PHYSICAL REVIEW E **96**, 060101(R) (2017)

Maximum of an Airy process plus Brownian motion and memory in Kardar-Parisi-Zhang growth

Pierre Le Doussal

CNRS-Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 24 rue Lhomond, 75231 Paris Cedex, France

(Received 28 September 2017; published 7 December 2017)

We obtain several exact results for universal distributions involving the maximum of the Airy_2 process minus a parabola and plus a Brownian motion, with applications to the one-dimensional Kardar-Parisi-Zhang (KPZ) stochastic growth universality class. This allows one to obtain (i) the universal limit, for large time separation, of the two-time height correlation for droplet initial conditions, e.g., $C_\infty = \lim_{t_2/t_1\to+\infty} \overline{h(t_1)h(t_2)}^c/\overline{h(t_1)^2}^c$, with $C_\infty \approx 0.623$, as well as conditional moments, which quantify ergodicity breaking in the time evolution; (ii) in the same limit, the distribution of the midpoint position $x(t_1)$ of a directed polymer of length t_2; and (iii) the height distribution in stationary KPZ with a step. These results are derived from the replica Bethe ansatz for the KPZ continuum equation, with a "decoupling assumption" in the large time limit. They agree and confirm, whenever they can be compared, with (i) our recent tail results for two-time KPZ with the work by de Nardis and Le Doussal [J. Stat. Mech. (2017) 053212], checked in experiments with the work by Takeuchi and co-workers [De Nardis *et al.*, Phys. Rev. Lett. **118**, 125701 (2017)] and (ii) a recent result of Maes and Thiery [J. Stat. Phys. **168**, 937 (2017)] on midpoint position.

DOI: 10.1103/PhysRevE.96.060101

Stochastic processes, such as the Brownian motion, are useful unifying mathematical tools to describe the universal behavior of complex systems. In recent years, the Airy_2 process, introduced in [1], appeared in several contexts in physics and mathematics. Its simplest definition (see e.g., [2]) involves the Dyson Brownian motion (DBM) [3]: consider a large Hermitian random matrix H whose independent entries (both real and imaginary parts) perform independent stationary Orstein-Ulhenbeck processes (i.e., Brownian motions equilibrated in a harmonic well). The Airy_2 process describes the evolution of the largest eigenvalue of H (centered and scaled). The Airy_2 process appears as a limit process in directed last passage percolation [4], nonintersecting Brownian bridges [1,4–10], random tilings [11], interacting particle transport in one dimension (1D) [12,13], quantum dynamics of fermions [14–16], and stochastic growth models, either discrete [1,17] or the continuum 1D Kardar-Parisi-Zhang (KPZ) equation [18,19] (for a review, see [20–22]). In fact, the Airy_2 process is a hallmark of the very broad 1D-KPZ universality class, which arises in all these models.

Models in the 1D-KPZ class usually allow for the definition of a height field $h(x,t)$, which undergoes stochastic growth. The prominent example is the continuum KPZ equation [18], where $h(x,t)$ is an interface height at point $x \in \mathbb{R}$, evolving as a function of time t as

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda_0}{2}[\partial_x h(x,t)]^2 + \sqrt{D}\xi(x,t) \qquad (1)$$

driven by unit white noise $\overline{\xi(x,t)\xi(x',t')} = \delta(x-x')\delta(t-t')$. For the curved (i.e., droplet) initial condition (IC) it is known (in some cases proved) that it converges at large time $t \to +\infty$ (rescaled and centered) to [1,4,22,23]

$$(\Gamma t)^{-1/3}[h_{\text{drop}}(x,t) - v_\infty t] \simeq \mathcal{A}_2(\hat{x}) - \hat{x}^2, \quad \hat{x} = A\frac{x}{2t^{2/3}}, \qquad (2)$$

where $\mathcal{A}_2(\hat{x})$ is the Airy_2 process, as an identity between processes (i.e., as $\hat{x}$ is varied). Since $\mathcal{A}_2$ is stationary (statistical translational and reflection invariant in $\hat{x}$) the $-\hat{x}^2$ term embodies the mean parabolic profile. We use units such that the nonuniversal constants $\Gamma = A = 1$, i.e., $\lambda_0 = D = 2$ and $\nu = 1$ for the KPZ equation (1), and set $v_\infty = 0$ (upon a shift of h). The equilibrium measure of the DBM being the Gaussian unitary ensemble (GUE) measure for H, the fluctuations of the Airy_2 process, hence of the KPZ height from (2), at any given point, e.g., $x = 0$, is the Tracy-Widom (TW) distribution for the largest eigenvalue of a GUE random matrix [24]. Its cumulative distribution function (CDF) is explicitly known as a Fredholm determinant

$$\text{Prob}\,[\mathcal{A}_2(0) < \sigma] = F_2(\sigma) = \text{Det}\,(I - P_\sigma K_{\text{Ai}}), \qquad (3)$$

where $K_{\text{Ai}}(u,v) = \int_0^{+\infty} dy\, \text{Ai}(y+u)\text{Ai}(y+v)$ is the Airy kernel, P_σ being here the projector on $[\sigma,+\infty[$. Furthermore, from properties of the DBM, the Airy_2 process is determinantal, i.e., any p-point joint CDF (JCDF) of $\mathcal{A}_2(\hat{x})$ can be written as $p \times p$ matrix Fredholm determinants, in terms of an extended Airy kernel [1,20].

Although much studied, and fully characterized by its determinantal structure, important open questions remain about the Airy process, with applications to the 1D KPZ class. First, for more general initial conditions $h(x,t=0)$, the value at a given point, e.g., $x = 0$, is obtained from the variational problem [25,26]

$$t^{-1/3}h(0,t) \simeq \max_{\hat{y}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \mathsf{h}_0(\hat{y})] \qquad (4)$$

when a limit exists for the rescaled IC $\mathsf{h}_0(\hat{y}) \simeq t^{-1/3}h(2t^{2/3}\hat{y},0)$. Droplet subclass ICs correspond to $\mathsf{h}_0(0) = 0$ and $\mathsf{h}_0(\hat{y}) = -\infty$ for $\hat{y} \neq 0$, recovering (2), while flat subclass ICs correspond to $\mathsf{h}_0(\hat{y}) = 0$. The CDF of $h(0,t)$ and of the argmax in (4) (i.e., the end-point distribution of a directed polymer; see below) for flat IC, and other results such as intermediate classes of IC, have been obtained from exact solutions of models in the KPZ class at large t, or from powerful methods directly on the Airy process which allow one to treat a large class of h_0 [20,23,26]. The latter, however, do not readily extend to *random initial conditions*, such as the Brownian IC, related to the important stationary KPZ subclass.

