Entropic nonadditivity, *H* theorem, and nonlinear Klein-Kramers equations

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(Received 13 September 2017; published 6 November 2017)

We use the H theorem to establish the entropy and the entropic additivity law for a system composed of subsystems, with the dynamics governed by the Klein-Kramers equations, by considering relations among the dynamics of these subsystems and their entropies. We start considering the subsystems governed by linear Klein-Kramers equations and verify that the Boltzmann-Gibbs entropy is appropriated to this dynamics, leading us to the standard entropic additivity, $S_{BG}^{(1\cup 2)} = S_{BG}^1 + S_{BG}^2$, consistent with the fact that the distributions of the subsystem are independent. We then extend the dynamics of these subsystems to independent nonlinear Klein-Kramers equations. For this case, the results show that the H theorem is verified for a generalized entropy, which does not preserve the standard entropic additivity for independent distributions. In this scenario, consistent results are obtained when a suitable coupling among the nonlinear Klein-Kramers equations is considered, in which each subsystem modifies the other until an equilibrium state is reached. This dynamics, for the subsystems, results in the Tsallis entropy for the system and, consequently, verifies the relation $S_q^{(1\cup 2)} = S_q^1 + S_q^2 + (1-q)S_q^1 S_q^2/k$, which is a nonadditive entropic relation.

DOI: 10.1103/PhysRevE.96.052109

I. INTRODUCTION

Entropy is one of the most universal tools used to obtain information for a system from the particles' microscopic dynamics details, which can be connected to macroscopic quantities and, consequently, with thermodynamics quantities. The first steps towards the concept of entropy started with Clausius's studies of thermal machines [1]. Afterwards, the Boltzmann and Gibbs works incorporated the concept of probability, leading us to the fundamentals of statistical mechanics [2–5].

One of the greatest success of the Boltzmann-Gibbs statistical mechanics is the agreement with the thermodynamics, which is essentially restricted to a class of additives and extensive phenomena. This feature may be directly related to the dynamics exhibited by these systems, which are, in general, characterized by short-range interactions [6] and Markovian processes. This scenario, typically additive and extensive, is suitably described in terms of the Boltzmann-Gibbs (BG) entropy ($S_{BG} = -k_B \sum_i \rho_i \ln \rho_i$), which verifies the entropic additivity $S_{BG}^{1\cup 2} = S_{BG}^1 + S_{BG}^2$ (in the sense of Penrose [2]), for two independent subsystems 1 and 2, when $\rho_{1,2} = \rho_1 \rho_2$ (independent probabilities). However, systems with long-range interaction [7–10], long-range correlations [11], and memory effects [12] may not preserve the additivity and/or extensivity properties. A suitable description for these systems has been investigated by different approaches [6] and, in particular, by extending the BG entropy in order to incorporate these scenarios. In this regard, the Tsallis entropy,

$$S_q = \frac{k}{q-1}\left(1 - \sum_i \rho_i^q\right), \tag{1}$$

where k is a constant and q is a parameter, has been successful applied in several contexts such as black holes [13,14], random chains of spins [15], standard maps [16], nonlinear Fokker-Planck equations [17–20], quantum problems [21], and quantum dissipation [22], which are not suitably described in terms of the standard statical mechanics. It is worth mentioning that Eq. (1) recovers the Boltzmann-Gibbs entropy for $q \to 1$, and the parameter q may be considered a measurement of the interactions [23–26]. Thus, it leads us to an extension of the Boltzmann factor, i.e., distributions of short- and long-tailed behaviors, which can be linked to a generalization of the central limit theorem [27,28]. Another important property related to this entropy concerns the entropic additivity law: $S_q^{(1\cup 2)} = S_q^1 + S_q^2 + (1-q)S_q^1 S_q^2/k$ for two independent subsystem (independent probabilities), which is nonadditive. Both entropies verify the H theorem [4,29,30], which represents one of the most important results of nonequilibrium statistical mechanics, by ensuring that a system will reach an equilibrium after a long-time evolution. Thus, this theorem shows that for each dynamics, there is one entropy. For example, the BG entropy leads to the standard Fokker-Planck (FP) and Klein-Kramers equations [31], which are governed by the same stochastic process. In the context of the Tsallis entropy, Refs. [32,33] have shown that the dynamics is given in terms of the nonlinear FP [34–38] and nonlinear Klein-Kramers equations [39–41]. Other situations have also been investigated with the H theorem by taking into account nonlinear extensions of the FP equation related to general entropies [37] such as Rényi [42] and Kaniadakis [43–45]. In this perspective, the H theorem establishes a connection between the dynamics and entropy, which may be used to investigate the dynamics behind the entropy additivity law of these entropies. In this sense, by considering the linear and nonlinear Klein-Kramers equations, the H theorem can be used to show how these entropic additivity laws can be obtained when a system composed of subsystems is considered.

