Effects of group velocity and multiplasmon resonances on the modulation of Langmuir waves in a degenerate plasma

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(Received 16 August 2017; published 28 November 2017)

We study the nonlinear wave modulation of Langmuir waves (LWs) in a fully degenerate plasma. Using the Wigner–Moyal equation coupled to the Poisson equation and the multiple scale expansion technique, a modified nonlocal nonlinear Schrödinger (NLS) equation is derived which governs the evolution of LW envelopes in degenerate plasmas. The nonlocal nonlinearity in the NLS equation appears due to the group velocity and multiplasmon resonances, i.e., resonances induced by the simultaneous particle absorption of multiple wave quanta. We focus on the regime where the resonant velocity of electrons is larger than the Fermi velocity and thereby the linear Landau damping is forbidden. As a result, the nonlinear wave-particle resonances due to the group velocity and multiplasmon processes are the dominant mechanisms for wave–particle interaction. It is found that in contrast to classical or semiclassical plasmas, the group velocity resonance does not necessarily give rise the wave damping in the strong quantum regime where $\hbar k \sim m v_F$ with $\hbar$ denoting the reduced Planck's constant, m the electron mass, and v_F the Fermi velocity; however, the three-plasmon process plays a dominant role in the nonlinear Landau damping of wave envelopes. In this regime, the decay rate of the wave amplitude is also found to be higher compared to that in the modest quantum regime where the multiplasmon effects are forbidden.

DOI: 10.1103/PhysRevE.96.053209

I. INTRODUCTION

Recently, there has been growing and considerable interest in investigating new aspects of Landau damping in plasmas. Although much attention has been paid to the classical regimes (see, e.g., Refs. [1–3]), there are many aspects which are still unexplored in quantum regimes. When the plasma density is increased, various quantum effects enter the picture [4–6]. This includes, e.g., the degeneracy effects [4] and suppression of classical particle trapping due to quantum effects [6]. Moreover, in a quantum plasma, we note that the photon momenta may be described using a distribution function, leading to the concept of photon Landau damping [5].

In the well-known standard theory of Landau damping, particles traveling with a speed v close to the phase velocity v_p of a wave (i.e., $v \simeq v_p$) feel almost a constant electric field, leading to a systematic acceleration of particles which results in an energy transfer from waves to resonant particles. A similar effect is also associated with the ponderomotive force acting on charged particles in an oscillating electromagnetic field. When these particles have a velocity close to the group velocity λ of wave envelopes (i.e., $v \simeq \lambda$), the accelerating field due to the ponderomotive force changes very slowly on the same time scale (to be more specific) as that for the evolution of wave envelopes. This leads to a magnified energy exchange between the propagating waves and the particles that are resonant with the ponderomotive force. We refer to this phenomenon as the group velocity resonance.

The group velocity resonance can be important in the diffusion of particles in velocity space (e.g., thermalization, heating, and acceleration) and transport of particle, momentum, and energy. Also, since in the nonlinear evolution of wave envelopes the transformation of wave energy from higher-frequency side bands to the lower-frequency side bands takes place due to this resonance process, there may be the possibility of the onset of weak/strong turbulence in nonlinear dispersive media [7].

Many of the more well-known aspects of Landau damping can be studied even for an infinite plane wave. Generalizing this setup to the more realistic case of propagating wave packet, the effects of group velocity resonances enter the picture [8–10]. In this context, the nonlinear theory of Landau damping due to group velocity resonance of wave envelopes has been developed in both classical [9] and semiclassical plasmas [10]. However, it has also been shown that going beyond the linear theory, wave damping can also take place due to multiplasmon resonances [11,12], which can be present even for an infinite plane wave.

In the present work, we start with the Wigner–Moyal equation, which accounts for the particle dispersive quantum properties (but ignores other quantum effects such as exchange effects [13]), coupled to the Poisson equation. While the general theory is applicable for an arbitrary degree of degeneracy of plasmas, we will, for simplicity, focus on the case of a fully degenerate plasma. A generalization of our results to the case of nonzero electron temperature can be made with the help of results from, e.g., Ref. [4]. In the fully degenerate system, the quantum effects can be seen to appear in a wide range of wave-number scales including, in particular, the weak quantum, the modest quantum, and the strongly quantum regimes. In the weak quantum regime in which the Langmuir wavelength is much larger than the typical de Broglie wavelength, the particle's resonant velocity still approaches the phase velocity of the wave as in classical [9] and semiclassical [10] theories. However, in the modest or strong quantum regimes, the resonant velocity in the linear theory is shifted due to finite momentum and energy of particles,

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i.e., $v_{\rm res} = \omega/k \pm \hbar k/2m \equiv v_p \pm v_q$, where $\omega(k)$ is the wave frequency (number), $v_p = \omega/k$ is the phase velocity, $h = 2\pi\hbar$ is the Planck's constant, m is the electron mass, and $v_q = \hbar k/2m$ is the velocity associated with the plasmon quanta. It has been shown that (see, e.g., Ref. [14]) in a fully degenerate plasma with the background distribution as corresponding to the zero-temperature Fermi-Dirac equilibrium in which particles can have maximum velocity v_F, the Fermi velocity, the linear Landau damping takes place for particles having velocity, $v_{\rm res} \leqslant v_F$ when $k > k_{\rm cr}$. Here k_r is a critical value to be determined from the linear dispersion relation [14,15].

The scenario changes significantly when one looks for the nonlinear evolution of waves. It has been shown in Refs. [11,12] that not only one plasmon resonances take place, but that there are also the possibilities of multi-plasmon resonances with velocities $v_{\rm res}^n = v_p \pm n v_q$, where $n = 1,2,3,\ldots,$ respectively, correspond to the one-plasmon (linear), two-plasmon, three-plasmon resonances, etc. Since the results for the one-plasmon resonance are known from the linear theory, we are, however, interested in the resonance processes for $n > 1$ in the regime of $k < k_{\rm cr}$ in which the linear damping (corresponding to one-plasmon resonance processes) is forbidden.

On the other hand, the nonlinear theory of electrostatic wave envelopes has been investigated using the Vlasov-Poisson system [9] as well as the semiclassical limit of the Wigner-Moyal-Poisson system [10]. It has been shown that the nonlinear evolution of wave envelopes can be described by a modified nonlinear Schrödinger (NLS) equation with a nonlocal nonlinearity which appears to be due to resonant particles moving with the group velocity of the wave envelope. The purpose of the present work is to consider this type of resonance as well as the multiplasmon resonances on the modulation and nonlinear evolution of Langmuir wave envelopes. We show that, in contrast to classical [9] and semiclassical [10] plasmas, the two- and three-plasmon resonances modify the cubic nonlinearity, and, moreover, the nonlocal nonlinear term is modified by the three-plasmon resonance. As a consequence, the wave damping is significantly enhanced due to the presence of multiplasmon resonances that effectively convert the wave energy to the particle's kinetic energy.

II. THE MODEL

We consider the nonlinear wave-particle interaction of Langmuir waves in a fully degenerate quantum plasma. Our starting point is the three-dimensional (3D) Wigner-Moyal equation for electrons, given by

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}} f + \frac{iem^3}{(2\pi)^3\hbar^4}\iint d^3\mathbf{r}' d^3\mathbf{v}' e^{im(\mathbf{v}-\mathbf{v}')\cdot\mathbf{r}'/\hbar} \times\left[\phi\left(\mathbf{r}+\frac{\mathbf{r}'}{2},t\right)-\phi\left(\mathbf{r}-\frac{\mathbf{r}'}{2},t\right)\right] f(\mathbf{r},\mathbf{v}',t) = 0, \quad (1)$$

where f is the Wigner distribution function; $e,m,\mathbf{v}$, respectively, are the charge, mass, and velocity of electrons, and ϕ is the self-consistent electrostatic potential which satisfies the Poisson equation

$$\nabla^2\phi = 4\pi e\left(\int f dv - n_0\right). \quad (2)$$

Here n_0 is the background number density of electrons and ions, where we, for simplicity, consider an electron proton plasma. Moreover, we consider a dense plasma with degenerate electrons in the low-temperature limit. The three-dimensional equilibrium distribution is given by

$$f^{(0)}(\mathbf{v}) = \begin{cases} 2m^3/(2\pi\hbar)^3, & |\mathbf{v}| \leqslant v_F \\ 0, & |\mathbf{v}| > v_F \end{cases}, \quad (3)$$

where $v_F = \sqrt{2E_F/m}$ is the speed of electrons on the Fermi surface and $E_F = \hbar^2(3\pi^2 n_0)^{2/3}/2m$ is the Fermi energy. Since we will consider the wave propagation in a single direction (which we choose to be along the x axis), it is convenient to compute the reduced one-dimensional (1D) distribution function to be obtained by projecting the 3D distribution (3) on the v_x axis, i.e., using the cylindrical coordinates in v_y and v_z, we obtain the reduced 1D distribution function as (replacing v_x by v)

$$F^{(0)}(v) = \iint f^{(0)}(\mathbf{v}) dv_y dv_z = 2\pi\int_0^{v_F^2-v^2}\frac{2m^3}{(2\pi\hbar)^3}u_\perp du_\perp = \begin{cases}[2\pi m^3/(2\pi\hbar)^3]\left(v_F^2-v^2\right), & |v| \leqslant v_F \\ 0, & |v| > v_F\end{cases}. \quad (4)$$