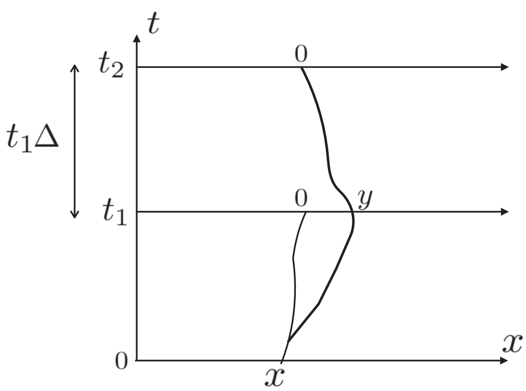

FIG. 1. Two directed polymers in a random potential with fixed end points, starting at $(x,0)$, ending, respectively, at $(0,t_1)$ and $(0,t_2)$. Minus their free energies maps to KPZ heights $h(0,t_1)$ and $h(0,t_2)$ with droplet initial condition centered at x. The PDF of the midpoint y, and the correlations between $h(0,t_1)$ and $h(0,t_2)$ are obtained here exactly in the limit $t_1,\Delta = \frac{t_2-t_1}{t_1} \to +\infty$.

The aim of this Rapid Communication is to study some properties of the optimization problem

$$\max_{\hat{y}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})], \tag{5}$$

where $B(\hat{y})$ is the two-sided unit Brownian motion. Equation (5) defines $\mathcal{A}_{\rm stat}(0)$, the Airy process associated to the stationary KPZ equation, at $\hat{x} = 0$. We first describe our three main results and applications, then give an explicit formula, and finally sketch the replica Bethe ansatz method.

Our first result is the probability distribution function (PDF) of the position $\hat{y}_m$ of the maximum in (5), i.e.,

$$\hat{y}_m = \underset{\hat{y}\in\mathbb{R}}{\operatorname{argmax}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})] \tag{6}$$

This distribution arises in the *midpoint probability of a directed polymer* (DP) in the white noise $d = 1 + 1$ random potential ξ. Recall that the partition sum $Z(x,t|y,0)$ of continuum directed paths from $(y,0)$ to (x,t) defined as

$$Z(x,t|y,0) := \int_{x(0)=y}^{x(t)=x} Dx\, e^{-\int_0^t d\tau\{1/4(dx/d\tau)^2 - \sqrt{2}\xi[x(\tau),\tau]\}} \tag{7}$$

equals $e^{h(x,t)}$ where $h(x,t)$ is the solution of (1) with droplet initial condition (centered at y). Consider a DP from $(0,0)$ to $(0,t_2)$ and ask about the PDF, $P_{t_1,t_2}(y)$, of the position $x(t_1) = y$ at intermediate time t_1 (see Fig. 1). In the limit of large times t_1,t_2, with $\hat{y} = y/(2t_1^{2/3})$,

$$\overline{P_{t_1,t_2}(y)}dy = \frac{\overline{Z(0,t_2|y,t_1)Z(y,t_1|0,0)}}{Z(0,t_2|0,0)}dy \to P_\Delta(\hat{y})d\hat{y}. \tag{8}$$

One finds (see below) that as $\Delta = \frac{t_2-t_1}{t_1} \to +\infty$, $P_{t_1,t_2}(y)$ concentrates on $\hat{y} = \hat{y}_m$ defined in (6), hence

$$\mathcal{P}(\hat{y})d\hat{y} := P_{+\infty}(\hat{y})d\hat{y} = \text{Prob}\,(\hat{y}_m \in [\hat{y},\hat{y} + dy[). \tag{9}$$

Here we calculate this distribution, which also arises in the study of the coalescence of optimal paths [27].

Our second result is the following joint CDF:

$$G(\sigma_1,\sigma_2) := \text{Prob}\,\{\mathcal{A}_2(0) < \sigma_1, \max_{\hat{y}\in\mathbb{R}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})] < \sigma_2\}. \tag{10}$$

It is important in the study of the so-called *persistence of correlations in the two-time KPZ problem for droplet initial conditions*, which exhibits an interesting memory effect, also called ergodicity breaking [28–32]. Indeed, consider the rescaled heights at t_1 and at $t_2 > t_1$, in the limit where both times are large, with $\Delta = (t_2 - t_1)/t_1$ fixed:

$$\begin{aligned} t_1^{-1/3}h(0,t_1) &\simeq \mathcal{A}_2(0),\\ t_1^{-1/3}h(0,t_2) &\simeq \max_{\hat{y}\in\mathbb{R}}\left\{\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \Delta^{1/3}\left[\tilde{\mathcal{A}}_2\left(\frac{\hat{y}}{\Delta^{2/3}}\right) - \frac{\hat{y}^2}{\Delta^{4/3}}\right]\right\}\\ &\underset{\Delta\to+\infty}{\simeq} \Delta^{1/3}\tilde{\mathcal{A}}_2(0) + \max_{\hat{y}\in\mathbb{R}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})] + O\left(\frac{1}{\Delta^{1/3}}\right), \end{aligned} \tag{11}$$

where $\mathcal{A}_2$ and $\tilde{\mathcal{A}}_2$ denote two independent Airy processes. The second line expresses that the height at t_2 is the sum of a first contribution from the time interval $[0,t_1]$ and the second from $[t_1,t_2]$ which, for a fixed intermediate point y, are independent (see Fig. 1). Optimization over $\hat{y}$ correlates them. Obtaining the resulting joint PDF (JPDF) of the two rescaled heights for arbitrary Δ is a difficult problem [28,29,32–37]. In the limit of well separated times, i.e., large Δ, using that the Airy process $\tilde{\mathcal{A}}_2$ is locally a Brownian, one obtains the third line in (11), where B and $\mathcal{A}_2$ are independent processes [38]. As is clear from (11), $h(0,t_1) \sim t_1^{1/3}$, $h(0,t_2) \sim t_2^{1/3}$ are quite different in magnitude (for large t_2/t_1), but remain correlated by the $O(t_1^{1/3})$ term. To measure this memory effect one usually defines the dimensionless ratio of the two covariances

$$C(t_1,t_2) = \frac{\overline{h(0,t_1)h(0,t_2)}^c}{\overline{h(0,t_1)^2}^c} \simeq C_\Delta \tag{12}$$

which, at large times $t_1,t_2 \to +\infty$, becomes a universal function C_Δ of Δ. From (11) the latter has a finite limit

$$C_\infty = \frac{\int d\sigma_1 d\sigma_2 \sigma_1\sigma_2 p(\sigma_1,\sigma_2)}{\kappa_2^{\rm GUE}}, \tag{13}$$

where $p(\sigma_1,\sigma_2) = \partial_{\sigma_1}\partial_{\sigma_2}G(\sigma_1,\sigma_2)$ is the JPDF associated to (10), obtained here exactly (here and below $\kappa_p^{\rm GUE}$ is the pth cumulant of the GUE-TW distribution).