Here we investigate by using the H theorem the suitable entropy for a system composed of the subsystems with dynamics governed by Klein-Kramers equations. We first consider the dynamics of the subsystems governed by linear independent

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Klein-Kramers equations, which are related to Markovian processes. We then extend this scenario to nonlinear Klein-Kramers equations, which can be related to non-Markovian processes and, consequently, anomalous relaxation processes. For the first case, we verify that the dynamics of the subsystems results in the BG entropy and, consequently, in the standard entropic additivity, which is consistent with the fact that the probability distribution of system should be the product of probabilities of each subsystems. We next extend this scenario to nonlinear Klein-Kramers equations and show that if they are independently considered, each subsystem satisfies the Tsallis entropy. However, the entropic additivity found for a system composed of these subsystems is inconsistent with the fact that the probability distribution of the system should be the product of the probabilities of each subsystem, as expected in a statistical mechanics when the equilibrium is reached. To overcome this point, we consider a coupling among the nonlinear Klein-Kramers equations, which lead us to the Tsallis entropy for the system composed by these subsystems and verify the relation $S_q^{(1\cup 2)} = S_q^1 + S_q^2 + (1-q)S_q^1 S_q^2/k$, which is a nonaddictive entropic relation. This result is consistent with the fact the probability distribution of system should be the product of probabilities of each subsystem when the equilibrium is reached. Afterwards, we extend this case by considering n subsystems, and we show that the H theorem can be used to obtain the Boltzmann-Gibbs and Tsallis entropies. These developments are performed in Sec. II and Sec. III. Our conclusions are presented in Sec. IV.

II. KLEIN-KRAMERS EQUATION: LINEAR CASE

Let us start our investigation about the thermodynamic proprieties of a system by considering, for simplicity, that it is composed of two subsystems (1 and 2) with the dynamics defined in terms of the following Klein-Kramers equations:

$$\frac{\partial \rho_1}{\partial t} + \frac{p_1}{m}\frac{\partial \rho_1}{\partial x_1} = -\frac{\partial}{\partial p_1}\left[\left(-\frac{\gamma}{m}p_1 + \mu_1\right)\rho_1\right] + \frac{\Gamma}{2m^2}\frac{\partial^2 \rho_1}{\partial p_1^2}, \tag{2}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{p_2}{m}\frac{\partial \rho_2}{\partial x_2} = -\frac{\partial}{\partial p_2}\left[\left(-\frac{\gamma}{m}p_2 + \mu_2\right)\rho_2\right] + \frac{\Gamma}{2m^2}\frac{\partial^2 \rho_2}{\partial p_2^2}, \tag{3}$$

in which $\mu_i = -d\phi_i/dx_i$ ($\phi_i(x_i)$ is a confining potential, where $i = 1$ and 2, for the corresponding subsystem) and $\rho_i = \rho_i(p_i,x_i,t)$. We also consider that the current probability densities, $\mathbf{J}_i = (J_{x_i}, J_{p_i})$, with $J_{x_i} = (p_i/m)\rho_i$ and $J_{p_i} = \mathcal{J}_i + \mu_i\rho_i$, where

$$\mathcal{J}_i = -\frac{\gamma}{m}p_i\rho_i - \frac{\Gamma}{2m^2}\frac{\partial}{\partial p_i}\rho_i, \tag{4}$$

related to these equations are subjected to the boundary conditions $J_{x_i}(\pm\infty,x_i,t) = J_{x_i}(p_i,\pm\infty,t) = 0$ and $J_{p_i}(\pm\infty,x_i,t) = J_{p_i}(p_i,\pm\infty,t) = 0$. It is worth mentioning that these subsystems are independent of each other and the probability distribution of the system can be written as $\rho_{1,2} = \rho_1\rho_2$, i.e., the product of probabilities of each one. The internal energy for this system, composed of two subsystems, is additive, $U = U_1 + U_2$, and it is given by the following expression:

$$U = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\rho_1(p_1,x_1,t)\rho_2(p_2,x_2,t)\mathcal{H}(p_1,p_2,x_1,x_2)dp_1\,dp_2\,dx_1\,dx_2, \tag{5}$$

in which $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ is the Hamiltonian of the system, with $\mathcal{H}_i = p_i^2/2m + \phi_i(x_i)$, respectively. By using these equations and the H theorem [46,47], we may obtain the suitable entropy for this system and, consequently, the entropic additivity behind the dynamics chosen for the subsystems. For this reason, we assume for the entropy the following expression:

$$S = k\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}g[\rho_1(p_1,x_1,t),\rho_2(p_2,x_2,t)]dp_1\,dp_2\,dx_1\,dx_2, \tag{6}$$

in which $g(\rho_1,\rho_2)$ is an arbitrary function of the probabilities. Thus, the dynamics will naturally lead to the suitable form for the entropy when the H theorem is verified, by determining the function $g(\rho_1,\rho_2)$.