III. DERIVATION OF THE NLS EQUATION

We consider the one-dimensional propagation (along the x axis) and the evolution of weakly nonlinear Langmuir wave envelopes in a fully degenerate plasma. So we introduce the multiple space–time scales as [9,10,16]

$$x \to x + \epsilon^{-1}\eta + \epsilon^{-2}\zeta \qquad t \to t + \epsilon^{-1}\sigma. \quad (5)$$

Here η, ζ, and σ are the coordinates stretched by a small parameter ϵ. Since the wave amplitude is infinitesimally small, so for $t > 0$ there will be a slight deviation of order ϵ from the uniform initial value. Thus, we expand

$$\phi(x,t) = \sum_{n=1}^{\infty}\epsilon^n\sum_{l=-\infty}^{\infty}\phi_l^{(n)}(\eta,\sigma,\zeta)\exp[il(kx-\omega t)],$$
$$f(\mathbf{v},x,t) = f^{(0)}(\mathbf{v}) + \sum_{n=1}^{\infty}\epsilon^n\sum_{l=-\infty}^{\infty}f_l^{(n)}(\mathbf{v},\eta,\sigma,\zeta) \times\exp[il(kx-\omega t)], \quad (6)$$

where the reality conditions, namely $f_{-l}^{(n)} = f_l^{(n)*}$, $\phi_{-l}^{(n)} = \phi_l^{(n)*}$, hold. In order to properly take into account the contributions of resonant particles, the harmonic components $f_l^{(n)}$ and $\phi_l^{(n)}$ are further expanded into Fourier-Laplace integrals as [16]

$$f_l^{(n)}(\mathbf{v},\eta,\sigma,\zeta) = \frac{1}{(2\pi)^2}\int_C d\Omega\int_{-\infty}^{\infty}dK\,\tilde{f}_l^{(n)}(\mathbf{v},K,\Omega,\zeta) \times\exp[i(K\eta-\Omega\sigma)],$$
$$\phi_l^{(n)}(\eta,\sigma,\zeta) = \frac{1}{(2\pi)^2}\int_C d\Omega\int_{-\infty}^{\infty}dK\,\tilde{\phi}_l^{(n)}(K,\Omega,\zeta) \times\exp[i(K\eta-\Omega\sigma)], \quad (7)$$

where the contour C is parallel to the real axis and lies above the coordinate of convergence. Equation (7) shows that the perturbations are in the form of waves propagating along the η direction with a speed Ω/K.

We substitute the stretched coordinates (5) and the expansions (6) and (7) into Eqs. (1) and (2) to obtain, respectively,

$$-il(\omega-kv)f_l^{(n)}+\frac{\partial f_l^{(n-1)}}{\partial\sigma}+v\frac{\partial f_l^{(n-1)}}{\partial\eta}+v\frac{\partial f_l^{(n-2)}}{\partial\zeta}+\frac{em}{2i\pi\hbar^2}\iint dx'd^3\mathbf{v}'\exp[im(v-v')x'/\hbar]$$
$$\times\phi_l^{(n)}f_0\{\exp(ikx'l/2)-\exp(-ikx'l/2)\}+\frac{em}{2i\pi\hbar^2}\iint dx'd^3\mathbf{v}'\exp[im(v-v')x'/\hbar]$$
$$\times\sum_{s=1}^{\infty}\sum_{l'=-\infty}^{\infty}\phi_{l-l'}^{(n-s)}f_{l'}^{(s)}\{\exp[ikx'(l-l')/2]-\exp[-ikx'(l-l')/2]\}\doteq 0, \tag{8}$$

$$(lk)^2\phi_l^{(n)}-2ilk\frac{\partial}{\partial\eta}\phi_l^{(n-1)}-i2lk\frac{\partial}{\partial\zeta}\phi_l^{(n-2)}-\frac{\partial^2}{\partial\eta^2}\phi_l^{(n-2)}-4\pi e\int f_l^{(n)}d^3\mathbf{v}=0, \tag{9}$$

where the symbol $\doteq$ is used to denote the equality in the weak sense, and we have removed the terms which contain $\phi_l^{(n-3)}$ and $\phi_l^{(n-4)}$ in Eq. (9). In the subsequent analysis, we determine the contributions of the resonant particles by solving the σ evolution of the components $f_l^{(n)}$ and $\phi_l^{(n)}$ as an initial value problem with the initial condition

$$f_0^{(n)}(v,\eta,\sigma=0,\zeta)\doteq 0,\ \ n\geqslant 1 \tag{10}$$

in the multiple space-time scheme corresponding to that on the distribution function

$$f_0^{(n)}(v,t=0)=0. \tag{11}$$

A. Harmonic modes with $n=1,l=1$: Linear dispersion law

From Eqs. (8) and (9), equating the coefficients of ϵ for $n=1,l=1$, we obtain the linear dispersion law:

$$D(k,\omega)\equiv 1+\frac{m\omega_p^2}{n_0\hbar k^3}\int_C\frac{f^{(0)}(\mathbf{v}+\mathbf{v}_q)-f^{(0)}(\mathbf{v}-\mathbf{v}_q)}{v_p-v}d^3\mathbf{v}=0. \tag{12}$$

Equation (12) can be rewritten as

$$1-\frac{\omega_p^2}{n_0k^2}\int_C\frac{f^{(0)}(\mathbf{v})}{(v_p-v)^2-v_q^2}d^3\mathbf{v}=0, \tag{13}$$

which, in one-dimensional geometry with the reduced distribution function (4), gives

$$1-\frac{\omega_p^2}{n_0k^2}\int_C\frac{F^{(0)}(v)}{(v_p-v)^2-v_q^2}dv=0. \tag{14}$$

Here C is the contour parallel to the real axis which do not need to consider the poles at $v=v_p\pm v_q$. Such an omission of the pole contribution is due to the fact that we are interested in the regime where the one-plasmon resonance is forbidden. Thus, to evaluate the integral in Eq. (14), we consider only the principal value which excludes the poles at $v=v_p\pm v_q$.

Next, considering the harmonic modes for $l\neq 0,n=1$ and the zeroth harmonic modes for $n=1,2;l=0$, we obtain the following conditions:

$$f_{\alpha,l}^{(1)}\doteq 0 \text{ and } \phi_l^{(1)}=0 \text{ for } |l|\geqslant 2, \tag{15}$$

together with the zeroth-order components, given by

$$f_{\alpha,0}^{(1)}\doteq 0,\ \ \phi_0^{(1)}=0. \tag{16}$$

B. Modes with $n=2$, $l=1$: Group velocity

For $n=2, l=1$, we have from Eqs. (8) and (9) the following compatibility condition for the group velocity:

$$\left\{\frac{\partial}{\partial\sigma}+\lambda\frac{\partial}{\partial\eta}\right\}\phi_1^{(1)}(\eta,\sigma;\zeta)=0, \tag{17}$$

where $\lambda\equiv\partial\omega/\partial k$ is the group velocity, given by $\lambda=\lambda_1/\lambda_2$ with

$$\lambda_1=2-\frac{4\pi e^2}{mk^2}\int_C\frac{v_p^2-v^2+v_q^2}{\{(v_p-v)^2-v_q^2\}^2}F^{(0)}(v)dv,$$
$$\lambda_2=-\frac{8\pi e^2}{mk^2}\int_C\frac{v_p-v}{\{(v_p-v)^2-v_q^2\}^2}F^{(0)}(v)dv. \tag{18}$$

Equation (17) determines the σ-η variation of the first-order perturbation, i.e.,

$$\phi_1^{(1)}(\eta,\sigma;\zeta)=\phi_1^{(1)}(\xi;\zeta), \tag{19}$$

with a new coordinate ξ, given by

$$\xi=\eta-\lambda\sigma=\epsilon(x-\lambda t). \tag{20}$$

It follows that the coordinate ξ in Eq. (20) establishes a clear relationship between the reductive perturbation theory and the multiple scale expansion scheme.

C. Second-harmonic modes with $n=l=2$

For the second-order quantities with $n=l=2$, we obtain from Eqs. (8) and (9) the expressions

$$f_2^{(2)}=-\frac{e}{2\hbar k(v_p-v)}[\{f_0(v+2v_q)-f_0(v-2v_q)\}\phi_2^{(2)}$$
$$+\{f_1^{(1)}(v+v_q)-f_1^{(1)}(v-v_q)\}\phi_1^{(1)}], \tag{21}$$

$$\phi_2^{(2)}=-\frac{1}{8}A(k,\omega)\phi_1^{(1)}\phi_1^{(1)}. \tag{22}$$

The expression for A is given in Appendix A.

