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Relativistic analysis of stochastic kinematics

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The relativistic analysis of stochastic kinematics is developed in order to determine the transformation of the effective diffusivity tensor in inertial frames. Poisson-Kac stochastic processes are initially considered. For one-dimensional spatial models, the effective diffusion coefficient measured in a frame Σ moving with velocity w with respect to the rest frame of the stochastic process is inversely proportional to the third power of the Lorentz factor $\gamma(w) = (1 - w^2/c^2)^{-1/2}$. Subsequently, higher-dimensional processes are analyzed and it is shown that the diffusivity tensor in a moving frame becomes nonisotropic: The diffusivities parallel and orthogonal to the velocity of the moving frame scale differently with respect to $\gamma(w)$. The analysis of discrete space-time diffusion processes permits one to obtain a general transformation theory of the tensor diffusivity, confirmed by several different simulation experiments. Several implications of the theory are also addressed and discussed.

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I. INTRODUCTION

The inclusion of stochastic processes within the formal structure of relativity (special or general) is a relevant issue in theoretical physics (field theory) with important implications in high-energy physics and cosmology [1,2]. It is a challenging issue, due to causality and to the formal constraints of the Minkowskian structure of the space-time, imposed by the boundedness of the propagation velocity of physical processes, which forces us to consider space and time variable on equal footing. Relativistic covariance lies in the background of extended thermodynamic theories of irreversible processes and transport phenomena [3–5]. The limit imposed by the constant value of the velocity of light *in vacuo* implies that any consistent relativistic stochastic process should possess bounded propagation velocity and, as a consequence of this, an almost-everywhere-smooth structure of the space-time trajectories.

The works by Dudley [6–8] and Hakim [9–11] elucidated further the subtleties of the relativistic formulation of stochastic processes. The most relevant constraint is the impossibility of a strictly Markovian process in the Minkowski space-time. As observed by Dudley [7], the structure of the Minkowsky space-time forces one to include information on the velocity in specifying the state of a stochastic process. This general and significant observation can be interpreted and followed in two conceptually different ways in order to define relativistic models of Brownian motion.

The first strategy is to build up a relativistic version of Ornstein-Uhlenbeck processes in the μ space [9], that is, the Cartesian product of the Minkowski space-time $\mathcal{M}_4$ times the $\mathbb{R}^4$ space of the 4-velocities. This is the strategy adopted in defining the so-called relativistic Ornstein-Uhlenbeck process by Debbasch *et al.* [12,13] and the relativistic Brownian motion by Dunkel and Hänggi [14,15]. These classes of models represent Ornstein-Uhlenbeck processes, i.e., Langevin equations for the position and the momentum of a particle driven by a stochastic force in the presence of a friction contribution. The stochastic force is modeled in the form of a Wiener process and the momentum-dependent prefactors accounting for the Lorentz covariance, modulating both the friction and the intensity of the stochastic perturbations, can be derived from the stationary relativistic velocity distribution, expressed by the Jüttner distribution [16] (or by the modified Jüttner distribution), corresponding to the stationary solution of the relativistic Boltzmann equation [17–20]. A central-limit theorem for these families of relativistic processes is developed in [21].

All these models describe a stochastic dynamics but not a stochastic kinematics, i.e., a stochastic process involving exclusively space-time coordinates that, according to the above mentioned observation by Dudley and Hakim, cannot be grounded in a strict Markovian model. The way to approach a relativistically consistent stochastic kinematic model is to consider a system of bounded velocities, the selection of which is controlled by a Markov-chain process. This is the essence of Poisson-Kac processes, introduced by Kac [22] and further elaborated by many authors [23–26] and extended in [27–30] to any space dimension via the concept of generalized Poisson-Kac processes. For a general review of the different approaches in relativistic stochastic analysis see [31].

The aim of this article is to analyze the (special) relativistic transformation of stochastic kinematics and specifically the transformation of the tensor diffusivity induced by a Lorentz boost. Although this issue is of general relevance in both theory and applications, discussion of it in the literature is lacking. A possible explanation of this lack stems from the fact that, for relativistic stochastic processes, a μ-space formulation has been followed and the resulting nonlinear Langevin equations in $\mathcal{M}_4 \times \mathbb{R}^4$ are fairly complex and not easily amenable to a closed-form analysis of the associated diffusivity tensor. In this article, using different approaches and relativistic stochastic processes (Poisson-Kac processes, discrete space-time diffusion models recently studied in [32,33], etc.), the relativistic transformation of the tensor diffusivity is derived and some of its implications are explored.

The article is organized as follows. Section II reviews briefly the class of Poisson-Kac processes considered and Sec. III analyzes their relativistic transformation. Section IV develops the moment analysis of the spatial one-dimensional Poisson-Kac process in order to obtain the expression for the

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effective diffusivity measured in two inertial frames in relative motion. Section V extends the analysis to higher dimensions, showing the occurrence of two different scalings for the longitudinal and transversal diffusivities. Scaling analysis and numerical simulations are developed in Secs. VI and VII, respectively. Section VIII discusses a consequence of the transformation analysis of tensor diffusivity, associated with the relativistic invariance of the stochastic action. Section IX addresses the general transformation theory of tensor diffusivity, by considering discrete space-time diffusion models as a prototypical example of stochastic processes amenable to closed-form analysis due to the homogenization theory developed in [32,33]. Finally, Sec. X discusses some implications of the theory, including also a brief analysis of the concept of deterministic vs stochastic motion in a Minkowskian space-time.

II. POISSON-KAC PROCESSES

Let Σ' (space-time coordinates x' and t') be a frame in which a Poisson-Kac process is at rest, i.e., it possesses vanishing effective (long-term) velocity; Σ' can be referred to as the rest frame of the process. In Σ' the free Poisson-Kac process is described by the stochastic equation

$$dx'(t') = b_0'(-1)^{\chi(t')}dt', \tag{1}$$

where $\chi(t')$ is the realization of a Poisson process possessing transition rate a_0', i.e., such that the probability density function $p_{\tau'}(\tau')$ for the switching times of the Poisson process is given by the exponential distribution $p_{\tau'}(\tau') = a_0' e^{-a_0'\tau'}$, $\tau' \in [0,\infty)$. The parameter b_0' is the characteristic local velocity of the Poisson-Kac perturbation $0 < b' \leqslant c$, where c is the light velocity *in vacuo*.

Let $p'(x',t')$ be the probability density function associated with the stochastic evolution (1) and $p'^{,\pm}(x',t')$ the partial probabilities characterizing the statistical evolution of the Poisson-Kac process (also referred to as partial probability waves). In Σ', the equations for $p'^{,\pm}(x',t')$ read

$$\begin{aligned} \frac{\partial p'^{,+}}{\partial t'} &= -b_0'\frac{\partial p'^{,+}}{\partial x'} - a_0' p'^{,+} + a_0' p'^{,-}, \\ \frac{\partial p'^{,-}}{\partial t'} &= b_0'\frac{\partial p'^{,-}}{\partial x'} + a_0' p'^{,+} - a_0' p'^{,-}, \end{aligned} \tag{2}$$

where $p' = p'^{,+} + p'^{,-}$. The parameter D_0 given by

$$D_0 = \frac{(b_0')^2}{2a_0'} \tag{3}$$

represents the diffusion coefficient of the Poisson-Kac process in the rest frame (rest diffusion coefficient). It is well known from the work by Kac [22] that in the limit of b_0' and a_0' tending to infinity, keeping fixed the ratio D_0, the solution of Eq. (2) converges in the long-term limit to that of a pure diffusion equation for $p'(x',t')$ characterized by the diffusivity D_0.

III. INERTIAL TRANSFORMATIONS

Let Σ be an inertial frame moving with constant velocity w, $|w| \leqslant c$, with respect to Σ' and (x,t) its space-time coordinates. Enforcing the Lorentz transformation between Σ and Σ',

$$x = \gamma(w)(x' - wt'), \quad t = \gamma(w)(t' - wx'/c^2), \tag{4}$$

where $\gamma(w) = (1 - w^2/c^2)^{-1/2}$ is the Lorentz factor, Eqs. (2) become

$$\begin{aligned} &\gamma(w)\left(1 - \frac{b_0' w}{c^2}\right)\frac{\partial p'^{,+}}{\partial t} \\ &\quad = -\gamma(w)(b_0' - w)\frac{\partial p'^{,+}}{\partial x} - a_0' p'^{,+} + a_0' p'^{,-}, \\ &\gamma(w)\left(1 + \frac{b_0' w}{c^2}\right)\frac{\partial p'^{,-}}{\partial t} \\ &\quad = \gamma(w)(b_0' + w)\frac{\partial p'^{,-}}{\partial x} + a_0' p'^{,+} - a_0' p'^{,-}. \end{aligned} \tag{5}$$

Let us define the transformed partial probability densities $p^+(x,t)$ and $p^-(x,t)$ in Σ as

$$\begin{aligned} p^+ &= \gamma(w)\left(1 - \frac{b_0' w}{c^2}\right)p'^{,+}, \\ p^- &= \gamma(w)\left(1 + \frac{b_0' w}{c^2}\right)p'^{,-}. \end{aligned} \tag{6}$$

The quantities $p^+(x,t)$ and $p^-(x,t)$ are the representation of the partial probability densities for the stochastic process (1) in Σ and their balance equations read

$$\begin{aligned} \frac{\partial p^+}{\partial t} &= -b^+\frac{\partial p^+}{\partial x} - a^+ p^+ + a^- p^-, \\ \frac{\partial p^-}{\partial t} &= b^-\frac{\partial p^+}{\partial x} + a^+ p^+ - a^- p^-, \end{aligned} \tag{7}$$

where

$$b^+ = \frac{b_0' - w}{1 - b_0' w/c^2}, \quad b^- = \frac{b_0' + w}{1 + b_0' w/c^2} \tag{8}$$

and

$$\begin{aligned} a^+ &= \frac{a_0'}{\gamma(w)(1 - b_0' w/c^2)}, \\ a^- &= \frac{a_0'}{\gamma(w)(1 + b_0' w/c^2)}. \end{aligned} \tag{9}$$

From Eqs. (7), the overall probability density $p = p^+ + p^-$ in Σ is a conserved quantity. Moreover, from the definitions (6) it follows that the transformation for the probability densities and fluxes in the two systems is given by

$$p = \gamma(w)\left(p' - \frac{w}{c^2}J'\right), \quad J = \gamma(w)(J' - wp'), \tag{10}$$

where $J' = b_0'(p'^{+} - p'^{-})$ and $J = b^+ p^+ - b^- p^-$ are the probability fluxes in the two reference systems. Equation (10) corresponds to the Lorentz boost for the two-dimensional (as we consider a $1+1$ Minkowski space-time) 4-vector $j_\nu' = (cp', J')$. Observe from Eqs. (8) that the transformations for the coefficient b^+ and b^- are consistent with the relativistic composition of the velocities.

IV. MOMENT ANALYSIS AND DIFFUSIVITY TRANSFORMATION

The statistical properties of the stochastic process considered in the moving reference system Σ can be conveniently approached by considering the associated moment hierarchy

$$m_n(t) = \int_{-\infty}^{\infty} x^n p(x,t)dx = m_n^+(t) + m_n^-(t), \tag{11}$$

where the partial moments

$$m_n^{\pm}(t) = \int_{-\infty}^{\infty} x^n p^{\pm}(x,t)dx \tag{12}$$

satisfy the system of equations

$$\begin{aligned} \frac{dm_n^+(t)}{dt} &= nb^+m_{n-1}^+(t) - a^+m_n^+(t) + a^-m_n^-(t),\\ \frac{dm_n^-(t)}{dt} &= -nb^-m_{n-1}^-(t) + a^+m_n^+(t) - a^-m_n^-(t). \end{aligned} \tag{13}$$

For $n = 0$ (zeroth-order moment) Eqs. (13) reduce to

$$\frac{dm_0^{\pm}(t)}{dt} = \mp a^+m_0^+(t) \pm a^-m_0^-(t). \tag{14}$$

In the long-term limit (here the concept of long-term or asymptotic property refers to times larger than the characteristic time scale characterizing the recombination dynamics between the two probability waves p^+ and p^- describing statistically the Poisson-Kac process and corresponds to time scales $t \gg \max\{1/a^+,1/a^-\}$), the zeroth-order moments converge to steady values satisfying the relation

$$m_0^+ = \frac{a^-}{a^+}m_0^-. \tag{15}$$

Enforcing the consistency condition $m_0^+ + m_0^- = 1$, one thus obtains

$$m_0^{\pm} = \frac{a^{\mp}}{a^+ + a^-} = \frac{1 \mp b_0'w/c^2}{2}. \tag{16}$$

Next consider the first-order moments, i.e., $n = 1$, for which Eqs. (13) provide

$$\begin{aligned} \frac{dm_1^+(t)}{dt} &= b^+m_0^+(t) - a^+m_1^+(t) + a^-m_1^-(t),\\ \frac{dm_1^-(t)}{dt} &= -b^-m_0^-(t) + a^+m_1^+(t) - a^-m_1^-(t). \end{aligned} \tag{17}$$

The long-term behavior of $m_1^{\pm}(t)$ is at most linear in time, i.e.,

$$m_1^{\pm}(t) = \mu_1^{\pm}t + \delta_1^{\pm}. \tag{18}$$

Substituting these expressions into Eqs. (17) and equating the coefficients of equal powers of t^p, $p = 0,1$, one obtains the algebraic relations

$$a^+\mu_1^+ - a^-\mu_1^- = 0 \tag{19}$$

and

$$\begin{aligned} \mu_1^+ &= b^+m_0^+ - a^+\delta_1^+ + a^-\delta_1^-,\\ \mu_1^- &= -b^-m_0^- + a^+\delta_1^+ - a^-\delta_1^-. \end{aligned} \tag{20}$$

