Energy decay in a tapped granular column: Can a one-dimensional toy model provide insight into fully three-dimensional systems?

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The decay of energy within particulate media subjected to an impulse is an issue of significant scientific interest, but also one with numerous important practical applications. In this paper, we study the dynamics of a granular system exposed to energetic impulses in the form of discrete taps from a solid surface. By considering a one-dimensional toy system, we develop a simple theory, which successfully describes the energy decay within the system following exposure to an impulse. We then extend this theory so as to make it applicable also to more realistic, three-dimensional granular systems, assessing the validity of the model through direct comparison with discrete particle method simulations. The theoretical form presented possesses several notable consequences; in particular, it is demonstrated that for suitably large systems, effects due to the bounding walls may be entirely neglected. We also establish the existence of a threshold system size above which a granular bed may be considered fully three dimensional.

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I. INTRODUCTION

Granular and particulate media are ubiquitous both in industry and our everyday lives, representing—aside from water—the most widely handled commodity on Earth [1]. Despite this fact, the dynamical behaviors of these materials remain incompletely understood [1–3] and thus, in many cases, impossible to accurately predict. This unpredictability carries many negative consequences, both in nature and in numerous diverse industries [4–7].

While we are currently witnessing a continual improvement in the speed and power of simulational techniques for modeling granular systems [8] as well as in the capabilities of experimental techniques, we are still a long way from being able to reliably analyze the behaviors of large, three-dimensional systems with precision. However, within the limitations of contemporary technology, we can successfully model small, often one-dimensional or quasi-two-dimensional toy systems to a high degree of precision, producing simulations and models that accurately and quantitatively represent observations from similar experimental systems [9–11].

Nonetheless, such systems are typically beset by a recurring question: can the behaviors of these simple models provide gainful insight into the behavior of more complex, larger, and/or fully three-dimensional systems more relevant to the majority of real-world processes?

In this paper, we consider a very simple toy model of a granular system: a one-dimensional column of dissipative particles subjected to a single discrete excitation event, or tap. The use of this system ensures simplicity not only in terms of geometry, but also in terms of the manner of excitation.

Tapped granular systems are of direct relevance to a number of contemporary applications, such as the creation of granular dampers used, for example, in the construction, aerospace and even medical sectors [12–15], and the compaction of particulate media [16]. Perhaps more importantly, however, these systems provide fundamental insight into the behaviors of excited granular media as a whole, in particular vibrated and vibrofluidized beds.

In the present work, we focus specifically on the dissipation of energy within the tapped granular systems studied. It is the dissipative nature of granular materials that predominantly separates them from classical solids, liquids, and gases, of which our understanding is far superior. It is hoped that by gaining an improved, predictive knowledge of said energy dissipation within the fundamental canonical systems studied here, we may take a vital early step towards an improved understanding of excited granular media as a whole.

Due to their value as a means through which to gain insight into the fundamental physics of particulate systems, one-dimensional granular columns have been widely studied. Falcon *et al.* studied the collision of a granular column with a static wall [9]. The paper focused predominantly on the force exerted upon the wall by the falling column, making the surprising observation that the maximal force experienced by the wall remained constant irrespective of the number of particles forming the column. However, more relevantly to the current work, Falcon *et al.* also provided insight into the dissipative behaviors of the column, which we touch upon in subsequent sections. Louge [17] utilized a simple granular column, impacted from above by a single particle, in order to better understand the invariance of particle packing fraction with depth in granular flows along inclined planes, which he hypothesized could be considered as a series of individual columns. Other authors have, in place of experimental systems and discrete particle simulations, utilized one-dimensional lattice models to explore the fundamental behaviors of various important processes within granular systems. The work of Brey

et al. [18], for example, uses such a model to simulate the compaction process of a repeatedly tapped granular system. The authors find that their simple one-dimensional model successfully captures the well-known (inverse) logarithmic relaxation of the system as observed in experiments. More recently, Lasanta *et al.* [19] and Manacorda *et al.* [20] explored a greatly simplified model in which not only is the system fully one dimensional, but particles are also constrained to fixed positions, able only to exchange momentum (which is conserved) and energy (which is not), i.e., there exists no mass transport within the system. Amazingly, in spite of the simplicity of this system, it is nonetheless able to recreate key features of a granular fluid, including the temporal evolution of the system's (granular) temperature, which forms the focus of our current work.

In this paper, we construct an elementary model capable of characterizing the energy decay within a granular column subsequent to its exposure to a discrete excitation. We show that, from this model, we may indeed gain valuable information regarding the behaviors of more complex, larger systems.

Specifically, we show that—as the dimensionality of our system is increased from one-dimensional, through quasi-two-dimensional to fully three-dimensional—the energy decay observed continues to adhere to certain seemingly universal trends. As such, our results suggest that although theories developed using such extremely simplified models may not be directly applicable to more complex systems, they can nonetheless elucidate important general relations applicable also to these cases. Thus, through the addition of further theoretical and/or empirical arguments, the fundamental models developed can be adapted and extended to accurately characterize systems, which, while still simplified, more closely represent those observed in the real world.

II. SIMULATIONS

A. Simulation details

The systems explored in this paper are simulated using the MERCURYDPM discrete particle method software package [21–24]. The simulations utilize a frictional spring-dashpot model with a linear elastic and a linear dissipative contribution to model the forces acting between particles. The normal (f_{ij}^n) and tangential (f_{ij}^t) forces acting between colliding particles are determined, respectively, as [25,26]:

$$f_{ij}^n = k^n \delta_{ij}^n \hat{\mathbf{n}}_{ij} - \zeta^n \mathbf{v}_{ij}^n \tag{1}$$

and

$$f_{ij}^t = -k^t \delta_{ij}^t - \zeta^t \mathbf{v}_{ij}^t. \tag{2}$$

In the above, $\mathbf{v}_{ij}^n$ and $\mathbf{v}_{ij}^t$ are, respectively, the normal and tangential components of the relative velocity between interacting particles and δ_{ij}^t is the elastic tangential displacement (described in detail in Ref. [27]). The relevant normal spring constants, k^n, and damping constants, ζ^n, are calculated, respectively, as:

$$k^n = m_{ij}\left[\left(\frac{\pi}{t_c}\right)^2 - \left(\frac{\log \varepsilon}{t_c}\right)^2\right] \tag{3}$$

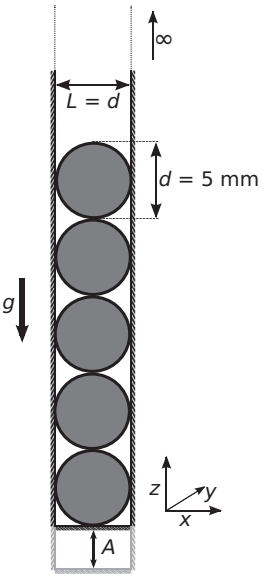

FIG. 1. Schematic diagram of the simple columnar granular system modeled in simulation. Here, g indicates the direction of gravitational acceleration, d the diameter of the particles modeled, L the separation of the walls constraining particles, and A the amplitude of the half-sinusoid motion of the system's base, which excites the granular assembly.