D. Zeroth harmonic modes with $n = 3$, $l = 0$

We consider the terms corresponding to $n = 3$, $l = 0$ from Eqs. (8) and (9) and use Eqs. (16) and (17) to obtain a set of reduced equations. These equations are then Fourier-Laplace transformed with respect to η and σ. Finally, the initial condition (10) is used to obtain

$$\hat{f}_0^{(2)}(v,k,\Omega,\zeta) = -\frac{K}{\Omega - Kv}\frac{e^2}{\hbar^2}I(v)H(K,\Omega), \tag{23}$$

where $H(K,\Omega)$ is defined as

$$\left|\phi_1^{(1)}(\eta-\lambda\sigma,\zeta)\right|^2 = \frac{1}{(2\pi)^2}\int d\Omega \int dK\, H(K,\Omega)\exp[i(K\eta-\Omega\sigma)] \tag{24}$$

with

$$H(K,\Omega) = 2\pi\delta(\Omega - K\lambda)\int dK'\phi_1^{(1)*}(K')\phi_1^{(1)}(K+K'). \tag{25}$$

The similar expression for $\hat{\phi}_0^{(2)}$ is not so required, since in the subsequent equations its coefficients appear to be vanished identically.

E. Harmonic modes with $n = 3$, $l = 1$: The NLS equation

Finally, for $n = 3$ and $l = 1$ and from Eqs. (8) and (9), we obtain the modified NLS equation

$$i\frac{\partial\phi}{\partial\tau} + P\frac{\partial^2\phi}{\partial\xi^2} + Q|\phi|^2\phi + \frac{R}{\pi}\mathcal{P}\int\frac{|\phi(\xi',\tau)|^2}{\xi-\xi'}\phi d\xi' = 0, \tag{26}$$

for the small- but finite-amplitude perturbation $\phi(\xi,\tau) \equiv \phi_1^{(1)}(\xi,\tau)$.

The coefficients of the dispersion (group velocity), cubic nonlinear (local), nonlocal nonlinear terms, respectively, are P,Q and R, given by $P \equiv (1/2)\partial^2\omega/\partial k^2 = \beta/\alpha, Q = \gamma/\alpha$ and $R = D/\alpha$, where

$$\alpha = -\frac{8\pi e^2}{mk}\int_C \frac{v_p - v}{\left[(v_p - v)^2 - v_q^2\right]^2}F^{(0)}(v)dv, \tag{27}$$

$$\beta = 1 + \frac{4\pi e^2}{\hbar k^3}\int_C \left[\frac{(v - v_q - \lambda)^2}{(v_p - v + v_q)^3} - \frac{(v + v_q - \lambda)^2}{(v_p - v - v_q)^3}\right] \times F^{(0)}(v)dv, \tag{28}$$

$$\gamma = \left(\frac{1}{4}\frac{AA_1}{\hbar} - \frac{1}{2\hbar^2}B + C\right)k^2, \tag{29}$$

$$D = -\frac{4\pi e^4}{m\hbar^2k^2}\int_\gamma \Bigg[\delta\{v - (v_p - 3v_q)\} \times \frac{v - \lambda + v_q}{(v_p - v - v_q)^3(v - \lambda + 2v_q)} + 2\delta(v - \lambda)v_q\frac{\{(v_p - v)^2 + v_q^2\}}{\{(v_p - v)^2 - v_q^2\}^3}\Bigg]F^{(0)}(v)dv. \tag{30}$$

The expressions for A, A_1, B, and C in γ with the distribution function $F^{(0)}$ are given in Appendix A. However, the reduced expressions for α,β,γ, and D describing the coefficients of the NLS equation P, Q, and R with the Fermi-Dirac distribution (4) are given in Appendix B.

IV. MULTIPLASMON AND GROUP VELOCITY RESONANCES

We investigate the coefficients of the NLS equation, especially their modifications due to the quantum effects arising those from the Wigner-Moyal equation and the background distribution of electrons being the Fermi-Dirac distribution, as well as, the effects due to the wave-particle resonances. We find that the integrands in the expressions of α and β do not have any pole except at $v = v_{\rm res}^l \equiv v_p - v_q$, which corresponds to the linear Landau damping and lies outside the regime of interest. The linear damping can be associated with some other factor with a positive sign, i.e., at $v = v_p + v_q$. However, these are also of less importance as the lower resonant velocity gives the wave damping more easily. Thus, α and β, and hence the group velocity dispersion P, do not have any resonance contribution in the regime of interest. The detailed discussion about the parameter regimes is given in the Sec. V A.

On the other hand, inspecting the local nonlinear coefficient Q of the NLS equation (26), and looking at the denominators of different expressions for A, A_1, B, and C (see Appendix A) in Q and factorizing them, we find that

$$(\omega - kv)^2 - k^2v_q^2 = (\omega - kv - kv_q)(\omega - kv + kv_q), \tag{31}$$

$$(\omega - kv - kv_q)^2 - k^2v_q^2 = (\omega - kv - 2kv_q)(\omega - kv), \tag{32}$$

$$(\omega - kv - kv_q)^2 - 4k^2v_q^2 = (\omega - kv - 3kv_q)(\omega - kv + kv_q). \tag{33}$$

Thus, in addition to the resonance at the phase velocity ($v = v_p$) and the linear resonance ($v = v_{\rm res}^l$), the two- and three-plasmon resonances also occur for $\omega - kv \pm nkv_q = 0$, i.e., at $v_{\rm res}^n = v_p - nv_q$ for $n = 2,3$. The other resonant velocities for $n = 4,5,\dots$ will not appear as those are associated with higher orders of ϵ than cubic, which is not the present case. Note that the resonances at $v = v_p$, $v = v_{\rm res}^l$ and at $v_{\rm res}^n = v_p + nv_q$ are not of interest in the present study as those fall in the regime of $k > k_{\rm cr}$, where k_{cr} is some critical value of k which can be shown to be $\lesssim 1$ in the strong quantum regime where $\hbar\omega_p \sim mv_F^2$ [14,15]. In fact, we have the relation for the resonant velocities $v_p - 3v_q < v_p - 2v_q < v_p - v_q < v_p < v_p + nv_q$. Thus, the nonlinear coefficient of the NLS equation is significantly modified by the resonance contributions from the two- and three-plasmon processes, which do not appear in classical [9] or semiclassical [10] plasmas. In what follows, looking at the nonlocal coefficient $R \propto D$, we find in D that the first term is the contribution from the three-plasmon resonance, while the second one is from the group velocity resonance.

Thus, from the above discussion, one concludes that in contrast to classical [9] or semiclassical [10] plasmas, while the local nonlinear coefficient Q contains resonance contributions from two- and three-plasmon processes, the nonlocal nonlinear coefficient R of the NLS equation appears to be modified by the group velocity, as well as the three-plasmon resonances.

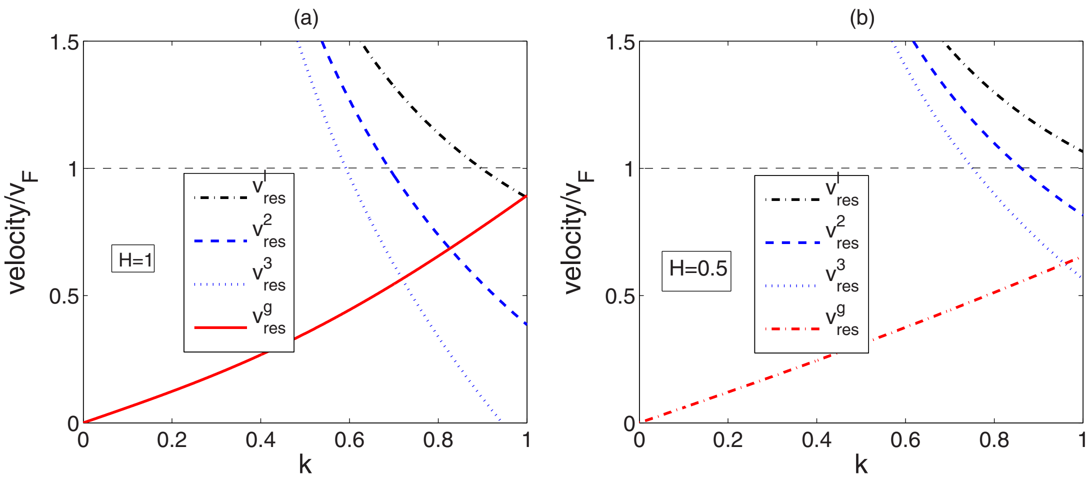

FIG. 1. The normalized resonant velocities ($\sim v_F$) are plotted against the normalized wave number $k(\sim\lambda_F^{-1})$ for two different values of the dimensionless quantum parameter H to show different parameter regimes, namely semiclassical (e.g., $0<k\ll 0.59$ for $H=1$), modest quantum ($0<k\lesssim 0.59$ for $H=1$ and $0<k\lesssim 0.75$ for $H=0.5$) and strong quantum ($0.591\lesssim k\lesssim 0.9$ for $H=1$ and $0.75\lesssim k\lesssim 1$ for $H=0.5$) regimes. In the legends, $v_{\rm res}^n$ denotes the resonant velocity, where $n=l,2,3,g$, respectively, correspond to the velocities for the linear, two-plasmon, three-plasmon, and group velocity resonances.

