Multichain models of conserved lattice gas

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Conserved lattice-gas models in one dimension exhibit absorbing state phase transition (APT) with simple integer exponents $\beta = 1 = \nu = \eta$, whereas the same on a ladder belong to directed percolation (DP) universality. We conjecture that additional stochasticity in particle transfer is a relevant perturbation and its presence on a ladder forces the APT to be in the DP class. To substantiate this we introduce a class of restricted conserved lattice-gas models on a multichain system ($M \times L$ square lattice with periodic boundary condition in both directions), where particles which have *exactly one vacant neighbor* are active and they move deterministically to the neighboring vacant site. We show that for odd number of chains, in the thermodynamic limit $L \to \infty$, these models exhibit APT at $\rho_c = \frac{1}{2}(1 + \frac{1}{M})$ with $\beta = 1$. On the other hand, for even-chain systems transition occurs at $\rho_c = \frac{1}{2}$ with $\beta = 1,2$ for $M = 2,4$, respectively, and $\beta = 3$ for $M \geqslant 6$. We illustrate this unusual critical behavior analytically using a transfer-matrix method.

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I. INTRODUCTION

In the study of absorbing state phase transition (APT) [1], directed percolation (DP) [2] has been considered to be the most robust universality class. Critical behavior encountered in many diverse problems, such as synchronization [3], damage spreading [4], depinning transition [5], catalytic reactions [6], forest fire [7], extinction of species [8], etc., belong to the DP universality class [9]. It has been conjectured [10] that in the absence of any special symmetry or quenched randomness, APT in systems following short-ranged dynamics, characterized by a non-negative fluctuating scalar order parameter, belongs to DP. The presence of additional conservation laws, such as particle-hole symmetry [11], conservation of parity [12], and symmetry between different absorbing states [13] lead to different universalities. Non-DP behavior has also been reported in sandpile models [14] where the order parameter itself does not have any additional symmetry but it is coupled to a conserved density or height field [15]. In fact, the existence of a conserved density field is not a sufficient criterion to characterize various universality classes; the noise in the order parameter field due to the dynamics plays a crucial role. There are many examples of systems belonging to the DP universality class in the presence of a conserved field, most important being the conserved Manna models [16–20]. These models have recently been claimed to belong to the DP class [21] contrary to the common belief that they exhibit non-DP critical behavior. Sticky sand-piles are another generic class of models [22] which show DP behavior in the presence of conserved fields.

APT in the presence of a conserved field [20,23,24] has been a subject of interest in recent years. The conserved lattice-gas (CLG) model [25,26] and some of its extensions [27,28] are exactly solvable in one dimension and they provide clear examples of non-DP behavior. These models are rather simple having trivial integer exponents. Some variations of CLG models also show continuously varying critical exponents or multicritical behavior [27,28]. CLG-like models in one dimension (1D) are known to flow to the DP universality class when density conservation is violated [29]. Thus, one might naively think that the non-DP behavior observed in aforesaid models are due to the presence of the conserved density. However, this is not true because the usual CLG dynamics on a ladder geometry, although density conserving, lead to an absorbing transition belonging to the DP class [21]. In this article, we propose that the reason for the CLG in 1D (1DCLG) belonging to a universality class different from DP is that the particle transfer is deterministic. We show that if the dynamics is restricted so that particles hop deterministically the CLG models on a ladder belong to the universality class of 1DCLG. A natural question is then, what is the nature of the absorbing transition in multichain systems?

The multichain systems introduced in this article can be solved using a transfer-matrix method by expressing the steady-state weights as the trace of the product of matrices formed by replacing each rung by a representative matrix. When the number of chains M is an odd integer, the system exhibits an APT at density $\frac{1}{2}(1 + \frac{1}{M})$ belonging to the 1DCLG universality class with the order parameter exponent $\beta = 1$. On the other hand, for even number of chains critical density turns out to be $\frac{1}{2}$ and the order parameter exponent for large $M > 4$ is $\beta = 3$, with an unusual finite size effect for small M : $\beta = 1,2$ for $M = 2,4$, respectively.

The article is organized as follows. In Sec. II we introduce the restricted CLG dynamics and study APT in these models on a ladder geometry. Here we introduce the transfer-matrix formalism and obtain the critical exponents β, ν, η. In Sec. III we generalize the model for $M > 2$ and study the odd and even M chains in separate subsections. Finally, in Sec. IV we conclude and discuss some important issues of the multichain systems of the conserved lattice-gas model with deterministic particle transfer.

II. THE MODEL

The conserved lattice-gas model in one dimension [23,25,26] is defined by the dynamics,

$$110 \to 101; 011 \to 101. \tag{1}$$

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The dynamics conserves the number of particles N or the density $\rho = \frac{N}{L}$. The first part of the dynamics $110 \to 101$, which corresponds to rightward hopping, is effectively a combination of $1100 \to 1010$ and $1101 \to 1011$, of which the former one destroys the consecutive zeros (00's) and consecutive ones (11's) if present in the system and the second part is 00- and 11-conserving. The same is true for left hop $011 \to 101$. Thus, the number of consecutive zeros (CZs) and consecutive 1's can only reduce as the system evolves. Once the system leaves a configuration with a higher number of CZs to another with a lower number of CZs (which necessarily have a lower number of 11's, as 11's are destroyed along with 00's), it never visits the same configuration again; for $\rho \leqslant \frac{1}{2}$ (i.e., when the number of 00's is larger than the number of 11's) the system eventually destroys all consecutive 1's and reaches an absorbing configuration which cannot evolve following dynamics (1). On the other hand, when $\rho > \frac{1}{2}$, all consecutive 0's are destroyed and the system remains active. From the exact results [26] one knows that the absorbing transition takes place at $\rho_c = \frac{1}{2}$ with critical exponents $\beta = 1 = \nu = \eta$.

To generalize the model to a ladder and multichain system, we notice that the dynamics of the 1DCLG model can be interpreted in two ways: (a) particles having one occupied neighbor are active and they move to the neighboring vacant site with unit rate, or (b) particles which have *exactly* one vacant neighbor move to *that* vacant site. A natural extension of (a) to a two-chain system (a ladder) would result in a dynamics, where particles having at least one occupied neighbor are active and they move to one of the available vacant sites. The dynamics is stochastic here, as each site on a ladder has three nearest neighbors and an active particle may have more than one vacant nearest neighbors where it must choose one of them randomly and independently and hops to that site. This model was studied in [21], which showed that CLG on a ladder exhibits an absorbing phase transition belonging to the DP universality class. Interpretation (b) can also be extended to a two-chain CLG model, where the particle-transfer dynamics would be deterministic; this is because, now each of the active particles has *exactly one vacant neighbor* and it hops deterministically to that site. In the following, we study these dynamics and show that with the deterministic particle-transfer dynamics, CLG on a ladder belongs to the universality class of 1DCLG, with exponents $\beta = 1 = \nu = \eta$. Moreover, we observe that these models on a multichain system show many interesting features, which we discuss in the next section. First we study the model for a two-chain system and show that this quasi-1D system with deterministic particle-transfer dynamics is not different from their one-dimensional counterpart.