and

$$\zeta^n = -2m_{ij}\left(\frac{\log \varepsilon}{t_c}\right) \tag{4}$$

based on a user-defined restitution coefficient, ε, and contact time, t_c [28]. The contact time is, for the results shown here, taken as $t_c = 1^{-5}$, a value previously shown to be suitable for systems similar to our own [29,30]. Test simulations were conducted using values of t_c an order of magnitude lower and higher than this value; the consistency in the observed results supports the suitability of the value implemented. The variable m_{ij} corresponds to the reduced mass of two colliding particles, i and j, determined as $m_{ij} = \frac{m_i m_j}{m_i + m_j}$. The tangential spring and damping constants are taken, respectively, as $k^t = \frac{2}{7}k^n$ and $\zeta^t = \zeta^n$.

In order to satisfy the inequality $f_{ij}^t \leqslant \mu f_{ij}^n$ (where μ represents the relevant frictional coefficient) as required by the Coulomb model of friction, a static yield criterion is applied so as to truncate the magnitude of δ_{ij}^t at the limiting value. All particles within the system are subject to a gravitational acceleration, g, which acts downwards in the negative z direction, as shown in Fig. 1, where the relevant coordinate system is defined.

We integrate the resulting force relations [Eqs. (1), (2)] in time using a velocity-Verlet time-stepping algorithm [31] with a step size $\delta t = \frac{t_c}{50}$ to model the evolution of the velocities and positions of the particles within the system.

For the interested reader, further details regarding the implementation of the discrete particle method in MERCURYDPM may be found in the appendixes of our Ref. [29], while a more detailed description of the method in general can be seen in Ref. [32].

B. Toy system

A granular system can be defined fundamentally as a conglomeration of discrete, macroscopic particles [1]. We attempt here to assemble the simplest possible representation of a granular system, namely a single one-dimensional column of particles (see Fig. 1).

The basic simulated system comprises a vertical column of $N \in [1,20]$ particles[1] each of identical diameter, $d = 5$ mm, and density, $\rho = 7850$ kgm^{-3}. The system is bounded by four frictionless vertical sidewalls, each opposing pair separated by a distance $L = d$, thus acting as a guide to the column of particles and hence maintaining their columnar structure during excitation. For simplicity and consistency, in the results presented here, interactions between particles are, unless otherwise stated, modeled as frictionless. It is worth noting, however, that preliminary tests using values of the interparticle friction coefficient μ in the range $[0,1.0]$ demonstrated no significant influence of this parameter on the system's behavior, due to the predominantly collisional nature of the interactions between particles. Although the known experimental value of the friction coefficient for steel-steel contacts ($\mu \sim 0.1$ [33]) lies toward the lower limit of this domain, it is nonetheless valuable to demonstrate that our model can also accommodate particles possessing a range of differing frictional properties. The vertical sidewalls of the system are modeled as perfectly elastic, such that dissipation within the system can be attributed solely to interparticle collisions. The system is not subject to an upper constraint in the vertical dimension, meaning that particle motion is not limited in the positive z direction.

The parameters detailed above are chosen due to the fact that prior investigations have shown these values to produce quantitative agreement with data acquired from experimental systems [11,34], ensuring the reliability and real-world relevance of the results presented here.

Due to the inherently dissipative nature of granular materials, they will remain perfectly motionless in a solidlike state unless exposed to some form of external excitation. Continuing in the vein of ensuring simplicity, we choose to excite our system via a single discrete tap in the vertical direction, thus avoiding any complex resonance effects or hysteretic behaviors that may be present in systems exposed to continuous vibration [34–36], or indeed repeated discrete excitations [37,38]. The form of the tap applied is a half-sinusoid of period f and peak amplitude A. For the results presented here, unless otherwise stated, the tap frequency is held fixed at a value $f = 30$ Hz. The amplitude of the tap is varied in the range $\frac{A}{d} \in [0.2,10]$ (where d is the particle diameter), thus allowing a range of excitation strengths to be explored. Before the excitation is applied, the system is initially allowed to fully relax under gravity, ensuring a consistent and realistic starting point for each system tested. A schematic representation of the system may be seen in Fig. 1.

[1]Trial simulations were conducted under various conditions for N up to 100, however, for the systems explored here, a maximum number of particles $N = 20$ was sufficient to capture all information of interest.

III. CONSTRUCTING A SIMPLE MODEL

The decay of energy within granular systems has long been a subject of interest, due both to the scientific interest of the topic as well as its relevance to various important real-world applications [39–42]. However, few studies have systematically studied the specific roles of crucial variables such as the number of particles, N, within the system or the elastic properties of particles on the rate of dissipation. Moreover, we provide a cogent, predictive model capable of relating this energy decay to N for systems. In this section, we attempt to formulate a simple model capable of quantitatively predicting the energy decay of a tapped granular column based on the main variables, which dominate the behavior of such a system: the number and material properties of the particles in the system, and the strength with which it is driven. The decay of energy within such a system is, as established by Rosato *et al.* [11], expected to follow the general form:

$$\tau_r = \frac{\alpha}{N^\beta} + \gamma. \tag{5}$$

We consider first the limiting case $N \to \infty$. In this situation, Eq. (5) will reduce simply to:

$$\tau_r^\infty = \gamma, \tag{6}$$

where, for our toy system, we assume α to be a function of particle elasticity and driving strength [$\alpha = \alpha(\varepsilon, f, A)$], the γ term to depend also on the driving parameters [$\gamma = \gamma(f, A)$] and the exponent β to be a constant. As in the original work, τ_r is defined as the time required for the energy, E, possessed by a granular column exposed to a single tap to decay to 10^{-5} of its original value.

In the limit $N \to \infty$, our system of particles may be expected to behave simply as a single, perfectly inelastic body [43–45]. As such we may assume that, in this case, the bed will simply be launched with an initial velocity equal to the peak velocity of the oscillating base, $v_{\text{peak}} = 2\pi f A$ and, upon recolliding with the base, instantaneously dissipate all kinetic energy. In other words, the constant γ may be expected to simply equal the free-flight time of our infinitely large bed, i.e.,

$$\gamma = \frac{4\pi f A}{g} \tag{7}$$

with $g = 9.81$ ms^{-2} the gravitational acceleration.

In order to determine the coefficient α, we consider next the case $N = 1$, for which Eq. (5) can be rewritten as:

$$\alpha = \tau_r^1 - \gamma. \tag{8}$$

Unlike the complex multibody problems encountered for large N, in the case $N = 1$, the decay time can—for a given ε—be determined analytically.

