Effect of the correlation between internal noise and external noise on logical stochastic resonance in bistable systems

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Some noisy nonlinear systems could be exploited to operate reliable logic operation in an optimal window of noise intensity, which is termed as logical stochastic resonance (LSR). We investigated the LSR phenomenon in bistable systems when internal noise and external noise are correlated. The LSR effect is evaluated by the success probability of the obtained desired output with various combinations of logic inputs. It is shown that the OR-NOR, AND-NAND, and Latch operations still can operate reliably with the correlated internal noise and external noise. A positive correlation strength tends to enhance OR-NOR logic and suppress AND-NAND logic. The negative correlation strength tends to suppress OR-NOR logic and enhance AND-NAND logic. The results provide possible corroboration for implementing reliable LSR when internal noise and external noise are correlated.

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The dynamics of nonlinear systems has attracted great attention during the last decades and it has been found that the constructive interplay between noise and nonlinearity can produce surprisingly rich behaviors. One of the most prominent examples is stochastic resonance (SR) [1,2]. In 1981, Benzi *et al.* first proposed the concept of SR [3]. It has been verified that the small periodic force can be amplified by some nonlinear systems, due to the constructive role of noise. Shortly after, Nicolis and Nicolis also independently proposed the same mechanism for climate change [4]. Over the past decades, the constructive interplay between noise and nonlinearity has been extensively explored. On the other hand, the effect of noise correlation on the SR phenomenon has received much attention. For example, Berdichevsky and Gitterman verified that SR can even occur in linear systems subjected to multiplicative colored noise [5]. Cabrera *et al.* investigated logical stochastic resonance (LSR) in nonlinear oscillation systems subjected to multiplicative Ornstein-Uhlenbeck noise [6]. It has been found that there is a correlation time for the optimal SR response. The interplay between the correlation time and the system's periodicity leads to the consequence. These studies verify that the correlation of noise plays an important role in the SR phenomenon.

Recently, the concept of logical stochastic resonance (LSR) was proposed by Murali *et al.* [7,8]. It is verified that some noisy nonlinear systems, in an optimal window of moderate noise, can operate correct logical operations. The studies of LSR have received much attention. The LSR phenomenon has been realized in many scenarios such as electronic systems, nanomechanical systems, optical systems, chemical systems, and biological systems [9–14].

Most of the recent studies on LSR have taken the assumption that the noise source is zero correlated. However, for many cases this assumption is insufficient and real fluctuations are always correlated. Recently, there has been a growing interest in studying the effects of the correlation time of noise on LSR. We studied the effects of external exponentially correlated noise on LSR in a bistable system [15]. It is shown that reliable logic operation cannot be obtained when correlation time is in the intermediate or long range. Although the reliability of the system versus correlation time displays a nonmonotonic behavior, the reliable logic operation cannot be realized by increasing correlation time. Zhang *et al.* investigated the LSR effect in triple-well potential systems under internal colored noise and external colored noise [16]. It is verified that reliable logic operation cannot be obtained under external colored noise, but the reliability of the system can be enhanced by adjusting the internal colored noise. Das investigated the LSR effect in energetic and entropic systems under external dichotomous colored noise [17]. It is shown that the input-output logic correspondence can be controlled by an external exponentially correlated dichotomous noise optimizing the logical response, which exhibits a maximum at an intermediate value of correlation time. Wang and Song investigated the effects of colored noise on set-reset latch operation. It has been shown that reliable latch operation can be obtained under external colored noise [18].

However, recent studies mainly focused on the effects of correlation time of internal or external noise independently. The effect of the correlation between internal noise and external noise on the LSR has not received much attention. For many nonlinear systems, noise usually arises from different sources that can be conveniently classified into the two cases: internal noise and external noise. Internal noise describes all the fluctuations generated by the nonlinear system itself. External noise refers to any fluctuations, which come from the extrinsic environment variation [19–21]. For traditional SR, it has already been verified that internal noise and external noise can affect each other to enhance the SR effect in many nonlinear systems [22,23]. For many cases, the internal noise, which usually arises from a noisy input, inevitably depends on the extrinsic environment variation; i.e., the correlation between internal noise and external noise plays an important role in the interplay of noise and nonlinearity. In actual digital logic circuits, it has been verified that LSR can occur for the case of external noise (i.e., additive noise), as well as internal noise (i.e., multiplicative noise) [12]. Internal noise is fairly common due to the fluctuations in circuit parameters, e.g., resistors. For biological LSR systems, internal noise describes all the fluctuations generated from the gene activity

TABLE I. True table of logic operations.