V. LANGMUIR ENVELOPES WITH ZERO-TEMPERATURE FERMI-DIRAC DISTRIBUTION

We consider the amplitude modulation and the nonlinear evolution of Langmuir envelopes in a fully degenerate plasma. The background distribution of electrons are assumed to be given by the Fermi–Dirac distribution at zero temperature [Eq. (4)]. In this situation, the linear dispersion law, the group velocity, as well as, the coefficients of the NLS equation (26) will be reduced. The dispersion relation (12) reduces to

$$1+\frac{3\omega_p^2}{4k^2v_F^2}\left(2-\sum_{j=\pm 1}\frac{j}{2v_qv_F}\{v_F^2-(v_p+jv_q)^2\}\right.$$
$$\left.\times\log\left|\frac{v_p+jv_q-v_F}{v_p+jv_q+v_F}\right|\right)=0, \tag{34}$$

where $\omega_p=\sqrt{4\pi e^2n_0/m}$ is the electron plasma oscillation frequency. Before we proceed to analyze the modulational instability and the nonlinear evolution of Langmuir waves, we first investigate some parameter regimes of interest which may correspond to semiclassical and quantum plasmas. These are discussed in the following Sec. V A.

Parameter regimes

We investigate different parameter regimes in the plane of particle's velocity (v) and the wave number (k) where the group velocity and/or multiplasmon resonance effects become significant. Since we consider the region of small wave numbers, i.e., $k\lambda_F\lesssim 1$, where $\lambda_F=v_F/\omega_p$ is the Fermi wavelength, for which the linear resonant velocity lies outside the background distribution, i.e., $v_p\pm v_q>v_F$, the dispersion relation (34) can further be reduced. So, expanding the log functions for small wave numbers and keeping terms up to $o(k^4)$ we obtain from Eq. (34) [14,17]

$$\omega^2=\omega_{pe}^2+\tfrac{3}{5}k^2V_{\rm Fe}^2+(1+\alpha)k^2v_q^2, \tag{35}$$

where $H=\hbar\omega_p/mv_F^2$ is the dimensionless quantum parameter and $\alpha=(48/175)m_e^2V_{\rm Fe}^4/\hbar^2\omega_{pe}^2=(48/175)H^2$. We note that H may be of the order of unity for metallic densities, which means that typically $\alpha<1$.

We plot the resonant velocities $v_{\rm res}^n=v_p-nv_q, n=l,2,3,$ in the vk plane for two different values of H: $H\sim 1$ and $H\sim 0.5$. Here, the velocity is normalized by the Fermi velocity v_F and k by the inverse of the Fermi wavelength λ_F^{-1}, and the expression for v_p is used from Eq. (35). From Fig. 1, it is clear that there are, in fact, two parameter regimes: one where both the multiplasmon and the group velocity resonances can be important, and the other where the group velocity resonance is only the damping mechanism. We note that the group velocity resonance (which occurs at $v_{\rm res}^g=\lambda$) curve is always within the region of $v<v_F$ throughout the interval $0<k\lesssim 1$. However, the multiplasmon resonance curves may or may not fall in the region of $v<v_F$ depending on the values of k and H. For $H\sim 1$, Fig. 1(a) shows that the regime of k for which the linear resonance is forbidden is $0<k\lesssim 0.9$. However, the two- and three-plasmon resonances disappear and only the group velocity resonance comes to the picture in $0<k\lesssim 0.59$. On the other hand, both the three-plasmon and the group velocity resonances can be effective in the regime $0.59\lesssim k\lesssim 0.6953$. Furthermore, the group velocity, as well as the two- and three-plasmon resonances, can be significant in $0.59\lesssim k\lesssim 0.9$. In these regimes of k, the magnitudes of the coefficients P, Q, and R of the NLS equation (26) are to be noticed. As will be shown later, these magnitudes essentially give some useful information for the estimation of frequency shift and the rate of energy transfer in the modulation of Langmuir waves, as well as the nonlinear evolution of envelope solitons. For example, at $H=1$, while the values of P and R increase, the values of $|Q|$ decrease with successive reduction of the values of k from $k=0.59$ to that in the regime $0<k\lesssim 0.59$. In the other regime, i.e., $0.59\lesssim k\lesssim 0.9$ with $H=1$, while the values of P and $|Q|$ decrease, the values of R increase with increasing values of k until the inequality $v_{\rm res}^3\gtrsim v_{\rm res}^g$ holds. An opposite

trend occurs with $v_{\rm res}^3 < v_{\rm res}^g$ where the values of P increase, but those of $|Q|$ and R decrease with increasing values of k.

On the other hand, as the value of the quantum parameter H is reduced [see, e.g., Fig. 1(b) for $H = 0.5$], the multiplasmon resonance curves tend to disappear from the region: $0 < k < 1$, $v < v_F$, and they completely disappear (not shown in the figure), e.g., at $H \sim 0.1$. The latter, in some sense, corresponds to a weak quantum regime. In this situation, only the group velocity resonance becomes significant as in classical or semiclassical plasmas [9,10]. Thus, from the consequences of Fig. 1, one can, in particular, define three different regimes of interest: (i) the semiclassical or weak quantum regime, (ii) the modest quantum regime, and (iii) the strong quantum regime, which we briefly discuss as follows:

Semiclassical regime. If the Langmuir wavelength is much larger than the typical de Broglie wavelength, i.e., $\hbar k \ll m v_F$, then the quantum effects associated with the terms $\propto \hbar k/m$, which appear due to the use of the Wigner equation rather than the Vlasov equation, are almost negligible. In this case, the quantum contributions are only due to the background distribution of electrons being a Fermi-Dirac distribution at zero temperature rather than a Maxwellian one. The coefficients of the NLS equation will be somewhat modified; however, the results will be similar to some previous works [10], because the resonance velocity is only the group velocity. The regimes of k can be sort of $0 < k \ll 0.59$ for $H \sim 1$. Furthermore, in this regime, the nonlocal nonlinear coefficient R of the NLS equation, which basically modifies the shape of a pulse profile, remains positive, implying that only the group velocity resonance gives rise to the nonlinear Landau damping of wave envelopes in the semiclassical regime.

Modest quantum regime. In this case, $\hbar k \sim m v_F$; however, the three-plasmon resonance velocity is slightly larger than the Fermi velocity, i.e., $v_{\rm res}^3 > v_F$. This means that the resonance contribution is still due to the group velocity [see Fig. 1(a)]. The results may be similar to the semiclassical case; however, the coefficients of the NLS equation will be modified by the quantum contributions from the Wigner equation, as well as from the background distribution of electrons. From Fig. 1, it is evident that the corresponding values of k are in $0 < k \lesssim 0.59$ for $H \sim 1$ and in $0 < k \lesssim 0.75$ for $H \sim 0.5$. In this case, R is also positive, and a similar conclusion can be drawn as for the semiclassical regime.

Strong quantum regime. The most important and interesting is the strong quantum regime where $\hbar k \sim m v_F$ still holds; however, the three-plasmon resonance velocity is smaller than the Fermi velocity, i.e., $v_{\rm res}^3 < v_F$. In this case, not only the group velocity resonance contributes but also the two- and three-plasmon resonances come to the picture as is evident from Fig. 1 (see dashed and dotted lines). Also, we note that the three-plasmon resonance contribution to the nonlocal coefficient R is proportional to the difference between v_F and $v_{\rm res}^3$. Furthermore, the resonance contributions from the two- and three-plasmon processes in the local nonlinear coefficient Q are also proportional to the difference $v_F - v_{\rm res}^n$ (for $n = 2,3$). Thus, the effects of the two- and three-plasmon resonances are to be important, and the inequality $v_F > v_{\rm res}^n$ must hold at least with a small margin. In this case, R remains not only positive in $0.591 \lesssim k \lesssim 0.9$, but also the contribution from the three-plasmon resonance becomes higher in magnitude as long as $v_{\rm res}^3 (< v_F)$ remains close to but slightly larger than $v_{\rm res}^g$. Thus, it follows that in the strong quantum regime, the group velocity resonance does not necessarily play a decisive to the wave damping as in the semiclassical and modest quantum regimes; however, the three-plasmon resonance plays a dominating role in the Landau damping process.

VI. THE NONLINEAR LANDAU DAMPING AND MODULATIONAL INSTABILITY

We consider the amplitude modulation of Langmuir wave envelopes in a degenerate plasma. To this end, we assume a plane wave solution of Eq. (26) of the form

$$\phi = \rho^{1/2} \exp\left(i \int^{\xi} \frac{\sigma}{2P} d\xi \right), \tag{36}$$

where ρ and σ are real functions of ξ and τ. Substitution of the solution (36) into Eq. (26) results in a set of equations which can be separated for the real and imaginary parts. These equations are then linearized by splitting up ρ and σ into their equilibrium (with suffix 0) and perturbation (with suffix 1) parts, i.e.,

$$\rho = \rho_0 + \rho_1 \cos(K\xi - \Omega\tau) + \rho_2 \sin(K\xi - \Omega\tau), \tag{37}$$

$$\sigma = \sigma_1 \cos(K\xi - \Omega\tau) + \sigma_2 \sin(K\xi - \Omega\tau), \tag{38}$$

where Ω and K are, respectively, the wave frequency and the wave number of modulation, to obtain the dispersion relation [9,16]

$$(\Omega^2 + 2\rho_0 P Q K^2 - P^2 K^4)^2 = -(2\rho_0 P R K^2)^2. \tag{39}$$

The equation (39) is, in general, complex, irrespective of the negative sign and/or the presence of the nonlocal coefficient R on the right-hand side, as Q contains pole contributions from the multiplasmon processes. In the semiclassical and modest quantum regimes, where the multiplasmon resonances are forbidden for which Q is real, the dispersion relation (39) can be complex due to the presence of R on the right-hand side, irrespective of whether $PQ > 0$ or $PQ < 0$ as in the case of an ordinary NLS equation.