As discussed above, upon its exposure to a tap, a single particle can be assumed to travel with an initial velocity $v_0 = v_{\text{peak}} = 2\pi f A$. The particle will then experience a period of free flight $\Delta t_0 = \frac{2}{g} v_0$, before recolliding with the base. Upon recollision, the particle can be assumed to lose a fraction of its energy corresponding to its elastic coefficient, i.e., after this interaction $v_1 = \varepsilon v_0$. The particle will then proceed to enter free flight again, this time for a duration $\Delta t_1 = \frac{2}{g} v_1 = \frac{2}{g} \varepsilon v_0$. This energy-dissipating process will repeat as the particle

bounces on the plate, meaning that the duration of the free-flight period following the ith collision can be determined as:

$$\Delta t_i = \frac{2}{g}\varepsilon^{i-1}v_0, \quad i = 0,1,2,\ldots. \tag{9}$$

The total time interval, T_n, between the initiation of a tap and the nth collision with the base can therefore be given as:

$$T_n = \sum_i^n \Delta t_i. \tag{10}$$

As noted above, in keeping with Ref. [11], the rest time t_r is defined as the duration, starting from the initiation of a tap, required for a particle's energy to decay to a value $E_r = 10^{-5}E_0$, where E_0 denotes the kinetic energy of the initial tap. Thus, in simulations, the entire time interval was reported, meaning that fits to the simulation data reported in [11] included a constant term γ. As also mentioned previously, a single bouncing particle can be expected to lose a fixed fraction of its energy upon each collision. Specifically, a particle's rebound velocity after its kth collision can be determined as $|v_k| = \varepsilon^k v_0$. Since kinetic energy is proportional to the square of velocity, the rebound energy, E_k, of a particle immediately after experiencing its kth collision can be written as:

$$E_k = E_0\varepsilon^{2k}, \quad k = 1,2,3,\ldots. \tag{11}$$

Under this assumption, the number, n_r, of collisions required for our system to come to rest can be determined as:

$$n_r = \frac{\ln(E_r) - \ln(E_0)}{\ln(\varepsilon)}, \tag{12}$$

i.e., $n_r = \frac{10^{-5}}{\ln(\varepsilon)}$. As such, the rest time for a single particle can be approximated as:

$$\tau_r^1 = \sum_i^{n_r} \Delta t_i \tag{13}$$

meaning that

$$\alpha = \tau_r^1 = \sum_i^{n_r} \Delta t_i - \gamma. \tag{14}$$

Noting that the zeroth term of the series is canceled by the coefficient γ, by combining Eqs. (9) and (14) and recalling that $v_0 = 2\pi f A$, we may express the coefficient α as:

$$\alpha = \frac{4\pi f A}{g}\sum_i^{n_r}\varepsilon^i = \frac{4\pi f A}{g}\frac{\varepsilon(1-\varepsilon^{n_r})}{1-\varepsilon}. \tag{15}$$

Since one of the aims of this paper is to demonstrate the potential value and effectiveness of even highly simple toy models, we determine our final coefficient through the highly simplified assumption that doubling the number of particles within the system will on average, for a given system energy, double the collision rate within the column and, hence, double also the dissipation rate. As such, the constant β can be simply taken as unity. This chosen value also agrees well with the experimental findings of Falcon *et al.* [9], who demonstrated empirically that a granular column impacting on a static wall possessed an effective restitution coefficient whose magnitude carried an approximate $\frac{1}{N}$ dependence.

Interestingly, the same simple $\frac{1}{N}$ scaling for the stopping time, τ_r, assumed above can also be obtained via a more rigorous theoretical approach. The approach we describe utilizes an approximate integropartial differential continuum model for particulate media known as the BSR model [46]. For tapped columns such as those explored here, this model can be expressed in the following form [46,47]:

$$\begin{aligned} u_t + uu_z &= -g + F(z,u) \\ &:= -g + \int_{B_r(z)} \rho(\eta,t)\Theta(z,\eta,u(z,t),u(\eta,t))d\eta \\ \rho_t + u\rho_z &= -\rho u_y. \end{aligned} \tag{16}$$

In the above, the former and latter equations represent, respectively, the momentum and continuity equations. In these equations, $u := \frac{dz}{dt}$ is the velocity in the vertical direction, g the gravitational acceleration, ρ represents density (with η the corresponding packing density), and F the material interaction force per unit mass. In the present paper, F is expressed in terms of an integral localized to a ball of radius r. Finally, Θ represents the interaction force kernel determined by the particular model chosen. By multiplying the momentum equation given in (16) by the vertical velocity, u, we obtain:

$$\frac{d}{dt}\left(\frac{1}{2}u^2 + gz\right) = \frac{dE_m}{dt} = uF, \tag{17}$$

which provides an expression for the rate of decrease of energy of a given particle as it moves subsequent to an applied tap. As the continuum interval can be considered as being partitioned into N particle intervals, it follows that the total energy of the system decays at a rate N times that of a single particle. In other words, one can expect the decay time for a column of N particles to be $\frac{1}{N}$ times that of a single particle – in agreement with our original, inductive argument.

Finally, before directly testing our model against simulation data, it is worthwhile to first ensure that the predictions of our model in the relevant limiting cases are not unphysical. First, we find that the model is dimensionally correct, as both α and β possess units of time, while $\frac{1}{N}$ is, naturally, dimensionless. Second, in the limit $\varepsilon \to 0$ (i.e., the case of completely inelastic particles), $\tau_r \to \gamma$. In other words, after a period of free flight of duration $\frac{2\pi f A}{g}$ the system will dissipate all energy immediately upon recollision with the base, as expected. Finally, in the limit $\varepsilon \to 1$, where particles are perfectly elastic, $\tau_r \to \infty$, i.e., the system will never decay, simply maintaining a constant energy for all t.

IV. RESULTS

A. Testing the model

Having established that the behavior predicted by our model remains physical in the extremal limits of N and ε, the next step is to test its validity against our simulated data sets. Figure 2 shows the rest time, τ_r (as defined in Sec. III) as a function of the number of particles within the one-dimensional simulated system described in Sec. II B. Data is shown for a range of particle elasticities $\epsilon \in [0.88,0.99]$ and for driving amplitudes