A. CLG model on a ladder with deterministic particle transfer

The two-chain model ($M = 2$) is defined on a periodic one-dimensional ladder of length L, i.e., the total number of sites is $2L$ labeled by $i = 1,2,\ldots,2L$. Each site i of the ladder is either vacant or occupied by at most one particle; correspondingly the site i is denoted by $s_i = 0,1$. A generic configuration of the system is thus represented by

$$C \equiv \begin{Bmatrix} \cdots & s_{i-1} & s_i & s_{i+1} & \cdots \\ \cdots & s_{i-1+L} & s_{i+L} & s_{i+1+L} & \cdots \end{Bmatrix}.$$

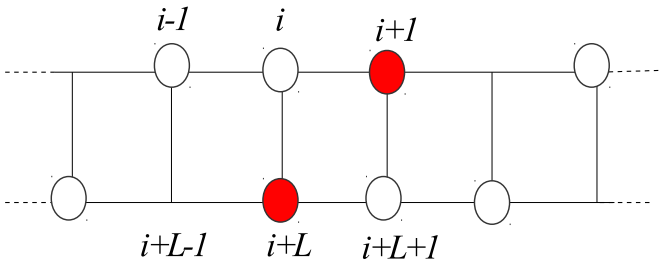

FIG. 1. Schematic description of the model: Particles having two occupied nearest neighbors and one vacant nearest neighbor are active (filled circle), whereas all other particles are inactive (open circle). The active particles hop to the *only* vacant nearest neighbor they have.

Particles are allowed to hop to the neighboring vacant site with a rate that depends upon the total number of occupied neighbors: those having exactly two occupied neighbors (out of three) hop with unit rate to the *only* vacant neighboring site they have. A schematic description of the dynamics is shown in Fig. 1. We follow a random sequential update rule. All the possible hopping scenarios are listed below, where active particles are shown with a $\hat{1}$ and sites marked $*$ can be in any state, vacant or occupied.

$$\begin{Bmatrix} *1* \\ 1\hat{1}0 \end{Bmatrix} \to \begin{Bmatrix} *1* \\ 101 \end{Bmatrix}; \quad \begin{Bmatrix} *1* \\ 0\hat{1}1 \end{Bmatrix} \to \begin{Bmatrix} *1* \\ 101 \end{Bmatrix},$$
$$\begin{Bmatrix} 1\hat{1}0 \\ *1* \end{Bmatrix} \to \begin{Bmatrix} 101 \\ *1* \end{Bmatrix}; \quad \begin{Bmatrix} 0\hat{1}1 \\ *1* \end{Bmatrix} \to \begin{Bmatrix} 101 \\ *1* \end{Bmatrix}, \qquad (2)$$
$$\begin{Bmatrix} 1\hat{1}1 \\ *0* \end{Bmatrix} \to \begin{Bmatrix} 101 \\ *1* \end{Bmatrix}; \quad \begin{Bmatrix} *0* \\ 1\hat{1}1 \end{Bmatrix} \to \begin{Bmatrix} *1* \\ 101 \end{Bmatrix}.$$

Note that, unlike CLG on a ladder studied in [21], here the dynamics is restricted to follow deterministic particle transfer. However, the dynamics has a stochastic component coming from the random sequential update rules.

It is evident from the dynamics that the total number of particles N or equivalently the particle density $\rho = \frac{N}{2L}$ is conserved. It can be understood that the number of active particles (i.e., which can hop) in the system depends on the density of particles ρ. For low densities all the particles will be able to organize themselves such that none of them are surrounded by two occupied nearest neighbors. Hence activity in the system will cease and the system is expected to fall into an absorbing state. On the other hand, for large densities, many particles would have more than one occupied neighbor and hence the system can remain active. Thus one expects an absorbing state phase transition (APT) to take place when the density of the system is decreased below a critical threshold ρ_c. Our aim is to characterize the critical behavior of this APT.

For CLG on a ladder with the deterministic particle-transfer dynamics (2), when an active particle at site i hops to the vacant nearest neighbor it creates a vacancy at i which is now surrounded by occupied sites. Thus particle hopping can never create additional consecutive 0's (CZs), either in horizontal or in vertical directions. The existing consecutive 0's, if present in the initial configuration, can only decrease with time. Thus, starting from any initial configuration the system would reach

a stationary state, with minimum number of CZs. Since for density $\rho < \frac{1}{2}$ all configurations must have some CZs, the stationary state is expected to be absorbing. On the other hand, when density $\rho \geqslant \frac{1}{2}$, the dynamics (2) is expected to get rid of all CZs present in the initial configuration and the stationary state, like 1DCLG [26], would be devoid of CZs. Thus the stationary configurations of the system are composed of rungs (the vertical supports) which do not have any CZs. Explicitly, among the four possible rungs $\{{}^0_0\}$, $\{{}^0_1\}$, $\{{}^1_0\}$, and $\{{}^1_1\}$, the stationary configurations of the system with $\rho \geqslant \frac{1}{2}$ are composed of only three; the rung $\{{}^0_0\}$ must be absent. To keep track of the number of particles, we denote the rungs by two indices n and k; n the number of particles in the rung, and $k = 1,2,\ldots,\kappa_n$ is a running index that distinguishes different rungs in a given n-particle sector. Here, $\kappa_1 = 2$, $\kappa_2 = 1$ denotes the number of rungs in $n = 1,2$ particle sectors, respectively.

The configurations in the stationary state are now

$$C \equiv \{n_1k_1,n_2k_2,\ldots,n_Lk_L\} \equiv \{n_ik_i\}. \tag{3}$$

However, any arbitrary combination of these three rungs i not allowed in the stationary state as they may produce CZs. For example, repetition of rungs $\{{}^0_1\}$ or $\{{}^1_0\}$, which create CZs in the horizontal direction, must be absent in the stationary state.