Summing the two expressions (20), a further linear equation in $\mu_1^{\pm}$ is obtained,

$$\mu_1^+ + \mu_1^- = b^+m_0^+ - b^-m_0^-. \tag{21}$$

Enforcing Eq. (16), it follows readily that

$$\mu_1 = \mu_1^+ + \mu_1^- = -w, \tag{22}$$

where μ_1 is the effective velocity measured in the moving reference system, $m_1(t) \sim \mu_1 t$. Equation (22) is physically straightforward, implying that the effective mean velocity of the stochastic process measured in the moving system is just the reverse velocity $-w$. Equations (19) and (21) provide a linear system in the two unknown $\mu_1^{\pm}$, yielding as a solution

$$\mu_1^{\pm} = \frac{a^{\mp}(b^+m_0^+ - b^-m_0^-)}{a^+ + a^-}. \tag{23}$$

Finally, in order to derive dispersion properties, consider the second-order moments $m_2^{\pm}(t)$, which satisfy the system of equations

$$\begin{aligned} \frac{dm_2^+(t)}{dt} &= 2b^+m_1^+(t) - a^+m_2^+(t) + a^-m_2^-(t),\\ \frac{dm_2^-(t)}{dt} &= -2b^-m_1^-(t) + a^+m_2^+(t) - a^-m_2^-(t). \end{aligned} \tag{24}$$

In the long-term limit, $m_2^{\pm}(t)$ attains a quadratic expression in t,

$$m_2^{\pm}(t) = \sigma_2^{\pm}t^2 + \zeta_2^{\pm}t + \eta_2^{\pm}. \tag{25}$$

The substitution of Eq. (25) into Eq. (24) provides the system of linear relations in the long-term coefficients

$$a^+\sigma_2^+ - a^-\sigma_2^- = 0, \tag{26}$$

$$\begin{aligned} 2\sigma_2^+ &= 2b^+\mu_1^+ - a^+\zeta_2^+ + a^-\zeta_2^-,\\ 2\sigma_2^- &= -2b^-\mu_1^- + a^+\zeta_2^+ - a^-\zeta_2^-, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \zeta_2^+ &= 2b^+\delta_1^+ - a^+\eta_2^+ + a^-\eta_2^-,\\ \zeta_2^- &= -2b^-\delta_1^- + a^+\eta_2^+ - a^-\eta_2^-. \end{aligned} \tag{28}$$

Summing together the two expressions (27) and using Eq. (26), a system of two linear equations for $\sigma_2^{\pm}$ is obtained, the solution of which is

$$\sigma_2^{\pm} = \frac{a^{\mp}}{a^+ + a^-}(b^+\mu_1^+ - b^-\mu_1^-). \tag{29}$$

We are mainly interested in the scaling of the mean square displacement $\sigma^2(t)$,

$$\begin{aligned} \sigma^2(t) &= m_2(t) - [m_1(t)]^2\\ &= m_2^+(t) + m_2^-(t) - [m_1^+(t) + m_1^-(t)]^2. \end{aligned} \tag{30}$$

Asymptotically, i.e., for time scales in which the recombination process between backward and forward probability waves has reached a stationary behavior, the expression for $\sigma^2(t)$ attains the form

$$\sigma^2(t) = A_2t^2 + \Sigma(v)t + \Delta, \tag{31}$$

where

$$\begin{aligned} A_2 &= \sigma_2^+ + \sigma_2 - (\mu_1^+ + \mu_1^-)^2, \\ \Sigma(v) &= (\zeta_2^+ + \zeta_2^-) - 2(\mu_1^+ + \mu_1^-)(\delta_1^+ + \delta_1^-), \\ \Delta &= (\eta_2^+ + \eta_2^-) - (\delta_1^+ + \delta_1^-)^2. \end{aligned} \tag{32}$$

From the expressions obtained for $\sigma_2^\pm$ and $\mu_1^\pm$ it follows identically that $A_2 = 0$, so that, as expected,

$$\sigma^2(t) \sim \Sigma(w)t. \tag{33}$$

Summing together the two equations (28), one obtains

$$\zeta_2^+ + \zeta_2^- = 2(b^+\delta_1^+ - b^-\delta_1^-). \tag{34}$$

This equation, together with Eq. (21), provides, after some algebra, the following expression for $\Sigma(w)$:

$$\Sigma(w) = 2(b^+ - \mu_1^+ - \mu_1^-)\left(\frac{\mu_1^- + b^- m_0^-}{a^+}\right). \tag{35}$$

Substituting the expressions for $\mu_1^\pm$, m_0^-, $b^\pm$, and a^+ into Eq. (35), one finally arrives at the compact expression

$$\frac{\Sigma(w)}{2} = \frac{(b_0')^2}{2a_0'}\left(1 - \frac{w^2}{c^2}\right)^{3/2}. \tag{36}$$

However, $\Sigma(v)/2$ is the diffusion coefficient D in the moving reference frame, while, from Eq. (3), $(b_0')^2/2a_0'$ is the rest diffusion coefficient D_0. Consequently, Eq. (36) can be expressed in a compact way as

$$D = D_0\left(1 - \frac{w^2}{c^2}\right)^{3/2} = D_0\gamma^{-3}(w), \tag{37}$$

which is the transformation connecting the diffusion coefficients in the two inertial frames.

V. EXTENSION TO HIGHER SPATIAL DIMENSIONS

The analysis developed in the preceding section is extended to higher spatial dimensions. Without loss of generality, two spatial coordinates are considered.

A. Kolesnik-Kac stochastic process

As a two-dimensional model of a Poisson-Kac stochastic process we consider that proposed by Kolesnik and Turbin [34] and Kolesnik [35] and therefore referred to as the Kolesnik-Kac model. It is a particular case of the class of generalized Poisson-Kac processes introduced in [27] and thoroughly studied in [28].

Consider a system of $N > 2$ uniform velocity vectors

$$\begin{aligned} \mathbf{b}'^{,(\alpha)} &= b_0'(\cos(2\pi\alpha/N), \sin(2\pi\alpha/N)) = (b_x'^{,(\alpha)}, b_y'^{,(\alpha)}), \\ \alpha &= 0, \dots, N-1, \end{aligned} \tag{38}$$

where b_0 is the reference velocity, since $|\mathbf{b}'^{,(\alpha)}| = b_0$ for any α, and the stochastic process

$$d\mathbf{x}'(t') = \mathbf{w}(t')dt', \tag{39}$$

where $\mathbf{x}' = (x', y')$ and $\mathbf{w}(t')$ is a stochastic velocity vector that changes its value, among the N possible states $\mathbf{b}'^{(\alpha)}$ in a equiprobable way, at random time intervals characterized by an exponential distribution defined by the transition rate $a_0' > 0$. The mathematical properties of this process have been studied by Kolesnik [34].

In the stationary system Σ' (space-time coordinates x', y', and t'), a system of N partial probability density functions $p'^{,(\alpha)}(x', y', t')$, $\alpha = 0, \dots, N$, fully describes the statistical properties of the Kolesnik-Kac model and the overall probability density function of the process is $p'(x', y', t') = \sum_{\alpha=0}^{N-1} p'^{,(\alpha)}(x', y', t')$. In the stationary frame the partial probabilities satisfy the system of hyperbolic equations

$$\begin{aligned} \frac{\partial p'^{,(\alpha)}}{\partial t'} &= -b_x'^{,(\alpha)}\frac{\partial p'^{,(\alpha)}}{\partial x'} - b_y'^{,(\alpha)}\frac{\partial p'^{,(\alpha)}}{\partial y'} \\ &\quad - a_0' p'^{,(\alpha)} + \frac{a_0'}{N}\sum_{\beta=0}^{N-1} p'^{,(\beta)}, \end{aligned} \tag{40}$$

where $\alpha = 0, \dots, N-1$. In an inertial frame Σ (space-time coordinates x, y, and t) moving with respect to Σ' at constant velocity $w\mathbf{e}_x$ along the x axis, the process is characterized by the N partial probability densities $p^{(\alpha)}(x, y, t), \alpha = 0, \dots, N-1$, defined as

$$p^{(\alpha)} = \gamma(w)\kappa(\alpha)p'^{,(\alpha)}, \quad \alpha = 0, \dots, N-1, \tag{41}$$

where

$$\kappa(\alpha) = 1 - \frac{b_x'^{,(\alpha)} w}{c^2}. \tag{42}$$

The balance equation for the partial probability system $\{p^{(\alpha)}\}_{\alpha=0}^{N-1}$ is obtained from Eq. (40) enforcing the Lorentz transform (4) and $y = y'$, leading to

$$\begin{aligned} \frac{\partial p^{(\alpha)}}{\partial t} &= -b_x^{(\alpha)}\frac{\partial p^{(\alpha)}}{\partial x} - b_y^{(\alpha)}\frac{\partial p^{(\alpha)}}{\partial y} \\ &\quad - a^{(\alpha)} p^{(\alpha)} + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} p^{(\beta)}, \end{aligned} \tag{43}$$

where $\alpha = 0, \dots, N-1$. The coefficients in these equations are defined by

$$b_x^{(\alpha)} = \frac{b_x'^{,(\alpha)} - w}{\kappa(\alpha)}, \quad b_y^{(\alpha)} = \frac{b_y'^{,(\alpha)}}{\gamma(w)\kappa(\alpha)}, \quad a^{(\alpha)} = \frac{a_0'}{\gamma(w)\kappa(\alpha)}, \tag{44}$$

where $\alpha = 0, \dots, N-1$.

B. Moment hierarchy

As in the one-dimensional spatial case previously analyzed, moment analysis provides the simplest tool to extract the statistical properties associated with Eq. (43). Let

$$m_{m,n}^{(\alpha)}(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n p^{(\alpha)}(x, y, t)dx\, dy, \tag{45}$$

$m, n = 0, 1, \dots$, and $\alpha = 0, \dots, N-1$ be the (m,n)th partial moment. The global moment hierarchy $\{m_{m,n}(t)\}_{m,n=0}^{\infty}$ defined with respect to $p(x, y, t)$ is readily given by the sum over α of the (m,n)th partial moments. From the definition (45) and from Eq. (43) it follows that the partial moments satisfy the linear system of differential equations

$$\frac{dm_{m,n}^{(\alpha)}(t)}{dt} = mb_x^{(\alpha)} m_{m-1,n}^{(\alpha)}(t) + nb_y^{(\alpha)} m_{m,n-1}^{(\alpha)}(t) - a^{(\alpha)} m_{m,n}^{(\alpha)}(t) + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} m_{m,n}^{(\beta)}(t), \tag{46}$$

where the regularity conditions at infinity for $p^{(\alpha)}(x,y,t)$ have been enforced.

C. Zeroth- and first-order moments

To begin with, consider the zeroth-order moments $m_{0,0}^{(\alpha)}(t)$, satisfying the system of equations

$$\frac{dm_{0,0}^{(\alpha)}(t)}{dt} = -a^{(\alpha)} m_{0,0}^{(\alpha)}(t) + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} m_{0,0}^{(\beta)}(t), \tag{47}$$

where $\alpha = 0, \ldots, N-1$. In the long-term (asymptotic) limit, i.e., once the recombination among the partial waves has reached a steady state $dm_{0,0}^{(\alpha)}(t)/dt = 0$, which implies $m_{0,0}^{(\alpha)}(t) = C'/a^{(\alpha)} = C\kappa(\alpha)$, where the constant C is determined by the probabilistic consistency condition $\sum_{\alpha=0}^{N-1} m_{0,0}^{(\alpha)}(t) = 1$, leading to the expression

$$m_{0,0}^{(\alpha)}(t) = \frac{\kappa(\alpha)}{N}, \quad \alpha = 0, \ldots, N-1. \tag{48}$$

Next consider the first-order moments. For $m = 1$ and $n = 0$, Eq. (46) becomes

$$\frac{dm_{1,0}^{(\alpha)}(t)}{dt} = b_x^{(\alpha)} m_{0,0}^{(\alpha)}(t) - a^{(\alpha)} m_{1,0}^{(\alpha)}(t) + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} m_{1,0}^{(\beta)}(t), \tag{49}$$

where $\alpha = 0, \ldots, N-1$. In the long-term limit Eq. (49) becomes a nonhomogeneous linear system driven by a constant contribution $b_x^{(\alpha)} m_{0,0}^{(\alpha)}$, where $m_{0,0}^{(\alpha)}$ is given by Eq. (48). Consequently, the asymptotic solution for $m_{1,0}^{(\alpha)}(t)$ grows linearly in time, i.e.,

$$m_{1,0}^{(\alpha)}(t) = \mu_x^{(\alpha)} t + \delta_x^{(\alpha)}, \tag{50}$$

where $\alpha = 0, \ldots, N-1$. Substituting these expressions into Eq. (49) and equating the coefficients of equal powers of t, one obtains the system of relations for the asymptotic coefficients $\mu_x^{(\alpha)}$ and $\delta_x^{(\alpha)}$,

$$a^{(\alpha)} \mu_x^{(\alpha)} = \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} \mu_x^{(\beta)}, \tag{51}$$

$$\mu_x^{(\alpha)} = b_x^{(\alpha)} m_{0,0}^{(\alpha)} - a^{(\alpha)} \delta_x^{(\alpha)} + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} \delta_x^{(\beta)}, \tag{52}$$

where $\alpha = 0, \ldots, N-1$ and $m_{0,0}^{(\alpha)}$ equals its long-term expression (48).