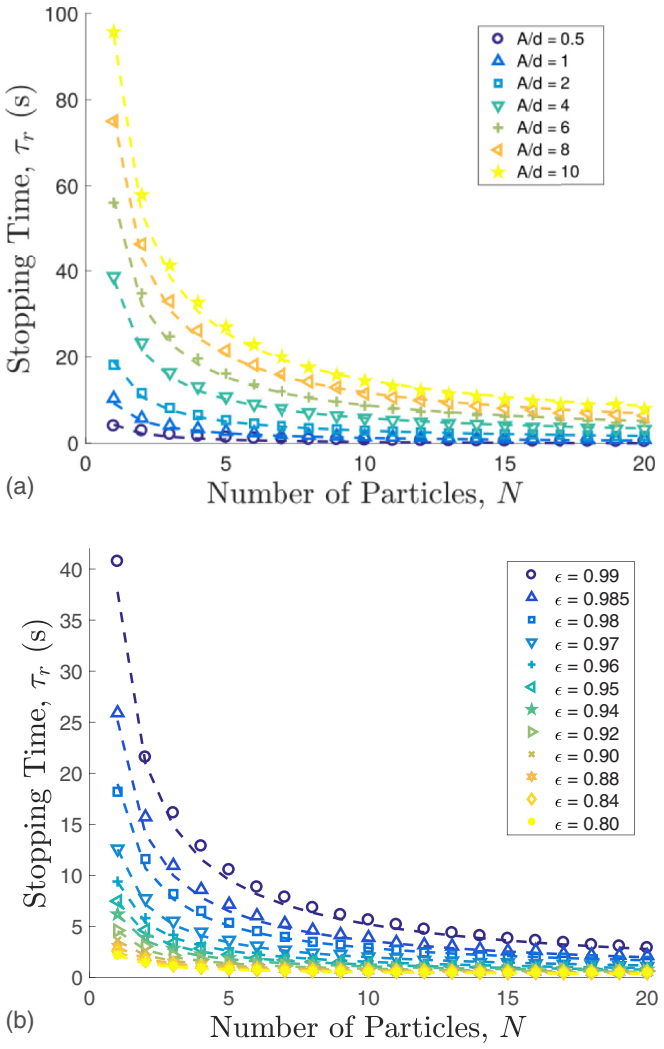

FIG. 2. The rest time, τ_r (defined in Sec. III), of a one-dimensional column of particles as a function of N, the number of particles by which the column is formed. In (a), data is shown for various driving amplitudes, A, at fixed particle elasticity $\varepsilon = 0.98$. In (b), data is shown for fixed driving amplitude $A/d = 2$ for varying values of ϵ. In both cases, all other system parameters are held constant. In all cases, open symbols correspond to simulated data sets, while the correspondingly-coloured dashed lines represent the appropriate theoretical curves corresponding to the model outlined in Sec. III.

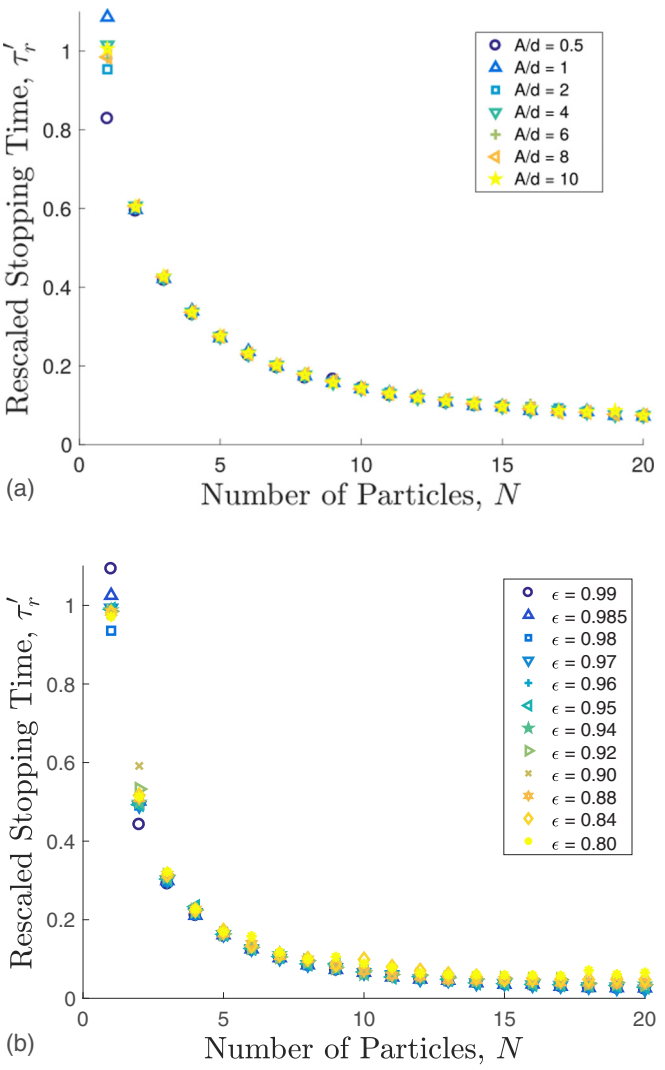

FIG. 3. Simulated stopping time vs particle number data from Fig. 2 rescaled as $\tau_r' = (\tau_r - \gamma)/\alpha$.

varied over two decades. It is first valuable to note that the functional form exhibited by our simulations agrees well with prior experimental observations and numerical results acquired using differing force models [11], lending strong support to the validity of our simulations.

In Fig. 3, we show the same data rescaled by the parameters α and γ derived in Sec. III. The collapse of the data points onto a single curve is, considering the highly simplistic nature of the model, remarkable. Nonetheless, there exist some notable deviations from the master curve, in particular for the case of varying ε.

In Fig. 4, we replot our data once more, this time additionally rescaled by an empirically derived exponent, β. Specifically, our value of β is acquired through a nonlinear least-squares analysis of all simulated data sets. Taking the average of the resultant β values obtained from all data sets, we achieve a value $\beta = 0.84 \pm 0.01$, as opposed to the value of unity originally implemented. While this scaling performs slightly better than the previous at large N, the improvement is not significant—as can be seen form a comparison of Figs. 3 and 4, i.e., our theoretical β value of unity is more than adequate to describe the form of the observed relations. Even with the empirical scaling, however, there remains a degree of inaccuracy for systems comprising only one or two particles. This inaccuracy likely arises from the assumption that, upon being subjected to a tap, a particle will achieve a velocity equal to the peak velocity, $v_{\text{peak}} = \frac{2\pi f}{g}$, of the base. In reality, the velocity at detaching of a given particle may vary depending on the specific driving details and elastic properties of the system in question [48–50]. Nonetheless, significant deviation is observed only for very small (one- and two-particle) systems, which are, to the community at large, of comparatively little interest. In fact, it has previously been shown in experiments and simulations [34] that, for larger systems, the assumption $v_{\text{peak}} = \frac{2\pi f}{g}$ does indeed hold to a reasonable accuracy.