Finally, our third main result is the CDF for the height $h(x,t)$ for an *IC equal to a Brownian plus a step*, corresponding to a rescaled IC $\mathsf{h}_0(\hat{y}) = \sqrt{2}B(\hat{y}) - \hat{H}\,\text{sgn}(\hat{y})$. It will be relevant for any KPZ system where two half spaces, each stationary, are put in contact at $t = 0$, with a mismatch in height $2\hat{H}t^{1/3}$.

Before displaying our explicit formula, it is important to recall some known results about the stationary KPZ IC subclass, which corresponds to $\mathsf{h}_0(\hat{y}) = \sqrt{2}B(\hat{y})$ where $B(x)$ is a two-sided unit Brownian $\langle dB(x)^2\rangle = dx$ with $B(0) = 0$. It is realized, e.g., by the solution $h_{\rm stat}(x,t)$ of the KPZ equation (1) with a unit two-sided Brownian initial condition $h(x,t=0)=B_0(x)$ [39] as

$$t^{-1/3}h_{\rm stat}(x,t) \simeq \mathcal{A}_{\rm stat}(\hat{x}) \tag{14}$$

$$= \max_{\hat{y}}[\mathcal{A}_2(\hat{y}-\hat{x})-(\hat{y}-\hat{x})^2+\sqrt{2}B(\hat{y})], \tag{15}$$

where here the last equality is only at fixed $\hat{x}$ (not as a process). Let us now define the two functions

$$\mathcal{B}_w(v) = e^{(1/3)w^3-vw} - \int_0^{+\infty} dy\, \mathrm{Ai}(v+y)e^{wy},$$

$$\mathcal{L}_{\hat{x}}(\sigma) = \sigma - 1 - \hat{x}^2 + \int_\sigma^{+\infty} dv[1-\mathcal{B}_{\hat{x}}(v)\mathcal{B}_{-\hat{x}}(v)]. \tag{16}$$

It is known that the one-point CDF of the $\mathcal{A}_{\rm stat}$ process is given by the extended Baik-Rains distribution [20,25,40–45], which has the following exact expression:

$$\mathrm{Prob}\,[\mathcal{A}_{\rm stat}(\hat{x})<\sigma] =: F_0(\sigma-\hat{x}^2,\hat{x}) = \partial_\sigma[F_2(\sigma)Y_{\hat{x}}(\sigma)] \tag{17}$$

in terms of the auxiliary function

$$Y_{\hat{x}}(\sigma) := 1+\mathcal{L}_{\hat{x}}(\sigma) - \mathrm{Tr}\big[P_\sigma K_{\rm Ai}(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma \mathcal{B}_{-\hat{x}}\mathcal{B}_{\hat{x}}^T\big], \tag{18}$$

where we denote AB^T the projector $AB^T(u,v)=A(u)B(v)$. For $\hat{x}=0$, the function $F_0(\sigma,0)=F_0(\sigma)=\partial_\sigma[F_2(\sigma)Y_0(\sigma)]$ is the CDF of the standard Baik-Rains (BR) distribution F_0.

The PDF of argmax. We now display our first result. Let $H(\hat{x})=\mathrm{Prob}(\hat{y}_m>\hat{x})$ be the CDF of the position $\hat{y}_m$ of the maximum defined by (6). Our method, detailed below, gives

$$\begin{aligned} H(-\hat{x}) = \int d\sigma\, F_2(\sigma)\big\{&Y_{\hat{x}}(\sigma)\mathrm{Tr}[(I-P_\sigma K_{\rm Ai})^{-1} \\ &\times P_\sigma(\mathrm{Ai}'+\hat{x}\mathrm{Ai})\mathrm{Ai}^T] \\ &+\big(\mathrm{Tr}\big[(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma \mathrm{Ai}\mathcal{B}_{\hat{x}}^T\big]-1\big) \\ &\times \mathrm{Tr}\big[(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma(\mathrm{Ai}'+\hat{x}\mathrm{Ai})\mathcal{B}_{-\hat{x}}^T\big]\big\}, \end{aligned} \tag{19}$$

where Ai′ is the derivative of the Airy function. Interestingly, in a recent work, Maes and Thiery [46] noted that the distribution of argmax $\hat{y}_m$ can be related, using the fluctuation-dissipation theorems (FDT), to the variance of the height in stationary KPZ, defined as

$$g(\hat{x}) = \langle\sigma^2\rangle_{F_0,\hat{x}} - \langle\sigma\rangle_{F_0,\hat{x}}^2, \tag{20}$$

where $\langle O(\sigma)\rangle_{F_0,\hat{x}} = \int d\sigma\, O(\sigma)\partial_\sigma F_0(\sigma-\hat{x}^2,\hat{x})$ denotes an average over the extended Baik-Rains distribution, which is an even function of $\hat{x}$. Note that the second term is simply $-\langle\sigma\rangle_{F_0,\hat{x}}^2=-\hat{x}^4$. As a consequence of [46], the scaled PDF of the midpoint probability is predicted as

$$\mathcal{P}(\hat{y}) = -H'(\hat{y}) = f_{\rm KPZ}(\hat{y}), \quad f_{\rm KPZ}(\hat{y}) := \tfrac{1}{4}g''(\hat{y}), \tag{21}$$

where the notation $f_{\rm KPZ}(\hat{y})$ for the second derivative of g in (20) was introduced in the context of the polynuclear growth model and totally asymmetric exclusion process models [47].