It is evident that in the absence of CZs, a particle that hops from site i to j would find that two of its neighbors at j are already occupied and thus, hopping of this active particle at site j to the vacant neighbor i is also allowed by the dynamics (2). So, the stationary dynamics satisfy detailed balance with steady-state weights given by

$$w(C) = \begin{cases} 0 & \text{if CZs are present} \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

In the following, we construct a transfer matrix T so that

$$w(\{n_1k_1,n_2k_2,\ldots,n_Lk_L\}) = \prod_{i=1}^{L} \langle n_{ik_i}|T|n_{i+1k_{i+1}}\rangle, \tag{5}$$

where the orthonormal basis vectors for the transfer matrix that correspond to different rungs are

$$\begin{Bmatrix}1\\0\end{Bmatrix} \equiv |1_1\rangle; \quad \begin{Bmatrix}0\\1\end{Bmatrix} \equiv |1_2\rangle; \quad \text{and} \quad \begin{Bmatrix}1\\1\end{Bmatrix} \equiv |2_1\rangle. \tag{6}$$

Here, again we use a notation $|n_k\rangle$, with n being the number of particles in the rung and $k = 1,2,\ldots$ is a running index that distinguishes different rungs in a given n-particle sector. To ensure that the weight of all those configurations that produce CZs in the horizontal direction are zero, we must set

$$\langle 1_1|T|1_1\rangle = 0 = \langle 1_2|T|1_2\rangle. \tag{7}$$

Explicitly, the 3×3 transfer matrix is given by

$$T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{8}$$

In fact the weights, as written in Eq. (5), ensure that the steady state of the CLG on a ladder has a matrix product form where each rung is represented by a matrix,

$$\begin{Bmatrix}1\\0\end{Bmatrix} \equiv |1_1\rangle\langle 1_1|T; \quad \begin{Bmatrix}0\\1\end{Bmatrix} \equiv |1_2\rangle\langle 1_2|T; \quad \begin{Bmatrix}1\\1\end{Bmatrix} \equiv |2_1\rangle\langle 2_1|T. \tag{9}$$

The steady-state probability of any configuration

$$P_N(\{n_ik_i\}) = \frac{w(\{n_ik_i\})}{Q_N}. \tag{10}$$

Here, Q_N is the canonical partition function,

$$Q_N = \sum_{\{n_ik_i\}} w(\{n_ik_i\})\delta\left(\sum_i n_i - N\right), \tag{11}$$

which, in this model, counts the number of recurring configurations of a system of size $2L$ containing N particles. It is convenient to work in the grand-canonical ensemble (GCE) where the density of the system can be tuned by a fugacity z. The partition function in GCE is $Z(z) = \sum_{N=0}^{\infty} z^N Q_N$; from Eqs. (5) and (11),

$$Z(z) = \mathrm{Tr}[C(z)^L]; \quad \text{where}$$
$$C(z) = \sum_{n=1}^{2} z^n \sum_{k=1}^{\kappa_n} |n_k\rangle\langle n_k|T = \begin{pmatrix} z^2 & z^2 & z^2 \\ z & 0 & z \\ z & z & 0 \end{pmatrix}. \tag{12}$$

The eigenvalues of $C(z)$ are

$$\lambda\pm = \frac{z}{2}(1 + z \pm \sqrt{z^2 + 6z + 1}); \quad \lambda = -z. \tag{13}$$

In the thermodynamic limit $L \to \infty$, the partition function gets the dominant contribution from λ_+, the largest eigenvalue of $C(z)$,

$$Z(z) \simeq \lambda_+(z)^L. \tag{14}$$

The average steady-state density of the system is then

$$\rho(z) = \frac{z}{2}\frac{\partial}{\partial z}\ln \lambda_+(z). \tag{15}$$

In Fig. 2(a) we plot ρ as a function of z; it approaches a finite value $\rho_c = \frac{1}{2}$ as for $z \to 0$. Hence, the critical density below which the system goes to an absorbing state is $\rho_c = \frac{1}{2}$. In this critical limit,

$$\lim_{z\to 0} \rho_z \simeq \tfrac{1}{2} + z - 4z^2 + \mathcal{O}(z^3) \Rightarrow z \simeq (\rho - \rho_c). \tag{16}$$

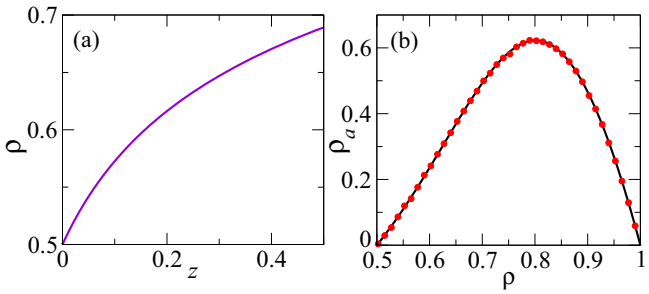

FIG. 2. (a) Plot of density ρ as a function of z : ρ approaches the critical value $\rho_c = \frac{1}{2}$ as $z \to 0$. (b) Plot of order parameter ρ_a (i.e., steady-state density of active particles) versus ρ. ρ_a becomes nonzero above critical density $\rho_c = \frac{1}{2}$. For $\rho = 1$, all the sites are occupied and thus $\rho_a = 0$.

Above critical density $\rho > \frac{1}{2}$, the system remains in active phase. To measure activity, the density of active particles ρ_a, as a function of tuning parameter ρ, we calculate the probability that an occupied site is active in the steady state. To determine whether an occupied site is active, one must check the occupancy status of all its neighbors; thus the activity ρ_a is the steady-state average of the following three-rung local configurations,

$$\rho_a = 2\left\langle\begin{matrix}110\\011\end{matrix}\right\rangle + 2\left\langle\begin{matrix}011\\110\end{matrix}\right\rangle + \left\langle\begin{matrix}111\\101\end{matrix}\right\rangle + \left\langle\begin{matrix}101\\111\end{matrix}\right\rangle + \left\langle\begin{matrix}111\\110\end{matrix}\right\rangle + \left\langle\begin{matrix}110\\111\end{matrix}\right\rangle + \left\langle\begin{matrix}011\\111\end{matrix}\right\rangle + \left\langle\begin{matrix}111\\011\end{matrix}\right\rangle. \quad (17)$$

A factor 2 in first two terms indicates that these local configurations have two active sites. Let us calculate the first term explicitly; others can be calculated in a similar way.

$$\left\langle\begin{matrix}110\\011\end{matrix}\right\rangle = \frac{1}{Z(z)}\mathrm{Tr}[z|1_1\rangle\langle 1_1|Tz^2|2_1\rangle\langle 2_1|Tz|1_2\rangle\langle 1_2|TC(z)^{L-3}] = \frac{z^3}{Z(z)}\langle 1_2|C(z)^{L-2}|1_1\rangle. \quad (18)$$

It is evident that the first two terms of (17) give rise to the lowest order terms in z, as these three-rung configurations have four particles in total, whereas the others have five (each particle contributes a factor z).