A general solution of Eq. (39) can be obtained by considering $\Omega = \Omega_r + i\Gamma$, $Q = Q_1 + iQ_2$ with Ω_r, Γ, Q_1, Q_2 being real and Q_2 is the resonance contribution from the multi-plasmon processes, as

$$\Omega_r = \pm\frac{|K|}{\sqrt{2}}[\{(P^2K^2 - 2\rho_0 P Q_1)^2 + [2\rho_0 P(R + Q_2)]^2\}^{1/2} + (P^2K^2 - 2\rho_0 P Q_1)]^{1/2}, \tag{40}$$

$$\Gamma = \mp\frac{|K|}{\sqrt{2}}[\{(P^2K^2 - 2\rho_0 P Q_1)^2 + [2\rho_0 P(R + Q_2)]^2\}^{1/2} - (P^2K^2 - 2\rho_0 P Q_1)]^{1/2}, \tag{41}$$

where we consider the upper (lower) sign for $K > 0 (K < 0)$. From Eqs. (40) and (41) it is clear that, in comparison with classical [9] or semiclassical [10] results, both the frequency shift and the energy transfer (from the wave energy to the

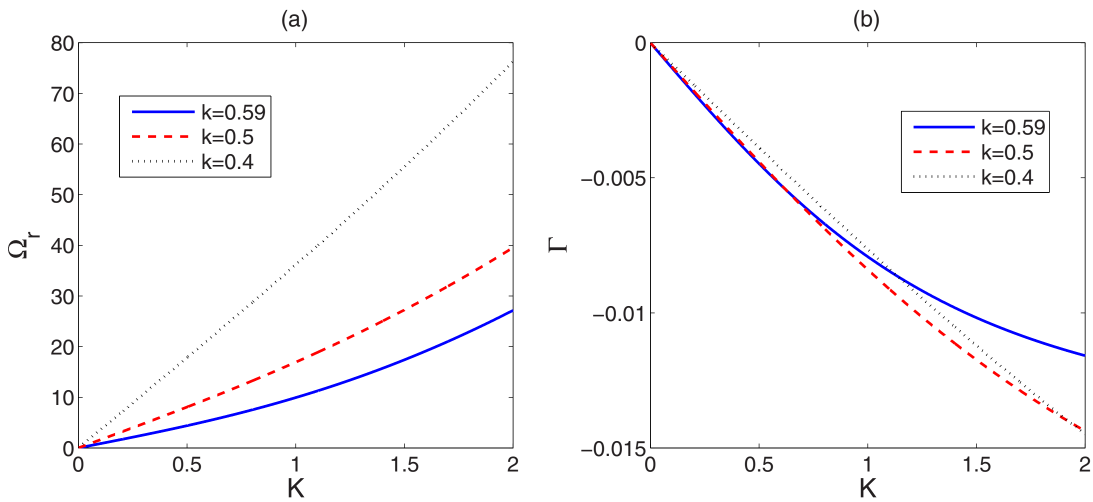

FIG. 2. The normalized frequency shift $\Omega_r(\sim\omega_p)$ [Eq. (40), panel (a)] and the energy transfer rate $\Gamma(\sim\omega_p)$ [Eq. (41), panel (b)] are plotted against the normalized wave number of modulation $K(\sim\lambda_F^{-1})$ for different values of the carrier wave number $k(\sim\lambda_F^{-1})$ that correspond to semiclassical and modest quantum regimes.

particle's kinetic energy) rate get modified by the imaginary part of Q associated with the multiplasmon resonances.

In what follows, we numerically analyze the properties of Ω_r and Γ for different values of the carrier wave number k which correspond to, especially the modest and strong quantum regimes as discussed before (since the semiclassical results are similar to the previous works [10], we skip those discussion in the present work). The results are displayed in Figs. 2 and 3. Panel (a) in each figure shows the plots of the frequency shift Ω_r, and panel (b) that for the energy transfer rate Γ against the dimensionless wave number of modulation $K(\lambda_F^{-1})$. Figure 2 shows the curves for Ω_r and Γ in the semiclassical and modest quantum regimes where the imaginary part of Q is zero. We choose a value of $k = 0.59$ (see the solid lines) at which the three-plasmon resonant velocity $v_{\rm res}^3$ marginally exceeds the Fermi velocity v_F, and the effects of the two- and three-plasmon resonances are thereby forbidden. So only the resonance effect comes from the group velocity. The values of k are then lowered from $k = 0.59$ to the values $k = 0.5$ and $k = 0.4$ (see the dashed and dotted lines). It is found that as one approaches the regimes of low wave numbers, the frequency shift increases gradually; however, the values of $|\Gamma|$ increase until k assumes the value $k = 0.5$, and then decrease as the value of k is further lowered from $k = 0.4$. The reason is that at $k = 0.59$, the contribution from the cubic nonlinearity Q becomes higher in magnitude than those of the group velocity dispersion P and the nonlocal nonlinearity R. However, as k is further lowered from $k = 0.59$ to $k = 0.5$, the magnitude of Q gets highly reduced, being lower than (but comparable to) that of P but still larger than R. As a result, both Ω_r and $|\Gamma|$ remain higher at $k = 0.59$ than those at $k = 0.5$. The magnitude of Q becomes significantly reduced at $k \lesssim 0.4$ in which the group velocity dispersion P dominates over Q and R. So, in the regime of low wave

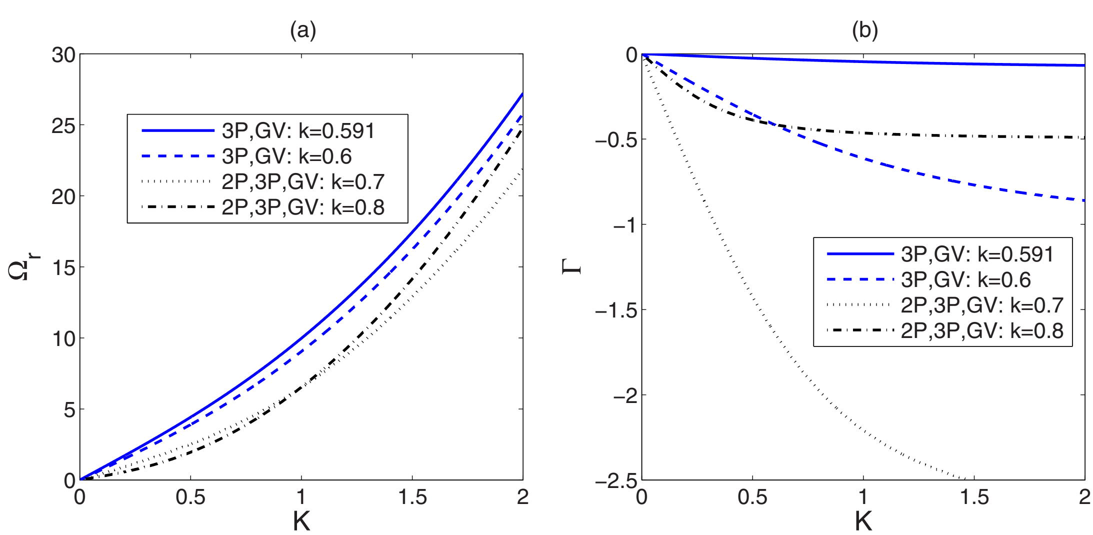

FIG. 3. The same as in Fig. 3 but in the strong quantum regime. In the legends, $2P$, $3P$, and GV, respectively, stand for two-plasmon, three-plasmon, and group velocity resonance effects.

numbers (below $k = 0.5$), though the frequency shift remains high, the magnitude of Γ gets highly reduced. This implies that the rate of transfer of wave energy to the particle's kinetic energy may not be faster as one approaches the semi-classical regimes where $\hbar k/mv_F \ll 1$ at some small wave number k is satisfied, and this transfer rate can be maximum near the point $k = 0.5$ where the three-plasmon resonant velocity slightly exceeds the Fermi velocity.