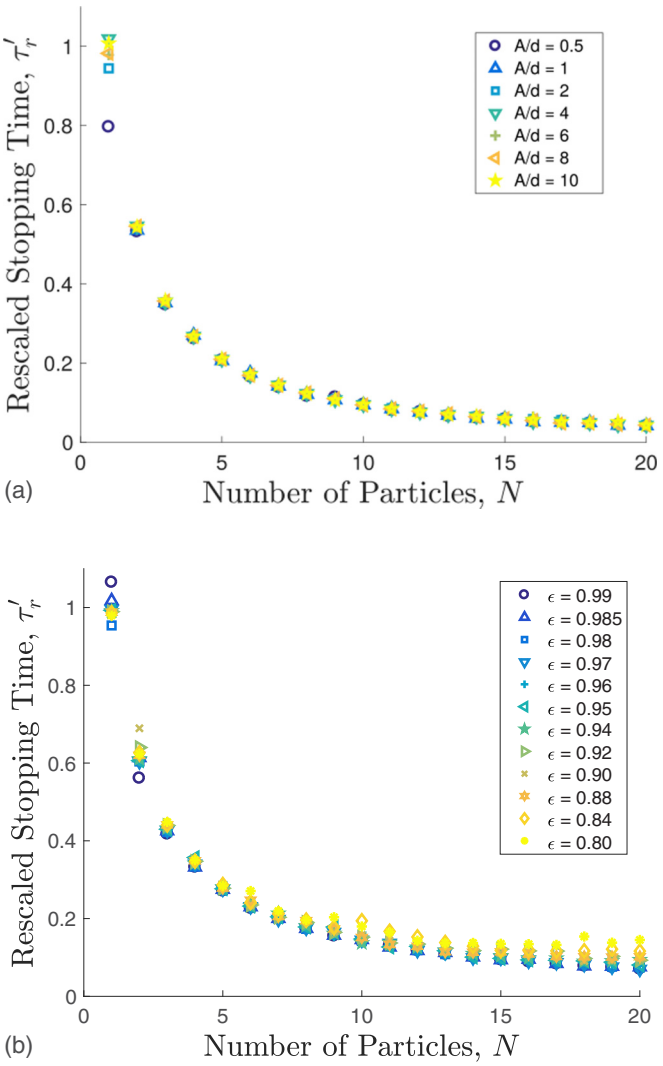

FIG. 4. Simulated stopping time vs particle number data from Fig. 2 rescaled as $[(\tau_r - \gamma)/\alpha]^{\frac{1}{\beta}}$.

So far, we have demonstrated that our very simple model can successfully capture the behaviors of a one-dimensional column of particles exposed to a single, discrete excitation. This in itself is perhaps surprising, considering the complex nonlinear nature of such systems. Nonetheless, in the following section we ask a further question: can such a model provide insight also into larger and increasingly complex systems more typical of those encountered in the real world?

B. Extending to higher dimensions

As we attempt to extend our one-dimensional model to the three-dimensional case, we must consider the increase in the degrees of translational freedom possessed by the system and, with this, the inherent introduction of additional energy-dissipating collisions in the horizontal directions.

For simplicity, we consider our three-dimensional system as a number $N_c = (L_1/d) \times (L_2/d)$ of individual, adjacent, one-dimensional columns, each of height $N_L = \frac{N}{N_c}$, where L_1 and L_2 are, respectively, the width and depth of the cuboidal containers explored and d is the particle diameter. Each of these columns, in isolation, can be expected to obey Eq. (5) in its original, one-dimensional form. For our three-dimensional systems, therefore, we can very simply rewrite our original equation as:

$$\tau_r = \frac{\alpha}{N_L^{\beta}} + \gamma. \tag{18}$$

Since, in the limits $N_L = 1$ and $N_L \to \infty$, the system's behavior can be expected to remain broadly similar to the same limits in the one-dimensional case, the forms of the coefficients α and γ should still hold in the three-dimensional case. Moreover, maintaining our treatment of the system as a series of columns we can still, as before, assume that doubling the number of particles within the system will double the mean collision rate and, hence, halve the decay time. In other words, a simple $\frac{1}{N}$ scaling should still represent a reasonable first approximation. Of course, in reality, the system will not exist as a set of distinct, noninteracting columns; rather, as mentioned above, each notional column will interchange particles with other columns and, more importantly, experience also collisions from adjacent particles.

Assuming motion within the system in its excited state to be random, we can make a further assumption that particles will, on average, experience an equal number of collisions in each of the three spatial dimensions now available to them. The feasibility of this assumption is also borne out by our simulations, which show the systems explored to rapidly achieve an effectively isotropic temperature state. In Fig. 5, we show an example of the decay of the ratio of the horizontal and vertical components of the granular

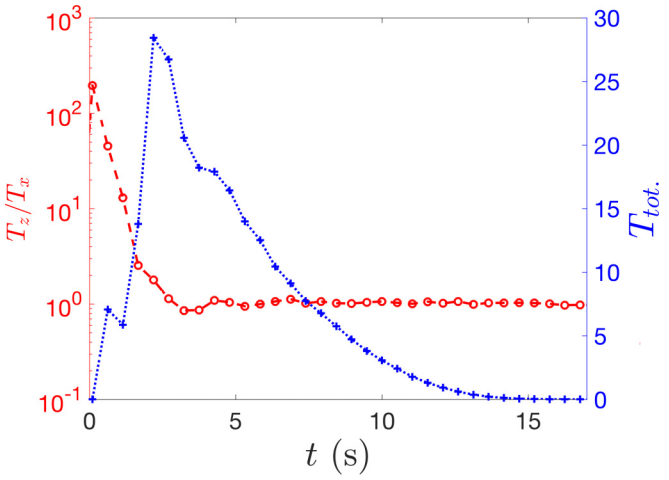

FIG. 5. Image showing the typical variation in time of the temperature anisotropy, $\frac{T_z}{T_x}$, (red circles and dashed line) and the total system temperature, $T_{tot.}$, for a three-dimensional system of particles exposed to a single tap. The data shown correspond to a simulated system of particles with $\varepsilon = 0.99$ housed within a system of base dimension $L_1/d \times L_2/d = 20 \times 20$ and exposed to a tap of amplitude $A/d = 4$. Here, $t = 0$ represents the point at which the initial excitation is applied to the system.

temperature [2], T, with time for a three-dimensional system exposed to a single discrete tap. Note that we choose here to analyze the temperature, as opposed to energy anisotropy as it is T, i.e., the fluctuant kinetic energy, which will act to determine the interparticle collision rate in which our interest lies. To put it differently, a system with an extremely high kinetic energy may, if all particles exhibit strongly correlated motion, experience very few particle collisions, whereas a system with a high temperature will inherently exhibit a proportionally high collision rate. In the example shown in Fig. 5, the initial free flight of the system lasts approximately 2.5 seconds, during which time the bed may be considered only as a solid body and energy dissipation largely neglected. From Fig. 5, we see that subsequent to this initial free flight the granular bed rapidly achieves temperature isotropy. Since, as mentioned above, the collision rate of particles within the system is largely determined by T, the assumption that particles will receive equivalent numbers of collisions from all directions seems reasonable. This rapid acquisition of a near-isotropic state is unsurprising, as the system is only instantaneously exposed to any form of directional excitation, and it is well known that granular systems will, following such a monodirectional excitation, rapidly redistribute the energy acquired to the remaining spatial dimensions [51–53].