All the terms of (17) can be calculated in a similar way, as in (18). The exact expression of ρ_a as a function of z is long and we do not present it here, but a parametric plot of $\rho_a(z)$ as a function of $\rho(z)$ is shown in Fig. 2(b). It clearly shows that ρ_a vanishes linearly as the density approaches the critical limit $\rho \to \rho_c = \frac{1}{2}$, i.e., $\rho_a \propto (\rho - \rho_c)$ and thus the order parameter exponent of the absorbing phase transition is $\beta = 1$. In the same figure, the points represent the value of ρ_a obtained from Monte Carlo simulation of the restricted CLG on a ladder, for a system size $L = 10^3$.

In fact, to obtain the order parameter exponent β, it is enough to calculate one of the first two terms in the expression of ρ_a in Eq. (17), which are the lowest order in z, because in the critical limit $z \to 0$ these terms, if they turn out to be nonzero, contribute dominantly. We consider,

$$\rho_a^* \equiv \left\langle\begin{matrix}110\\011\end{matrix}\right\rangle = \frac{z^3}{Z(z)}\langle 1_2|C(z)^{L-2}|1_1\rangle. \quad (19)$$

To the lowest order (for the system with even number of sites), from Eq. (12) we have

$$\langle 1_2|C(z)^{L-2}|1_1\rangle = z^{L-2}\langle 1_2|T|1_1\rangle\langle 1_1|T|1_2\rangle\cdots$$
$$\langle 1_2|T|1_1\rangle = z^{L-2}.$$

and $Z(z) \sim z^L$ [from Eq. (14)]. Thus, $\rho_a^* \simeq z$. Again, from Eq. (16), $z \propto (\rho - \rho_c)$, implying $\rho_a^* \propto (\rho - \rho_c)$ and thus $\beta = 1$. In Fig. 4 we have shown the plot of ρ_a^* as a function of ρ (solid line), along with the same obtained from Monte Carlo simulations of a system of size $L = 1000$.

In the critical limit the total activity $\rho_a \simeq z \simeq (\rho - \rho_c)$ can also be obtained directly from the Taylor series expansion of ρ_a. However, the number of active-three-rung configurations that contribute to ρ_a rapidly increase for larger M chains, and it is convenient to calculate β from ρ_a^*, rather than from ρ_a.

Now we turn our attention to the density correlation function, which for a fixed $r < \frac{L}{2}$, is the probability that the $(i + r)$th site is occupied (i.e., $s_{i+r} = 1$) given that $s_i = 1$. Without the loss of generality we take the ith site is in the lower rung. Now we can choose two matrices D and E such that ET and DT represent the occupancy of a site i in the lower rung, $s_i = 0,1$, respectively;

$$D = |1_2\rangle\langle 1_2| + |2_1\rangle\langle 2_1|; \quad E = |1_1\rangle\langle 1_1|. \quad (20)$$

Thus, the average density of the system is $\rho(z) = \frac{\mathrm{Tr}[DC(z)^L]}{\mathrm{Tr}[C(z)^L]}$; it is straightforward to show that in the thermodynamic limit this expression is equivalent to Eq. (15). The density correlation function is now

$$g(r) = \langle s_i s_{i+r}\rangle - \rho^2 = \frac{\mathrm{Tr}[DC(z)^r DC(z)^{L-r}]}{\mathrm{Tr}[C(z)^L]}. \quad (21)$$

In the thermodynamic limit

$$g(r) \propto \left(\frac{\lambda_-}{\lambda_+}\right)^r = e^{-r/\xi}; \quad \xi^{-1} = \left|\ln\frac{\lambda_+}{\lambda_-}\right|. \quad (22)$$

From Eq. (13), it is evident that the correlation length ξ diverges in the critical limit $z \to 0$,

$$\xi \simeq \frac{1}{z} = (\rho - \rho_c)^{-\nu}; \quad \text{with exponent } \nu = 1. \quad (23)$$

Any rung-rung correlation function, or the correlation functions for activity also decay exponentially (not shown here) with the same length scale ξ. Since *at* the critical point one expects power-law correlation, $g(r) \sim r^{2-D-\eta}$, we conclude that for this quasi-1D system $\eta = 1$.

The critical exponents that we obtained for the restricted CLG model on a ladder are thus characterized by the critical exponent $\beta = 1 = \nu = \eta$, which is the same as the CLG model in 1D. Previous studies of CLG models on the ladder [21] exhibit absorbing transition in DP universality class due to the fact that the dynamics of that model was essentially stochastic, in the sense that the active particles there may have more than one vacant neighbor, and then it must choose one of them randomly as the target site, and hop there. Once the stochastic particle transfer is ceased, in the present model, the critical behavior of the absorbing transition becomes the same as that of 1DCLG.

In the following section we discuss a multichain system and calculate the critical exponents of the absorbing transitions there. We see that the odd and even number of chains exhibit different universal features.

III. MULTICHAIN SYSTEM

The multichain models are a straightforward generalization of the restricted CLG on a ladder discussed in the previous section, but their critical behavior depends on M, the number of chains. Formally we start with a $M \times L$ rectangular lattice where each site $i = 1,2,\ldots,ML$ is either vacant ($s_i = 0$) or occupied by one particle ($s_i = 1$). Further, we assume periodic boundary conditions in both the x and y directions. The dynamics of the system for $M > 2$ is similar to that defined on a ladder ($M = 2$): sites which have *exactly* one vacant neighbor (i.e.,

three other neighbors are occupied) can hop to the vacant site with unit rate. The rightward hop of an active particle is then

$$\begin{Bmatrix} \cdots \\ *11 \\ 1\hat{1}0 \\ *11 \\ \cdots \end{Bmatrix} \rightarrow \begin{Bmatrix} \cdots \\ *11 \\ 101 \\ *11 \\ \cdots \end{Bmatrix}, \tag{24}$$

where * represents an arbitrary occupancy, vacant or occupied, and the active particle is marked with a hat. Similarly, the active particle can also hop to the left or upward or downward when these sites are the only vacant neighbors of a particle.

The dynamics conserves the total number of particles N or density $\rho = \frac{N}{ML}$ and, like the dynamics on a ladder, cannot create consecutive 0's but destroy the ones present in the system. Thus one expects that consecutive 0's are absent in the steady state [30]. Thus we work in the $\rho > \frac{1}{2}$ regime and assume to start with an initial configuration which does not have any consecutive 0's. Since the system, once transit from a configuration with a higher number of CZs, will never come back to visit it again (as the creation of CZs is not allowed by the dynamics), it would be economic in terms of simulation time to start with an initial configuration (IC) which does not have any consecutive 0's. For these models, we call such ICs natural initial configurations (natural ICs) and it is certainly possible to create such configurations for $\rho \geqslant \frac{1}{2}$; we choose to discuss the $\rho > \frac{1}{2}$ case in more detail and show that the critical density is $\rho_c \geqslant \frac{1}{2}$ for all M. These models for $M > 2$ have some subtle features for $\rho > \frac{1}{2}$ which were not present in $M = 1$ [26] or $M = 2$ (previous section); we will discuss these issues in Sec. IV in some detail.