Summing over α in Eq. (52), the expression for the effective velocity V_x,

$$m_{1,0}(t) \sim V_x t, \tag{53}$$

is obtained

$$V_x = \sum_{\alpha=0}^{N-1} \mu_x^{(\alpha)} = \sum_{\alpha=0}^{N-1} b_x^{(\alpha)} m_{0,0}^{(\alpha)} = -w, \tag{54}$$

and from Eqs. (51) and (54) it follows that the expression for $\mu_x^{(\alpha)}$ takes the form

$$\mu_x^{(\alpha)} = -\frac{w\kappa(\alpha)}{N}, \tag{55}$$

where $\alpha = 0, \ldots, N-1$. From Eq. (52) one also obtains the functional form of $\delta_x^{(\alpha)}$, which plays a central role in the estimate of diffusional properties. Setting $C_x = \sum_{\alpha=0}^{N_1} a^{(\alpha)} \delta_x^{(\alpha)}$, Eq. (52) can be rewritten as

$$\delta_x^{(\alpha)} = \frac{1}{a^{(\alpha)}}\left[b_x^{(\alpha)} m_{0,0}^{(\alpha)} - \mu_x^{(\alpha)} + \frac{C_x}{N} \right], \tag{56}$$

where $\alpha = 0, \ldots, N-1$. Substituting the long-term expressions (48) and (55), one finally arrives at

$$\delta_x^{(\alpha)} = \frac{\gamma(w)}{Na_0}\big(b_x'^{,(\alpha)} - w\big)\kappa(\alpha) + \frac{w\gamma(w)}{Na_0}\kappa^2(\alpha) + \frac{\gamma(w)C_x}{Na_0}\kappa(\alpha), \tag{57}$$

where $\alpha = 0, \ldots, N-1$. The expressions for $\delta_x^{(\alpha)}$ contain the unknown constant C_x. As we will see in the next section, this constant is totally immaterial in the estimate of the diffusional properties.

Finally, consider the other family of first-order partial moments $m_{0,1}^{(\alpha)}(t)$. The analysis is completely specular to that developed above for $m_{1,0}^{(\alpha)}(t)$, so only the final results are reported. In the long-term limit, $m_{0,1}^{(\alpha)}(t)$ attains a linear expression analogous to Eq. (50), namely,

$$m_{0,1}^{(\alpha)}(t) = \mu_y^{(\alpha)} t + \delta_y^{(\alpha)}, \tag{58}$$

where $\alpha = 0, \ldots, N-1$. From the balance equations it follows that

$$\mu_y^{(\alpha)} = 0, \tag{59}$$

where $\alpha = 0, \ldots, N-1$, which implies for $V_y, m_{0,1}(t) \sim V_y t$, that

$$V_y = 0. \tag{60}$$

As regards the factors $\delta_y^{(\alpha)}$ in the asymptotic functional form of these first-order partial moments, one obtains

$$\delta_y^{(\alpha)} = \frac{1}{a^{(\alpha)}}\left[b_y^{(\alpha)} m_{0,0}^{(\alpha)} + \frac{C_y}{N} \right] = \frac{1}{Na_0}\beta_y^{(\alpha)}\kappa(\alpha) + \frac{\gamma(w)C_y}{Na_0}\kappa(\alpha), \tag{61}$$

where $\alpha = 0, \ldots, N-1$. In deriving Eq. (61), Eq. (59) has been used. The coefficients $\delta_y^{(\alpha)}$ contain the constant $C_y = \sum_{\alpha=0}^{N-1} a^{(\alpha)} \delta_y^{(\alpha)}$, which is immaterial in the estimate of the long-term diffusional properties, which are addressed in the next section.

D. Second-order moments and diffusion tensor

To begin with, consider $m_{2,0}^{(\alpha)}(t)$, i.e., $m=2$ and $n=0$ in Eq. (46). Equation (46) specializes to

$$\frac{dm_{2,0}^{(\alpha)}(t)}{dt} = 2b_x^{(\alpha)} m_{1,0}^{(\alpha)}(t) - a^{(\alpha)} m_{2,0}^{(\alpha)}(t) + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} m_{2,0}^{(\beta)}(t), \tag{62}$$

where $\alpha = 0,\ldots,N-1$. In the long-term limit, since the forcing term $b_x^{(\alpha)} m_{1,0}^{(\alpha)}(t)$ is linear in time, $m_{2,0}^{(\alpha)}(t)$ should be, at most, a quadratic function of t,

$$m_{2,0}^{(\alpha)}(t) = K_{x,x}^{(\alpha)} t^2 + g_{x,x}^{(\alpha)} t + \zeta_{x,x}^{(\alpha)}, \tag{63}$$

where $\alpha = 0,\ldots,N-1$. Substituted into Eq. (62), it provides the following relations among the expansion coefficients:

$$a^{(\alpha)} K_{x,x}^{(\alpha)} = \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} K_{x,x}^{(\beta)}, \tag{64}$$

$$2K_{x,x}^{(\alpha)} = 2b_x^{(\alpha)} \mu_x^{(\alpha)} - a^{(\alpha)} g_{x,x}^{(\alpha)} + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} g_{x,x}^{(\beta)}, \tag{65}$$

$$g_{x,x}^{(\alpha)} = 2b_x^{(\alpha)} \delta_x^{(\alpha)} - a^{(\alpha)} \zeta_{x,x}^{(\alpha)} + \frac{1}{N}\sum_{\beta=0}^{N-1} a^{(\beta)} \zeta_{x,x}^{(\beta)}, \tag{66}$$

where $\alpha = 0,\ldots,N-1$. Summing over α in Eq. (65) and enforcing Eq. (55), it follows that

$$\sum_{\alpha=0}^{N-1} K_{x,x}^{(\alpha)} = \sum_{\alpha=0}^{N-1} b_x^{\alpha} \mu_x^{(\alpha)} = w^2, \tag{67}$$

which, together with Eq. (64), provides for $K_{x,x}^{(\alpha)}$ the expression

$$K_{x,x}^{(\alpha)} = \frac{w^2 \kappa(\alpha)}{N}, \tag{68}$$

where $\alpha = 0,\ldots,N-1$. The variance $\sigma_{x,x}^2(t)$ along the x coordinate is given by

$$\sigma_{x,x}^2(t) = \sum_{\alpha=0}^{N-1} m_{2,0}^{(\alpha)}(t) - \left[\sum_{\alpha=0}^{N-1} m_{1,0}^{(\alpha)}(t)\right]^2. \tag{69}$$

Substituting the long-term expressions and enforcing Eqs. (55) and (68), it follows that $\sigma_{x,x}^2(t)$ is asymptotically a linear function of time

$$\sigma_{x,x}^2(t) \sim 2D_{x,x} t, \tag{70}$$

where the diffusivity along x is given by the expression

$$2D_{x,x} = \sum_{\alpha=0}^{N-1} g_{x,x}^{(\alpha)} + 2w \sum_{\alpha=0}^{N-1} \delta_x^{(\alpha)}. \tag{71}$$

The first sum in Eq. (71) can be explicated by summing Eq. (66) over α, providing the compact expression for $D_{x,x}$,

$$D_{x,x} = \sum_{\alpha=0}^{N-1} \big(b_x^{(\alpha)} + w\big) \delta_x^{(\alpha)}, \tag{72}$$

where

$$b_x^{(\alpha)} + w = \frac{b_x^{\prime,(\alpha)}}{\gamma^2(w)\kappa(\alpha)} \tag{73}$$

where $\alpha = 0,\ldots,N-1$. From the expressions (72) and (73) it follows that the term containing the unknown constant C_x in the expression (56) gives a vanishing contribution in the estimate of $D_{x,x}$. Therefore,

$$\begin{aligned} D_{x,x} &= \frac{1}{\gamma^2(w)} \sum_{\alpha=0}^{N-1} \frac{b_x^{\prime,(\alpha)}}{\kappa(\alpha)} \left[\frac{\gamma(w)}{N a_0^{\prime}} \big(b_x^{\prime,(\alpha)} - w\big)\kappa(\alpha) \right. \\ &\quad \left. + \frac{w\gamma(w)}{N a_0^{\prime}} \kappa^2(\alpha) \right] \\ &= \frac{1}{\gamma(w) N a_0^{\prime}} \left[\sum_{\alpha=0}^{N-1} b_x^{\prime,(\alpha)} \big(b_x^{\prime,(\alpha)} - w\big) + w \sum_{\alpha=0}^{N-1} b_x^{\prime,(\alpha)} \kappa(\alpha) \right] \\ &= \frac{1}{\gamma(w) N a_0^{\prime}} \left(1 - \frac{w^2}{c^2}\right) \sum_{\alpha=0}^{N-1} \big[b_x^{\prime,(\alpha)}\big]^2 = \frac{(b_0^{\prime})^2}{2a_0^{\prime}} \gamma^{-3}(w), \end{aligned} \tag{74}$$

where we have used the identities $\sum_{\alpha=0}^{N-1} b^{\prime,(\alpha)} = 0$ and $\sum_{\alpha=0}^{N-1} [b_x^{\prime,(\alpha)}]^2 = (b_0^{\prime})^2 N/2$. Equation (74) implies the transformation of the diffusion coefficient $D_{x,x}$ parallel to the frame-velocity direction

$$D_{x,x} = D_{\parallel} = D_0 \gamma^{-3}(w), \tag{75}$$

consistently with the corresponding expression (37) derived for the one-dimensional spatial Poisson-Kac process.

Consider now the other family $m_{0,2}^{(\alpha)}(t)$, corresponding to $m=0$ and $n=2$ in Eq. (46). Details are skipped as the algebra is identical to the $m_{2,0}^{(\alpha)}$ case. In the long-term regime, $m_{0,2}^{(\alpha)}(t)$ is quadratic in time

$$m_{0,2}^{(\alpha)}(t) = K_{y,y}^{(\alpha)} t^2 + g_{y,y}^{(\alpha)} t + \zeta_{y,y}^{(\alpha)}, \tag{76}$$

where $\alpha = 0,\ldots,N-1$. From the moment balance equation one obtains

$$K_{y,y}^{(\alpha)} = 0, \tag{77}$$

where $\alpha = 0,\ldots,N-1$. As regards $\sigma_{y,y}^2(t) = m_{0,2}(t) - [m_{0,1}(t)]^2$, one obtains

$$\sigma_{y,y}^2(t) \sim 2D_{y,y} t, \tag{78}$$

where the diffusion coefficient $D_{y,y}$ is given by

$$2D_{y,y} = \sum_{\alpha=0}^{N-1} g_{y,y}^{(\alpha)} = 2\sum_{\alpha=0}^{N-1} b_y^{(\alpha)} \delta_y^{(\alpha)}. \tag{79}$$

Substituting the expressions (44) and (61) for $b_y^{(\alpha)}$ and $\delta_y^{(\alpha)}$ into Eq. (79), one gets

$$\begin{aligned} D_{y,y} &= \sum_{\alpha=0}^{N-1} \frac{b_y^{\prime,(\alpha)}}{\gamma(w)\kappa(\alpha)} \left[\frac{1}{N a_0^{\prime}} b_y^{\prime,(\alpha)} \kappa(\alpha) + \frac{\gamma(w) C_y}{N a_0^{\prime}} \kappa(\alpha) \right] \\ &= \frac{1}{\gamma(w) N a_0^{\prime}} \sum_{\alpha=0}^{N-1} \big[b_y^{\prime,(\alpha)}\big]^2 = \frac{(b_0^{\prime})^2}{2a_0^{\prime}} \gamma^{-1}(w), \end{aligned} \tag{80}$$

which implies that

$$D_{y,y} = D_\perp = D_0 \gamma^{-1}(w) = D_0 \sqrt{1 - \frac{w^2}{c^2}} \tag{81}$$

i.e., the effective diffusion coefficient perpendicular to the frame direction of motion is proportional to the reciprocal of the Lorentz factor.

Finally, consider the mixed moments $m_{1,1}^{(\alpha)}(t)$, i.e., $m = n = 1$ in Eq. (46). Also in this case, in the long term

$$m_{1,1}^{(\alpha)}(t) = K_{x,y}^{(\alpha)} t^2 + g_{x,y}^{(\alpha)} t + \zeta_{x,y}^{(\alpha)}, \tag{82}$$

where $\alpha = 0, \ldots, N-1$. From the moment balance equation it follows that

$$K_{x,y}^{(\alpha)} = 0, \tag{83}$$

where $\alpha = 0, \ldots, N-1$, and

$$\sigma_{x,y}^2(t) = m_{1,1}(t) - m_{1,0}(t) m_{0,1}(t) \sim 2 D_{x,y} t, \tag{84}$$

where

$$\begin{aligned} 2D_{x,y} &= \sum_{\alpha=0}^{N-1} g_{x,y}^{(\alpha)} - \sum_{\alpha=0}^{N-1} \mu_x^{(\alpha)} \sum_{\beta=0}^{N-1} \delta_x^{(\beta)} \\ &= \sum_{\alpha=0}^{N-1} \left(b_x^{(\alpha)} + w\right) \delta_y^{(\alpha)} + \sum_{\alpha=0}^{N-1} b_y^{(\alpha)} \delta_x^{(\alpha)}. \end{aligned} \tag{85}$$

Using the expressions previously derived for the quantities in Eq. (85), one derives, after some algebra, that

$$D_{x,y} = 0, \tag{86}$$

which completes the diffusional analysis of the Kolesnik-Kac process.

VI. SCALING ANALYSIS

The transformations of the diffusion coefficient in inertial systems can be physically interpreted on the basis of time-dilation and length-contraction phenomena using the classical Einstein scaling for the diffusion coefficient. The analysis is not mathematically rigorous and it is aimed at highlighting the physical origin of the diffusivity transformations, at least in these simple cases. The diffusivity D' is proportional to the ratio of the mean square displacement $\langle (\Delta x')^2 \rangle$ divided by the time scale $\Delta t'$,

$$D' = \frac{1}{2} \frac{\langle (\Delta x')^2 \rangle}{\Delta t'} = D_0, \tag{87}$$

which is the Einstein equation for diffusion. Similarly, in the reference frame Σ, the diffusion coefficient is given by

$$D = \frac{1}{2} \frac{\langle (\Delta x)^2 \rangle}{\Delta t}. \tag{88}$$

However, Δt and Δx are related to $\Delta x'$ and Δt via the Lorentz transformation, which implies length contraction

$$\Delta x = \Delta x' \sqrt{1 - \frac{w^2}{c^2}} = \gamma^{-1}(w) \Delta x' \tag{89}$$

and time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{w^2}{c^2}}} = \gamma(w) \Delta t'. \tag{90}$$

Therefore,

$$\langle (\Delta x)^2 \rangle = \gamma^{-2}(w) \langle (\Delta x')^2 \rangle. \tag{91}$$

Substituting Eqs. (90) and (91) into Eq. (88), Eq. (37) follows. Therefore, the scaling of D with the third power of $\gamma^{-1}(w)$ can be viewed as a direct consequence of the space-time contraction and dilation properties of the Lorenz transformation.