The scenario changes significantly when the three-plasmon resonance effect starts playing a role in the strong quantum regime $0.591 \lesssim k \lesssim 0.9$. Figure 3 shows that at $k = 0.591$, the three-plasmon velocity is close to but smaller than v_F for which the contribution from the three-plasmon resonance in R becomes smaller than that from the group velocity resonance. As a result, the frequency shift remains high; however, $|\Gamma|$ attains its minimum value. As are seen from the solid and dashed lines that the effect of the three-plasmon process is to decrease the values of Ω_r but to increase the values of $|\Gamma|$. The similar results are also found with the effects of the two-plasmon resonance while combined with the effects of the three-plasmon process in the regime $0.6953 \lesssim k \lesssim 0.9$ as long as the inequality $v^3_{\rm res} \gtrsim v^g_{\rm res}$ holds (see the dotted lines). In this restriction, the values of both P and $|Q|$ decrease, but those of R increase. However, as we further increase the value of k, i.e., $k = 0.8$ such that $v^3_{\rm res} < v^g_{\rm res}$ holds, the reduction of $|Q|$ becomes significantly high, the values of P start increasing and those of R decreasing. As a result, while the frequency shift starts increasing, the values of $|\Gamma|$ decrease with decreasing values of k (see the dash-dotted lines). Thus, it follows that the effect of the three-plasmon resonance is to reduce the frequency shift but to enhance the energy transfer rate whenever the corresponding resonant velocity remains greater than the group velocity resonance.

VII. NONLINEAR LANDAU DAMPING OF SOLITARY WAVE SOLUTION

It is to be noted that in the absence of the nonlocal coefficient R, the NLS equation (26) possesses an infinite number of conservation laws. The first three conserving quantities are namely the mass $I_1 = \int |\phi|^2 d\xi$, the momentum $I_2 = (2i)^{-1}\int (\phi^*\partial_\xi\phi - \phi\partial_\xi\phi^*)d\xi$, and the wave energy $I_3 = \int (|\partial_\xi\phi|^2 - (|Q/2P|)|\phi|^4)d\xi$. However, the similar quantities for the modified NLS equation (26) with nonzero R satisfy the following equations:

$$\frac{\partial I_1}{\partial\tau} = 0, \tag{42}$$

$$\frac{\partial I_2}{\partial\tau} + \frac{R}{\pi}\mathcal{P}\iint \frac{1}{\xi-\xi'}|\phi(\xi',\tau)|^2\frac{\partial}{\partial\xi}|\phi(\xi,\tau)|^2 d\xi d\xi' = 0, \tag{43}$$

$$\frac{\partial I_3}{\partial\tau} + i\frac{R}{\pi}\mathcal{P}\iint \frac{1}{\xi-\xi'}|\phi(\xi',\tau)|^2 \times \frac{\partial}{\partial\xi}\left(\phi\frac{\partial^2}{\partial\xi^2}\phi^* - \phi^*\frac{\partial^2}{\partial\xi^2}\phi\right)d\xi d\xi' = 0. \tag{44}$$

From Eq. (44), on using the fact that the integral over ξ is a convolution of the functions $\mathcal{P}[1/(\xi'-\xi)]$ and $\partial_\xi\varphi(\xi,\tau)$, where $\phi\partial_\xi^2\phi^* - \phi^*\partial_\xi^2\phi = \partial_\xi(\phi\partial_\xi\phi^* - \phi^*\partial_\xi\phi) \equiv \partial_\xi\varphi(\xi,\tau)$, and noting that the Fourier inverse transform of $i\,\mathrm{sgn}(s) = -(1/\pi)\mathcal{P}(1/\xi)$, we obtain

$$\int \frac{\partial\varphi(\xi,\tau)}{\partial\xi}\mathcal{P}\frac{1}{\xi'-\xi}d\xi = \frac{1}{2}\int \exp(is\xi')|s|\hat{\varphi}(s,\tau)ds. \tag{45}$$

So, performing the integral over ξ' as a Fourier transform of $|\phi(\xi',\tau)|^2$ we obtain

$$\mathcal{P}\iint \frac{1}{\xi-\xi'}|\phi(\xi',\tau)|^2\frac{\partial\varphi(\xi,\tau)}{\partial\xi}d\xi d\xi' = \frac{1}{2}\int |s|\hat{\varphi}(s,\tau)\hat{\phi}(-s,\tau)|^2 ds, \tag{46}$$

where the "hat" denotes the Fourier transform with respect to ξ or ξ'. Furthermore, using $\hat{\varphi}(s,\tau) \equiv -2is|\hat{\phi}(s,\tau)|^2$, we obtain from Eq. (44)

$$\frac{\partial I_3}{\partial\tau} = -\frac{R}{\pi}\int s^2|\hat{\phi}(s,\tau)|^2|\hat{\phi}(-s,\tau)|^2 ds. \tag{47}$$

The left-hand side of Eq. (47) represents the rate of change of the wave energy, and the integral on the right-hand side is a positive definite. Thus, it follows that the wave amplitude decreases or increases depending on whether $R > 0$ or < 0. In the former case, we have the inequality (the equality holds for $\phi = 0\forall\xi$)

$$\frac{\partial I_3}{\partial\tau} \leqslant 0, \tag{48}$$

implying that an initial perturbation (e.g., in the form a plane wave) will decay to zero with time τ, and hence a steady-state solution with $|I_3| < \infty$ of the NLS equation (26) may not exist in presence of the nonlocal term $\propto R$. In this situation, an approximate soliton solution of the NLS equation (26) with a small effect of the nonlinear Landau damping ($\propto R$) can be obtained whose amplitude is of the form [9,10]

$$\phi(\xi,\tau) \propto \sqrt{\phi_0(\xi,0)}\left(1 - i\frac{\tau}{\tau_0}\right)^{-1/2}, \tag{49}$$

where τ_0 is some constant inversely proportional to R and $\phi_0(\xi,0)$ is the value of ϕ at $\tau = 0$ (for details, see, e.g., Ref. [9]).

A careful examination reveals that the coefficient R of the NLS equation (26) is always positive in the parameter regimes as shown in Fig. 1. However, the contribution from the three-plasmon resonance becomes higher as the value of $v^3_{\rm res}$ is gradually lowered from v_F until the relation $v^3_{\rm res} \gtrsim v^g_{\rm res}$ holds. A qualitative plot of the decay rate $DR \equiv |(1 - i\tau/\tau_0)^{-1/2}|$ is shown in Fig. 4 in the modest and strong quantum regimes to show the relative importance of the group velocity (solid and dashed lines) and three-plasmon (dotted and dash-dotted lines and as indicated in the figure) resonances. We find that in the regime where the resonance contribution is only from the group velocity, as the wave number decreases and hence the group velocity, the decay rate of the wave amplitude becomes higher or the magnitude of the wave amplitude gets highly reduced. However, the decay rate can be higher or the magnitude of the wave amplitude can be minimized in presence of the three-plasmon effect. The black-dashed line

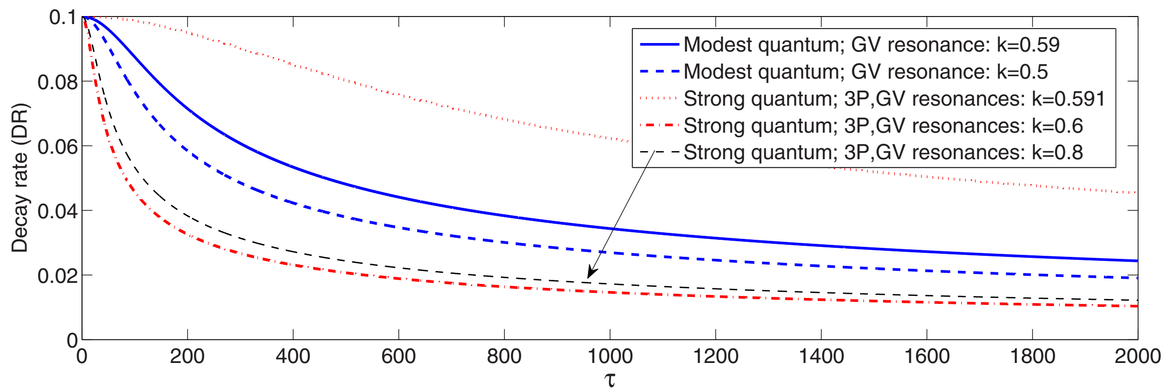

FIG. 4. The absolute value of the decay rate $DR \equiv |(1 - i\tau/\tau_0)^{-1/2}|$ is shown against the normalized time variable $\tau(\omega_p^{-1})$ in different parameter regimes as in the legend.

shows that as the restriction $v_{\rm res}^3 \gtrsim v_{\rm res}^g$ is relaxed at a higher value of k, the decay rate is further reduced or the magnitude of DR is increased (compare red dash-dotted and black dashed lines). This is a consequence to the fact that as the carrier wave number k is increased to $k = 0.8$ or above, the effects of both the cubic nonlinearity and nonlocal nonlinearity get significantly diminished; however, those of the group velocity dispersion are enhanced.