It is easy to see that in the absence of CZs, if the dynamics allows a transition from any configuration C to another one C' it also allows the reverse transition $C' \rightarrow C$. Since any such transition occurs with unit rate, the steady state must satisfy detailed balance, with steady-state weight $w(C) = 1$ for all C which are devoid of CZs. Thus, all the configurations (devoid of CZs) in the supercritical regime are equally likely. Our first step is to enumerate such configurations.

Any M-chain system of size L consists of L rungs, which are the vertical supports. Since we want to construct configurations which are devoid of CZs, we must primarily ensure that every rung must not contain any CZs. Let d_M be the number of such rungs; clearly, d_M is the same as the number of allowed configurations in the steady state of the 1DCLG model [26] on system size M with $\frac{M}{2}$ or more particles. This is because, for a density larger than $\frac{1}{2}$ the 1DCLG model leads to a steady state where CZs are absent. The steady-state weights of these models can be expressed in a matrix product form [26]; the grand-canonical partition function with a fugacity z that controls the particle density of the 1D chain for a system size M is given by

$$Z_{1\mathrm{D}}(z) = \mathrm{Tr}\left[\begin{pmatrix} z & 1 \\ z & 0 \end{pmatrix}^M\right]. \tag{25}$$

For $z = 1$, the partition function counts all possible configurations of the system irrespective of its density. Thus,

$$d_M = Z_{1\mathrm{D}}(1) = \mathrm{Tr}\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^M\right] \tag{26}$$

$$= \frac{1}{2^M}[(\sqrt{5}+1)^M + (\sqrt{5}-1)^M]. \tag{27}$$

In fact, the matrix that appears in Eq. (26) is simply the transfer matrix which is used to construct a binary string which does not possess CZs. Also note that the asymptotic form of d_M is

$$d_M \simeq \phi^M \quad \text{where } \phi = \frac{\sqrt{5}+1}{2} \tag{28}$$

is the golden ratio.

The M-chain system is composed of d_M different kinds of rungs, but any arbitrary arrangement of rungs is not allowed in the steady state. This is because the rungs themselves do not contain any CZs, but any arbitrary placement of rungs could generate CZs on horizontal bonds. Our aim would be to construct a transfer matrix considering each of the rungs as basis vectors, which would automatically take care of the forbidden arrangements. Let us categorize the collection of d_M rungs with respect to the number of particles they have; in the n particle sector we have, say, κ_n rungs labeled by $k = 1,2,\ldots,\kappa_n$. Thus,

$$\sum_{n=\nu}^{M} \kappa_n = d_M, \tag{29}$$

where ν is the minimum number of particles in a rung. Since the rungs do not contain CZs in the vertical direction, the minimum number of particles in the rung is

$$\nu = \left\lfloor \frac{M+1}{2} \right\rfloor = \begin{cases} M/2 & \text{for } M = \text{even} \\ (M+1)/2 & \text{for } M = \text{odd}, \end{cases} \tag{30}$$

and the maximum number is M. The exact value of κ_n (number of rungs that contain exactly n particles and, of course, $M - n$ vacant sites) is the coefficient of z^n in the Taylor series expansion of $Z_{1\mathrm{D}}(z)$ about $z = 0$,

$$\kappa_n = \{z^n\} Z_{1\mathrm{D}}(z). \tag{31}$$

For some n it is straightforward to calculate κ_n. For example, for $n = M$ we have $\kappa_M = 1$ (the rung is filled with 1's), for $n = \nu$, $\kappa_\nu = 2$ when M is even (alternative sites are occupied, starting with 0 or 1) and $\kappa_\nu = M$ for odd M (with $n = \frac{M+1}{2}$ particles, one of the M vertical bonds of the rung must have consecutive 1's).

At this stage, we use a systematic ordering of the rungs, which can act as the basis vectors for the transfer matrix. We represent the rungs by $\{n_k\}$ where n,k are integers: n varies in the range (ν,M), and k, for a given n, varies in the range $(1,\kappa_n)$. The standard basis for the transfer matrix is a set of orthonormal vectors

$$\begin{aligned} \{|n_k\rangle\} \equiv \{&|\nu_1\rangle, |\nu_2\rangle, \ldots, |\nu_{\kappa_\nu}\rangle \\ &\cdots \\ &|n_1\rangle, |n_2\rangle, \ldots |n_{\kappa_n}\rangle, \ldots \\ &\cdots \\ &|M_1\rangle\}. \end{aligned} \tag{32}$$

In this basis, the elements of the transfer matrix are nonzero, $\langle n_k|T|n'_{k'}\rangle = 1$, when two rungs $|n_k\rangle$ and $|n'_{k'}\rangle$ as neighbors do not produce any CZs in the horizontal direction, i.e., if one of the rungs has 0's at certain positions, the other must have 1's at that position.

$$\langle n_k|T|n'_{k'}\rangle = \begin{cases} 1 & \text{if } |n_k\rangle, |n'_{k'}\rangle \text{ do not generate CZs} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to obtain the transfer matrix manually for small M, but the dimension of the matrix $d_M \sim \phi^M$ grows exponentially, and quickly the calculation becomes tedious. However, it can be computed numerically noticing the fact that for any two M-bit binary strings s and s' which do not have consecutive zeros, the operation $\tilde{s}\&\tilde{s'}$, where & and $\sim$represent bit-wise AND and NOT operations, respectively, gives a nonzero value only when there is at least one spatial position where both strings have a 0.

It is easy to see that T^L generates all possible configurations devoid of CZs, irrespective of the number of particles (1's). To describe the $M \times L$ system with a conserved particle number N (or conserved density $\rho = \frac{N}{ML}$) we introduce a fugacity z and write the partition function in grand-canonical ensemble as

$$Z(z) = \mathrm{Tr}[C(z)^L]; \quad \langle n_k|C(z)|n'_{k'}\rangle = z^n \langle n_k|T|n'_{k'}\rangle. \tag{33}$$

Since the minimum number of particles in any of the rungs is ν, we can expand $C(z)$ as follows:

$$C(z) = \sum_{n=\nu}^{M} z^n C_n, \tag{34}$$

where matrices C_n are independent of z. The description of the grand-canonical ensemble is incomplete, unless we specify the density as a function of fugacity z. Density of the $M \times L$ system can be calculated by taking trace ($\mathrm{Tr}[\cdot]$) over all configurations where one specified site of the system is occupied. Since the rung n_{ik_i} at site i is only a binary string $\{s_i, s_{i+L}, \ldots, s_{i+(M-1)L}\}$ with $\sum_{j=0}^{M-1} s_{i+jL} = n_i$, we can associate a unique decimal value $\mathcal{D}(n_k) = \sum_{j=0}^{M-1} 2^j s_{i+jM}$ to it; the decimal value is an odd integer if the first site of the rung is occupied. Thus by defining a diagonal matrix,

$$D = \sum_{n=\nu}^{M} \sum_{k=1}^{\kappa_n} |n_k\rangle\langle n_k| \delta[1 - \mathcal{D}(n_k) \mathrm{mod} 2], \tag{35}$$

we get the density of the system as

$$\rho(z) = \frac{1}{Z(z)} \mathrm{Tr}[DC(z)^L]. \tag{36}$$

Of course, one standard way to calculate the density is as follows. If the largest eigenvalue of $C(z)$ is $\lambda(z)$, in the thermodynamic limit $Z(z) \simeq \lambda(z)^L$ and the density is

$$\rho(z) = \frac{z}{M} \frac{d}{dz} \ln \lambda(z). \tag{37}$$

However, when the dimension of the transfer matrix is large (which is indeed the fact as the dimension $d_M \sim \phi^M$) it is advantageous to calculate $\rho(z)$ numerically, using Eq. (36).