In higher-dimensional spaces

$$D_{x_i,x_i} = \frac{1}{2} \frac{\langle (\Delta x_i)^2 \rangle}{\Delta t'}, \tag{92}$$

where $i = 1, \ldots d$, with $d = 2,3$, two cases should be considered separately depending on whether x_i is a spatial coordinate in the direction of the frame velocity or not. In the first case, namely, for a frame velocity directed along x_i, $D_{x_i,x_i} = D_\parallel$, $\langle (\Delta x_i)^2 \rangle$ satisfies Eq. (91), and Eq. (75) is recovered for $D_\parallel$. Conversely, if x_i is a coordinate in a direction orthogonal to the frame velocity $D_{x_i,x_i} = D_\perp$, then $\langle (\Delta x_i)^2 \rangle = \langle (\Delta x_i')^2 \rangle$, so only time dilation contributes to $D_\perp$, returning $D_\perp = D_0 \gamma^{-1}(w)$ as derived in preceding section. For generic stochastic processes, scaling analysis is not sufficient to provide the expression for the complete transformation of the tensor diffusivity. This analysis is developed in Sec. IX.

VII. NUMERICAL EXAMPLES

It is useful to illustrate the main results developed in the previous sections with the aid of numerical examples of stochastic dynamics. Throughout this section, a normalized light velocity is assumed, i.e., $c = 1$.

To begin with, consider the one-dimensional (1D) spatial Poisson-Kac process (1). Set $b_0' = c = 1$ and $a_0' = 1$ so that the rest diffusivity equals $D_0 = 1/2$. Figure 1 depicts a portion

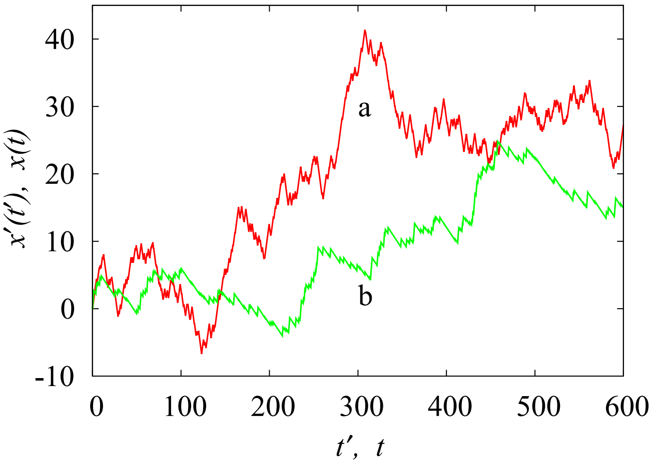


FIG. 1. Graph of the evolution of a realization of a Poisson-Kac process ($b_0' = c = 1$ and $a_0' = 1$) in the rest frame Σ' [line a, i.e., $x'(t')$] and in the inertial system Σ moving with constant velocity $w/c = 0.8$ with respect to Σ' [line b, i.e., $x(t)$].

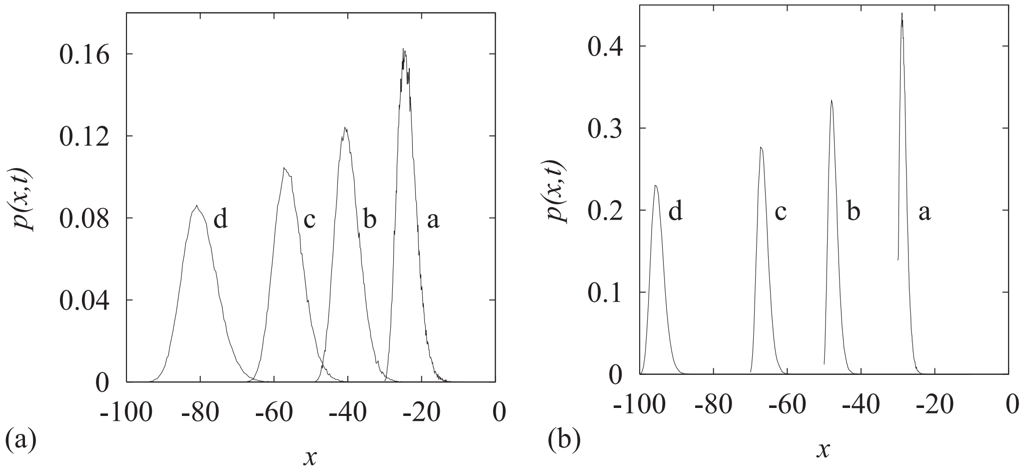


FIG. 2. Overall probability density function $p(x,t)$ vs x of a Poisson-Kac process ($b_0' = c = 1$ and $a_0' = 1$) in the frame Σ at several time instants for (a) $w/c = 0.8$ and (b) $w/c = 0.95$. Lines a–d refer to $t = 30,50,70,100$, respectively.

of a realization of this process in the rest frame Σ' and in an inertial frame moving with respect to Σ' at constant velocity w. Consider an ensemble of N_p realizations of the Poisson-Kac process. Each realization can be viewed as a stochastic particle, the dynamics of which follows the microscopic dynamics (1). At time $t' = t = 0$ set $x' = 0$ for all the elements of the ensemble. Figure 2 depicts the overall probability density function (PDF) $p(x,t)$ vs the spatial coordinate x of the moving inertial frame obtained numerically from the stochastic simulations over this ensemble for different values of t and for two different relative velocities of the frame Σ: $w/c = 0.8$ [Fig. 2(a)] and $w/c = 0.95$ [Fig. 2(b)]. All the simulations refer to a statistics over $N_p = 10^5$ particles. As expected, the PDFs $p(x,t)$ are characterized by a mean value that equals $-wt$ and by a variance $\sigma^2(t)$ that becomes narrower as w increases up to the limit value $c = 1$ [compare the corresponding curves in Figs. 2(a) and 2(b)].

The graph of the mean square displacement $\sigma^2(t)$ vs t for different values of the relative frame velocity w is depicted in Fig. 3. The arrow in this figure indicates increasing values of w, from $w = 0$ (top curve) up to $w = 0.8$ (bottom curve). For $w = 0$, $D_0 = 1/2$, as expected from Eq. (3), i.e., $\sigma^2(t) = t$, while as w increases the diffusion coefficient D, corresponding to half of the slope of the asymptotic linear plot of $\sigma^2(t)$ vs t, decreases.

Observe that the concept of long-term properties regarding the recombination dynamics among the partial probability waves $p^+(x,t)$ and $p^-(x,t)$ refers to time scales $t \gg t_{\min}$, where $t_{\min} = 1/a^-$. For instance, $t_{\min} = 3$ at $w = 0.8$, corresponding to a very fast relaxation towards the asymptotic properties compared to the time scales reported on the abscissa of Fig. 3.

The behavior of the diffusion coefficient D vs the relative frame velocity w is depicted in Fig. 4 and it is in perfect agreement with Eq. (37). Next consider higher-dimensional stochastic processes. The first example is given by the two-dimensional Kolesnik-Kac model (38) and (39). We have chosen $N = 5$, $b_0' = c = 1$, and $a_0' = 1$, by considering an ensemble of $N_p = 10^5$ stochastic particles. Figure 5(a) shows the behavior of the diagonal entries $D_{x,x}$ and $D_{y,y}$, obtained from stochastic simulations confirming the theoretical expressions (75) and (81). The off-diagonal entry is not depicted for the sake of brevity, but is vanishing for any value of the frame velocity w.

Finally, let us consider a three-dimensional model, namely, the Poisson-Kac process

$$dx_i'(t') = b_0'(-1)^{\chi_i(t')} dt', \quad i = 1,2,3, \tag{93}$$

where $\chi_i(t')$, $i = 1,2,3$, are three Poisson processes, statistically independent of each other, characterized by the same values of the reference velocity b_0' and of the transition rate a_0'. This model, statistically described by means of eight partial probability density functions, converges, for $b_0', a_0' \to \infty$, keeping fixed the ratio $(b_0')^2/a_0'$, to an isotropic three-dimensional parabolic diffusion equation for the overall PDF $p(\mathbf{x}',t')$ with diffusion coefficient equal to D_0 given by Eq. (3).

Let Σ be an inertial frame moving with a constant velocity along the direction x_1' with respect to Σ'. It is expected that

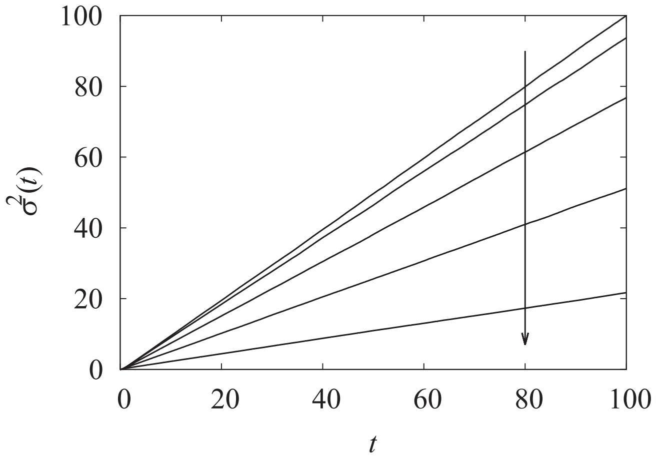


FIG. 3. Plot of $\sigma^2(t)$ vs t for a 1D Poisson-Kac process ($b_0' = c = 1$ and $a_0' = 1$) in the frame Σ for different values of the frame velocity w. The arrow indicates increasing values of $w/c = 0,0.2,0.4,0.6,0.8$.

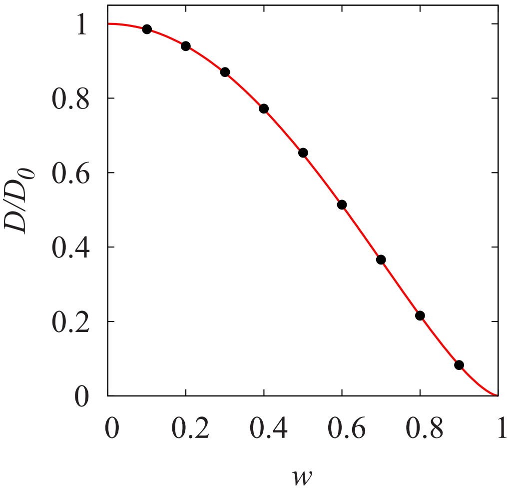


FIG. 4. Plot of D/D_0 vs w ($c = 1$) for a 1D Poisson-Kac process ($b_0' = c = 1$ and $a_0' = 1$). Circles represent the results of stochastic simulations and the solid line is the curve $D/D_0 = \gamma^{-3}(w) = (1 - w^2)^{3/2}$.

Eqs. (75) and (81) apply also in this case, providing for the diffusivity tensor D_{x_i,x_j} in Σ the expression

$$\left(D_{x_i,x_j}\right) = D_0 \begin{pmatrix} \gamma^{-3}(v) & 0 & 0 \\ 0 & \gamma^{-1}(v) & 0 \\ 0 & 0 & \gamma^{-1}(v) \end{pmatrix}. \tag{94}$$

The results of stochastic simulations over an ensemble of $N_p = 10^5$ particles are depicted in Fig. 5(b), confirming the theoretical prediction. Off-diagonal entries (not shown) prove to be vanishing.

VIII. STOCHASTIC ACTION INVARIANCE

There is a striking analogy between the results expressed by Eq. (94) and the scaling of the longitudinal and transversal masses. Although the concepts of longitudinal and transversal masses, introduced by Einstein [36] in the early days of relativity theory (see also [37]), are nowadays considered obsolete, their use in the present context is convenient in order to derive an interesting implication of Eq. (94).