Applying these definitions in the Helmholtz free energy of the system, $F = U - TS$, it is possible to show that

$$\begin{aligned}\frac{dF}{dt} &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dp_1\,dp_2\,dx_1\,dx_2\left(\rho_2\mathcal{H} - kT\frac{\partial g}{\partial\rho_1}\right)\frac{\partial\rho_1}{\partial t}\\ &\quad + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dp_1\,dp_2\,dx_1\,dx_2\left(\rho_1\mathcal{H} - kT\frac{\partial g}{\partial\rho_2}\right)\frac{\partial\rho_2}{\partial t}\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dp_1\,dp_2\,dx_1\,dx_2\left(\rho_2\frac{p_1}{m} - kT\frac{\partial\rho_1}{\partial p_1}\frac{\partial^2 g}{\partial\rho_1^2}\right)\mathcal{J}_1(p_1,x_1,t)\\ &\quad + \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dp_1\,dp_2\,dx_1\,dx_2\left(\rho_1\frac{p_2}{m} - kT\frac{\partial\rho_2}{\partial p_2}\frac{\partial^2 g}{\partial\rho_2^2}\right)\mathcal{J}_2(p_2,x_2,t)\end{aligned}$$

$$- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 \left(\rho_2 \mathcal{H} - kT \frac{\partial g}{\partial \rho_1}\right)\left(\frac{p_1}{m}\frac{\partial \rho_1}{\partial x_1} + \mu_1 \frac{\partial \rho_1}{\partial p_1}\right)$$
$$- \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 \left(\rho_1 \mathcal{H} - kT \frac{\partial g}{\partial \rho_2}\right)\left(\frac{p_2}{m}\frac{\partial \rho_2}{\partial x_2} + \mu_2 \frac{\partial \rho_2}{\partial p_2}\right). \tag{7}$$

From the previous equation, we can define the quantity

$$\Pi_{1(2)} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 \left(\rho_{2(1)} \mathcal{H} - kT \frac{\partial g}{\partial \rho_{1(2)}}\right)\left(\frac{p_{1(2)}}{m}\frac{\partial \rho_{1(2)}}{\partial x_{1(2)}} + \mu_{1(2)} \frac{\partial \rho_{1(2)}}{\partial p_{1(2)}}\right)$$
$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 \rho_{2(1)} \left(\phi_{1(2)} \frac{p_{1(2)}}{m}\frac{\partial \rho_{1(2)}}{\partial x_{1(2)}} + \frac{p_{1(2)}^2}{2m} \mu_{1(2)} \frac{\partial \rho_{1(2)}}{\partial p_{1(2)}}\right), \tag{8}$$

which, after performing integration by parts on variables $x_{2(1)}$ and $p_{2(1)}$, allows us to show that $\Pi_{1(2)} = 0$. By using this result in Eq. (7), it is possible to show that

$$\frac{dF}{dt} = -\frac{1}{\gamma} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\mathcal{J}_1^2 \frac{\rho_2}{\rho_1} + \mathcal{J}_2^2 \frac{\rho_1}{\rho_2}\right) dp_1\, dp_2\, dx_1\, dx_2 \leqslant 0, \tag{9}$$

with $\gamma = \Gamma/(2m^2 kT)$, for a general entropy, in which $g(\rho_1,\rho_2)$ should simultaneously satisfy the following equations:

$$\frac{1}{\rho_2}\frac{\partial^2 g}{\partial \rho_1^2} = -\frac{1}{\rho_1} \quad \text{and} \quad \frac{1}{\rho_1}\frac{\partial^2 g}{\partial \rho_2^2} = -\frac{1}{\rho_2}. \tag{10}$$