In the following we see that the critical density ρ_c where the $M \times L$ system undergoes a nonequilibrium phase transition from an active to an absorbing state is

$$\rho_c = \lim_{z \to 0} \rho(z) \tag{38}$$

and the critical behavior of the system depends on how the partition function and other observables depend on the $z \to 0$ limit; in this regime, contributions from matrices C_ν and $C_{\nu+1}$ are most important.

A. Steady state in matrix product form

The steady state average of different observables can be calculated easily, if we write the steady-state weights of the configurations in a matrix product form. Every configuration of the system is composed of L rungs. Denoting a rung n_k by a matrix $R(n_k)$ (in total there are d_M number of different matrices) the steady-state probability of a configuration $\{n_i k_i\}$ can be written in a matrix product form using a matrix product ansatz,

$$P(\{n_1k_1, n_2k_2 \ldots n_Lk_L\} = \frac{1}{Q_N} \mathrm{Tr}\left[\prod_{i=1}^{L} R(n_{ik_i})\right] \times \delta\left(\sum_{i=1}^{L} n_i - N\right),$$

where the δ function ensures conservation of the number of particles N, and Q_N is the canonical partition function

$$Q_N = \sum_{\{n_i=\nu\}}^{M} \sum_{\{k_i=1\}}^{\kappa_{n_i}} \mathrm{Tr}\left[\prod_{i=1}^{L} R(n_{ik_i})\right] \delta\left(\sum_{i=1}^{L} n_i - N\right).$$

The grand-canonical partition function is then

$$Z(z) = \sum_{N=0}^{\infty} z^N Q_N = \mathrm{Tr}\left[\left(\sum_{n=\nu}^{M} \sum_{k=1}^{\kappa_n} z^n R(n_k)\right)^L\right].$$

Comparing this with Eq. (33), we get matrices

$$C(z) = \sum_{n=\nu}^{M} z^n \sum_{k=1}^{\kappa_n} R(n_k) \tag{39}$$

$$\text{and} \quad R(n_k) = |n_k\rangle\langle n_k|T. \tag{40}$$

Equation (40) is very important to us, as any explicit matrix representation is useful for the calculation of observables. For example, the steady-state average of a particular rung $\bar{n}_{\bar{k}}$ is

$$\langle \bar{n}_{\bar{k}} \rangle = \frac{\mathrm{Tr}[z^{\bar{n}} R(\bar{n}_{\bar{k}}) C(z)^{L-1}]}{\mathrm{Tr}[C(z)^L]} = \frac{\langle \bar{n}\bar{k}|C(z)^L|\bar{n}\bar{k}\rangle}{\mathrm{Tr}[C(z)^L]}. \tag{41}$$

A comparison of Eqs. (39) and (34) gives

$$C_n = \left[\sum_{k=1}^{\kappa_n} |n_k\rangle\langle n_k|\right] T = \Pi_n T, \tag{42}$$

where Π_n is the projection operator, defined by the term within brackets $[\cdot]$, which projects out all the rungs having exactly n particles.

One important observable we would be interested in is the order parameter of the absorbing phase transition, namely, activity. Writing a matrix representation for it is not that simple, as constructing all possible arrangements of the rungs

that can create active sites is not possible for general M; for $M = 2$, as we have discussed in the previous section, there are eight three-rung configurations which have at least one active site. However, one can infer about the behavior of the order parameter at the critical point easily by considering only any of the three-rung configurations which have the minimum number of particles which contribute to the lowest order in z. However, we have already mentioned, the critical behavior of the system with an odd number of chains is different from the same with even M; we discuss these two cases separately in the following two sections.

B. CLG on odd number of chains

For odd number of chains, $M = 2m + 1$, the minimum number of particles on a rung (which does not have CZs) is $\nu = m + 1$. There are exactly M rungs which have $(m + 1)$ particles (1's) and m holes (0's), thus each one contains exactly one consecutive 1 in the vertical direction. We denote these rungs as

$$|\nu_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, |\nu_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}, \ldots, |\nu_{k_\nu}\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \quad (43)$$

Our first aim is to calculate the critical density ρ_c for the CLG dynamics (with deterministic particle transfer) on a system with an odd number of chains. In fact, since one can construct configurations of L rungs (without any CZs) using only the rungs containing ν particles (like $\{\nu_1, \nu_2, \nu_1, \nu_2 \ldots\}$) the steady-state density of the system cannot decrease below ν/M, and one expects the critical density to be $\rho_c \geqslant \frac{m+1}{2m+1}$. We show below that $\rho_c = \lim_{z\to 0} \rho(z) = \frac{m+1}{2m+1}$.

In the $z \to 0$ limit, the partition function is

$$\begin{aligned} Z(z) &= z^{\nu L} \mathrm{Tr}\{[C_\nu + zC_{\nu+1} + \mathcal{O}(z^2)]^L\} \\ &= z^{\nu L}\left(\mathrm{Tr}\big[C_\nu^L\big] + z\sum_{k=0}^{L-1} \mathrm{Tr}\big[C_\nu^k C_{\nu+1} C_\nu^{L-1-k}\big] + \mathcal{O}(z^2)\right). \end{aligned}$$

Thus the critical density is

$$\rho_c = \lim_{z\to 0} \rho(z) = \frac{\mathrm{Tr}\big[DC_\nu^L\big]}{\mathrm{Tr}\big[C_\nu^L\big]}. \quad (44)$$

Now, since the M rungs (vectors) in the $(m + 1)$-particle sector are related to each other by a rotational symmetry (with respect to the position of a *single* consecutive 1 in the vertical direction), $\langle \nu_k|C_\nu|\nu_k\rangle = \langle \nu_1|C_\nu|\nu_1\rangle$ for any $k = 1, 2, \ldots, \kappa_\nu = M$. Thus

$$\mathrm{Tr}\big[C_\nu^L\big] = \kappa_\nu \langle \nu_1|C_\nu^L|\nu_1\rangle. \quad (45)$$