It is thus important to check whether our result (19), obtained through an independent and completely different route, agrees with this prediction. Using the identities [48]

$$\begin{aligned} \partial_\sigma Y_{\hat{x}}(\sigma) &= \big(\mathrm{Tr}\big[(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma \mathrm{Ai}\mathcal{B}_{\hat{x}}^T\big]-1\big) \\ &\quad\times\big(\mathrm{Tr}\big[(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma \mathrm{Ai}\mathcal{B}_{-\hat{x}}^T\big]-1\big), \\ \partial_\sigma F_2(\sigma) &= \mathrm{Tr}[P_\sigma(I-P_\sigma K_{\rm Ai})^{-1}P_\sigma \mathrm{Ai}\mathrm{Ai}^T] \end{aligned} \tag{22}$$

a few algebraic manipulations [48] show that (19) can indeed be rewritten as

$$H(\hat{x}) = \tfrac{1}{2}-\tfrac{1}{4}g'(\hat{x}), \tag{23}$$

where, we recall, $g'(\hat{x})$ is odd, and $g'(\pm\infty)=\pm 2$. Hence our result (19) provides an equivalent, although different form for the midpoint probability $\mathcal{P}(\hat{y})=H(\hat{y})$. This provides a test of our method (the decoupling assumption; see below) and of the FDT for the KPZ problem. Note that the result of [46] extends to finite time, while our method deals with large times.

Joint PDF of Airy and Airy minus parabola plus Brownian and persistent KPZ two-time correlations. We now give our result for the JCDF (10). We find, for $\sigma_1\leqslant\sigma_2$,

$$\begin{aligned} G(\sigma_1,\sigma_2) = {}& F_2(\sigma_1)Y_0(\sigma_1)\mathrm{Tr}\big[\big(I-P_{\sigma_1}K_{\rm Ai}\big)^{-1} \\ &\times P_{\sigma_1}\mathrm{Ai}_{\sigma_2-\sigma_1}\mathrm{Ai}_{\sigma_2-\sigma_1}^T\big] \\ &+F_2(\sigma_1)\big(\mathrm{Tr}\big[(I-P_{\sigma_1}K_{\rm Ai})^{-1}P_{\sigma_1}\mathrm{Ai}_{\sigma_2-\sigma_1}\mathcal{B}_0^T\big] \\ &-1\big)^2, \end{aligned} \tag{24}$$

where $\mathrm{Ai}_\sigma(u)=\mathrm{Ai}(u+\sigma)$ and $G(\sigma_1,\sigma_2)=F_0(\sigma_2)$ for $\sigma_1\geqslant\sigma_2$. An extended result for $\hat{x}\neq 0$ is displayed in [48]. It is easy to check [48] the continuity, $G(\sigma,\sigma)=F_0(\sigma)$ using the identities (22). It is also easy to see that the marginal CDF of σ_1, $G(\sigma_1,+\infty)=F_2(\sigma_1)$, is the GUE-TW and the marginal CDF of σ_2, $G(+\infty,\sigma_2)=F_0(\sigma_2)$, is the BR distribution.

We now apply this result to the large time separation limit of the two-time correlation in the 1D KPZ class. Using integration by parts one obtains [48]

$$\langle(\sigma_2-\sigma_1)^2\rangle = 2\int_{-\infty}^{+\infty}d\sigma_2\int_{-\infty}^{\sigma_2}d\sigma_1[F_2(\sigma_1)-G(\sigma_1,\sigma_2)], \tag{25}$$

where here and below $\langle\cdots\rangle$ denotes averages with respect to $p(\sigma_1,\sigma_2)=\partial_{\sigma_1}\partial_{\sigma_2}G(\sigma_1,\sigma_2)$, the associated JPDF. This allows us to compute the two-time persistent dimensionless covariance ratio (13) as

$$C_\infty = \frac{\langle\sigma_2\sigma_1\rangle^c}{\langle\sigma_1^2\rangle} = \frac{\langle\sigma_2\sigma_1\rangle}{\kappa_2^{\rm GUE}} = \frac{\langle\sigma_1^2\rangle+\langle\sigma_2^2\rangle-\langle(\sigma_2-\sigma_1)^2\rangle}{2\kappa_2^{\rm GUE}}$$

$$= \frac{1}{2}+\frac{\big(\kappa_1^{\rm GUE}\big)^2+\kappa_2^{\rm BR}}{2\kappa_2^{\rm GUE}} - \frac{\langle(\sigma_2-\sigma_1)^2\rangle}{2\kappa_2^{\rm GUE}} \tag{26}$$

$$= 3.135\,98 - \frac{\langle(\sigma_2-\sigma_1)^2\rangle}{2\kappa_2^{\rm GUE}} \approx 0.6225\pm 0.0015 \tag{27}$$

using the known GUE-TW and BR cumulants, i.e., $\langle\sigma_2\rangle=\kappa_1^{\rm BR}=0$, $\kappa_1^{\rm GUE}=-1.771\,086\,8$, $\kappa_2^{\rm GUE}=0.813\,19$, and $\kappa_2^{\rm BR}=1.150\,39$, and evaluating (25) numerically (see Sec IV.6 in [48]). Equation (27) compares quite well with recent numerical simulations and experiments [49,50].

Let us recall our recent study [28,29] and reexamine the observables defined there, using our new exact results. There we defined the variables $h_1 := h(0,t_1)/t_1^{1/3}$ and the scaled height difference

$$h := [h(0,t_2) - h(0,t_1)]/(t_2 - t_1)^{1/3}. \tag{28}$$

Defining the (unknown) exact JPDF $P_\Delta(\sigma_1,\sigma) := \lim_{t_1,t_2=t_1(1+\Delta)\to+\infty} \overline{\delta(h_1 - \sigma_1)\delta(h - \sigma)}$, we derived an approximation of it, denoted $P_\Delta^{(1)}(\sigma_1,\sigma)$, conjectured to be exact to leading order in large positive σ_1, for any fixed σ and Δ. It was shown in [29] to be a good enough approximation to fit experiments and numerics in a broad range of values $\sigma_1 > \langle\sigma_1\rangle = \kappa_1^{\rm GUE}$. It is thus of great importance to check whether our present exact result, valid only for large Δ, but for any σ_1,σ, confirms these predictions.