Thus, for this system, the H theorem is established through Eq. (9), which enables us to obtain the entropy related to the dynamic of the subsystems by solving the previous set of equations. In particular, after performing some calculations, it is possible to show that

$$g(\rho_1,\rho_2) = -\rho_2\rho_1 \ln \rho_1 - \rho_1\rho_2 \ln \rho_2 = -\rho_2\rho_1 \ln (\rho_1\rho_2) \tag{11}$$

when conditions $g(1,1) = g(1,0) = 0$ and $g(0,1) = g(0,0) = 0$ are required. By substituting Eq. (11) in Eq. (6), we obtain the Boltzmann-Gibbs entropy,

$$S_{BG}(\rho_{1,2}) = -k_B \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 (\rho_{1,2} \ln \rho_{1,2}), \tag{12}$$

in which $\rho_{1,2} = \rho_1\rho_2$ and k_B is Boltzmann constant. This analysis shows that the entropy of the system is determined by the dynamics present in the subsystems, i.e., (2) and (3), and it is consistent with the fact that the probability distribution of the system is the product of the probabilities related to the subsystems. It also reveals for the system the entropic additivity law behind the dynamic present in the subsystems, which, in this case, corresponds to the standard entropic additivity, $S_{BG}^{(1\cup 2)} = S_{BG}^1 + S_{BG}^2$. These results, when the equilibrium is reached, may be connected to equilibrium distribution $\rho = \exp(-\beta\mathcal{H})/\mathcal{Z}$, in which $\mathcal{Z} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp(-\beta\mathcal{H}) dp_1\, dp_2\, dx_1\, dx_2$, and with the principle of maximum entropy [48,49]. In next section, we extend our analysis by considering nonlinear Klein-Kramers equations for the subsystems.

III. KLEIN-KRAMERS EQUATION: NONLINEAR CASE

In this section, we extend the previous scenario by considering that the dynamics of the subsystems is governed by nonlinear Klein-Kramers equations [39,40]. In this context, we analyze two possible situations involving the dynamic of the subsystems. In the first one, we consider that the dynamics of each subsystem is independent each other. In the second one, we consider a coupling between the subsystems, by implying that the dynamic of each one is influenced by the other.

A. First case: Independent dynamics

In this case, we consider that the subsystems are governed by the following nonlinear Klein-Kramers equations:

$$\frac{\partial \rho_1}{\partial t} + \frac{p_1}{m}\frac{\partial \rho_1}{\partial x_1} = -\frac{\partial}{\partial p_1}\left[\left(-\frac{\gamma}{m}p_1 + \mu_1\right)\rho_1\right] + \frac{\Gamma}{2m^2}\frac{\partial^2 \rho_1^q}{\partial p_1^2}, \tag{13}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{p_2}{m}\frac{\partial \rho_2}{\partial x_2} = -\frac{\partial}{\partial p_2}\left[\left(-\frac{\gamma}{m}p_2 + \mu_1\right)\rho_1\right] + \frac{\Gamma}{2m^2}\frac{\partial^2 \rho_2^q}{\partial p_2^2}, \tag{14}$$

with $\mu_i = -d\phi_i/dx_i$, in which ϕ_i is a confining potential, as in previous section. The current densities related to these equations are $J_{x_i} = (p_i/m)\rho_i$ and $J_{p_i} = \mathcal{J}_{i,q} + \mu_i\rho_i$, with

$$\mathcal{J}_{i,q} = -\frac{\gamma}{m}p_i\rho_i - \frac{\Gamma}{2m^2}\frac{\partial}{\partial p_i}\rho_i^q. \tag{15}$$

These equations are subjected to the boundary conditions $J_{x_i}(\pm\infty,x_i,t) = J_{x_i}(p_i,\pm\infty,t) = 0$ and $J_{p_i}(\pm\infty,x_i,t) = J_{p_i}(p_i,\pm\infty,t) = 0$, as in the previous case. A system composed of subsystems governed by these equations [Eqs. (13) and (14)] verifies the H theorem for an general entropic form given by Eq. (6), which implies a set of equations

$$\frac{1}{\rho_2}\frac{\partial^2 g}{\partial \rho_1^2} = -\frac{q}{\rho_1^{2-q}} \quad \text{and} \quad \frac{1}{\rho_1}\frac{\partial^2 g}{\partial \rho_2^2} = -\frac{q}{\rho_2^{2-q}} \tag{16}$$

which are simultaneously satisfied. The solution for these equations can be found as a superposition of particular solutions. In particular, it is possible to show that the solution is given by

$$g(\rho_1,\rho_2) = -\rho_2\frac{\rho_1-\rho_1^q}{1-q} - \rho_1\frac{\rho_2-\rho_2^q}{1-q} \tag{17}$$

and it satisfies the conditions $g(1,1) = g(1,0) = 0$ and $g(0,1) = g(0,0) = 0$. From this result, by simple integrations of Eq. (17), we obtain that the entropy of the system is the sum of entropies of each subsystem, $S^{1\cup 2} = S_q^1 + S_q^2$, where the entropy of each of the subsystems is given in term of the Tsallis entropy. It is worth mentioning that this result does not preserve the additivity in Penrose sense [2]. Thus, the entropy for the system, when the equilibrium is reached, does not verify the condition $S(\rho_{1,2}) = S(\rho_1\rho_2)$ usually required for a system composed of independent subsystems. Consequently, it is not possible to verify for a system composed of these subsystems an equilibrium scenario for $S^{1\cup 2}$ in connection with the thermodynamics as a single system as in the previous case, with the probability of the system defined as $\rho_{1,2} = \rho_1\rho_2$.