VIII. DISCUSSION AND CONCLUSION

We have investigated the nonlinear wave modulation of Langmuir waves in a fully degenerate plasma. Starting from the Wigner-Moyal equation coupled to the Poisson equation and using the multiple scale expansion technique, we have derived a modified NLS equation with a nonlocal nonlinearity. It is shown that, in contrast to classical and semiclassical results [9,10], both the local and nonlocal terms of the NLS equation get modified due to the multiplasmon processes if the dimensionless quantum parameter $H = \hbar\omega_p/mv_F^2$ or the dimensionless wave number $\hbar k/mv_F$ is not too small. In the regime of short wavelengths such that multiplasmon processes are allowed, but still $k < k_{\rm cr}$ such that one-plasmon resonances are forbidden, it is found that the three-plasmon processes play the dominant role for wave damping due to wave-particle interaction. Moreover, we note that the multiplasmon process can affect the modulation of the wave envelope in decreasing the frequency shift and increasing the energy transfer rate, as described by the contributions of the nonlinear coefficients Q and R of Eq. (26).

To discuss whether the frequency up-shift or down-shift occurs in the modulation of wave envelopes, we note that in the process of modulation of a plane wave solution [Eq. (36)] of the NLS equation [Eq. (26)] by plane-wave perturbations [Eqs. (37) and (38)], the solution (36), in fact, describes a three-wave interaction (see for details, e.g., Ref. [9]) of the unperturbed carrier wave (ω_0,k_0) and two side bands with wave numbers $k_0 \pm \epsilon|K|$ and frequencies $\omega_0 \pm \lambda|K| \pm \epsilon^2\Omega$, where ϵ is some scaling parameter and λ is the group velocity. Here, we take the upper (lower) sign for $K > 0 (K < 0)$. Thus, for $K > 0$, the frequency of the carrier pump wave ω_0 is up-shifted or down-shifted according to when Ω_r is positive or negative. In the present theory, we find that the expression for the frequency shift Ω_r changes significantly due to the presence of the nonlocal nonlinearity ($\propto R$ associated with the group velocity and three-plasmon resonances) in the NLS equation. From Eq. (40) it is clear that Ω_r can never be negative as it explicitly depends on K but not on P (the group velocity dispersion). However, Ω_r may be zero if the carrier-wave frequency can turn over with the group velocity dispersion going to zero and then to negative values (see, e.g., Ref. [9]). This is not the case in our present work. On the other hand, in the absence of the nonlocal nonlinearity, the expression for Ω [i.e., Eq. (36) with $R = 0$] explicitly depends on P, in which case the frequency shift (in the case of stable wave oscillation for a certain $K > K_c$; otherwise, the wave is unstable for $K < K_c$ with K_c denoting some critical wave number of modulation) can be positive or negative depending on whether $P > 0$ or $P < 0$. Thus, in our present analysis, the nonlinear effects (especially the nonlocal nonlinearity) lead Ω_r to be positive for $K > 0$, resulting in a frequency up-shift of the pump wave.

One of the earliest investigations of P. A. Sturrock [18] on nonlinear Langmuir waves may be discussed and compared with the present work. However, the work of P. A. Sturrock is mainly concerned with the coherent and incoherent interactions of Langmuir waves in electron fluid plasmas, especially interaction of three waves. It was shown that the coherent interaction is responsible only for a frequency shift associated with each wave number of the dispersion relation in which no exchange of energy between wave numbers takes place. However, the incoherent interaction is responsible for spectral decay in which the redistribution energy takes place in wave number space. Here, the damping mechanism is due to the particle–particle collisions quite distinctive from the Landau damping mechanism (wave-particle interaction) in collisionless plasmas as in our present theory. From our quantum kinetic theory, one cannot exactly recover the classical results, because even though the quantum effects associated with the terms $\propto \hbar k/m$, which appear to be due to the Wigner equation rather than the Vlasov equation can be negligible; however, the quantum contribution due to the background distribution (Fermi–Dirac at zero temperature) of electrons rather than the Maxwellian cannot be ignored. What we can say is that in the semiclassical limit, the frequency is up-shifted and remains high; however, the rate of transfer of wave energy from high-frequency side bands to lower ones is greatly reduced.

Furthermore, one important question may be raised in this context: Does the group velocity of a plasma wave give rise to a phase velocity of a nonlinearly driven wave? The answer is no, i.e., not the case in our present theory. This could be an important mechanism, in principle, e.g., for wake field generation for short pulses, similar to the well-known generation mechanism in the laser wake field scheme. However, we do not focus on the regime of very short Langmuir pulses and thereby such effects are forbidden.

The present approach may be generalized to cover the case of a finite temperature plasma, which is a project for future work.

ACKNOWLEDGMENTS

This work was supported by UGC-SAP (DRS, Phase III) with Sanction Order No. F.510/3/DRS-III/2015(SAPI) and UGC-MRP with F. No. 43-539/2014 (SR) and FD Diary No. 3668.

APPENDIX A: EXPRESSIONS FOR A, A_1, B, AND C IN γ

$$A=-\frac{16\pi e^3}{A_0m^2k^3}\int_C\frac{(v_p-v)\left[(v_p-v)^2+\frac{v_q^2}{2}\right]}{\left\{(v_p-v)^2-v_q^2\right\}^2\left\{(v_p-v+v_q)^2-v_q^2\right\}\left\{(v_p-v-v_q)^2-v_q^2\right\}}F^{(0)}(v)dv, \tag{A1}$$

where

$$A_0=1-\frac{\pi e^2}{k^2m}\int_C\frac{F^{(0)}(v)}{(v_p-v)^2-(2v_q)^2}dv. \tag{A2}$$

Also,

$$A_1=-12\pi e^3\frac{\hbar}{k^2m^2}\left[\int_C\frac{1}{\left\{(v_p-v+v_q)^2-4v_q^2\right\}\left\{(v_p-v-v_q)^2-4v_q^2\right\}dv}\right.$$
$$\left.+\frac{3}{2}\int_C\frac{\left[(v_p-v)^2+v_q^2\right]}{\left\{(v_p-v)^2-4v_q^2\right\}\left\{(v_p-v+2v_q)^2-v_q^2\right\}\left\{(v_p-v-2v_q)^2-v_q^2\right\}}\right]F^{(0)}(v)dv,$$

$$B=\frac{4\pi e^4}{k^4m}\int_C\left[\frac{1}{\{v_p-v+2v_q\}\{v_p-v+v_q\}\left\{(v_p-v+2v_q)^2-v_q^2\right\}}\right.$$
$$\left.+\frac{1}{\{v_p-v-2v_q\}\{v_p-v-v_q\}\left\{(v_p-v-2v_q)^2-v_q^2\right\}}-\frac{2}{\left\{(v_p-v)^2-v_q^2\right\}^2}\right]F^{(0)}(v)dv,$$

$$C(k,\omega;\lambda)=-\frac{4\pi e^4}{m\hbar^2k^2}\int_C\frac{1}{(v_p-v)^2-v_q^2}\frac{I(v)}{v-\lambda}dv=-\frac{4\pi e^4}{m\hbar^2k^4}\int_C\left[\frac{v-\lambda-v_q}{\left\{(v_p-v+2v_q)^2-v_q^2\right\}(v_p-v+v_q)^2(v-\lambda-2v_q)}\right.$$
$$-\frac{v-\lambda+v_q}{\left\{(v_p-v-2v_q)^2-v_q^2\right\}(v_p-v-v_q)^2(v-\lambda+2v_q)}$$
$$\left.-2v_q\frac{\left\{(v_p-v)^2+v_q^2\right\}}{(v-\lambda)\left\{(v_p-v)^2-v_q^2\right\}^3}-4v_q\frac{v_p-v}{\left\{(v_p-v)^2-v_q^2\right\}^3}\right]F^{(0)}(v)dv,$$

where

$$I(v)=\frac{1}{k^2}\left[(v-\lambda+v_q)\frac{f^{(0)}(v+2v_q)-f^{(0)}(v)}{\{v_p-(v+v_q)\}^2}+(v-\lambda-v_q)\frac{f^{(0)}(v)-f^{(0)}(v-2v_q)}{\{v_p-(v-v_q)\}^2}\right]. \tag{A3}$$

APPENDIX B: REDUCED EXPRESSIONS FOR α, β, γ, AND D WITH THE FERMI DISTRIBUTION AT ZERO TEMPERATURE

$$\alpha=-\frac{8m\omega_p^2}{3\hbar k^2v_F^3}\sum_{j=\pm1}(v_p+jv_q)\log\left|\frac{v_p+jv_q-v_F}{v_p+jv_q+v_F}\right|, \tag{B1}$$

$$\beta = 1 - \frac{3m\omega_p^2}{2\hbar k^3 v_F^2}\sum_{j=\pm 1}\left[\{2(v_p + jv_q) + (v_p - \lambda)\}\log\left|\frac{v_p + jv_q - v_F}{v_p + jv_q + v_F}\right| - \left\{v_F^2 - (v_p + jv_q)^2 - 2(v_p + jv_q)(v_p - \lambda)\right\}\frac{v_F}{v_F^2 - (v_P + jv_q)^2} + (v_p - \lambda)\frac{v_F(v_p + jv_q)}{v_F^2 - (v_p + jv_q)^2}\right], \tag{B2}$$

$$\gamma = \left(\frac{1}{4}\frac{AA_1}{\hbar} - \frac{1}{2\hbar^2}B + C\right)k^2, \tag{B3}$$