Consider the equation of motion of a particle of rest mass $m_0 = E_0/c^2$ in an inertial frame Σ moving with velocity w along the x_1 axis. Introduce the diagonal mass tensor $\mathbf{m} = \text{diag}(m_\perp, m_\parallel, m_\parallel)$, where $m_\perp$ and $m_\parallel$ are the longitudinal and transversal masses, respectively (the mass tensor enters the three-dimensional expression of the relativistic Newton equation). The relativistic scaling of $\mathbf{m}$ is expressed by

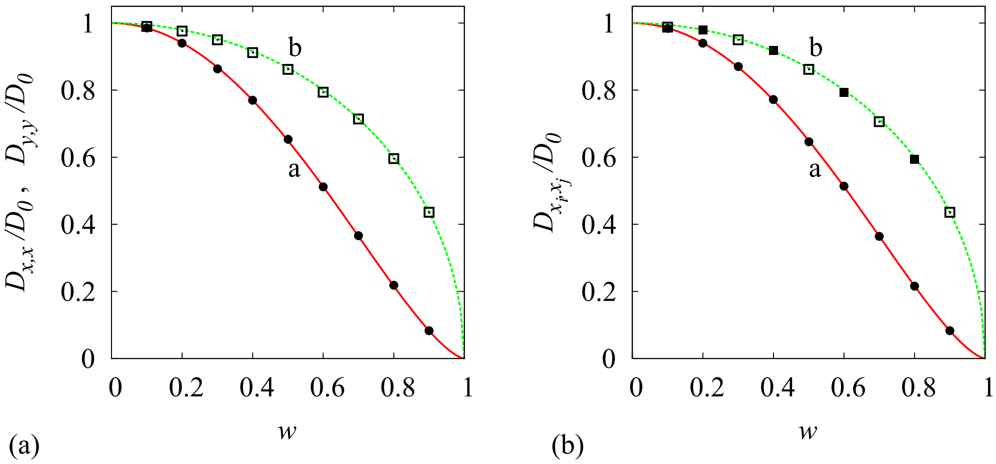
$$\mathbf{m} = m_0 \begin{pmatrix} \gamma^3(w) & 0 & 0 \\ 0 & \gamma(w) & 0 \\ 0 & 0 & \gamma(w) \end{pmatrix}. \tag{95}$$

From Eq. (94) it follows that the product $\mathbf{mD}$ of a particle performing purely diffusive stochastic motion is an isotropic tensor

$$\mathbf{mD} = m_0 D_0 \mathbf{I} \tag{96}$$

and the nonvanishing diagonal entries are relativistically invariant, i.e., they do not depend on the velocity w. This observation can be expressed as follows: If a particle possessing rest mass m_0 evolves according to a relativistically stochastic process admitting in a given inertial frame Σ' no convective contribution and an isotropic diffusivity tensor $\mathbf{D}' = D_0\mathbf{I}$ (this specific inertial frame can be referred to as the rest frame for the stochastic motion), then in all the inertial frames Σ moving with respect to Σ' at constant velocity w, say, along the x_1 coordinate, the product $\mathbf{mD}$ of the mass tensor times the diffusivity tensor is invariant with respect to w and equal to $m_0 D_0 \mathbf{I}$.

The quantity $h_m = m_0 D_0$ has the physical dimension of kgm^2/s, i.e., it corresponds to an action, so Eq. (97) implies that the stochastic action h_m is relativistically invariant. It does not depend on the relative frame velocity, but eventually it can depend on the particle rest mass.



(a) (b)

FIG. 5. (a) Plot of $D_{x,x}/D_0$ (● and line a) and $D_{y,y}/D_0$ (□ and line b) vs the relative frame velocity w for a two-dimensional Kolesnik-Kac process ($N = 5$, $b_0' = c = 1$, and $a_0' = 1$). Line a represents the function $\gamma^{-3}(w)$ [Eq. (75)] and line b the function $\gamma^{-1}(w)$ [Eq. (81)]. Symbols ● and □ correspond to the results of stochastic simulations for $D_{x,x}$ and $D_{y,y}$, respectively. (b) Diagonal diffusivities for the three-dimensional Poisson-Kac process (93) with $b_0' = c = 1$ and $a_0 = 1$. Symbols represent the results of stochastic simulations: ●, D_{x_1,x_1}; □, D_{x_2,x_2}; and ■, D_{x_3,x_3}. Lines a and b are the same as in (a).

We can further develop this concept by introducing quantum mechanical considerations. From the well known correspondence between the Schrödinger equation and stochastic processes [38,39], usually obtained via an analytic continuation of the time coordinate towards the imaginary axis, the quantum mechanical representation of the kinetic energy of a free particle corresponds to an effective quantum diffusivity D_{quantum} expressed by

$$D_{\text{quantum}} = \frac{\hbar}{2m_0}. \tag{97}$$

Therefore, for a free quantum particle, $D_0 = D_{\text{quantum}}$ and Eqs. (96) and (97) provide

$$h_m = \frac{\hbar}{2}, \tag{98}$$

i.e., the stochastic action is not only relativistically invariant but also independent of the particle rest mass. It indicates that in a stochastic representation of quantum motion, the basic constraint induced by quantization and by the Lorentz transformation is the relativistic invariance of the product of the mass tensor times the effective diffusivity tensor that, in any inertial frame, returns an isotropic tensor with eigenvalues equal to $\hbar/2$,

$$\mathbf{mD} = \frac{\hbar}{2}\mathbf{I}. \tag{99}$$

Equation (99) can be viewed as a stochastic quantization rule emerging from special relativity. Its implication will be explored elsewhere. However, a qualitative indication emerging from Eq. (99) is that, even in the low-energy limit, i.e., for the Schrödinger equation (not speaking of the Dirac counterpart), a stochastic interpretation of its formal structure could be properly grounded on a relativistic covariant framework (see [40]). This implies that the source of stochasticity, originating quantum uncertainty, should also possess covariant properties. The natural candidate possessing all these requirements is the zero-point energy fluctuations of the electromagnetic field [41,42]. A thorough analysis of this issue is left for future work.

IX. ANALYSIS OF THE DIFFUSIVITY TENSOR: SPACE-TIME DIFFUSION

The results obtained for Poisson-Kac processes are confirmed and generalized by the study of discrete stochastic space-time dynamics, which is addressed in this section. This extension provides a general expression for the relativistic transformation of the diffusivity tensor.

A. Space-time diffusion

Discrete space-time diffusion (STD) processes in $\mathbb{R}^N$ were originally introduced in [32] in order to describe in a compact way particle transport in periodic arrays of obstacles or localized repulsive potentials. A STD process is defined by the quintuple $(N,S,\boldsymbol{\pi},\{\mathbf{A}_\alpha\}_{\alpha=1}^S,\boldsymbol{\tau})$, where N is the space dimension, S is the number of states the random space-time displacements can attain, $\boldsymbol{\pi} = (\pi_1,\ldots,\pi_S)$ is an S-dimensional probability vector (with $\pi_\alpha > 0, \alpha = 1,\ldots,S$, and $\sum_{\alpha=1}^S \pi_\alpha = 1$), $\mathbf{A}_\alpha$ $(\alpha = 1,\ldots,S)$ are given (constant) space displacements in $\mathbb{R}^N$, and $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_S)$ are the corresponding time intervals (time displacements) $\tau_\alpha > 0$.

Consider an inertial system Σ defined by the space-time coordinates $(\mathbf{x},t)$. The dynamics of a STD process in Σ is defined with respect to a discrete iteration time $n = 0,1,\ldots$ as

$$(\mathbf{x}_{n+1},t_n) = (\mathbf{x}_n + \mathbf{A}_\alpha, t_n + \tau_\alpha), \tag{100}$$

with probability π_α, where $\alpha = 1,\ldots,S$. Suppose that the process is defined at $n = 0$ such that $\mathbf{x}_0 = 0$ and $t_0 = 0$ so that no issues of simultaneity arise. Equation (100) can be viewed as a stroboscopic sampling of a stochastic process in the reference system Σ.

From the theory of STD processes developed in [32], the long-term evolution for the probability density $p(\mathbf{x},t)$, $\mathbf{x} = (x_1,\ldots,x_N)$, associated with Eq. (100) converges to the solution of an effective constant-coefficient advection-diffusion equation

$$\partial_t p(\mathbf{x},t) = -\sum_{k=1}^N v_k \partial_{x_k} p(\mathbf{x},t) + \sum_{h,k=1}^N D_{h,k}\partial_{x_h}\partial_{x_k} p(\mathbf{x},t), \tag{101}$$

where $\partial_t = \partial/\partial t$, $\partial_{x_k} = \partial/\partial x_k$, and $\mathbf{v} = (v_1,\ldots,v_N)$ and $\mathbf{D} = (D_{h,k})_{h,k=1}^N$ represent the constant effective velocity vector and tensor diffusivity, respectively. Introducing the quantities

$$\begin{aligned}
V_t^{(n)} &= \sum_{\alpha=1}^S \pi_\alpha \tau_\alpha, \quad V_k^{(n)} = \sum_{\alpha=1}^S \pi_\alpha A_{\alpha,k}, \quad k = 1,\ldots,N\\
D_t^{(n)} &= \frac{1}{2}\left[\sum_{\alpha=1}^S \pi_\alpha \tau_\alpha^2 - \left(V_t^{(n)}\right)^2\right],\\
D_{t,k}^{(n)} &= \frac{1}{2}\left[\sum_{\alpha=1}^S \pi_\alpha \tau_\alpha A_{\alpha,k} - V_t^{(n)}V_k^{(n)}\right], \quad k = 1,\ldots,N\\
D_{h,k}^{(n)} &= \frac{1}{2}\left[\sum_{\alpha=1}^S \pi_\alpha A_{\alpha,h}A_{\alpha,k} - V_h^{(n)}V_k^{(n)}\right], \quad h,k = 1,\ldots,N,
\end{aligned} \tag{102}$$

where $A_{\alpha,k}$ is the kth entry of the space-displacement vector $\mathbf{A}_\alpha$, the effective transport parameters attain the expression

$$v_k = \frac{V_k^{(n)}}{V_t^{(n)}}, \quad k = 1,\ldots,N \tag{103}$$

and

$$D_{h,k} = \frac{D_t^{(n)}V_h^{(n)}V_k^{(n)}}{\left(V_t^{(n)}\right)^3} - \frac{\left[D_{t,h}^{(n)}V_k^{(n)} + D_{t,k}^{(n)}V_h^{(n)}\right]}{\left(V_t^{(n)}\right)^2} + \frac{D_{h,k}^{(n)}}{V_t^{(n)}}, \tag{104}$$

where $h = k = 1,\ldots,N$. Observe that the quantities expressed by Eqs. (102) represent the effective space-time velocity and diffusivities parametrized with respect to the discrete iteration time n.

B. Relativistic analysis

Let Eq. (100) be the dynamic description of a stochastic process in Σ and let Σ' be another inertial frame, the space-time coordinate of which is $(\mathbf{x}',t')$, moving with respect to Σ with constant relative velocity w along the x_1 axis. Set $c=1$ for the light speed *in vacuo* so that $w \in (-1,1)$. Enforcing the requirements of special relativity, the velocities of the STD process (100) should be bounded by c, which implies, since $c=1$, that

$$|\mathbf{A}|_\alpha < \tau_\alpha, \quad \alpha = 1,\dots,S. \tag{105}$$

For simplicity, consider the case $N=2$, i.e., a spatial two-dimensional model. In Σ' the STD process (100) is described by the evolution equation

$$(\mathbf{x}'_{n+1},t'_n) = (\mathbf{x}'_n + \mathbf{A}'_\alpha, t_n + \tau'_\alpha) \tag{106}$$

with probability π_α, where $\alpha = 1,\dots,S$, expressed with respect to the space-time coordinates of Σ', where the displacements $\mathbf{A}'_\alpha$ and τ'_α are related to $\mathbf{A}_\alpha$ and τ_α by a Lorentz boost

$$\begin{aligned} \tau'_\alpha &= \gamma(w)(\tau_\alpha - wA_{\alpha,1}), \\ A'_{\alpha,1} &= \gamma(w)(A_{\alpha,1} - w\tau_\alpha), \\ A'_{\alpha,2} &= A_{\alpha,2}. \end{aligned} \tag{107}$$

Given $\mathbf{A}'_\alpha$ and τ'_α, $\alpha = 1,\dots,S$, Eqs. (102)–(104) can be applied to derive the effective transport parameters measured in Σ'.

As regards the effective parameters with respect to the iteration time n, elementary algebra provides

$$\begin{aligned} &V_t^{\prime(n)} = \gamma(w)\big(V_t^{(n)} - wV_1^{(n)}\big), \quad V_1^{\prime(n)} = \gamma(w)\big(V_1^{(n)} - wV_t^{(n)}\big), \\ &V_2^{\prime(n)} = V_2^{(n)} \end{aligned} \tag{108}$$

and

$$\begin{aligned} D_t^{\prime(n)} &= \gamma^2(w)\big[D_t^{(n)} - 2wD_{t,1}^{(n)} + w^2D_{1,1}^{(n)}\big], \\ D_{t,1}^{\prime(n)} &= \gamma^2(w)\big[(1+w^2)D_{t,1}^{(n)} - w\big(D_t^{(n)} + D_{1,1}^{(n)}\big)\big], \\ D_{t,2}^{\prime(n)} &= \gamma(w)\big[D_{t,2}^{(n)} - wD_{1,2}^{(n)}\big], \\ D_{1,1}^{\prime(n)} &= \gamma^2(w)\big[D_{1,1}^{(n)} - 2wD_{t,1}^{(n)} + w^2D_t^{(n)}\big], \\ D_{1,2}^{\prime(n)} &= \gamma(w)\big[D_{1,2}^{(n)} - wD_{t,2}^{(n)}\big], \\ D_{2,2}^{\prime(n)} &= D_{2,2}^{(n)}. \end{aligned} \tag{109}$$

For the effective velocities v'_k measured in Σ', from Eqs. (103) and (108) one obtains

$$v'_1 = \frac{V_1^{\prime(n)}}{V_t^{\prime(n)}} = \frac{v_1 - w}{1 - wv_1}, \quad v'_2 = \frac{V_2^{\prime(n)}}{V_t^{\prime(n)}} = \frac{v_2\sqrt{1-w^2}}{1-wv_1}, \tag{110}$$

which correspond to the velocity transformations of special relativity.