B. Second case: Coupled dynamics

The result obtained in previous section evidences that a coupling between these subsystems needs to be considered in order to verify a consistent entropic additivity law with a thermodynamical equilibrium for a system composed of these subsystems. Consequently, the suitable result should preserve the conditions required for the system to be considered as a single system in the thermodynamics sense, with the entropic additivity determined by the dynamics of the subsystems. For this, we consider that the subsystems are governed by the following equations:

$$\frac{\partial\rho_1}{\partial t} + \frac{p_1}{m}\frac{\partial\rho_1}{\partial x_1} = -\frac{\partial}{\partial p_1}\left[\left(-\frac{\gamma}{m}p_1 + \mu_1\right)\rho_1\right] + \mathcal{D}_1(t)\frac{\Gamma}{2m^2}\frac{\partial^2\rho_1^q}{\partial p_1^2}, \tag{18}$$

$$\frac{\partial\rho_2}{\partial t} + \frac{p_2}{m}\frac{\partial\rho_2}{\partial x_2} = -\frac{\partial}{\partial p_2}\left[\left(-\frac{\gamma}{m}p_2 + \mu_2\right)\rho_2\right] + \mathcal{D}_2(t)\frac{\Gamma}{2m^2}\frac{\partial^2\rho_2^q}{\partial p_2^2}, \tag{19}$$

with

$$\mathcal{D}_1(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_2\, dx_2\rho_2^q, \quad \text{and}$$
$$\mathcal{D}_2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dp_1\, dx_1\rho_1^q. \tag{20}$$

For $q \to 1$, these equations recover the standard Klein-Kramers equations given by Eqs. (2) and (3), which were analyzed in connection with the Boltzmann-Gibbs entropy. It is worth mentioning that the dynamics of the subsystems are coupled to each other only for $q \neq 1$ by the diffusive term, exhibiting a time dependence before reaching the equilibrium. In addition, this coupling between these equation enables us to make a connection with the zeroth law of thermodynamics presented in Refs. [50–52], as we will discuss later. From these nonlinear Klein-Kramers equations, for the subsystems, we may define the current density J_{p_i} as $J_{p_i} = \bar{\mathcal{J}}_{i,q} + \mu_i\rho_i$, with

$$\bar{\mathcal{J}}_{i,q}(p_i,x_i,t) = -\frac{\gamma}{m}p_i\rho_i - \mathcal{D}_i(t)\frac{\Gamma}{2m^2}\frac{\partial}{\partial p_i}\rho_i^q, \tag{21}$$

subjected to the boundary conditions $J_{p_i}(\pm\infty,x_i,t) = J_{p_i}(p_i,\pm\infty,t) = 0$, which recovers the previous structure of the current probability density connected to Eqs. (2) and (3) for $q \to 1$. The component J_{x_i} is the same as the previous ones. In this scenario, we assume for the system the general entropic form

$$S = k\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dp_1\, dp_2\, dx_1\, dx_2 g(\rho_1\rho_2), \tag{22}$$

in which $g(\rho_{1,2})$ (with $\rho_{1,2} = \rho_1\rho_2$) is an arbitrary function, by taking into account the fact that the probability distribution related to independent processes is given by the product of them.

For this case, by taking the time derivative of the Helmholtz free energy, we obtain that

$$\frac{dF}{dt} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2 \times\left(\mathcal{H} - kT\frac{dg}{d\rho_{1,2}}\right)\frac{\partial}{\partial t}(\rho_1\rho_2). \tag{23}$$

Equation (23) can be written as

$$\frac{dF}{dt} = \Upsilon_1 + \Upsilon_2, \tag{24}$$

with

$$\Upsilon_1 = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\left(\mathcal{H}\rho_2 - kT\rho_2\frac{dg}{d\rho_{1,2}}\right)\frac{\partial\rho_1}{\partial t}$$
$$= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\left(\mathcal{H}\rho_2 - kT\rho_2\frac{dg}{d\rho_{1,2}}\right) \times\left\{-\frac{p_1}{m}\frac{\partial\rho_1}{\partial x_1} - \frac{\partial}{\partial p_1}\left[\left(-\frac{\gamma}{m}p_1 + \mu_1\right)\rho_1\right] + \frac{\Gamma}{2m^2}\mathcal{D}_1(t)\frac{\partial^2\rho_1^q}{\partial p_1^2}\right\}$$