where

$$A = -\frac{4em^2\omega_p^2}{A_0\hbar^3 k^6 v_F^3}\sum_{j=\pm 1}\left[kv_F v_q + \frac{\omega_p}{6v_q}\left\{\left(v_F^2 - (v_p + jv_q)^2\right)\left(4 - \frac{jkv_q}{2\omega_p}\right) - 6v_q(v_p + jv_q)\left(1 - \frac{jkv_q}{2\omega_p}\right)\right\}\log\left|\frac{v_p + jv_q - v_F}{v_p + jv_q + v_F}\right| + \frac{\omega_p}{3v_q}\left\{v_F^2 - (v_p + j2v_q)^2\right\}\left(1 - \frac{jkv_q}{4\omega_p}\right)\log\left|\frac{v_p + j2v_q - v_F}{v_p + j2v_q + v_F}\right| + i\pi v_q\omega_p\left\{v_F^2 - (v_p - 2v_q)^2\right\}\left(1 + \frac{kv_q}{4\omega_p}\right)\right], \tag{B4}$$

with

$$A_0 = 1 - \frac{3\omega_p^2}{16v_F^2k^2}\left[2 - \sum_{j=\pm 1}\frac{jv_q}{4v_F}\left\{v_F^2 - (v_p + jv_q)^2\right\}\log\left|\frac{v_p + j2v_q - v_F}{v_p + j2v_q + v_F}\right|\right] - i\frac{3\pi\omega_p^2}{64v_F^3 v_q k^2}\left\{v_F^2 - (v_p - 2v_q)^2\right\}. \tag{B5}$$

Also,

$$A_1 = -\frac{e}{m}\frac{9\omega_p^2 m^3}{4\hbar^2 v_F^3 k^5}\sum_{j=\pm 1}\left[\frac{j}{4}\left\{v_F^2 - (v_p + jv_q)^2\right\}\log\left|\frac{v_p + jv_q - v_F}{v_p + jv_q + v_F}\right| - \frac{2j}{3}\left\{v_F^2 - (v_p + j3v_q)^2\right\}\log\left|\frac{v_p + j3v_q - v_F}{v_p + j3v_q + v_F}\right| + j\left\{v_F^2 - (v_p + j2v_q)^2\right\}\log\left|\frac{v_p + j2v_q - v_F}{v_p + j2v_q + v_F}\right|\right] - i\frac{9\pi\omega_p^2}{16v_F^3k^3}\frac{e}{v_q^2}\left[\frac{2}{3}\left\{v_F^2 - (v_p - 3v_q)^2\right\} - \left\{v_F^2 - (v_p - 2v_q)^2\right\}\right], \tag{B6}$$

$$B = -\frac{e^2}{k^7}\frac{3\omega_p^2 m^3}{2v_F^3\hbar^3}\sum_{j=\pm 1}\left[16v_F v_q + 4\left\{v_F^2 - (v_p + j2v_q)^2\right\}\log\left|\frac{v_p + j2v_q - v_F}{v_p + j2v_q + v_F}\right| + \left\{v_F^2 - (v_p + j3v_q)^2\right\}\log\left|\frac{v_p + j3v_q - v_F}{v_p + j3v_q + v_F}\right| - \left\{v_F^2 - (v_p + jv_q)^2 - 8jv_q(v_p + jv_q)\right\}\log\left|\frac{v_p + jv_q - v_F}{v_p + jv_q + v_F}\right|\right] - i\frac{3\pi e^2\omega_p^2}{4k^4v_F^3}\left[-\frac{1}{v_q^3}\left\{v_F^2 - (v_p - 2v_q)^2\right\} + \frac{1}{4v_q^3}\left\{v_F^2 - (v_p - 3v_q)^2\right\}\right], \tag{B7}$$

$$C = -\frac{3}{4}\frac{e^2}{\hbar^2k^4}\frac{\omega_p^2}{v_F^3}\sum_{j=\pm 1}\left[-\frac{1}{8v_q^3}\frac{v_p + j2v_q - \lambda}{v_p + jv_q - \lambda}\left\{v_F^2 - (v_p + j3v_q)^2\right\}\log\left|\frac{v_p + j3v_q - v_F}{v_p + j3v_q + v_F}\right| + jM_j\log\left|\frac{v_p + jv_q - v_F}{v_p + jv_q + v_F}\right| + jN_j\frac{2v_F}{v_F^2 - (v_p + jv_q)^2} - \left(\frac{1}{2v_q}\frac{v_p - \lambda}{v_p - jv_q - \lambda} - \frac{1}{2}\frac{1}{v_p + jv_q - \lambda} - \frac{1}{2v_q}\right)\frac{2v_F(v_p + jv_q)}{v_F^2 - (v_p + jv_q)^2} - v_q\frac{v_F^2 - (\lambda + jv_q)^2}{(v_p - jv_q - \lambda)^3(v_p + jv_q - \lambda)}\log\left|\frac{\lambda + jv_q - v_F}{\lambda + jv_q + v_F}\right| + \frac{2v_q}{k^2}\frac{\left\{(\omega - k\lambda)^2 + \frac{\hbar^2k^4}{4m^2}\right\}\left(v_F^2 - \lambda^2\right)}{(v_p + v_q - \lambda)^3(v_p - v_q - \lambda)^3}\log\frac{\lambda - v_F}{\lambda + v_F}\right], \tag{B8}$$

with

$$M_{1,-1} = \frac{1}{4v_q^2(v_p - \lambda \mp 2v_q)^2}\left[(v_p - \lambda \mp 3v_q)\left\{v_F^2 - 3(v_p \pm v_q)^2 + 2(\lambda \pm v_q)(v_p \pm v_q)\right\} \mp 4v_q(v_p - \lambda \mp v_q)(3v_p - \lambda \pm 2v_q) - 2(v_p \pm v_q)(v_p - \lambda)(v_p - \lambda \mp 3v_q) + \left\{2(v_p - \lambda \mp v_q) + (v_p - \lambda \mp 3v_q) \pm 2(v_q - \lambda)(v_q - \lambda \mp 3v_q)^2\left(\frac{1}{2v_q} \mp \frac{1}{v_p - \lambda \mp 2v_q}\right)\right\}\left\{v_F^2 - (v_p \pm v_q)^2\right\}\right]$$

$$+\frac{1}{4v_q^2(v_p-\lambda\pm v_q)^2}\Big[5(v_p-\lambda\pm v_q)\{v_F^2-(v_p\pm v_q)^2\}\pm 3v_q\{v_F^2-(v_p\pm v_q)^2\}$$
$$\mp 4v_q^2(v_p\pm v_q)-(3v_p-3\lambda\pm 5v_q)\{v_F^2\mp 2v_q(v_p\pm v_q)-(v_p\pm v_q)^2\}+2v_q^2\frac{v_F^2-(v_p\pm v_q)^2}{v_p-\lambda\pm v_q}\Big]\pm\frac{v_p}{v_q^2}, \tag{B9}$$

$$N_{1,-1}=\pm\frac{1}{4v_q^2(v_p-\lambda\mp v_q)^2}\big[2v_q(v_p-\lambda\mp v_q)\{v_F^2-3(v_p\pm v_q)^2-2(v_p-\lambda)(v_p\pm v_q)\}$$
$$+\{v_F^2-(v_p\pm v_q)^2\}(v_p-\lambda)(v_p-\lambda\mp 3v_q)\big]$$
$$\mp\frac{1}{4v_q(v_p-\lambda\mp v_q)^2}\big[(v_p-\lambda\mp 3v_q)\{v_F^2-(v_p\pm v_q)^2\}\pm 4v_q(v_p\pm v_q)(v_p-\lambda\pm v_q)\big]$$
$$\pm\frac{1}{4v_q^2}\big[v_p\pm v_q-\{v_F^2-(v_p\pm v_q)^2\}\big], \tag{B10}$$

$$D=\frac{3e^2\pi\omega_p^2}{4k\hbar m v_F^3}\left[\left(v_F^2-\lambda^2\right)\frac{(v_p-\lambda)^2+v_q^2}{\{(v_p-\lambda)^2-v_q^2\}^3}+\frac{1}{8v_q^4}\{v_F^2-(v_p-3v_q)^2\}\frac{v_p-\lambda-2v_q}{v_p-\lambda-v_q}\right]. \tag{B11}$$

This expression (B11) of D is obtained by using the following relations:

$$\lim_{\nu_g\to 0}\frac{1}{\Omega-Kv+i\nu_g}=\frac{1}{\Omega-Kv}-i\pi\frac{1}{|K|}\delta\left(v-\frac{\Omega}{K}\right),$$
$$\lim_{\nu_3\to 0}\frac{1}{\omega-kv-3kv_q+i\nu_3}=\frac{1}{\omega-kv-3kv_q}-i\pi\frac{1}{|K|}\delta(v-v_p+3v_q), \tag{B12}$$

and we have made use of $\Omega/K\to\lambda$. The infinitesimal quantities $|\nu_g|$ and $|\nu_3|$ are taken to anticipate the Landau damping terms associated with the group velocity and three-plasmon resonances.

Thus, the reduced expressions of P, Q, and R can be obtained from the relations $P=\beta/\alpha$, $Q=\gamma/\alpha$, and $R=D/\alpha$.

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