Logic inputs	OR	AND	NOR	NAND
0,0	0	0	1	1
0,1	1	0	0	1
1,0	1	0	0	1
1,1	1	1	0	0

(e.g., transcription, translation, and degradation). External noise refers to any random fluctuations, especially extrinsic environmental variations, acting on biological systems [19]. For optical LSR systems, the source of internal noise is electrical and due to feedback, while the source of external noise is magnetic [13].

Because of the presence of the correlation between internal noise and external noise, thorough research is needed on how the correlation affects the LSR effect. In this paper, motivated by recent studies, we simulate the LSR phenomenon in bistable systems when internal noise and external noise are correlated. The reliability of the logic system is measured by the success probability of the logic operation, with various combinations of inputs. Our focus will be the effect of the correlation between internal noise and external noise on the LSR in bistable systems. We will show that when certain conditions change the correlation can enhance or suppress the LSR effect.

The system can be formally represented by the Langevin equation,

$$\dot{x} = -U'(x) + I + b + xD_M\xi(t) + D_A\eta(t). \tag{1}$$

where $U(x)$ is a symmetric two-well potential, I is the sum of the two square pulses encoding two logic inputs, and b is a bias parameter to control asymmetry of the two-well potential. If the random term $\eta(t)$ in the equation does not depend on the state of the system [i.e., on $x(t)$], we call it external or additive noise. On the other hand, if the random term $\xi(t)$ depends on the state of the system $x(t)$, then the noise term is called internal or multiplicative. The external noise $\eta(t)$ and internal noise $\xi(t)$ are the additive Gaussian white noise with noise intensities D_A and D_M. The correlation between internal noise and external noise is represented as follows:

$$\begin{aligned}\langle\xi(t)\xi(t')\rangle &= \delta(t-t'),\\ \langle\eta(t)\eta(t')\rangle &= \delta(t-t'),\\ \langle\eta(t)\xi(t')\rangle &= \langle\xi(t)\eta(t')\rangle = C\delta(t-t').\end{aligned} \tag{2}$$

where C is the correlation strength with the value $-1 < C < 1$.

For OR-NOR and AND-NAND logic operation, the input signal I is the sum of two aperiodic square waves, $I = I_1 + I_2$. With no loss of generality, the value of the two inputs I_1 and I_2 is taken as 0.5 when the logic input is 1, and –0.5 when the logic input is 0. The input $I = I_1 + I_2$ is encoded as a three-level square wave: –1.0 corresponding to input set (0,0), 0 corresponding to input set (0,1)/(1,0), and 1.0 corresponding to input set (1,1). According to Table I which shows true logic relationships, the output for the threshold value is 0. Representing the output $x > 0$ as logic 1, and $x < 0$ as logic 0 yields a clean OR-AND operation. Representing $x > 0$ as logic 0, and $x < 0$ as logic 1 yields a clean NOR-NAND operation. One is able to obtain different logic gates OR-NOR or AND-NAND by adjusting bias parameter b.

TABLE II. True table of Latch operations.

Set (I_1)	Reset (I_2)	Latch
0	**0**	No change (maintain previous state)
0	1	0
1	0	1

For the latch operation, we need to modify the encoding of the input values, so that we can distinguish between (1,0) and (0,1) states. The input encoding is represented as follows: The first input I_1 takes the value –0.5 when logic input is 0 and 0.5 when the logic input is 1, while the second input I_2 takes the value 0.5 when the logic input is 0 and –0.5 when the logic input is 1. By applying a NOT operation to the second input I_2, one can obtain the true latch; see Table II. Representing the output $x > 0$ as logic 1 and $x < 0$ as logic 0 yields a clean latch operation [10]. We now explicitly demonstrate the LSR phenomenon with the correlated internal noise and external noise, in a cubic nonlinear system,

$$\dot{x} = 2x - 4x^3 + I_1 + I_2 + b + xD_M\xi(t) + D_A\eta(t). \tag{3}$$

The success probability P(logic) of obtaining the desired output is calculated as follows: Each input set (I_1,I_2) drives the system over some reasonably long time interval, and then switches to another set; different input streams are fed to the system and its output is checked according to the true table of basic logic relationships, with each run being a permutation of three or four such input sets. Only when the outputs of all input sets in the run match the correct logic results, the run is considered as a success, and a failure otherwise. Actually, any partial success, where certain combinations of inputs fail to output correct logic operation, yield $P = 0$, since reliable logic operation means that all random combinations are correct. From the viewpoint of real application, any success probability below 1 is not meaningful, and the measurement reflects this stringent requirement [3,16].We simulated the system for a sequence of 1000 such runs.