$$= -\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_1\, dx_1 \frac{1}{\rho_1}\left\{\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_2 dx_2\left(\frac{p_1}{m}\rho_2\rho_1 - kT\rho_2^2\rho_1\frac{d^2g}{d\rho_{1,2}^2}\frac{\partial\rho_1}{\partial p_1}\right)\right]\right.$$
$$\left.\times\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_2\, dx_2\left(\frac{\gamma}{m}p_1\rho_1\rho_2 + \frac{\Gamma}{2m^2}\rho_2^q q\rho_1^{q-1}\frac{\partial\rho_1}{\partial p_1}\right)\right]\right\}$$
$$= -\frac{1}{\gamma}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\left(\frac{\rho_2}{\rho_1}\right)\bar{\mathcal{J}}_{1,q}^2, \tag{25}$$

and

$$\Upsilon_2 = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\left(\mathcal{H}\rho_1 - kT\rho_1\frac{dg}{d\rho_{1,2}}\right)$$
$$\times\left\{-\frac{p_2}{m}\frac{\partial}{\partial x_2}\rho_2 - \frac{\partial}{\partial p_2}\left[\left(-\frac{\gamma}{m}p_2 + \mu_2\right)\rho_2\right] + \frac{\Gamma}{2m^2}\mathcal{D}_2(t)\frac{\partial^2\rho_2^q}{\partial p_2^2}\right\},$$
$$= -\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_2\, dx_2 \frac{1}{\rho_2}\left\{\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_1\, dx_1\left(\frac{p_1}{m}\rho_2\rho_1 - kT\rho_1^2\rho_2\frac{d^2g}{d\rho_{1,2}^2}\frac{\partial\rho_2}{\partial p_2}\right)\right]\right.$$
$$\left.\times\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} dp_1\, dx_1\left(\frac{\gamma}{m}p_1\rho_1\rho_2 + \frac{\Gamma}{2m^2}\rho_1^q q\rho_2^{q-1}\frac{\partial\rho_2}{\partial p_2}\right)\right]\right\}$$
$$= -\frac{1}{\gamma}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\left(\frac{\rho_1}{\rho_2}\right)\bar{\mathcal{J}}_{2,q}^2, \tag{26}$$

where $\Gamma/(2m^2) = \gamma kT$. Equations (25) and (26) imply that $\Upsilon_1 \leqslant 0$ and $\Upsilon_2 \leqslant 0$ and, consequently, prove the *H* theorem, for a function $g(\rho_{1,2})$ which verifies the equation

$$\frac{d^2g}{d\rho_{1,2}^2} = -q\rho_{1,2}^{q-2}, \tag{27}$$

under the condition $g(0) = g(1) = 0$. By simple integrations, it is possible to show that the solution for this equation results in the Tsallis entropy,

$$S_q(\rho_{1,2}) = \frac{k}{q-1}\left(1 - \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{\infty} dp_1\, dp_2\, dx_1\, dx_2\rho_{1,2}^q\right), \tag{28}$$

for the system. By substituting $\rho_{1,2}$ in terms of ρ_1 and ρ_2, we have that

$$S_q^{(1\cup 2)} = S_q^1 + S_q^2 + (1-q)S_q^1 S_q^2/k, \tag{29}$$

implying that for $q \neq 1$ it is nonadditive. This result shows that by considering a suitable coupling between the nonlinear Klein-Kramers equations an entropic additivity law related to the thermodynamical equilibrium can be found. Thus, the Tsallis entropy [6] is verified as the appropriate entropy when the dynamics of the subsystems is governed by Eqs. (18) and (19) and the entropic nonadditivity appears as a consequence of the dynamics exhibited by the subsystems, enabling a thermodynamical equilibrium for a system composed of these subsystems as a single system. In this sense, it is interesting to mention that the zeroth law was introduced by considering the maximum principle entropy condition on the entropy of the system (composed of two subsystems) at the thermal state of equilibrium. A relation was found which should be satisfied by the subsystems in order to verify the zeroth law of thermodynamic [50–52]. We may establish a connection of the developments performed above with the results found in Ref. [50] by assuming that

$$\frac{\mathcal{D}_1(t)}{\mathcal{D}_2(t)} = \left(\frac{\mathcal{Z}_1}{\mathcal{Z}_2}\right)^{q-1}, \tag{30}$$

with $\mathcal{Z}_i^{1-q} = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dp_i \rho_i^q(x_i,p_i,t)$ and $\mathcal{D}_i(t)^{-1} = \beta_i(t)$.