Again matrix D, defined in Eq. (35), projects out only those rungs which have 1 in the first position irrespective of the total number of particles. Thus,

$$\mathrm{Tr}\big[DC_\nu^L\big] = {\sum_k}' \langle \nu_k|C_\nu^L|\nu_k\rangle = \nu_{\mathrm{odd}} \langle \nu_1|C_\nu^L|\nu_1\rangle, \quad (46)$$

where the prime indicates that the sum is restricted to consider only those k for which $\mathcal{D}(\nu_k)$ is an odd integer. The number of such rungs in the $(m + 1)$-particle sector is $\nu_{\mathrm{odd}} = m + 1$. Thus the critical density, Eqs. (45) and (46), is

$$\rho_c = \frac{m+1}{2m+1} = \frac{1}{M}\left\lfloor \frac{M+1}{2} \right\rfloor. \quad (47)$$

Further, in $z \to 0$ limit, using Eqs. (34) and (36) we get

$$\rho(z) \simeq \frac{\mathrm{Tr}[D(C_\nu + zC_{\nu+1})^L]}{\mathrm{Tr}[(C_\nu + zC_{\nu+1})^L]} \simeq \rho_c + \gamma z + \mathcal{O}(z^2), \quad (48)$$

where γ is a nonzero constant independent of z. Thus, in this critical limit,

$$z \propto (\rho - \rho_c). \quad (49)$$

We now proceed to calculate the order parameter $\rho_a(z)$, namely, the density of activity. To know that a particle at a given site is active, one needs to check that all except one of its neighbors is occupied. Since for $M > 2$ every site has four nearest neighbors, the active particle must have three occupied neighbors and one vacant neighbor; thus, one must consider *three* consecutive rungs to verify the occupancy of neighbors. One can place three different rungs in several possible ways to construct active configurations (having at least one active particle) which are devoid of CZs; we will not enumerate all these configurations. To know the behavior of activity $\rho_a(z)$ in the critical limit $z \to 0$, we need to consider only one of the active three-rung configurations with the minimum number of particles, because these configurations, being lowest order in z, contribute dominantly as $z \to 0$. In other words, if $\rho_a(z) = \sum_j c_j z^{\alpha_j}$ with $\alpha_1 < \alpha_2 < \cdots$, an active three-rung configuration leads to the dominant contribution at $z = 0$,

$$\rho_a(z) \propto z^{\alpha_1}. \quad (50)$$

Comparing Eqs. (49) and (50) we obtain the order parameter exponent β,

$$\rho_a \propto (\rho - \rho_c)^\beta, \quad \text{where } \beta = \alpha_1. \quad (51)$$

For odd M chain, the minimum number of particles in three rungs is $3\nu = 3(m + 1)$, i.e., when each rung has the minimum ν number of particles; however, one cannot create an active configuration only with these rungs. We show that an active configuration can be obtained with one extra particle, i.e., when one of the three rungs contains a $\nu + 1$ particle. There are many such active configurations with $3\nu + 1 = 3m + 4$ particles; a systematic construction for generic M follows. This construction is not unique, but a proof that the steady-state average of any such configuration is nonzero and it varies as z^{α_1} in $z \to 0$ limit is enough to determine the critical exponent β.

Let us take the $|\nu_2\rangle$ rung from Eq. (43) and put an extra particle on the first vacant site on this rung; this new rung

belongs to the $(\nu+1)$ particle sector and we denote it as $|(\nu+1)_1\rangle$. Let us take the active three-rung configurations as

$$\{\nu_1,(\nu+1)_1,\nu_3\} = \begin{pmatrix} 111 \\ 110 \\ 011 \\ 101 \\ 010 \\ 101 \\ 010 \\ \vdots\vdots \end{pmatrix}. \tag{52}$$

The steady-state average of this configuration for a given $M = 2m+1$ is

$$\rho_a^*(z) = \langle\{\nu_1,(\nu+1)_1,\nu_3\}\rangle = \frac{1}{Z(z)}\langle\nu_1|C(z)|(\nu+1)_1\rangle \times \langle(\nu+1)_1|C(z)|\nu_3\rangle\langle\nu_3|C(z)^{L-2}|\nu_1\rangle. \tag{53}$$

In the $z\to 0$ limit,

$$\rho_a^* = \frac{z^\nu z^{\nu+1} z^{\nu(L-2)}\langle\nu_3|C_\nu^{L-2}|\nu_1\rangle}{z^{\nu L}\mathrm{Tr}\left[C_\nu^L\right]} = Az, \tag{54}$$

where A is a positive constant. This is because C_ν is a positive symmetric matrix and $\langle\nu_3|C_\nu^{L-2}|\nu_1\rangle \geqslant \langle\nu_3|C_\nu|\nu_1\rangle^{L-2} = 1$. Thus, for any odd $M = 2m+1$ chain, the order parameter ρ_a, like ρ_a^*, approaches 0 continuously as

$$\rho_a \sim (\rho-\rho_c)^\beta; \quad \beta = 1. \tag{55}$$

In Fig. 3(a) we have shown a parametric plot of $\rho_a^*(z)$ as a function of $\rho(z)$ for different $M = 3, 5, 7, 9, 11$. The data points for $M = 3, 5, 7$ in the same plot show ρ_a^* obtained from a Monte Carlo simulation of the restricted CLG dynamics for different densities. The simulation was done on a system of size $L = 1000$ and starting from a natural initial configuration. Clearly, the critical density for $M = 2m+1$ is $\rho_c = \frac{m+1}{2m+1}$ and it approaches $\frac{1}{2}$ as M increases. In Fig. 3(b) we plot ρ_a^* as a function of $(\rho-\rho_c)$ in log scale to obtain the order parameter exponent $\beta = 1$.