Based on the above assumption, we can therefore expect that a given particle in a three-dimensional system will, per unit time, be subject to a threefold increase in the number of collisions experienced as compared to an equivalent, one-dimensional system.

This simple inductive argument is once again supported by a comparison with the more rigorous mathematics of the BSR model. In three dimensions, the model can be written as:

$$\begin{aligned} \boldsymbol{u}_t + \langle \boldsymbol{u}, \nabla \rangle \boldsymbol{u} &= -g\boldsymbol{\epsilon}_v + \boldsymbol{F} \\ &:= -g\boldsymbol{\epsilon}_v + \int_{B_r(x)} \rho(z,t)\boldsymbol{\Theta} \\ &\quad \times [\boldsymbol{x}, z, \boldsymbol{u}(\boldsymbol{x},t), \boldsymbol{u}(z,t)] dz. \end{aligned} \tag{19}$$

Here, $\boldsymbol{x}, \boldsymbol{y}$ represent position vectors and $\boldsymbol{u} := \dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}}{dt}$ denotes a velocity vector. $\boldsymbol{\epsilon}_v$ is the unit vertical coordinate vector and $\langle \cdot, \cdot \rangle$ is the standard inner product in the particular Euclidean space. For further information regarding this equation and its origins and its individual terms, the reader may refer to our Refs. [46,47]. Taking the inner product of the momentum equation given in (19) we obtain:

$$\frac{dE_m}{dt} = \frac{d}{dt}\left(\frac{1}{2}|\boldsymbol{u}|^2 + gz\right) = \langle \boldsymbol{F}, \boldsymbol{u} \rangle. \tag{20}$$

Following a similar set of arguments to those provided in our discussion of the one-dimensional BSR model in Sec. III, the formula given in Eq. (20) can be interpreted as the decrease in energy per unit mass rate per particle. Hence, if we are dealing with a k-dimensional system, the number of particles is $O(N^k)$, meaning that the collision rate will scale accordingly, it naturally follows that the energy loss within the system can be expected to show a similar dependence on the dimensionality of the system.

As is clear from the discourse presented in Sec. III, the increased collision rate discussed above must be accounted for in the dissipation term of our equation. Specifically, we implement the increased energy loss due to the higher dimensionality of the system by replacing each ε term in our original formulation with ε^3. Thus, our final modified equation can be written, in full, as:

$$\tau_r = \frac{4\pi f A}{g} \frac{\varepsilon^{\mathcal{D}}(1-\varepsilon^{\mathcal{D}n_r})}{1-\varepsilon^{\mathcal{D}}} \frac{L_1 L_2}{d^2 N} + \frac{2\pi f A}{g}, \tag{21}$$

where $\mathcal{D}$ represents the dimensionality of the system. Note that, for the case of a single column ($\frac{L_1 L_2}{d} = 1, \mathcal{D} = 1$), Eq. (21) reduces to its one-dimensional form, as one would expect.

In order to test the validity of our assumptions, we extend also the simulations described in Sec. II B to the three-dimensional case. The three-dimensional systems remain largely identical to their one-dimensional counterparts described previously. The major difference is that now the horizontal spatial dimensions, L_1 and L_2, may now be altered and, as such, the particles within the system are no longer constrained (as in the one-dimensional columnar case) to a fixed point in the horizontal directions but may travel freely throughout the three-dimensional system. To further increase the realism of the system, and its relevance to real-world granular beds where ordered packings are rarely achievable in practice [54–56], the initial packing of the particles is randomized. Each three-dimensional stopping time presented is averaged over five repeated data sets, each with a differing random initial packing. It is interesting, and perhaps valuable, to note that the initial packing of the system seemingly has only a highly limited influence on the decay time, as there typically exists a less than 2% variation across the τ_r values acquired from repeated data sets with unequal initial packings.

Figure 6 shows the rescaled stopping time $\frac{\tau_r - \gamma}{\alpha}$ plotted as a function of the rescaled particle number, $N_L = \frac{Nd^2}{L_1 L_2}$ for a number of system sizes ($L_{1,2}/d \in [10,80]$) and particle numbers $N \in [100,96000]$. The clear collapse of the curves for all system sizes shows strong support for the validity of our model, and its apparent generality to three-dimensional systems of significantly varying sizes.

It is finally worth noting that our results can also provide valuable insight into a question that is very often posed in the granular field: When can a system be considered truly three-dimensional? While Fig. 6 unambiguously shows systems of $L/d \geqslant 10$ [where $L = L_1 = L_2$)] to follow a clear, single trend, systems of smaller spatial extent were found to deviate from this master curve. As such, our results—specifically the apparent uniformity of the energy-dissipating behaviors of systems with $L/d \geqslant 10$—imply that this system size represents the lower bound above which a system such as ours can be safely assumed to be fully three dimensional. Considering the significant number of studies conducted using similar systems, this is a potentially valuable finding for future research.

[2]Note that, throughout this paper, any reference to temperature should be taken to refer not to the classical, thermodynamic temperature, but to the granular temperature, which represents an ensemble average of the fluctuations of particles' kinetic energies about their mean values [57–59].

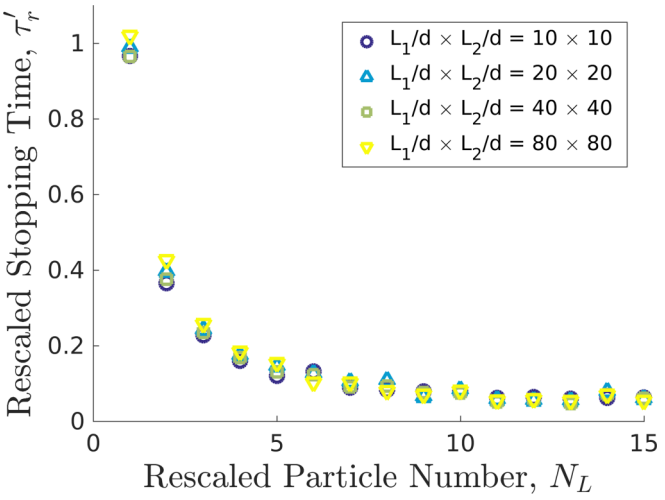

FIG. 6. Simulation data showing the rescaled stopping time, $(\tau_r - \gamma)/\alpha$, as a function of the rescaled particle number, $N_L = Nd^2/L_1L_2$ for a series of three-dimensional systems with elasticity $\varepsilon = 0.98$ driven at an amplitude $A/d = 4$ possessing differing spatial extents.