The success probability P(logic) is the ratio of the number of correct runs divided by the total number of runs.

$$p(\text{success}) = \frac{\text{the number of correct runs}}{\text{the total number of runs}}. \tag{4}$$

We simulated the system in Eq. (3) by using the method in Ref. [4] with the time step $\Delta t = 0.01$. The transience time is chosen to be equal to 10% of each time interval when the input is fed into the system. The output is measured over the subsequent 90% of each time interval [3,4].

To obtain the potential function $U(x)$, we use the Stratonovich prescription to write the the Fokker-Planck equation of the system as [15–17]

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t) &= -\frac{\partial}{\partial x}[a(x)P(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b(x)P(x,t)]\\ &= -\frac{\partial}{\partial x}\left(f(x) + \frac{1}{4}\left[2D_M^2x + 2CD_MD_Nx\right]\right)P(x,t)\\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[D_M^2x^2 + 2CD_MD_Nx + D_N^2\right]P(x,t),\end{aligned} \tag{5}$$

with the following drift and diffusion terms $a(x)$ and $b(x)$:

$$a(x) = 2x - 4x^3 + \frac{1}{4}\left[2D_M^2 x + 2C D_M D_N x\right], \tag{6}$$

and

$$b(x) = D_M^2 x^2 + 2C D_M D_N x + D_N^2. \tag{7}$$

The steady state distribution is

$$P_s(x) = N \exp[-U(x)], \tag{8}$$

where N is a normalization constant. The potential $U(x)$ can be formed as

$$U(x) = -2\int^x \frac{a(y)}{b(y)} dy + \ln[b(x)]. \tag{9}$$

It can be seen that the correlation strength C affects the shape of the potential function.

For the OR-NOR logic operation, the system output under different correlation strengths C is plotted (Fig. 1). For the negative correlation strength, it is observed that the particle never overcomes the barrier and hops to the correct wells. When the internal noise and external noise are uncorrelated, i.e., $C = 0$, the trajectories can almost follow the rules of basic OR-NOR logic operation. The internal and external noise are chosen with relatively not-large values, and the particle sometimes cannot overcome the barrier and hop to the correct wells. For the positive correlation strength, the result evidently shows that the trajectories can strictly follow the rules of basic OR-NOR logic operation. The results suggest that the positive correlation tends to enhance the OR-NOR logic operation and the negative correlation tends to suppress the OR-NOR logic operation.

For the AND-NAND logic operation, the system output under different correlation strengths C is plotted (Fig. 2). For the negative correlation strength, it is observed that the particle can always overcome the barrier and hop to the correct wells. The trajectories can strictly follow the rules of basic AND-NAND logic operation. When the internal noise and external noise are uncorrelated, the trajectories can almost follow the rules of basic AND-NAND logic operation, but the particle sometimes cannot overcome the barrier and hop to the correct wells. For the positive correlation strength, the result evidently shows that the trajectories fail to follow any rules of basic AND-NAND logic operation. The results suggest that the positive correlation tends to suppress the AND-NAND logic operation and the negative correlation tends to enhance the AND-NAND logic operation.

The success probability as a function of noise intensity D_M is displayed with different correlation strength C .To obtain a nonmonotonic resonance effect, we fix the noise intensity D_A at a relatively small value. For the OR-NOR logic operation, the results show that the success probability P first increases and then decreases with increasing D_M (Fig. 3). For different correlation strengths C, the reliable logic operation can be obtained in an optimal window of noise intensity D_M. With increasing C, the optimal noise window shifts towards the right and becomes wider. For the AND-NAND logic operation, the results show that the success probability P versus D_M still shows a nonmonotonic behavior (Fig. 4). With increasing C, the optimal noise window becomes narrower.