At large times, the height difference, from (11), takes the form, up to $O(\Delta^{-2/3})$ terms,

$$h \simeq \tilde{\mathcal{A}}_2(0) + \Delta^{-1/3}(\sigma_2 - \sigma_1), \tag{29}$$

where we denote (with a slight abuse of notations) the two random variables

$$\sigma_1 = \mathcal{A}_2(0), \quad \sigma_2 = \max_{\hat{y}\in\mathbb{R}}[\mathcal{A}_2(\hat{y}) - \hat{y}^2 + \sqrt{2}B(\hat{y})] \tag{30}$$

and we recall that $\tilde{\mathcal{A}}_2(0)$ is a GUE-TW random variable independent of the $O(\Delta^{-1/3})$ term. The first consequence, averaging (29), is that

$$\overline{h} = \kappa_1^{\rm GUE}\left(1 - \frac{1}{\Delta^{1/3}}\right) + O(\Delta^{-1}) \tag{31}$$

since $\langle\sigma_2\rangle = \kappa_1^{\rm BR} = 0$ and $\langle\sigma_1\rangle = \kappa_1^{\rm GUE}$, in agreement with the general formula for $\overline{h}$ for any Δ [see (48) in [28]]. Important quantities, introduced in [28,29], are the *conditional averages* of h, either for a fixed value of $h_1 = \sigma_1$, $\overline{h}_{h_1=\sigma_1}$, or, for a value larger than some threshold $h_1 = \sigma_1 > \sigma_{1c}$, $\overline{h}_{h_1=\sigma_1>\sigma_{1c}}$. From the above, one predicts

$$\overline{h}_{h_1=\sigma_1} \simeq \kappa_1^{\rm GUE} + \frac{1}{\Delta^{1/3}}\langle\sigma_2 - \sigma_1\rangle_{\sigma_1} + o\left(\frac{1}{\Delta^{1/3}}\right), \tag{32}$$

where the conditional average with respect to p, denoted as

$$\langle\sigma_2 - \sigma_1\rangle_{\sigma_1} = \frac{1}{F_2'(\sigma_1)}\int_{-\infty}^{+\infty} d\sigma_2(\sigma_2 - \sigma_1)p(\sigma_1,\sigma_2) \tag{33}$$

can be calculated from (24). For large positive σ_1 one shows from (24) that (see [48] where next order is also displayed)

$$p(\sigma_1,\sigma_2) \simeq -2\partial_{\sigma_2}K_{\rm Ai}(\sigma_1,\sigma_2) - {\rm Ai}(\sigma_2)^2 \tag{34}$$

which leads to, for large positive σ_1,

$$\langle\sigma_2 - \sigma_1\rangle_{\sigma_1} \simeq R_{1/3}(\sigma_1), \tag{35}$$

$$R_{1/3}(\sigma_1) := \frac{\left[\int_{\sigma_1}^{+\infty} dy\,{\rm Ai}(y)\right]^2 - \int_{\sigma_1}^{+\infty} dy\,K_{\rm Ai}(y,y)}{K_{\rm Ai}(\sigma_1,\sigma_1)} \tag{36}$$

which is precisely the prediction obtained in [28]. This is encouraging evidence that the method of [28] is good enough to capture, as claimed there, the tail of the two-time JPDF. We can thus calculate the conditional averages beyond the large positive σ_1 regime. We show in Fig. 2 the leading order of

$$\overline{h}_{h_1>\sigma_{1c}} = \kappa_1^{\rm GUE} + \Delta^{-1/3}\langle\sigma_2 - \sigma_1\rangle_{\sigma_1>\sigma_{1c}} + O(\Delta^{-2/3}) \tag{37}$$

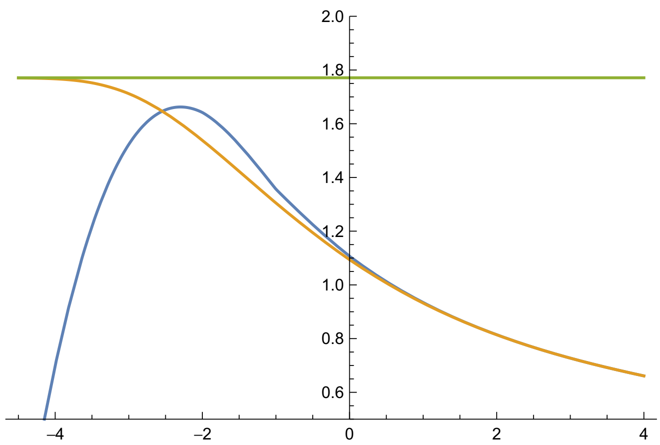

FIG. 2. Conditional average $\langle\sigma_2 - \sigma_1\rangle_{\sigma_1>\sigma_{1c}}$ (y axis) as a function of σ_{1c} (x axis), which describes the averaged scaled KPZ height difference h at large $\Delta = \frac{t_2-t_1}{t_1}$ [see (37)]. (i) Orange: exact result obtained here. (ii) Blue: prediction obtained in [28,29], which becomes exact for large positive σ_{1c} [indistinguishable from (i) for $\sigma_{1c} > 0$]. The limit $\sigma_{1c} \to -\infty$ (unconditioned mean), exact from (31) and to which our result (i) converges, is the orange horizontal line at $-\kappa_1^{\rm GUE} = 1.771\,09$.

evaluated numerically [48] from (24), a quantity which can be measured accurately in experiments and numerics.

Finally, the conditional covariance ratio was introduced and measured in [28,29]

$$C_\Delta(\sigma_{1c}) := \frac{\overline{h_1 h_2}^c_{h_1>\sigma_{1c}}}{\overline{h_1^2}^c_{h_1>\sigma_{1c}}}. \tag{38}$$

We obtain here its large Δ limit,

$$C_\infty(\sigma_{1c}) = \frac{\langle\sigma_2\sigma_1\rangle^c_{\sigma_1>\sigma_{1c}}}{\langle\sigma_1^2\rangle^c_{\sigma_1>\sigma_{1c}}} \tag{39}$$

a function of σ_{1c} which interpolates between $C_\infty(\sigma_{1c} = -\infty) = C_\infty$ the unconditioned two-time covariance ratio obtained in (27) and $C_\infty(\sigma_{1c} = +\infty) = 1$. It is evaluated numerically [48] from (24) and plotted in Fig. 3.