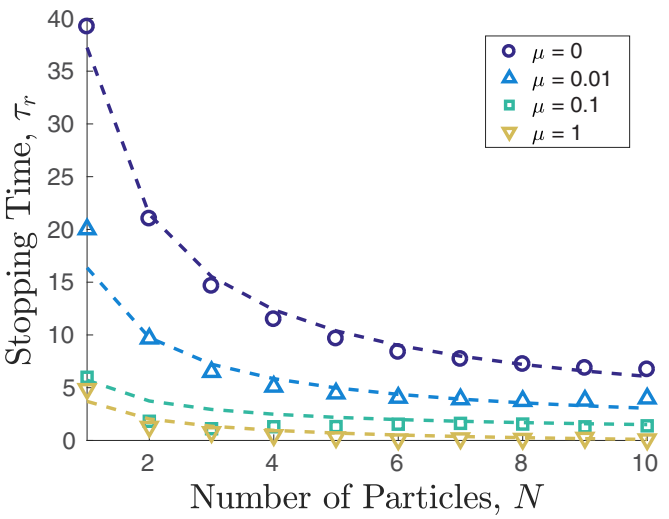

FIG. 7. Simulated (symbols) and theoretical (dashed lines) stopping times for a three-dimensional system of extent 100 mm × 100 mm with $\varepsilon = 0.99$ and $A/d = 4$. Systems are identical other than for their frictional coefficients, which are varied over a range of more than two decades.

C. Toward more complex systems

The theoretical model formulated in this work so far captures the influence of the quantities most fundamental to the behaviors of a strongly excited (i.e., collisionally dominated) tapped granular system—the strength of the excitations to which the system is exposed, the size and dimensionality of the system, and the characteristic elastic properties of the particles involved.

In the final part of this section, we demonstrate how the framework provided by this model may, through the introduction of additional empirical or theoretical arguments, be further extended to successfully characterize the behaviors of more complex systems. For the purpose of this demonstration, we choose to explore the influence of the frictional coefficient, μ, and the wall restitution, ε_w, on the energy decay of our system.

1. *Influence of friction*

While in the one-dimensional case, as discussed in Sec. II B, collisions between particles involve almost exclusively normal contact, with little or no relative tangential motion, the same cannot be safely assumed for higher-$\mathcal{D}$ systems.

Figure 7 illustrates the variation in the observed decay behavior of a three-dimensional tapped bed as the frictional coefficient, μ, is varied by two orders of magnitude. Since our interest in this current work lies in the energy decay of the studied systems, interparticle friction can be considered simply as a source of energy dissipation, akin to the collisional dissipation related to particle restitution. As such, in order to include the effects of friction in our model we can replace the ε term of Eq. (21) with an effective dissipation coefficient, $\tilde{\varepsilon} = \varepsilon \cdot f(\mu)$. Since the basic form of the decay is known from our prior derivations, the determination of $f(\mu)$ simply requires a least-squares analysis of a range of curves such as those shown in Fig. 7 possessing differing μ values.

From our analysis, we find that the function $f(\mu)$ can be well described by the form $f(\mu) = \frac{1-\mu^{k_1}k_2}{1+\mu^{k_1}k_2}$ where the constants k_1 and k_2 are equal, respectively, to approximately $\frac{1}{3}$ and $\frac{1}{20}$. Since our objective at present is simply to demonstrate the ability to empirically extend our model, we do not hypothesize as to the significance of this functional form or the specific coefficients arising from our calculations. The collapse of the curves from Fig. 7 when rescaled according to our empirical update to the model can be seen in Fig. 8.

2. *Influence of sidewall elasticity*

Until now, we have considered only the case of a system possessing perfectly elastic boundaries. In this section, we aim to briefly address two important factors: first, we show that the general form of our model can be expected to hold even when this constraint is relaxed. Second, we demonstrate that, in the

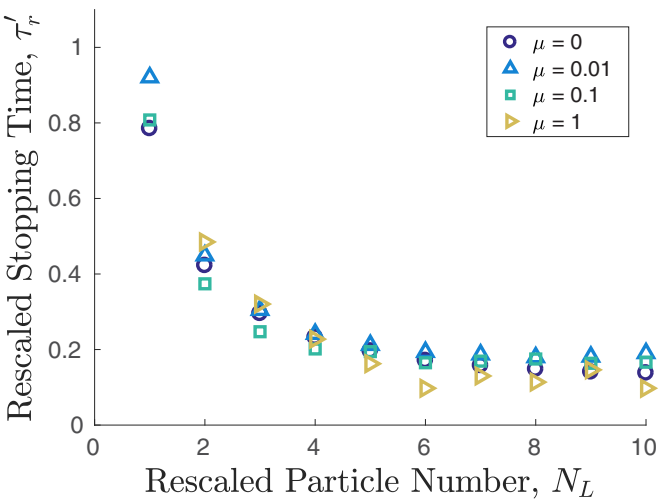

FIG. 8. Simulation data from Fig. 7 rescaled by the empirical relation outlined in the main text.

limit of large system size, our model can in fact be expected to hold even without consideration of sidewall dissipation.

Naturally, the degree of influence on the system's behavior stemming from sidewall dissipation will depend sensitively on a variety of parameters, making its full characterisation a matter for a separate study. For the purposes of the current work, our interest lies simply in ensuring that the general form of our theoretical model holds also for the case of inelastic boundaries.

We introduce the sidewall inelasticity to our model in a similar manner to the addition of frictional effects in the previous section, i.e., through an effective dissipation coefficient, $\tilde{\varepsilon}$. For the case $\varepsilon_w \neq \varepsilon_p$, the degree of dissipation within the system will naturally depend on the relative number of particle-wall and particle interactions within the system. As such, we assume an effective dissipation of the form:

$$\tilde{\varepsilon} = \frac{Z_{pp}\varepsilon_{pp} + Z_{pw}\varepsilon_{pw}}{Z_{pp} + Z_{pw}} = \frac{\varepsilon_{pp} + \frac{Z_{pw}}{Z_{pp}}\varepsilon_{pw}}{1 + \frac{Z_{pw}}{Z_{pp}}}, \tag{22}$$

where Z_{pp} and Z_{pw} are, respectively, the mean particle-particle and particle-wall collision rates.

Under the assumption that, during its excited phase, our system behaves as an ideal gas, if we consider a vertical section of our system, dz, and a duration of time, dt, both of which are adequately small that the current, local number density, n, may be assumed constant, we may estimate the particle-particle and particle-wall collision rates as, respectively:

$$Z_{pp} = \left(\frac{\sqrt{2}}{2}\pi d^2 \langle v \rangle n^2\right) L^2 dz \tag{23}$$

and

$$Z_{pw} = \tfrac{1}{4}\langle v \rangle n 4 L dz, \tag{24}$$

assuming in the above, for simplicity, a square system in which $L_1 = L_2 = L$, as used throughout this work.

