. Lett. **94**, 185002 (2005).
[28] J. Goree *et al.*, in *Frontiers in Dusty Plasmas*, edited by Y. Nakamura, T. Yokota, and P. K. Shukla (Elsevier Science, Amsterdam, 2000).
[29] L. Bonsall and A. A. Maradudin, Some static and dynamical properties of a two-dimensional Wigner crystal, Phys. Rev. B **15**, 1959 (1977).
[30] M. Antlanger, M. Mazars, L. Šamaj, G. Kahl, and E. Trizac, Taking one charge off a two-dimensional Wigner crystal, Mol. Phys. **112**, 1336 (2014).
[31] S. Deutschländer, A. M. Puertas, G. Maret, and P. Keim, Specific Heat in Two-Dimensional Melting, Phys. Rev. Lett. **113**, 127801 (2014).
[32] C. L. Mehta, *Random Matrices* (Academic, New York, 1967).
[33] J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices, J. Math. Phys. **6**, 440 (1965).
[34] T. Can, P. J. Forrester, G. Téllez, and P. Wiegmann, Exact and asymptotic features of the edge density profile for the one component plasma in two dimensions, J. Stat. Phys. **158**, 1147 (2015).
[35] M. P. Allen and D. J. Tildesley, *Computer Simulation of Liquids* (Clarendon, Oxford, 1989).
[36] B. Jancovici, G. Manificat, and C. Pisani, Coulomb systems seen as critical systems: Finite-size effects in two dimensions, J. Stat. Phys. **76**, 307 (1994).
[37] S. Johannesen and D. Merlini, On the thermodynamics of the two-dimensional jellium, J. Phys. A: Math. Gen. **16**, 1449 (1983).
[38] L. Šamaj and J. K. Percus, A functional relation among the pair correlations of the two-dimensional one-component plasma, J. Stat. Phys. **80**, 811 (1995).
[39] DiagHam, http://www.phys.ens.fr/∼regnault/diagham/.