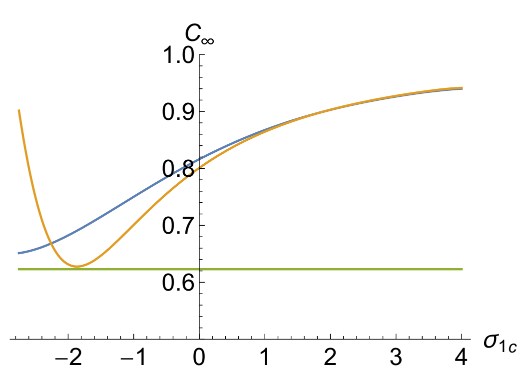

FIG. 3. Conditional covariance ratio $C_\infty(\sigma_{1c})$, Eq. (39). Orange: asymptotic prediction for large positive σ_{1c} from [28,29]. Blue: exact result from (24), converging to (27) (orange horizontal line) for $\sigma_{1c} = -\infty$.

We also explored the case where the longer polymer in Fig. 1 is constrained to pass to the right of 0, $y > 0$: we find $C_\infty \approx 0.6925$, i.e., that the correlations are increased.

Stationary KPZ in presence of a step. Finally, the height $h(x,t)$ in the KPZ class with a step at $x = 0$ in the initial condition, and independent Brownian initial condition on each side, takes the scaling form at large t:

$$t^{-1/3} h(x = 2t^{2/3}\hat{x},t) \approx \hat{h}(\hat{x}) := \max_{\hat{y}}[\mathcal{A}_2(\hat{x} - \hat{y}) - (\hat{x} - \hat{y})^2 + \sqrt{2}B(y) - \hat{H}\,\mathrm{sgn}(\hat{y})]. \quad (40)$$

Defining $G_{\hat{H}}(\sigma_L) = \mathrm{Prob}\,[\hat{h}(\hat{x}) - \hat{H} + \hat{x}^2 < \sigma_L]$, we obtain

$$G_{\hat{H}}(\sigma_L) = F_2(\sigma_L) \times \big(Y_{\hat{x}}(\sigma_L) e^{2\hat{x}\hat{H}} \mathrm{Tr}\big[(I - P_{\sigma_L} K_{\mathrm{Ai}})^{-1} P_{\sigma_L} \mathrm{Ai}_{2\hat{H}} \mathrm{Ai}^T\big] + \big(\mathrm{Tr}\big[(I - P_{\sigma_L} K_{\mathrm{Ai}})^{-1} P_{\sigma_L} \mathrm{Ai} \mathcal{B}_{\hat{x}}^T\big] - 1\big) \times \big(e^{2\hat{x}\hat{H}} \mathrm{Tr}\big[(I - P_{\sigma_L} K_{\mathrm{Ai}})^{-1} \mathrm{Ai}_{2\hat{H}} \mathcal{B}_{-\hat{x}}^T\big] - 1\big)\big). \quad (41)$$

Method. The method is based on the replica Bethe ansatz, which led to exact solutions for one-time observables for various initial conditions [2,43,51–60]. Since it is an extension of the calculation in [61] we only sketch the idea, with details in Secs. II and III of [48]. We define, jointly in the same noise realization, $h_1(x,t) = \ln Z(x,t|0,0)$ and $h_{L,R}(x,t)$ the solutions of the KPZ equation with ICs, respectively, droplet (1), and Brownian on $y < 0$ (L) and on $y > 0$ (R). We define the joint Laplace transform

$$\hat{g}_t(\sigma_1,\sigma_L,\sigma_R;\hat{x}) = \overline{e^{-\sum_{b=1,L,R} u_b Z_b(x,t)}} \quad (42)$$

with $u_b = e^{-t^{1/3}(\sigma_b - \hat{x}^2)}$. Its large time limit gives

$$\hat{g}_\infty(\sigma_1,\sigma_L,\sigma_R;\hat{x}) = G_{\hat{x}}(\sigma_1,\sigma_L,\sigma_R;\hat{x}) := \mathrm{Prob}\,[\mathcal{A}_2(-\hat{x}) < \sigma_1, \hat{h}_L(\hat{x}) + \hat{x}^2 < \sigma_L, \hat{h}_R(\hat{x}) + \hat{x}^2 < \sigma_R] \quad (43)$$

with

$$\hat{h}_{L,R}(\hat{x}) = \max_{\hat{y}<0,\hat{y}>0} [\mathcal{A}_2(\hat{y} - \hat{x}) - (\hat{y} - \hat{x})^2 + \sqrt{2}B(\hat{y})] \quad (44)$$

containing all desired information (and more). To compute $\hat{g}_t$ we expand the exponential in (42), and write the joint moments as quantum-mechanical expectations

$$\overline{\prod_{b=1,L,R} Z_b(x,t)^{n_b}} = \langle x,\ldots x| e^{-H_n t} |n_L,n_1,n_R\rangle, \quad (45)$$

where

$$H_n = -\sum_{\alpha=1}^{n} \frac{\partial^2}{\partial x_\alpha^2} - 2 \sum_{1\leqslant\alpha<\beta\leqslant n} \delta(x_\alpha - x_\beta) \quad (46)$$

is the Hamiltonian of the attractive Lieb-Liniger δ-Bose gas model [62], and $|n_L,n_1,n_R\rangle$ is a state with n_1 bosons at $y = 0$, and $n_{L,R}$ in $y < 0$ and $y > 0$, respectively, with $n = n_1 + n_L + n_R$. One inserts in (45) the known Bethe ansatz eigenstates, each consisting of $1 \leqslant n_s \leqslant n$ strings (bound states) of $m_j \geqslant 1$ bosons with rapidities $\lambda_{j,a} = k_j - \frac{i}{2}(m_j + 1 - 2a)$, $a = 1,\ldots,m_j$, and $\sum_{j=1}^{n_s} m_j = n$. For the Brownian (and flat) IC the overlap of $|n_L,n_1,n_R\rangle$ with any Bethe state can be expressed explicitly, although as a complicated sum over products of gamma functions, extending as in [61] the combinatoric method introduced in [54]. It can be simplified in the large t limit through the *decoupling assumption* (which sets all interstring double products to unity) as done in [2,54–60]. Summing over the eigenstates becomes possible and leads to a Fredholm determinant formula for $\hat{g}_\infty$ given in [48]. For regularization the calculation includes finite drifts $w_{L,R} > 0$ in the Brownian, and the (delicate) limit $w_{L,R} = 0^+$ converts the Fredholm determinant into a final expression for $G_{\hat{x}}(\sigma_1,\sigma_L,\sigma_R;\hat{x})$ given in [48]. Specializing to $\sigma_L = \sigma_R = \sigma_2$ leads to the JCDF $G_{\hat{x}}(\sigma_1,\sigma_2)$ for general $\hat{x}$, given in [48], which reduces to (24) for $G = G_0$ for $\hat{x} = 0$. Specializing instead to $\sigma_1 = +\infty$, one obtains (i) the result (19) for the CDF of the argmax of Airy minus parabola plus Brownian and (21) for the limiting midpoint DP probability, from