The previous result obtained for two subsystems can be extended to a system composed of n subsystems. For this, we consider that the ith subsystem is governed by the equation

$$\frac{\partial\rho_i}{\partial t} + \frac{p_i}{m}\frac{\partial\rho_i}{\partial x_i} = -\frac{\partial}{\partial p_i}\left[\left(-\frac{\gamma}{m}p_i + \mu_i\right)\rho_i\right] + \mathcal{D}_i(t)\frac{\Gamma}{2m^2}\frac{\partial^2\rho_i^q}{\partial p_i^2}, \tag{31}$$

with

$$\mathcal{D}_i(t) = \int_{\forall j}\prod_{j=1}^n dp_j dx_j \rho_j^q, \tag{32}$$

and $j \neq i$, yielding the Tsallis for the system,

$$S_q(t) = \frac{k}{q-1}\left[1 - \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^n dp_i\, dx_i\rho_i^q(x_i,p_i,t)\right], \tag{33}$$

after applying the previous procedure. Equation (33) recovers, for $q \to 1$, the BG entropy

$$S_{BG}(t) = -k_B\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{i=1}^n dp_i\, dx_i\rho_i(x_i,p_i,t) \times \ln\rho_i(x_i,p_i,t), \tag{34}$$

emphasizing that both results, Eqs. (33) and (34), preserve the conditions required by a thermodynamical equilibrium.

IV. CONCLUSION

We have analyzed, by using the H theorem, the entropies and the entropic additivity laws which emerge from a system composed of subsystems, with the dynamics governed by linear and nonlinear Klein-Kramers equations. For the subsystems governed by a linear Klein-Kramers equations, the H theorem has shown that the suitable entropy to describe the behavior of the system is the Boltzmann-Gibbs entropy (which is additive) with $\rho_{1,2} = \rho_1 \rho_2$, i.e., the probability appears as a product of probabilities related to the subsystems, which is consistent with a thermodynamical equilibrium scenario. Then we have analyzed the case for which the subsystems are governed by uncoupled nonlinear Klein-Kramers equations. This case has led us to results which are not consistent with an equilibrium scenario, where the stationary distribution for the system is the product of the probabilities related to the subsystems systems. In order to overcome this situation, we have shown that a suitable description, by accomplishing an equilibrium scenario connected to a thermodynamical context, can be obtained by coupling these equations. In particular, we have considered that the equations are coupled by the diffusive term. This feature implies that the dynamics of one subsystem is influenced by the other until the equilibrium be reached. This scenario can be related to the Tsallis entropy, in which the nonadditive propriety appears as a consequence of the dynamics considered for the subsystems. Other couplings between the Klein-Kramers equations may lead us to a different scenarios, where different entropies may be obtained with different entropic additivities. Finally, we hope that the results can be useful in the analysis of the entropy and entropic additivity of systems composed of subsystems, when nonequilibrium processes are considered.

ACKNOWLEDGMENT

This work was partially supported by Brazilian agency CNPq [National Institute of Science and Technology of Complex Systems (INCT-SC/CNPq)].