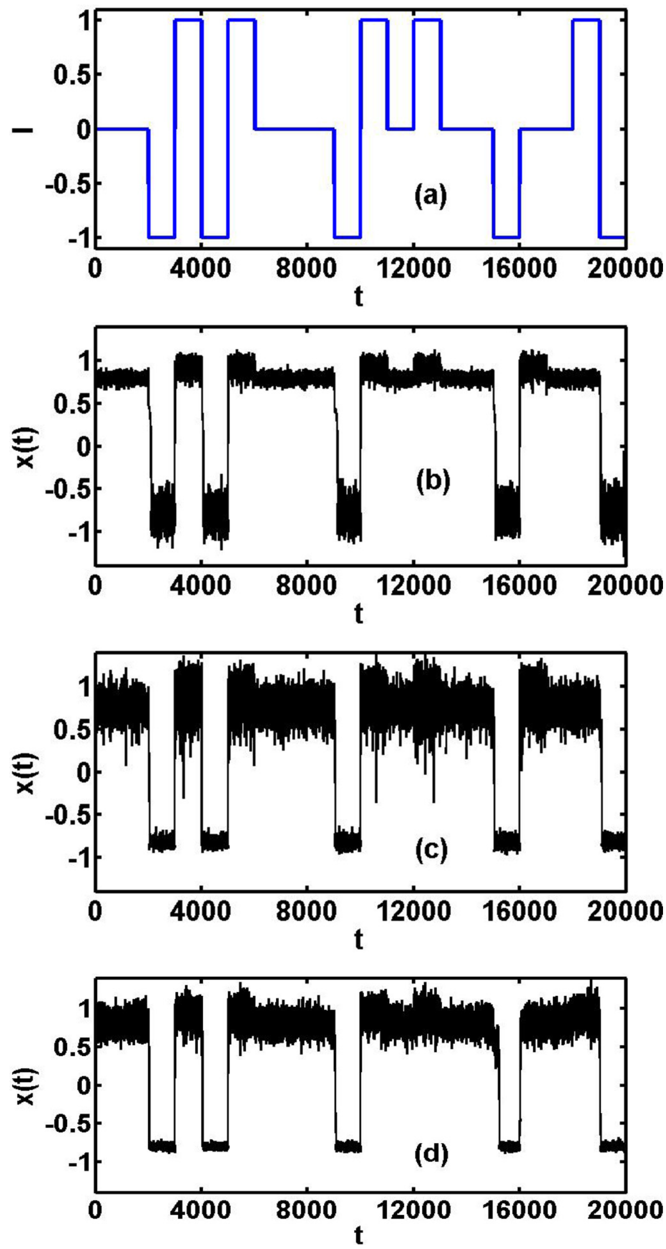

FIG. 1. For the OR-NOR logic operation, from top to bottom, panel (a) shows the stream of input $I = I_1 + I_2$; panels (b–d) show the output of the system with different correlation $C = -0.9$, $C = 0$, and $C = 0.9$, respectively. The parameter values $D_M = 0.25$, $D_A = 0.3$, $b = 0.5$ are used.

For different correlation strength C, the success probability P which is equal to 1 is displayed with noise intensity D_M and bias b (Fig. 5). For the OR-NOR logic operation, it can be seen that the optimal parameter region with $P = 1$ becomes larger with increasing C. For the AND-NAND logic operation, it is observed that the optimal parameter region with $P = 1$ becomes smaller with increasing C. Compared with the case when internal noise and external noise are uncorrelated, the positive correlation strength C tends to enhance the OR-NOR logic and suppress the AND-NANd logic, and the negative correlation strength C tends to suppress the OR-NOR logic and enhance the AND-NAND logic.