$$H(-\hat{x}) = \int_{-\infty}^{+\infty} d\sigma_R [\partial_{\sigma_R} \hat{g}_\infty(+\infty,\sigma_L,\sigma_R;\hat{x})]|_{\sigma_L=\sigma_R^-}$$

and (ii) the CDF of the KPZ height in the presence of a step IC: setting $\sigma_R = \sigma_L + \hat{H}$ one obtains $G_{\hat{H}}(\sigma_L) = \hat{g}_\infty(+\infty,\sigma_L,\sigma_R;\hat{x})$ leading to the result (41).

In conclusion, from a replica Bethe ansatz calculation, using a decoupling assumption, we obtained several distributions involving the maximum of the Airy process minus parabola plus Brownian. This leads to exact universal results for two-time KPZ in the large time separation limit $\Delta = \frac{t_2 - t_1}{t_1} \gg 1$, which correctly match, and nicely complement, our recent tail results [28,29] for any Δ, putting both methods on firmer ground. Taken together they should also lead to further accurate comparisons with experiments and numerics in the universal large time limit, and allow one to test other observables, e.g., the effect of the end-point position $\hat{x} \neq 0$ as predicted here and in [28].

We thank J. de Nardis, K. Takeuchi, and T. Thiery for stimulating discussions and collaborations, and A. Borodin, I. Corwin, T. Halpin-Healy, S. Majumdar, J. Quastel, and G. Schehr for interesting remarks.

[1] M. Prähofer and H. Spohn, Scale invariance of the PNG droplet and the Airy process, J. Stat. Phys. **108**, 1071 (2002); Universal Distributions for Growth Processes in 1+1 Dimensions and Random Matrices, Phys. Rev. Lett. **84**, 4882 (2000).

[2] S. Prolhac and H. Spohn, The one-dimensional KPZ equation and the Airy process, J. Stat. Mech. (2011) P03020.

[3] F. J. Dyson, A Brownian-motion model for the eigenvalues of a random matrix, J. Math. Phys. **3**, 1191 (1962).

[4] K. Johansson, Discrete polynuclear growth and determinantal processes, Commun. Math. Phys. **242**, 277 (2003).

[5] C. Tracy and H. Widom, Nonintersecting Brownian excursions, Ann. Appl. Probab. **17**, 953 (2007).

[6] G. Schehr, S. N. Majumdar, A. Comtet, and J. Randon-Furling, Exact Distribution of the Maximal Height of p Vicious Walkers, Phys. Rev. Lett. **101**, 150601 (2008).

[7] N. Kobayashi, M. Izumi, and M. Katori, Maximum distributions of bridges of non-colliding Brownian paths, Phys. Rev. E **78**, 051102 (2008).

[8] P. J. Forrester, S. N. Majumdar, and G. Schehr, Non-intersecting Brownian walkers and Yang-Mills theory on the sphere, Nucl. Phys. B **844**, 500 (2011).

[9] G. Schehr, S. N. Majumdar, A. Comtet, and P. J. Forrester, Reunion probability of N vicious walkers: Typical and large fluctuations for large N, J. Stat. Phys. **150**, 491 (2013).

[10] I. Corwin and A. Hammond, Brownian Gibbs property for Airy line ensembles, Invent. Math. **195**, 441 (2014).

[11] K. Johansson, The arctic circle boundary and the Airy process, Ann. Probab. **33**, 1 (2005).

[12] P. L. Ferrari, The universal Airy_1 and Airy_2 processes in the Totally Asymmetric Simple Exclusion Process, in Proceedings of Integrable Systems and Random Matrices: In Honor of Percy Deift [Contemp. Math. **458**, 321 (2008)].

[13] T. Kriecherbauer and J. Krug, A pedestrian's view on interacting particle systems, KPZ universality and random matrices, J. Phys. A: Math. Theor. **43**, 403001 (2010).

[14] V. Eisler and Z. Rácz, Full Counting Statistics in a Propagating Quantum Front and Random Matrix Spectra, Phys. Rev. Lett. **110**, 060602 (2013).

[15] N. Allegra, J. Dubail, J.-M. Stéphan, and J. Viti, Inhomogeneous field theory inside the arctic circle, J. Stat. Mech. (2016) 053108.

[16] P. Le Doussal, S. N. Majumdar, and G. Schehr, Periodic Airy process and equilibrium dynamics of edge fermions in a trap, Ann. Phys. (NY) **383**, 312 (2017).

[17] P. L. Ferrari and H. Spohn, Random growth models, in *The Oxford Handbook of Random Matrix Theory*, edited by G. Akemann and J. Baik, Ph. di Francesco (Oxford University Press, Oxford, 2011).

[18] M. Kardar, G. Parisi, and Y.-C. Zhang, Dynamic Scaling of Growing Interfaces, Phys. Rev. Lett. **56**, 889 (1986).

[19] I. Corwin and A. Hammond, KPZ line ensembles, Probab. Theory Relat. Fields **166**, 67 (2016).

[20] J. Quastel and D. Remenik, *Airy Processes and Variational Problems*, Topics in Percolative and Disordered Systems (Springer, New York, 2014).

[21] J. Baik, P. Deift, and T. Suidan, Combinatorics and random matrix theory, *Graduate Studies in Mathematics Vol.* 172 (American Mathematical Society, Providence, 2016).

[22] I. Corwin, The Kardar-Parisi-Zhang equation and universality class, Random Matrices Theor. Appl. **1**, 1130001 (2012).

[23] K. Matetski, J. Quastel, and D. Remenik, The KPZ fixed point, arXiv:1701.00018.

[24] C. A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, Commun. Math. Phys. **159**, 151 (1994).