More interesting is the transformation of the effective tensor diffusivity, i.e., how $\mathbf{D}'$ measured in Σ' is related to $\mathbf{D}$. To begin with, consider $D'_{1,1}$. Equation (104), expressed in Σ', can be written in the form of an Euclidean scalar product $\langle\cdot,\cdot\rangle$, as

$$D'_{1,1} = \frac{1}{\big(V_t^{\prime(n)}\big)^3}\langle \widetilde{\mathbf{V}}', \widetilde{\mathbf{D}}'\widetilde{\mathbf{V}}'\rangle, \tag{111}$$

where

$$\widetilde{\mathbf{V}}' = \begin{pmatrix} V_1^{\prime(n)} \\ V_t^{\prime(n)} \end{pmatrix}, \quad \widetilde{\mathbf{D}}' = \begin{pmatrix} D_t^{\prime(n)} & -D_{t,1}^{\prime(n)} \\ -D_{t,1}^{\prime(n)} & D_{1,1}^{\prime(n)} \end{pmatrix}. \tag{112}$$

Here $\widetilde{\mathbf{V}}'$ is related to $\widetilde{\mathbf{V}} = (V_1^{(n)}, V_t^{(n)})$ by a Lorentz boost $\widetilde{\mathbf{V}}' = \widehat{\mathcal{L}}_w\widetilde{V}$,

$$\widehat{\mathcal{L}}_w = \begin{pmatrix} \gamma & -\gamma w \\ -\gamma w & \gamma \end{pmatrix}. \tag{113}$$

The transformation for the entries of $\widetilde{\mathbf{D}}'$ stemming from Eq. (109) is compactly expressed by

$$\begin{pmatrix} D_t^{\prime(n)} \\ -D_{t,1}^{\prime(n)} \\ -D_{t,1}^{\prime(n)} \\ D_{1,1}^{\prime(n)} \end{pmatrix} = \gamma^2(w)\begin{pmatrix} 1 & w & w & w^2 \\ w & \frac{1+w^2}{2} & \frac{1+w^2}{2} & w \\ w & \frac{1+w^2}{2} & \frac{1+w^2}{2} & w \\ w^2 & w & w & 1 \end{pmatrix} \times \begin{pmatrix} D_t^{(n)} \\ -D_{t,1}^{(n)} \\ -D_{t,1}^{(n)} \\ D_{1,1}^{(n)} \end{pmatrix}. \tag{114}$$

The latter transformation can be expressed in tensorial form as $\widetilde{D}'_{h,k} = \Lambda_{h,k}^{m,n}\widetilde{D}_{m,n}$, where the fourth-order tensor $\Lambda_{h,k}^{m,n}$ accounts for the transformation (114) and Einstein summation notation has been adopted. Consequently, Eq. (111) becomes

$$D'_{1,1} = \frac{1}{\gamma^3(w)(1-wv_1)^3}\frac{M_{h,k}^{p,q}\widetilde{V}^h\widetilde{V}^k\widetilde{D}_{p,q}}{\big(V_t^{(n)}\big)^3}, \tag{115}$$

where $\widetilde{V}^h$ and $\widetilde{D}_{p,q}$ are the entries of $\widetilde{\mathbf{V}}$ and $\widetilde{\mathbf{D}}$, defined above, and $M_{h,k}^{p,q} = \widehat{\mathcal{L}}_{w,h}^p\Lambda_{p,q}^{h,k}\widehat{\mathcal{L}}_{w,k}^q$, $\widehat{\mathcal{L}}_{w,h}^p$ being the entries of the Lorentz boost (113). Developing the algebra, Eq. (115) yields the following expression for $D'_{1,1}$:

$$\begin{aligned} D'_{1,1} = {} & \frac{1}{\gamma^3(w)(1-wv_1)^3} \\ & = \left[\frac{D_t^{(n)}\big(V_1^{(n)}\big)^2 - 2D_{t,1}^{(n)}V_1^{(n)}V_t^{(n)} + D_{1,1}^{(n)}\big(V_t^{(n)}\big)^2}{\big(V_t^{(n)}\big)^3}\right]. \end{aligned} \tag{116}$$

The term in square brackets in Eq. (116) is just $D_{1,1}$ as measured in Σ [Eq. (104)] for $h=k=1$. Consequently, the transformation for $D_{1,1}$ attains the compact expression

$$D'_{1,1} = \frac{D_{1,1}}{\gamma^3(w)(1-wv_1)^3}. \tag{117}$$

Next consider $D'_{1,2}$, which in Σ' is given by

$$D'_{1,2} = \frac{D_t^{\prime(n)} V_1^{\prime(n)} V_2^{\prime(n)} - D_{t,1}^{\prime(n)} V_2^{\prime(n)} V_t^{\prime(n)} - D_{t,2}^{\prime(n)} V_1^{\prime(n)} V_t^{\prime(n)} + D_{1,2}^{\prime(n)} \left(V_t^{\prime(n)}\right)^2}{\left(V_t^{\prime(n)}\right)^3}. \tag{118}$$

Substituting the expressions (108) and (109) for the transport parameters in Σ' referring to the iteration time n as a function of the corresponding quantities in Σ, elementary algebra provides the expression

$$D'_{1,2} = \frac{D_{1,2} + w Q_{1,2}}{\gamma^2(w)(1 - w v_1)^3}, \tag{119}$$

where

$$Q_{1,2} = \frac{D_{1,1}^{(n)} V_2^{(n)} V_t^{(n)} - D_{1,2}^{(n)} V_1^{(n)} V_t^{(n)} - D_{t_1}^{(n)} V_1^{(n)} V_2^{(n)} + D_{t,2}^{(n)} \left(V_1^{(n)}\right)^2}{\left(V_t^{\prime(n)}\right)^3}. \tag{120}$$

Equation (104) can be used to express $D_{1,1}^{(n)}$ and $D_{1,2}^{(n)}$ as a function of $D_{1,1}$ and $D_{1,2}$. In this way, the expression for $Q_{1,2}$ greatly simplifies, providing $Q_{1,2} = D_{1,1} v_2 - D_{1,2} v_1$, and consequently the transformation relation for $D'_{1,2}$ becomes

$$D'_{1,2} = \frac{D_{1,2} + w(D_{1,1} v_2 - D_{1,2} v_1)}{\gamma^2(w)(1 - w v_1)^3}. \tag{121}$$

Finally, consider $D'_{2,2}$. From its definition

$$D'_{2,2} = \frac{D_t^{\prime(n)} \left(V_2^{\prime(n)}\right)^2 - 2 D_{t,2}^{\prime(n)} V_2^{\prime(n)} V_t^{\prime(n)} + D_{2,2}^{\prime(n)} \left(V_t^{\prime(n)}\right)^2}{\left(V_t^{\prime(n)}\right)^3}, \tag{122}$$

which can be rearranged in the form

$$D'_{2,2} = \frac{D_{2,2} + 2 w P_{2,2} + w^2 R_{2,2}}{\gamma(w)(1 - w v_1)^3}, \tag{123}$$

where

$$\begin{aligned} P_{2,2} &= \frac{D_{1,2}^{(n)} V_2^{(n)} V_t^{(n)} - D_{2,2}^{(n)} V_1^{(n)} V_t^{(n)} - D_{t,1}^{(n)} \left(V_2^{(n)}\right)^2 + D_{t,2}^{(n)} V_1^{(n)} V_2^{(n)}}{\left(V_t^{\prime(n)}\right)^3}, \\ R_{2,2} &= \frac{D_{1,1}^{(n)} \left(V_2^{(n)}\right)^2 - 2 D_{1,2}^{(n)} V_1^{(n)} V_2^{(n)} + D_{2,2}^{(n)} \left(V_1^{(n)}\right)^2}{\left(V_t^{\prime(n)}\right)^3}. \end{aligned} \tag{124}$$

Using Eq. (104) to express $D_{h,k}^{(n)}$ as a function of the diffusivities $D_{h,k}$ expressed with respect to the physical time, the expressions for $P_{2,2}$ and $R_{2,2}$ simplify to obtain for $D'_{2,2}$ the transformation

$$D'_{2,2} = \frac{D_{2,2} + 2w(D_{1,2} v_2 - D_{2,2} v_1) + w^2 \left(D_{1,1} v_2^2 - 2 D_{1,2} v_1 v_2 + D_{2,2} v_1^2\right)}{\gamma(w)(1 - w v_1)^3}. \tag{125}$$

C. Three-dimensional case

The extension of the transformations developed in the preceding section to three-dimensional spatial coordinates in straightforward. As before, suppose that Σ' moves relatively to Σ with a constant velocity w along the x_1 axis.

As regards $D'_{1,1}$, $D'_{1,h} = D'_{h,1}$, and $D'_{h,h}$, with $h = 2,3$, the expressions follow from Eqs. (117), (121), and (125), namely,

$$\begin{aligned} D'_{1,1} &= \frac{D_{1,1}}{\gamma^3(w)(1 - w v_1)^3}, \quad D'_{1,h} = \frac{D_{1,h} + w(D_{1,1} v_h - D_{1,h} v_1)}{\gamma^2(w)(1 - w v_1)^3}, \quad h = 2,3 \\ D'_{h,h} &= \frac{D_{h,h} + 2w(D_{1,h} v_h - D_{h,h} v_1) + w^2 \left(D_{1,1} v_h^2 - 2 D_{1,h} v_1 v_h + D_{h,h} v_1^2\right)}{\gamma(w)(1 - w v_1)^3}, \quad h = 2,3. \end{aligned} \tag{126}$$

The entry $D'_{2,3} = D'_{3,2}$ requires some additional calculations that, following the same approach outlined in the preceding section, return the expression

$$D'_{2,3} = \frac{D_{2,3} + w(D_{1,2} v_3 + D_{1,3} v_2 - 2 D_{2,3} v_1) + w^2 \left(D_{1,1} v_2 v_3 - D_{1,2} v_1 v_3 - D_{1,2} v - 1 v_3 + D_{2,3} v_1^2\right)}{\gamma(w)(1 - w v_1)^3}. \tag{127}$$

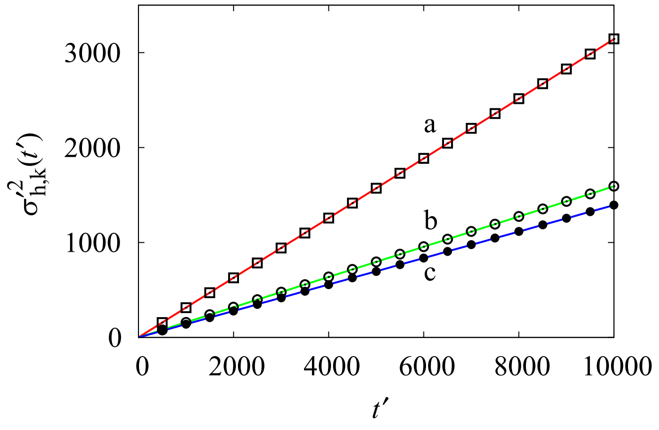


FIG. 6. Behavior of the second-order central moments $\sigma'^2_{h,k}(t')$ measured in Σ' vs t' for the two-dimensional spatial model described in the text. Symbols are the results of numerical simulation of the stochastic dynamics and solid lines are the theoretical predictions based on the Einstein scaling $\sigma'^2_{h,k}(t') = 2D'_{h,k}t'$, where $D'_{h,k}$ are given by Eqs. (117), (121), and (125): line a and □, $\sigma'^2_{1,1}$; line b and ○, $\sigma'^2_{1,2}$; and line c and ●, $\sigma'^2_{2,2}$.

This completes the analysis of the Lorentzian transformation of the tensor diffusivity referring to two inertial frames in relative motion.

D. Numerical simulations

In this section, a numerical validation of the transformation theory for the effective tensor diffusivities is provided. Consider a spatially two-dimensional STD model, $N = 2$ and $S = 3$, with $\boldsymbol{\pi} = (0.5,0.1,0.3)$, $\boldsymbol{\tau} = (1,0.6,0.5)$, $\mathbf{A}_1 = (0.8,0.4)$, $\mathbf{A}_2 = (-0.4,0.3)$, and $\mathbf{A}_3 = (0.48,-0.05)$.

Numerical simulations of STD dynamics have been performed by considering an ensemble of 10^7 particles, initially located at the same space-time point $(\mathbf{x}_0, t_0 = 0)$ evolving according to Eq. (100). Using the Lorentz boost, the corresponding coordinates $\mathbf{x}'_n$ and t'_n in Σ' can be derived, and from the linear scaling of the first and second-order (central) moments with respect to t', the values v'_k and $D'_{h,k}$ can be estimated.

Figure 6 depicts the scaling of the second-order central moments $\sigma'^2_{h,k}(t') = \langle [x'_h(t') - \langle x_h(t')\rangle][x'_k(t') - \langle x_k(t')\rangle]\rangle$ for $w = 0.7$. Solid lines represent the theoretical predictions $\sigma'^{,2}_{h,k}(t') = 2D'_{h,k}t'$, where $D'_{h,k}$ are given by Eqs. (117), (121), and (125), while symbols refer to the results of the numerical simulations of the stochastic STD model.

The review of the value of the effective transport parameters measured in Σ' vs the relative velocity w can be found in Fig. 7. Figure 7(a) refers to the effective velocity entries and Fig. 7(b) to the effective tensor diffusivities.

Apart from the excellent agreement between theory (lines) and simulations (symbols), the behavior of $D'_{h,k}$ vs w is highly nontrivial and the effective diffusivities display a local maximum at some values $w^*_{h,k}$ of the relative velocity that depend on h and k. This phenomenon is a consequence of the presence in the STD model considered of an advective contribution, accounted for by the effective velocity v_1, that is significantly greater than zero. Consequently, the factor $(1 - wv_1)^3$ in the denominator of the expressions for $D'_{h,k}$ modulates their behavior, determining nonmonotonic effects vs w. For example, in the case of $D'_{1,1}$, the abscissa $w^*_{1,1}$ of the local maximum equal v_1 itself, in the present case $v_1 \simeq 0.726$, and $D'_{1,1}(w^*_{1,1})$ is almost three times larger than $D_{1,1}$. For $w \to 1$, all the diffusivities $D'_{h,k}$ decay to zero, as $D'_{1,1} \sim \gamma^{-3}(w)$, $D'_{2,2} \sim \gamma^{-1}(w)$, and $D'_{1,2} \sim \gamma^{-2}(w)$.