From the above, for our given volume of the system, the ratio of particle-wall to particle-particle collision rates can be determined as:

$$\frac{Z_{pw}}{Z_{pp}} = \frac{2}{\sqrt{2}\pi d^2 L n} = \frac{d}{3\sqrt{2}L\eta}, \tag{25}$$

where η is the system's packing density. For the square-based, monodisperse system considered, both L and d can be assumed constant with respect to both varying height and time. As such, for the system as a whole, the ratio of particle-wall to particle-particle interaction rates within the system can, from the above, be approximated as:

$$\frac{Z_{pw}}{Z_{pp}} = \frac{d}{3\sqrt{2}L\langle \eta \rangle}. \tag{26}$$

Here. $\langle \eta \rangle$ represents the an appropriate average over space and time of the system's packing density. As $\langle \eta \rangle$ represents a complex function of A, f, ε_{pp}, and indeed ε_{pw}, for our current purposes, i.e., demonstrating that our model is consistent with the presence of dissipative sidewalls, it is sufficient to treat this quantity simply as a fitting parameter. In order to demonstrate the continued applicability of our model, we choose a range of parameter space for which L remains comparatively small (i.e., wall effects can be expected to remain significant) and consider relatively small perturbations in ε_w at fixed f, A, N, such that $\langle \eta \rangle$ can be assumed to remain approximately constant. The relatively high value of A and small N_L implemented are chosen so as to create a relatively dilute system, thereby further maximizing the influence of wall inelasticity in order to ensure a rigorous test.

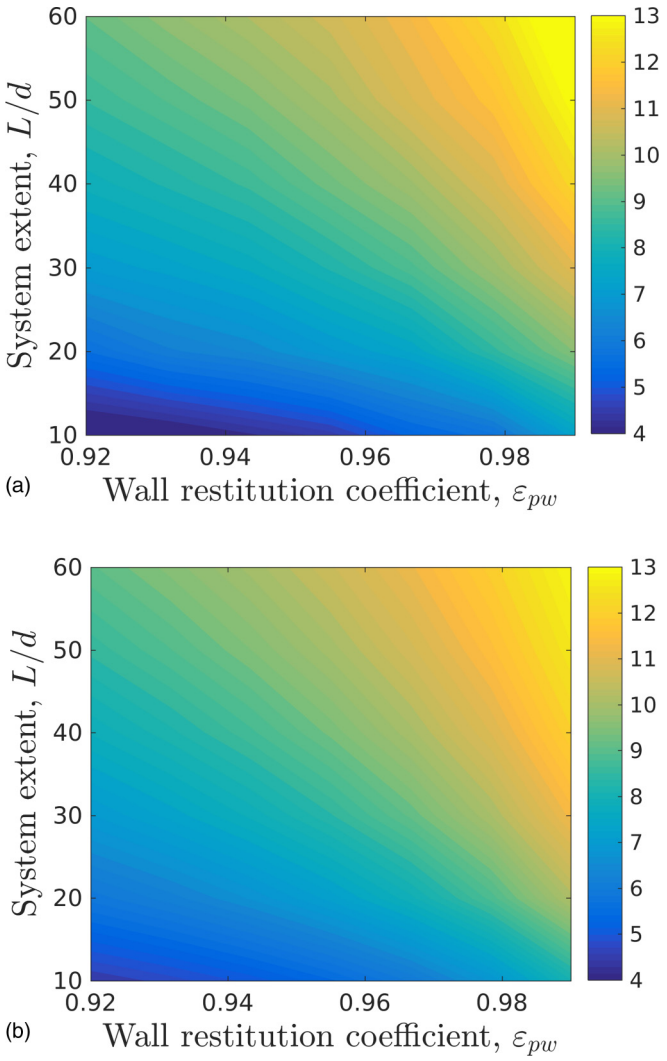

FIG. 9. Phase diagram showing the variation of the stopping time τ_r with the sidewall restitution coefficient, ε_{pw}, and the system's lateral extent, L. Data is shown for both (a) numerical simulations and (b) theoretical predictions. In all cases, the driving amplitude is held constant at 20 mm and the particle elasticity maintained at $\varepsilon_{pp} = 0.99$.

Figure 9 shows the variation of the decay time τ_r in the two-dimensional ε_w-L parameter space, demonstrating the dual influences of sidewall inelasticity and system size discussed above. The similarity in the observed trends for our numerical simulations [Fig. 9(a)] and theoretical predictions [Fig. 9(b)] provides a good indication that the assumed form provides a reasonable description of the true physics.

This proposed form carries the notable consequence that as $\frac{L}{d} \to \infty$, $\frac{Z_{pw}}{Z_{pp}} \to 0$, i.e., $\tilde{\varepsilon} \to \varepsilon$. In other words, for large (e.g., industrial-scale) systems where the system size greatly exceeds

the particle size, our equation returns to its original form in which elastic boundaries are assumed, further emphasising the value of this simple model.

In addition to the parameters μ and ε, specifically discussed above, a similar approach may be taken to account for the effects of a variety of other quantities, which may be of academic, or indeed industrial, interest. The considerable scope for the extension of the model to a diverse range of systems beyond the relatively simple cases explored here provides the potential for significant future research.

V. CONCLUSIONS

In this work, we have presented a simple analytical model to describe the dependence of the energy decay within a granular system driven by discrete excitations on key parameters such as particle number, particle inelasticity, particle friction, and system excitation strength. Using discrete particle method simulations, we have validated the model for a variety of system parameters. In doing so, we have demonstrated that even simple models developed using extremely simplified toy systems can, if correctly adapted and extended, provide valuable insight into larger, more complex cases that, while remaining somewhat idealized, more closely approximate real-world systems. This may prove particularly valuable in predicting and characterizing the behaviors of systems where the numbers of particles or the physical extent of the system may render full parameter studies via simulation or experiment unfeasible. Since, due to the limitations of contemporary computational hardware and experimental techniques, a majority of industrial particle-handling systems currently fall within this category [8], this approach may indeed prove highly useful.

In addition, the work performed has provided insight into the situations in which systems such as those studied here may—and, importantly, may not—be considered fully three dimensional. This is a potentially significant outcome, providing an approximate benchmark against which the validity of the assumption of three dimensionality may be tested. Our results may thus prove useful to the design of future studies, where experimental and simulational geometries may be deliberately designed so as to ensure generality to larger systems. Moreover, these findings may also give cause to other researchers to reevaluate their interpretations of existing studies conducted in systems where the threshold for full three dimensionality is not definitively exceeded.

Another notable consequence of the theoretical form proposed is that for systems whose size greatly exceeds that of its constituent particles (i.e., $\frac{L}{d} \gg 1$), the elastic properties of the walls bounding the systems are seemingly inconsequential. This is a potentially highly valuable result for those wishing to understand, model, and predict the behaviors of large experiments or industrial-scale systems.

Finally, our findings provide considerable scope for future work. Using the examples of interparticle friction and sidewall restitution, we have demonstrated manners through which the effects of other system parameters may be additionally incorporated into the model presented. Further, by changing certain boundary conditions of the model presented, it may be possible to successfully extend its application to other systems; for instance systems in closed containers, systems in the absence of gravity or vibrated and vibrofluidized systems.

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