[25] I. Corwin, Z. Liu, and D. Wang, Fluctuations of TASEP and LPP with general initial data, Ann. Appl. Probab. **26**, 2030 (2016).

[26] J. Quastel and D. Remenik, How flat is flat in random interface growth?, arXiv:1606.09228.

[27] L. P. R. Pimentel, Duality between coalescence times and exit points in last-passage percolation models, Ann. Probab. **44**, 3187 (2016); S. I. López and L. P. R. Pimentel, Geodesic forests in the last-passage percolation, Stochastic Processes Appl. **127**, 304 (2017).

[28] J. de Nardis and P. Le Doussal, Tail of the two-time height distribution for KPZ growth in one dimension, J. Stat. Mech. (2017) 053212.

[29] J. De Nardis, P. Le Doussal, and K. A. Takeuchi, Memory and Universality in Interface Growth, Phys. Rev. Lett. **118**, 125701 (2017).

[30] K. A. Takeuchi and M. Sano, Evidence for geometry-dependent universal fluctuations of the Kardar-Parisi-Zhang interfaces in liquid-crystal turbulence, J. Stat. Phys. **147**, 853 (2012).

[31] T. Halpin-Healy and K. A. Takeuchi, A KPZ cocktail- shaken, not stirred: Toasting 30 years of kinetically roughened surfaces, J. Stat. Phys. **160**, 794 (2015).

[32] P. L. Ferrari and H. Spohn, On time correlations for KPZ growth in one dimension, SIGMA **12**, 074 (2016).

[33] V. Dotsenko, Two-time free energy distribution function in (1+1) directed polymers, J. Stat. Mech. (2013) P06017.

[34] V. Dotsenko, Two-time free energy distribution function in the KPZ problem, arXiv:1507.06135.

[35] V. Dotsenko, On two-time distribution functions in (1+1) random directed polymers, J. Phys. A: Math. Theor. **49**, 27LT01 (2016).

[36] K. Johansson, Two time distribution in Brownian directed percolation, Commun. Math. Phys. **351**, 441 (2017).

[37] I. Corwin and A. Hammond (unpublished).

[38] J. Hagg, Local Gaussian fluctuations in the Airy and discrete PNG processes, Ann. Probab. **36**, 1059 (2008).

[39] $\sqrt{2}B$ is obtained from rescaling of B_0.

[40] J. Baik and E. M. Rains, Limiting distributions for a polynuclear growth model with external sources, J. Stat. Phys. **100**, 523 (2000).

[41] J. Baik, P. L. Ferrari, and S. Péché, Limit process of stationary TASEP near the characteristic line, Commun. Pure Appl. Math. **63**, 1017 (2010).

[42] J. Baik and Z. Liu, On the average of the Airy process and its time reversal, Electron. Commun. Probab. **18**, 89 (2013).

[43] T. Imamura and T. Sasamoto, Exact Solution for the Stationary Kardar-Parisi-Zhang Equation, Phys. Rev. Lett. **108**, 190603 (2012); Stationary correlations for the 1D KPZ equation, J. Stat. Phys. **150**, 908 (2013).

[44] A. Borodin, I. Corwin, P. L. Ferrari, and B. Vető, Height fluctuations for the stationary KPZ equation, Math. Phys., Anal. Geom. **18** (2015).

[45] P. L. Ferrari and H. Spohn, Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process, Commun. Math. Phys. **265**, 1 (2006).

[46] C. Maes and T. Thiery, Midpoint distribution of directed polymers in the stationary regime: Exact result through linear response, J. Stat. Phys. **168**, 937 (2017).

[47] M. Prahofer and H. Spohn, Exact scaling functions for one-dimensional stationary KPZ growth, J. Stat. Phys. **115**, 255 (2004).

[48] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevE.96.060101 for details of the derivations of the main results of the paper.

[49] The estimate $C_\infty \approx 0.6$ was quoted in [29] as being consistent with numerical data obtained there, with the theory of [29], and with previous experimental data [K. A. Takeuchi (private communication)].

[50] The estimate $C_\infty = 0.626 \pm 0.003$ was obtained early this year by T. Halpin-Healy (private communication) simulating the Pt-Pt 1+1 DPRM subject to exponential site disorder.

[51] P. Calabrese, P. Le Doussal, and A. Rosso, Free-energy distribution of the directed polymer at high temperature, Europhys. Lett. **90**, 20002 (2010).

[52] V. Dotsenko, Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers, Europhys. Lett. **90**, 20003 (2010); V. Dotsenko and B. Klumov, Bethe ansatz solution for one-dimensional directed polymers in random media, J. Stat. Mech. (2010) P03022.

[53] P. Calabrese and P. Le Doussal, Exact Solution for the Kardar-Parisi-Zhang Equation with Flat Initial Conditions, Phys. Rev. Lett. **106**, 250603 (2011); P. Le Doussal and P. Calabrese, The KPZ equation with flat initial condition and the directed polymer with one free end, J. Stat. Mech. (2012) P06001.

[54] V. Dotsenko, Replica Bethe ansatz derivation of the GOE Tracy-Widom distribution in one-dimensional directed polymers with free boundary conditions, J. Stat. Mech. (2012) P11014.

[55] S. Prolhac and H. Spohn, Two-point generating function of the free energy for a directed polymer in a random medium, J. Stat. Mech. (2011) P01031.

[56] V. Dotsenko, Two-point free energy distribution function in (1+1) directed polymers, J. Phys. A **46**, 355001 (2013).

[57] T. Imamura, T. Sasamoto, and H. Spohn, On the equal time two-point distribution of the one-dimensional KPZ equation by replica, J. Phys. A: Math. Theor. **46**, 355002 (2013).

[58] V. Dotsenko, Distribution function of the endpoint fluctuations of one-dimensional directed polymers in a random potential, J. Stat. Mech. (2013) P02012.

[59] P. Le Doussal, Crossover from droplet to flat initial conditions in the KPZ equation from the replica Bethe ansatz, J. Stat. Mech. (2014) P04018.

[60] I. Corwin, J. Quastel, and D. Remenik, Renormalization fixed point of the KPZ universality class, J. Stat. Phys. **160**, 815 (2015).

[61] P. Le Doussal, Crossover between various initial conditions in KPZ growth: Flat to stationary, J. Stat. Mech. (2017) 053210.

[62] E. H. Lieb and W. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, Phys. Rev. **130**, 1605 (1963).