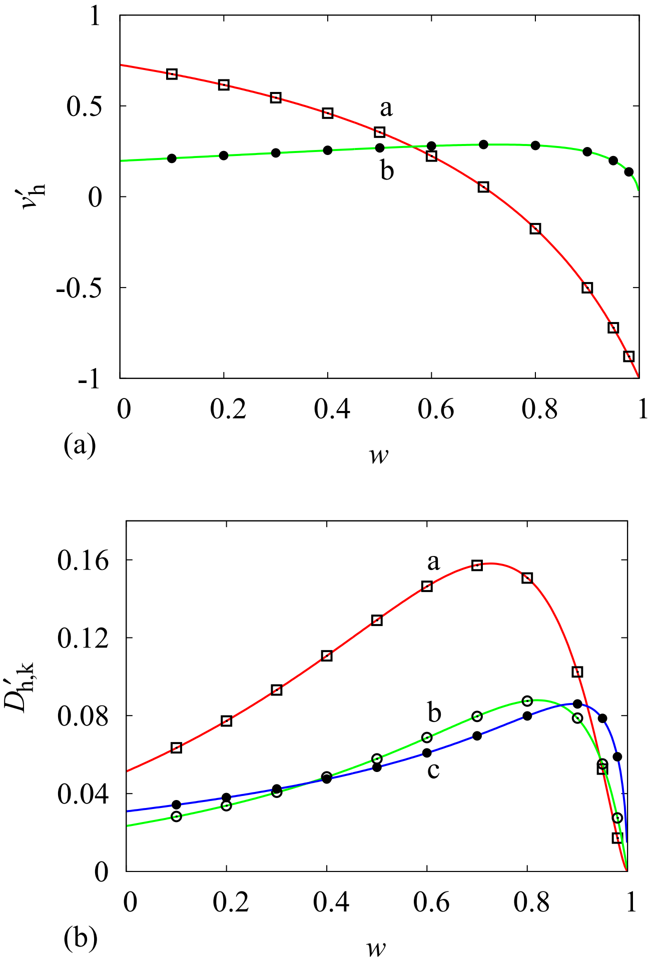


FIG. 7. Effective transport parameters of the spatial two-dimensional STD model described in the text measured in Σ' as a function of the relative frame velocity w. (a) Effective velocity entries: line a and □, v'_1; line b and ○, v'_2. (b) Effective tensor diffusivities: line a and □, $D'_{1,1}$; line b and ○, $D'_{1,2}$; and line c and ●, $D'_{2,2}$.

The behavior of the tensor diffusivities in Σ' for a spatially three-dimensional STD model ($N = 3$) is depicted in Fig. 8. The STD model considered admits $S = 6$ states with $\boldsymbol{\pi} = (0,1,0.2,0.3,0.1,0.2,0.1)$, $\boldsymbol{\tau} = (1,2,0.5,5,3,0.8)$, $\mathbf{A}_1 = (0.5,-0.3,0.2)$, $\mathbf{A}_2 = (1,-0.4,0.2)$, $\mathbf{A}_3 = (1,0.05,-0.05)$, $\mathbf{A}_4 = (-1.5,0.4,0.2)$, $\mathbf{A}_5 = (1.5,1,0.6)$, and $\mathbf{A}_6 = (0.3,-0.2,0.3)$. Also in this case, the agreement of the theoretical predictions (solid lines) based on Eqs. (126) and (127) with respect to the stochastic simulation data (symbols) is excellent.

X. DISCUSSION AND IMPLICATIONS

In this section, some implications and observations related to the transformation theory of tensor diffusivities are addressed.

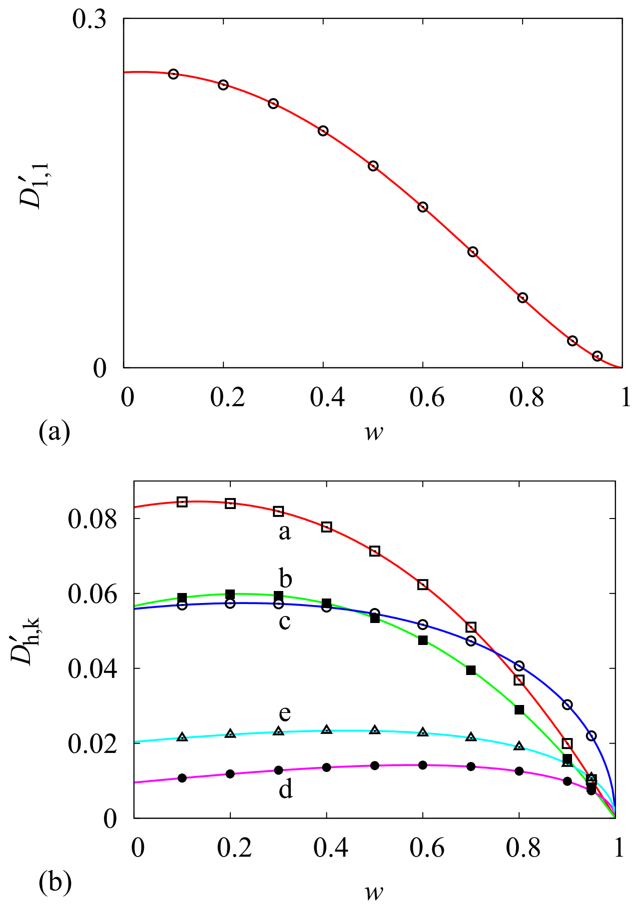


FIG. 8. Effective tensor diffusivities $D'_{h,k}$ vs the relative frame velocity w for the spatial three-dimensional model described in the text: (a) $D'_{1,1}$ vs w and (b) $D'_{h,k}$ vs w. Line a and □ denote $D'_{1,2}$; line b and ■, $D'_{1,3}$; line c and ○, $D'_{2,2}$; line d and ●, $D'_{2,3}$; and line e and △, $D'_{3,3}$.

A. Observation 1: Generality of the transformation

Although we have considered specific stochastic kinematics, i.e., Poisson-Kac processes and STD dynamics, Eqs. (126) and (127) are of general validity. To support and confirm this claim let us consider a totally different problem, fairly unusual in a relativistic context.

Consider a classical Langevin equation in $\mathbb{R}^3$,

$$dx_h(t) = v_h dt + \sqrt{2D_h} dw_h(t), \quad h = 1,2,3, \tag{128}$$

where $w_h(t)$, $h = 1,2,3$, are the realizations of three independent Wiener processes and v_h and D_h are constant. This model is obviously relativistically inconsistent as, by definition, a Wiener process possesses infinite propagation velocity [and so do the $x_h(t)$ defined by Eq. (128)], due to its Gaussian distribution of increments. Therefore, in order to use Eq. (128) in the present analysis, this process should be somehow rectified. The correction we apply is conceptually similar to the classical Wong-Zakai mollification of Brownian motion [43,44] (see also [45]).

Let $\{\widetilde{\mathbf{x}}_\alpha(t)\}_{\alpha=1}^{N_p}$ be an ensemble of N_p particles moving according to the stochastic kinematics (128), starting from $\mathbf{x}_\alpha(t=0) = 0$, and let $\{\widetilde{\mathbf{x}}_\alpha^{(n)} = \mathbf{x}_\alpha(nT)\}_{\alpha=1}^{N_p}$, $n = 0,1,\ldots$, be their stroboscopic sampling at multiples of the time interval $T > 0$. In order to make the sampling $\{\widetilde{\mathbf{x}}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$ of the stochastic dynamics relativistically consistent, particle positions should be filtered in order to ensure a propagation velocity less than c ($c = 1$ in the present analysis). Let $\{\mathbf{x}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$ be the filtered stroboscopic sampling of $\{\widetilde{\mathbf{x}}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$, obtained in the following way: (i) $\mathbf{x}_\alpha^{(0)} = \widetilde{\mathbf{x}}_\alpha^{(0)} = 0$ for all $\alpha = 1,\ldots,N$; (ii) choose a reference maximum velocity $v_{\max} < c$, say, $v_{\max} = 0.99$; (iii) if $\delta_\alpha^{(n)} = \|\widetilde{\mathbf{x}}_\alpha^{(n)} - \mathbf{x}_\alpha^{(n-1)}\| > v_{\max} T$, i.e., if the relativistic velocity constraint could be locally violated, set for $\mathbf{x}_\alpha^{(n)}$ the value

$$\mathbf{x}_\alpha^{(n)} = \frac{v_{\max} T}{\delta_\alpha^{(n)}} \widetilde{\mathbf{x}}_\alpha^{(n)} + \left(1 - \frac{v_{\max} T}{\delta_\alpha^{(n)}}\right) \mathbf{x}_\alpha^{(n-1)}, \tag{129}$$

corresponding to a local propagation velocity equal to $v_{\max}$. By definition, the filtered stroboscopic sampling $\{\mathbf{x}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$, $n = 0,1,2,\ldots$, is relativistically consistent and can be viewed as a form of Wong-Zakai mollification of the original process, by considering the stochastic trajectories between time instants $(n-1)T$ and nT represented by straight lines connecting $\mathbf{x}_\alpha^{(n-1)}$ to $\mathbf{x}_\alpha^{(n)}$.

Consequently, $\{\mathbf{x}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$ can be viewed as an ensemble of realizations of a relativistically plausible stochastic process defined in an inertial system Σ, out of which its transport parameters can be estimated. If the velocities and the diffusivities are small enough, the effective transport parameters estimated in Σ practically coincide with v_h and D_h, i.e., with the velocity and diagonal entries of the diffusivity tensor in Eq. (128). Enforcing the Lorentz boost, the effective transport parameters estimated in a reference Σ' moving with respect to Σ with constant relative velocity $w < 1$ along the x_1 axis can be obtained.

Consider for the velocities and diffusivities in Eq. (129) the values $v_1 = a$, $v_2 = -0.2$, $v_3 = 0$, $D_1 = 0.05$, $D_2 = 0.03$, and $D_3 = 0.01$, where a is a parameter, and let $d_{h,h}(w) = D'_{h,h}(w)/D_h$ for $h = 1,2,3$ and $d_{h,k}(w) = D'_{h,h}(w)$ for $h \neq k$. Figures 9(a)–9(f) depict the six independent entries of $d_{h,k}(w)$ as a function of the frame velocity w at four different values of the parameter $a = 0,0.2,0.4,0.6$ controlling convective particle motion along the x_1 direction.

Simulation results refer to the ensemble $\{\mathbf{x}_\alpha^{(n)}\}_{\alpha=1}^{N_p}$ obtained from Eq. (128) using the Wong-Zakai filtering discussed above with a sampling time $T = 20$ and $N_p = 10^5$. Stochastic trajectories have been obtained from Eq. (128) using a classical Euler-Langevin algorithm with a time step $\Delta t = 10^{-3}$. In the present simulations, involving a rather small ensemble of particles, Eq. (129) has never been used, and only the Wong-Zakai linear interpolation between $\mathbf{x}_\alpha^{(n-1)}$ and $\mathbf{x}_\alpha^{(n)}$ has been applied in order to obtain the particle position at a constant value of time t' measured in Σ'. Solid lines in Fig. 9 refer to the theoretical predictions based on Eqs. (126) and (127), where $D_{h,k} = D_h d_{h,k}$, with D_h the diagonal diffusivity in Eq. (128). Excellent agreement between theory and stochastic simulations can be observed, confirming the general validity of Eqs. (126) and (127).

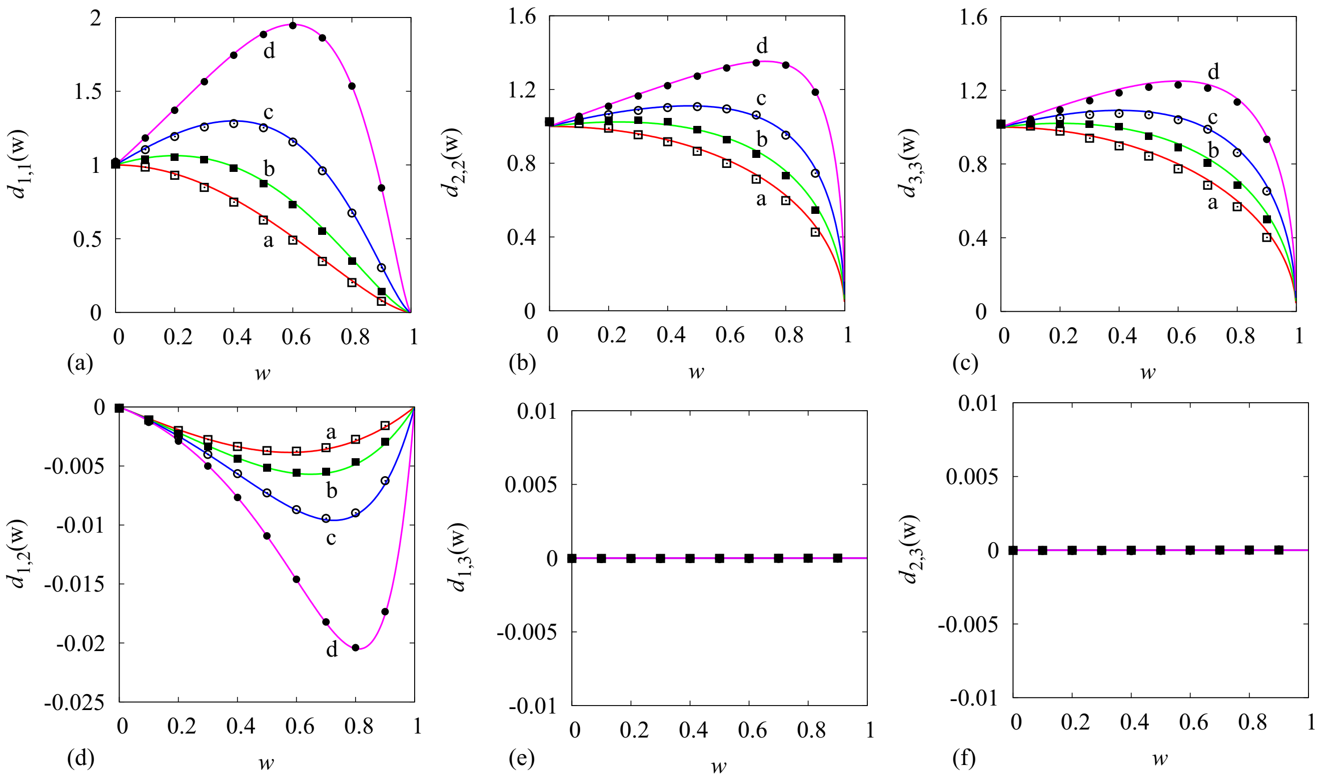


FIG. 9. Plot of $d_{h,k}(w)$ vs w for the filtered Wong-Zakai mollification of the process (128) sampled at $T = 20$: (a) $d_{1,1}(w)$, (b) $d_{2,2}(w)$, (c) $d_{3,3}(w)$, (d) $d_{1,2}(w) = d_{2,1}(w)$, (e) $d_{1,3}(w) = d_{3,1}(w)$, and (f) $d_{2,3}(w) = d_{3,2}(w)$. Lines refer to Eqs. (126) and (127) and symbols to numerical simulations: line a, $a = 0$; line b, $a = 0.2$; line c, $a = 0.4$; and line d, $a = 0.6$.