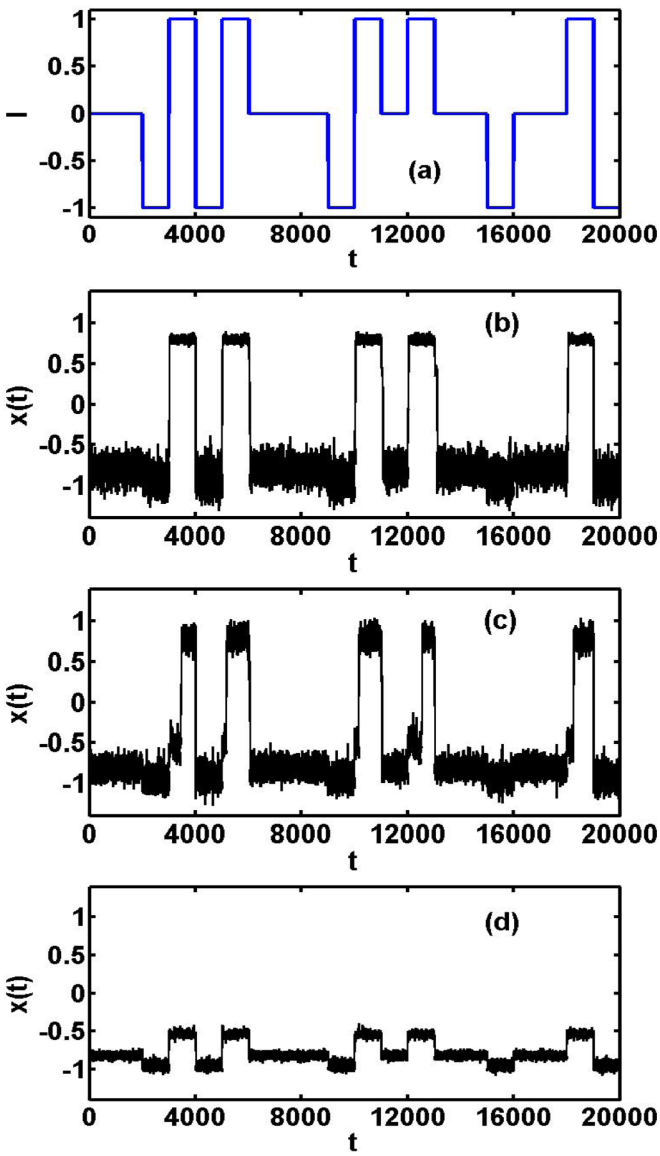

FIG. 2. For the AND-NAND logic operation, from top to bottom, panel (a) shows the stream of input $I = I_1 + I_2$; panels (b–d) show the output of the system with different correlation $C = -0.9$, $C = 0$, and $C = 0.9$, respectively. The parameter values $D_M = 0.25$, $D_A = 0.3$, $b = -0.5$ are used.

We also investigated the latch operation in the bistable system (Fig. 6). It is shown that the latch operation can even operate with a large bias, due to the coexistence of the internal noise and external noise. For a given noise intensity D_M, the optimal window of bias moves towards the right with increasing C. The system can robustly operate the latch operation when the internal noise and external noise are correlated. The change of the correlation strength C only slightly affects the range of the optimal bias.

Internal noise and external noise usually coexist in many nonlinear systems. For classic SR theory, it has been verified that internal noise and external noise can affect each other to enhance the SR effect. In this paper, we have studied the LSR phenomenon in a bistable system when internal noise and external noise are correlated. We demonstrated that the OR-NOR, AND-NAND, and latch operation still can exist robustly with the correlated internal noise and external noise. However, the correlation strength plays a different role in operating the OR-NOR and AND-NAND logic. Positive correlation strength tends to enhance OR-NOR logic and suppress AND-NAND logic. Negative correlation strength tends to suppress OR-NOR logic and enhance AND-NAND logic. The results provide possible corroboration for implementing reliable logic operation when internal noise and external noise are correlated.

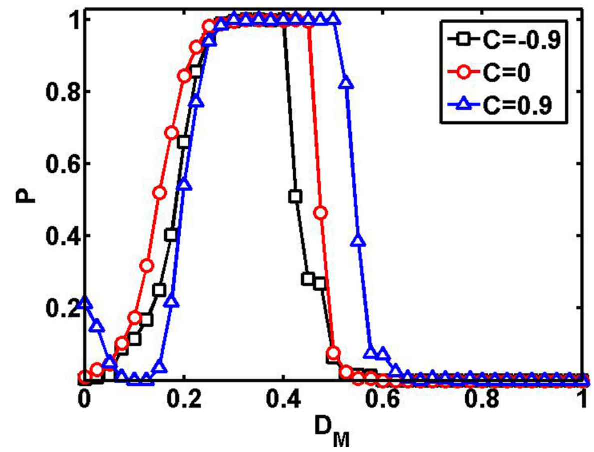

FIG. 3. The success probability P of the OR-NOR logic versus noise intensity D_M are plotted with different correlation C. The bias b is set to 0.475 ($C = -0.9$), 0.5($C = 0$), 0.525($C = 0.9$), respectively. The parameter value $D_A = 0.1$ is used.