[1] R. Clausius, *The Mechanical Theory of Heat: with Its Applications to the Steam-engine and to the Physical Properties of Bodies* (J. Van Voorst, London, 1867).
[2] O. Penrose, *Foundations of Statistical Mechanics: A Deductive Treatment* (Pergamon Press, Oxford, UK, 1970).
[3] S. G. Brush, *Kinetic Theory. Irreversible Processes* (Pergamon Press, Oxford, 1966).
[4] R. C. Tolman, *The Principles of Statistical Nechanics* (Oxford University Press, Oxford, 1938).
[5] J. W. Gibbs, *Elementary Principles in Statistical Mechanics* (Yale University Press, New Haven, 1902).
[6] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[7] D. Bagchi and C. Tsallis, Phys. Rev. E **93**, 062213 (2016).
[8] H. Christodoulidi, C. Tsallis, and T. Bountis, Europhys. Lett. **108**, 40006 (2014).
[9] R. Singh, R. Moessner, and D. Roy, Phys. Rev. B **95**, 094205 (2017).
[10] B. Neyenhuis, J. Smith, A. C. Lee, J. Zhang, P. Richerme, P. W. Hess, Z. X. Gong, A. V. Gorshkov, and C. Monroe, Sci. Adv. **3**, e1700672 (2017).
[11] M. P. de Albuquerque, A. Esquef, R. G. Mello, and M. P. de Albuquerque, Pattern Recognition Lett. **25**, 1059 (2004).
[12] M. Gell-Mann and C. Tsallis, *Nonextensive Entropy: Interdisciplinary Applications* (Oxford University Press, New York, 2004).
[13] C. Tsallis and L. J. L. Cirto, Eur. Phys. J. C **73**, 2487 (2013).
[14] R. F. Aranha, I. D. Soares, and E. V. Tonini, Phys. Rev. D **81**, 104005 (2010).
[15] A. Saguia and M. S. Sarandy, Phys. Lett. A **374**, 3384 (2010).
[16] U. Tirnakli and E. P. Borges, Sci. Rep. **6**, 23644 (2016).
[17] T. D. Frank, *Nonlinear Fokker-Planck Equations: Fundamentals and Applications* (Springer, Berlin, 2005).
[18] E. K. Lenzi, L. C. Malacarne, R. S. Mendes, and I. T. Pedron, Physica A **319**, 245 (2003).
[19] E. K. Lenzi, M. A. F. dos Santos, F. S. Michels, R. S. Mendes, and L. R. Evangelista, Entropy **15**, 3931 (2013).
[20] V. Schwämmle, F. D. Nobre, and C. Tsallis, Eur. Phys. J. B **66**, 537 (2008).
[21] F. D. Nobre, M. A. Rego-Monteiro, and C. Tsallis, Phys. Rev. Lett. **106**, 140601 (2011).
[22] J. R. Choi, Sci. Rep. **4**, 3911 (2014).
[23] C. Tsallis, Entropy **13**, 1765 (2011).
[24] C. Tsallis, Braz. J. Phys. **39**, 337 (2009).
[25] J. Du, Phys. Lett. A **329**, 262 (2004).
[26] P. Grigolini, C. Tsallis, and B. West, Chaos, Solitons Fractals **13**, 367 (2002).
[27] L. G. Moyano, C. Tsallis, and M. Gell-Mann, Europhys. Lett. **73**, 813 (2006).
[28] S. Umarov, C. Tsallis, and S. Steinberg, Milan J. Math. **76**, 307 (2008).
[29] H. R. Brown, W. Myrvold, and J. Uffink, Studies History Phil. Sci. B **40**, 174 (2009).
[30] J. A. S. Lima, R. Silva, and A. R. Plastino, Phys. Rev. Lett. **86**, 2938 (2001).
[31] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, Berlin, 1996).
[32] V. Schwämmle, E. M. F. Curado, and F. D. Nobre, Eur. Phys. J. B **58**, 159 (2007).
[33] M. S. Ribeiro, G. A. Casas, and F. D. Nobre, J. Phys. A **50**, 065001 (2017).
[34] M. A. F. dos Santos, E. K. Lenzi, and M. K. Lenzi, Chin. J. Phys. **55**, 1294 (2017).
[35] G. Sicuro, P. Rapcan, and C. Tsallis, Phys. Rev. E **94**, 062117 (2016).
[36] V. Schwämmle, F. D. Nobre, and E. M. F. Curado, Phys. Rev. E **76**, 041123 (2007).
[37] G. A. Casas, F. D. Nobre, and E. M. F. Curado, Phys. Rev. E **86**, 061136 (2012).
[38] G. Kaniadakis, Phys. Lett. A **288**, 283 (2001).
[39] P. H. Chavanis, Eur. Phys. J. B **62**, 179 (2008).
[40] P. H. Chavanis, Phys. Rev. E **68**, 036108 (2003).
[41] G. A. Mendes, M. S. Ribeiro, R. S. Mendes, E. K. Lenzi, and F. D. Nobre, Phys. Rev. E **91**, 052106 (2015).

[42] A. Rényi, in *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* (University of California Press, Berkeley, 1961).
[43] F. Clementi, M. Gallegati, and G. Kaniadakis, Eur. Phys. J. B **57**, 187 (2007).
[44] G. Kaniadakis, Physica A **296**, 405 (2001).
[45] G. Kaniadakis, Phys. Rev. E **66**, 056125 (2002).
[46] J. L. Lebowitz and P. G. Bergmann, Ann. Phys. **1**, 1 (1957).
[47] P. G. Bergmann and J. L. Lebowitz, Phys. Rev. **99**, 578 (1955).
[48] P. Penfield, Jr., *Principle of Maximum Entropy* (Massachusetts Institute of Technology, 2003).
[49] J. Shore and R. Johnson, IEEE Trans. Inf. Theory **26**, 26 (1980).
[50] S. Martínez, F. Pennini, and A. Plastino, Physica A **295**, 416 (2001).
[51] T. S. Biró and P. Ván, Phys. Rev. E **83**, 061147 (2011).
[52] Q. A. Wang and A. Le Méhauté, Chaos Solitons Fractals **15**, 537 (2003).