B. Observation 2: Poisson-Kac processes and the limit for $w \to c$

The analysis developed for STD processes is in agreement with the results obtained for the relativistic kinematics of Poisson-Kac processes. In the latter case $D_{1,2} = 0$ and the convective contributions are absent, i.e., $v_1 = v_2 = 0$. Correspondingly, Eqs. (126) and (127) predict the scaling of the longitudinal $D'_{1,1}$ and transversal $D'_{2,2}$ diffusivities given by Eq. (94).

Particularly interesting is the limit of these equations for w approaching $c = 1$. In this case, the measured diffusivities in Σ' vanish identically. For an observer that moves close to the velocity of light, the contribution of external stochastic perturbations, the intensity of which is related to the diffusivities $D'_{h,k}$, becomes progressively irrelevant as $w \to c$. In point of fact, the vanishing properties of $D'_{h,k}$ for $w \to c$ could have a deeper physical meaning: This indicates that in a reference system, moving with a velocity approaching that of light, all the external dissipative and irreversible processes associated with stochastic fluctuations decay to zero.

C. Observation 3: Relativity of stochasticity and determinism

There is another interesting implication of the transformation theory of the tensor diffusivity. From Eqs. (126) and (127) it follows that the effective diffusivities $D'_{h,k}$ measured in Σ' depend on the convective velocities v_h. Apart from the term $(1 - wv_1)^{-3}$, this dependence enters as factors multiplying the $O(w)$ and $O(w^2)$ terms in the expressions for $D'_{h,k}$. The only diffusivity entry that is not influenced by these convective contributions is the longitudinal diffusivity since $D'_{1,1} = D_{1,1}/\gamma^3(w)(1 - wv_1)^3$.

This observation suggests that it may happen that a process that is regarded as fully deterministic in a reference system Σ appears to possess a stochastic nature in Σ' and vice versa. To clarify this concept it is convenient to consider a simple example. Consider a STD process in Σ (referred to as model I) with $N = 2$ and $S = 2$ characterized by the parameters

$$\pi_1 = \pi_2 = \frac{1}{2}, \quad \tau_1 = \tau_2 = 1,$$
$$\mathbf{A}_1 = \begin{pmatrix} 0.8 \\ 0.5 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} -0.8 \\ 0.5 \end{pmatrix}. \tag{130}$$

For this process $v_1 = 0$, $v_2 = 0.5$, $D_{1,1} = 0.32$, and $D_{1,2} = D_{2,1} = D_{2,2} = 0$. Let $x = x_1$ and $y = x_2$. In the long-term limit, the dynamics of this process in Σ is described by the probability density function $p(x,y,t)$ that approaches the solution of the parabolic transport equation

$$\partial_t p(x,y,t) = -v_2 \partial_y p(x,y,t) + D_{1,1} \partial_x^2 p(x,y,t). \tag{131}$$

The marginal probability density $p_y(y,t) = \int_{-\infty}^{\infty} p(x,y,t)dx$ of the y process satisfies a strictly deterministic advection equation

$$\partial_y p_y(y,t) = -v_2 \partial_y p_y(y,t), \tag{132}$$

which follows directly from the inspection of the structure of the space-time displacements (130) characterizing this model.

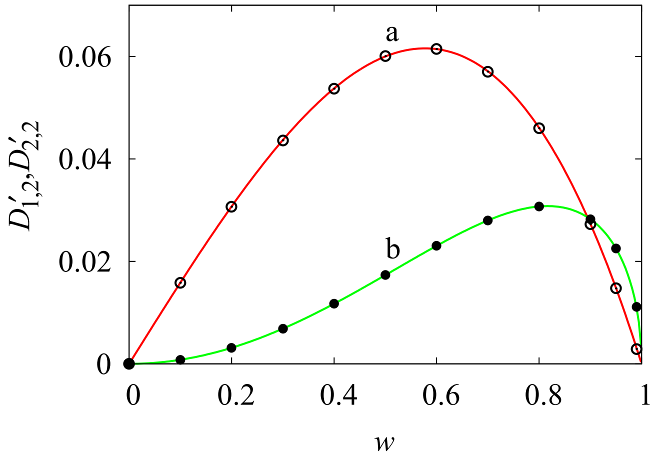


FIG. 10. Diffusivities $D'_{1,2}$ and $D'_{2,2}$ measured in Σ' vs the relative frame velocity w for model I discussed in the text: line a and ○, $D'_{1,2}$; line b and •, $D'_{2,2}$. Solid lines represent the theoretical predictions (126) and (127) and symbols the results of the numerical simulations of the stochastic STD dynamics (130).

Viewed from Σ, the evolution of the y dynamics defines a strictly deterministic process.

Consider the same process viewed by Σ' moving with respect to Σ with constant relative velocity $w > 0$ along the x axis. In this case the entries $D'_{1,2}$ and $D'_{2,2}$ of the diffusivity tensor are different from zero, specifically

$$D'_{2,2} = \frac{D_{1,1} w^2 v_2^2}{\gamma(w)(1 - w v_1)^3} > 0. \tag{133}$$

This phenomenon is depicted in Fig. 10, where the theoretical expressions for $D'_{1,2}$ and $D'_{2,2}$ (solid lines) are compared with numerical simulations of the STD (130). Therefore, the marginal probability density function $p'_{y'}(y',t')$ for the transversal y' process in Σ' approaches the solution of the advection-diffusion equation

$$\partial_{t'} p'_{y'}(y',t') = -v'_2 p_{y'} + D'_{2,2}\partial^2_{y'} p'_{y'}(y',t'), \tag{134}$$

corresponding to the evolution of a stochastic process characterized by an effective diffusivity $D'_{2,2} > 0$.

The reverse is also true, by adopting the same argument. Consider a stochastic process in Σ, for which $D_{2,2} \neq 0$. By tuning the relative velocity w of Σ', it can happen that $D'_{2,2} = 0$. Consequently, what appears in Σ as a stochastic dynamics is qualified in Σ' as strictly deterministic. This phenomenon is depicted in Fig. 11, where v_h and $D_{h,k}$ are given by $v_1 = -0.5$, $v_2 = 0.433$, $D_{1,1} = 0.208$, $D_{1,2} = 0.060$, and $D_{2,2} = 0.0173$ (referred to as model II), corresponding to the values of v'_h and $D'_{h,k}$ obtained in the STD model depicted in Fig. 9 at $w = 0.5$, and Σ' moves with respect to Σ with a negative relative velocity w.

D. Observation 4: Diffusivity and Markovian processes in the Minkowski space-time

The concept of tensor diffusivity for a relativistic stochastic process is a long-term emerging property, exactly as for Poisson-Kac processes that are characterized by an effective diffusivity for time scales much larger than the characteristic recombination time among their partial probability waves. A local (pointwise) diffusivity (possessing the dimension of m^2/s) cannot be defined in a Minkowski space-time $\mathcal{M}_4$, as it would be necessarily associated with fluctuations possessing a local almost-everywhere-nondifferentiable structure as a function of time and consequently an unbounded local propagation velocity.

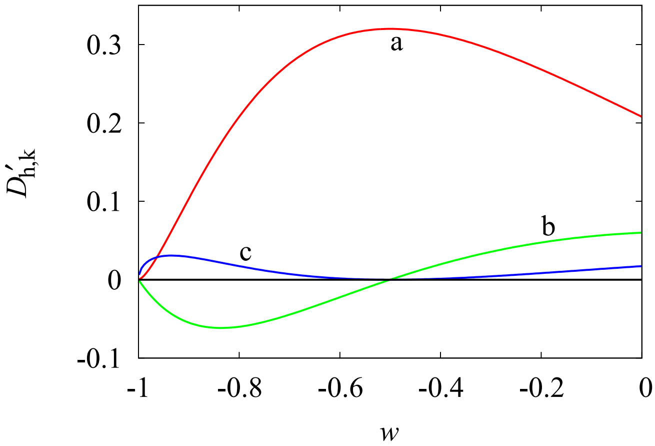


FIG. 11. Diffusivities $D'_{h,k}$ vs the relative frame velocity w measured in Σ' for model II discussed in the text: line a, $D'_{1,1}$; line b, $D'_{1,2}$; and line c, $D'_{2,2}$.

There is another general observation arising from the analysis developed in Sec. IX related to the Markovian nature of a stochastic process in a Minkowski space-time. From the works by Dudley and Hakim quoted in the Introduction, the impossibility of defining a strictly Markovian stochastic kinematics (continuous in time) in $\mathcal{M}_4$ follows. The case of STD processes introduced in Sec. IX provides a concrete example of a random dynamics for which the choice of a discrete-time parametrization (the iteration time n) makes it possible to define a strictly Markovian process in $\mathcal{M}_4$, where space and time variables are treated on equal footing by defining the space-time displacements $(\mathbf{A}_h,\tau_h)$. The example of STD processes does not contradict the Dudley-Hakim condition, as the extension with respect to a continuous-time variable of STD processes, associated with the concept of hyperbolic homogenization [33], leads to non-strictly Markovian processes in $(\mathbf{x},t)$ analogous to Poisson-Kac and generalized Poisson-Kac processes.

E. Observation 5: Skewed structure of the transformation

Given a stochastic process characterized by bounded propagation velocity less than c, let $\mathbf{v}$ and $\mathbf{D}$ be its effective (long-term) transport properties in a reference frame Σ, and $\mathbf{v}'$ and $\mathbf{D}'$ the corresponding quantities measured in Σ', moving with respect to Σ with constant velocity w along the x_1 axis. Here $\mathbf{D}$ and $\mathbf{D}'$ are the diffusivity tensors in the two reference frames.

As regards the effective velocity, the transformation from $\mathbf{v}$ to $\mathbf{v}'$ is the classical velocity transformation of special relativity

$$\mathbf{v}' = \mathcal{V}_w[\mathbf{v}] \tag{135}$$

for a relative velocity w. For the tensor diffusivity the transformation expressed by Eqs. (126) and (127) can be compactly indicated as

$$\mathbf{D}' = \mathcal{D}_w[\mathbf{D},\mathbf{v}]. \tag{136}$$

Observe the skew-product structure of Eq. (136) in which the transformation for the diffusivity tensors depends on $\mathbf{v}$. In a more compact form, Eqs. (135) and (136) can be summarized by the complete transformation of the effective transport parameters $(\mathbf{v},\mathbf{D})$,

$$(\mathbf{v}',\mathbf{D}') = \mathcal{T}_w[(\mathbf{v},\mathbf{D})]. \tag{137}$$

Obviously,

$$\mathcal{T}_0 = \mathcal{I}\,, \quad \mathcal{T}_w^{-1} = \mathcal{T}_{-w}, \tag{138}$$

where $\mathcal{I}$ denotes the identity operator. Furthermore, the skew product nature of Eq. (137) implies

$$\mathcal{D}_{-w}[\mathcal{D}_w[\mathbf{D},\mathbf{v}],\mathcal{V}_w[\mathbf{v}]] = \mathbf{D} \quad \forall \mathbf{v} \in (-c,c)^3. \tag{139}$$

XI. CONCLUSION

The purely kinematic investigation of stochastic processes in the Minkowski space-time opens up interesting perspectives in the analysis of the relativistic implications of stochasticity and determinism. Using Poisson-Kac processes first and STD dynamics subsequently, the relativistic transformation of the tensor diffusivity has been derived. The relativistic concept of tensor diffusivity is essentially an emergent physical property that makes sense in the long-term large-distance limit [46]. Particularly interesting are the conceptual implications involving the meaning of stochasticity in a relativistic framework, which in some sense is frame dependent, i.e., it depends on the observer's velocity.

Two main observations should be pointed out. The first is the relativistic invariance of a quantity having the dimension of an action, for a particle of rest mass m_0 moving of purely diffusive motion, i.e., in the case where the effective convective contributions are vanishing. Further analysis will clarify whether this is just a nice coincidence or it admits more fundamental quantum mechanical implications associated with the definition of the Planck constant h. The second observation is the fading of diffusivities measured in inertial systems Σ' moving with a relative velocity w approaching that of light, i.e., $\lim_{w\to c} D'_{h,k}(w) = 0$. For an observer in Σ', all the effects associated with stochasticity (e.g. irreversibility and dissipation) are suppressed for $w \to c$. In this framework, the concept of light velocity seems to acquire a different thermodynamic meaning as the threshold velocity at which external stochastic irreversible processes lose their dissipative nature and approach a strictly deterministic dynamics. The extension of this purely kinematic analysis of stochastic processes within the Riemannian space-time of general relativity is left for future work.

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