Our results verify that the presence of such correlation between internal noise and external noise can enhance or suppress the LSR effect. The effect of the correlation intensity is not the same for different logic functions. There are two particular cases: When $C \sim 0$, the internal noise and external noise are completely irrelevant; when $C \sim 1$, the internal noise and external noise actually originate from the same source. Actually, internal noise and external noise are more or less correlated [12]. In actual digital logic circuits, the

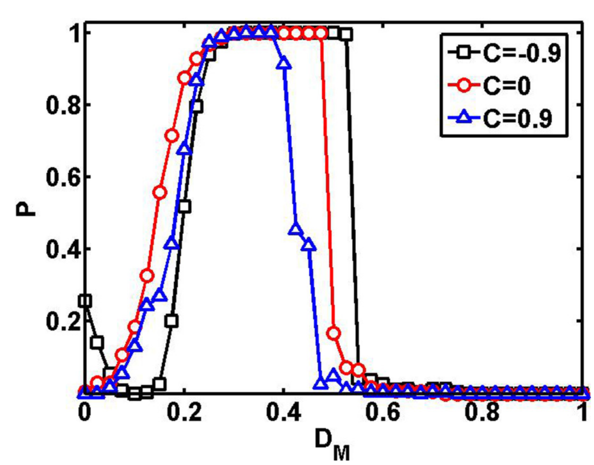

FIG. 4. The success probability P of the AND-NAND logic versus noise intensity D_M are plotted with different correlation C. The bias b is set to -0.525 ($C = -0.9$), -0.5 ($C = 0$), -0.475 ($C = 0.9$), respectively. The parameter value $D_A = 0.1$ is used.

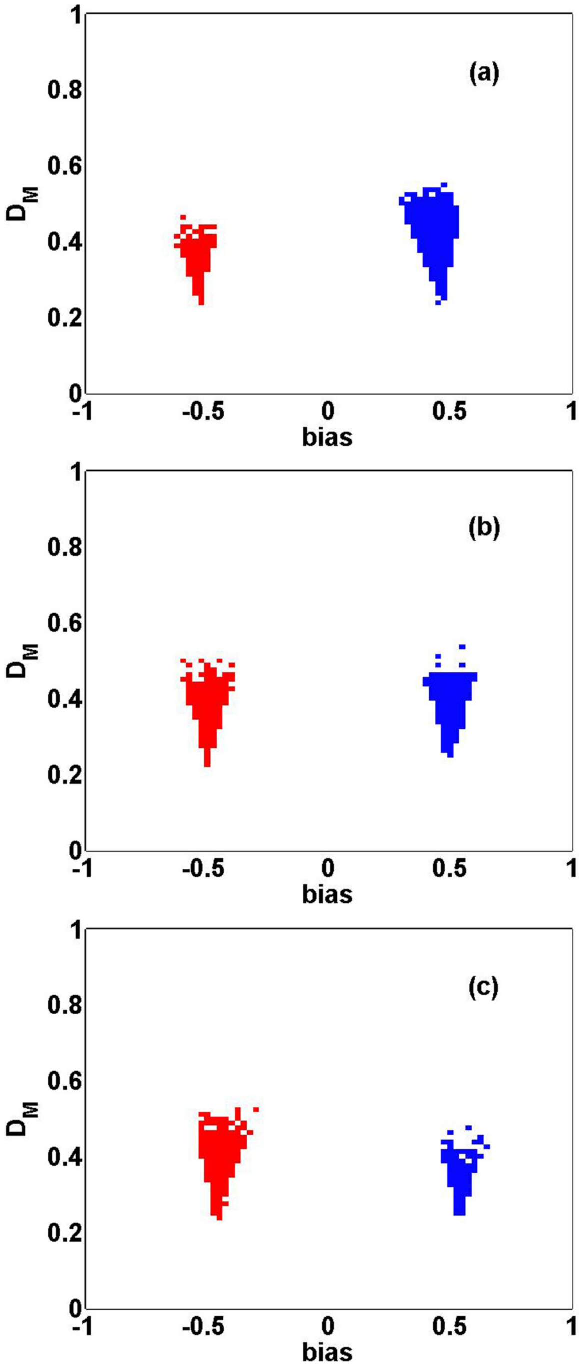

FIG. 5. Points marking where the OR-NOR logic (red) and AND-NAND logic (blue) are obtained with probability 1 for different correlation strengths: (a) $C = -0.9$, (b) $C = 0$, and (c) $C = 0.9$. The parameter value $D_A = 0.1$ is used.

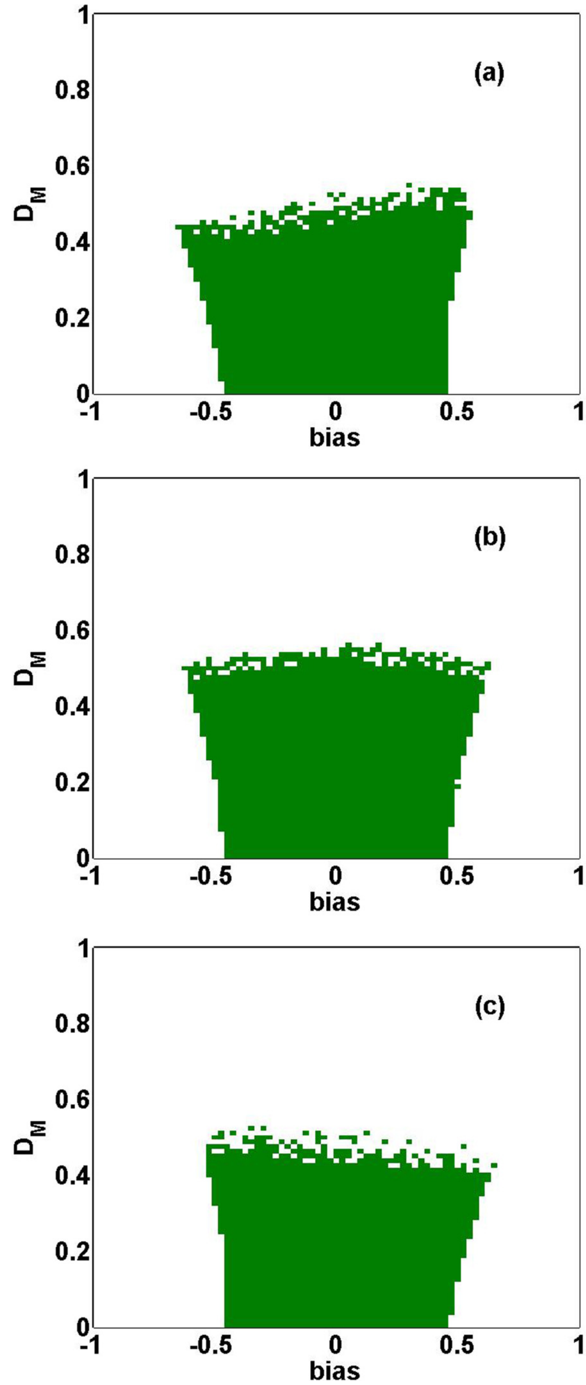

FIG. 6. Points marking where the latch operation (green) are obtained with probability 1 for different correlation strengths: (a) $C = -0.9$, (b) $C = 0$, and (c) $C = 0.9$.The parameter value $D_A = 0.1$ is used.

internal noise and external noise are commonly subjected to the same source. The exterior envionment variation inevitably affects the internal noise. For a biological LSR system, Dari *et al.* realized LSR in genetic networks [19]. In such a system, the correlation between internal noise and external noise can be traced to two processes: when the production of genetic networks dynamics acts back on its input, or when a molecule is produced to the detriment of other molecules influencing the input signal. For a quantum system, Pfeffer *et al.* realized LSR by coupling two quantum dots [24]. The cross correlation between internal noise and external noise also plays an important role in a quantum LSR system. Actually, for most cases, the internal and external noise are inevitably correlated. From the viewpoint of LSR, the results can find wide applications in digital logic circuits, biological systems, and optical systems, as well as quantum systems, etc.

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