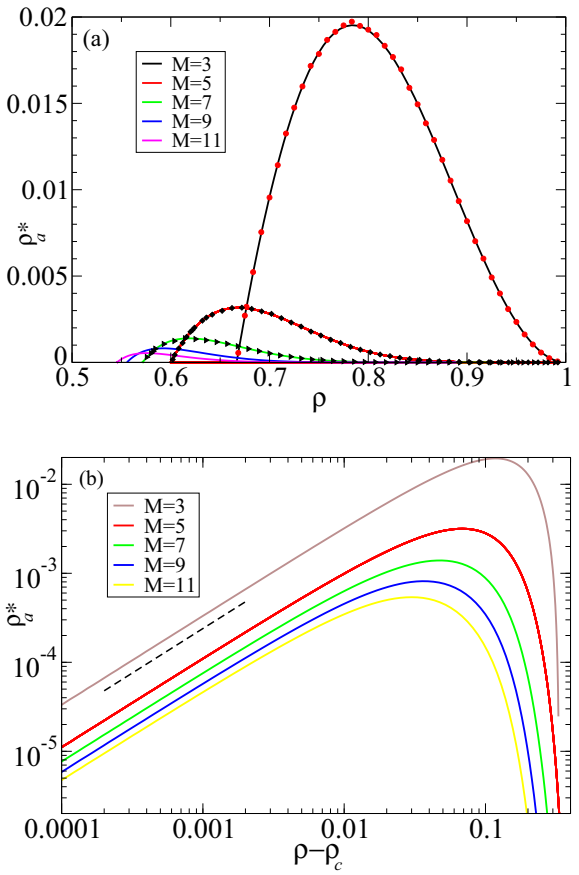

FIG. 3. The order parameter ρ_a is the sum of the steady-state averages of several three-rung configurations, of which one of the terms, which is lowest order in z, is ρ_a^* given by Eq. (52). (a) A parametric plot of $\rho_a^*(z)$ as a function of $\rho(z)$, calculated following the transfer-matrix method (solid line) for $M = 3, 5, 7, 9, 11$ (top to bottom) is compared with the same obtained for different densities (symbols) using Monte Carlo simulations of the restricted CLG dynamics (density conserving) on a $M \times L$ system, with $L = 1000$ and $M = 3, 5, 7$. Clearly, ρ_a^* vanish at $\rho_c = \frac{1}{M}\lfloor\frac{M+1}{2}\rfloor$. (b) Log scale plot of ρ_a^* as a function of $(\rho-\rho_c)$, along with a line with unit slope (dashed line) indicates that $\rho_a^* \sim (\rho-\rho_c)^\beta$ with $\beta = 1$.

C. CLG on even number of chains

A special case of the even $M = 2m$ chain is the ladder $(M = 2)$ which is discussed in Sec. II. There, we have explicitly calculated the density $\rho(z)$ and the activity $\rho_a(z)$ and found the order parameter exponent $\beta = 1$. Given that any odd M chain undergoes an absorbing transition with exponent $\beta = 1$, one expects that the same must be true for all even M; however, this is not true. Note that, a ladder is a very special case where open and periodic boundary conditions in the vertical direction results in the same lattice structure. Further, unlike any $M > 2$ system where every site has four nearest neighbors, the ladder has only three. We will see below that $M = 4$ is also a special case and it results in $\beta = 2$, whereas any even chain with $M > 4$ results in an absorbing transition with exponent $\beta = 3$.

For the even $M = 2m$, the minimum number of particles in the rungs that does not contain consecutive 0's is $\nu = m$. There are exactly two rungs which have ν particles, i.e., $\kappa_\nu = 2$,

$$|\nu_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad |\nu_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{pmatrix}. \tag{56}$$

We use Eqs. (44), (45), and (46), which also holds true when M is even (can be checked easily) to calculate the critical density,

$$\rho_c = \lim_{z\to 0}\rho(z) = \frac{\mathrm{Tr}\left[DC_\nu^L\right]}{\mathrm{Tr}\left[C_\nu^L\right]} = \frac{\nu_{\mathrm{odd}}}{\kappa_\nu} = \frac{1}{2}. \tag{57}$$

In fact, since the rungs are devoid of CZs, the minimum density of a configuration is $\rho = \frac{1}{2}$, obtained from, say, $\{\nu_1,\nu_2,\nu_1,\nu_2\ldots\}$ and one expects the critical density $\rho_c \geqslant \frac{1}{2}$. However, all configurations for density $\rho > \frac{1}{2}$ are not active and one needs to check explicitly that the minimum density is the critical density.

Next we focus on the order parameter ρ_a. Here too, we need to know three consecutive rungs to identify whether a particle at a given site is active, i.e., the active site must have three occupied and one vacant neighbor. Of all such three-rung configurations, what contributes near the critical point is the active three-rung configuration that has the minimum number of 1's. Unlike odd M, one cannot create an active three-rung configuration with $3\nu+1$ particles; we need at least $3\nu+2$. We start with $M=4$, which is the first even chain system where the lattice sites have four nearest neighbors, and extend it to $M = 6,8,\ldots$. Let us take the $|\nu_1\rangle = (1,0,1,0,1,0,\ldots)$, put a particle at the first vacant site, and move the particle from third to fourth position and denote this rung as $|(\nu+1)_1\rangle = (1,1,0,1,1,0,1,0,\ldots)$. Let us put a particle at the second vacant site of $|\nu_2\rangle = (0,1,0,1,\ldots)$ and denote it as $|(\nu+1)_2\rangle = (0,1,1,1,0,1,0,1,\ldots)$. Thus one of the active three-rung configurations with minimum number of particles is $\{(\nu+1)_1,(\nu+1)_2,\nu_1\}$,

$$M=4:\begin{pmatrix}101\\110\\011\\110\end{pmatrix};\quad M>4:\begin{pmatrix}101\\110\\011\\110\\101\\010\\101\\010\\\vdots\end{pmatrix}. \tag{58}$$

The steady-state average of this configuration for a given $M = 2m$ is

$$\begin{aligned}\rho_a^*(z) &= \langle\{(\nu+1)_1,(\nu+1)_2,\nu_1\}\rangle\\ &= \frac{1}{Z(z)}\langle(\nu+1)_1|C(z)|(\nu+1)_2\rangle\langle(\nu+1)_2|C(z)|\nu_1\rangle\\ &\quad\times\langle\nu_1|C(z)^{L-2}|(\nu+1)_1\rangle\\ &= z^{2\nu+2}\frac{\langle\nu_1|C(z)^{L-2}|(\nu+1)_1\rangle}{\mathrm{Tr}[C(z)^L]}.\end{aligned} \tag{59}$$

If $|\psi\rangle,\langle\psi|$ are, respectively, the right and left normalized eigenvector of $C(z)$ corresponding to the largest eigenvalue $\lambda_{\max} = z^\nu\lambda(z)$, in the thermodynamic limit $L\to\infty$ one can write ρ_a^* as

$$\rho_a^*(z) = \frac{z^2}{\lambda(z)^2}\langle\nu_1|\psi\rangle\langle\psi|(\nu+1)_1\rangle. \tag{60}$$

This expression, being independent of system size L, is very useful in evaluating $\rho_a^*(z)$. The results for different M are shown in Fig. 4. For $M=4$, we have $d_M = 7$ dimensional matrix $C(z) = z^2C_2 + z^3C_3 + z^4C_4$. Since we are interested in the $z\to 0$ limit it is sufficient to take an approximation $C(z)\simeq z^2[C_2+zC_3]$ and now the largest eigenvalue is $\lambda_{\max}(z) = z^2\lambda(z)$ where

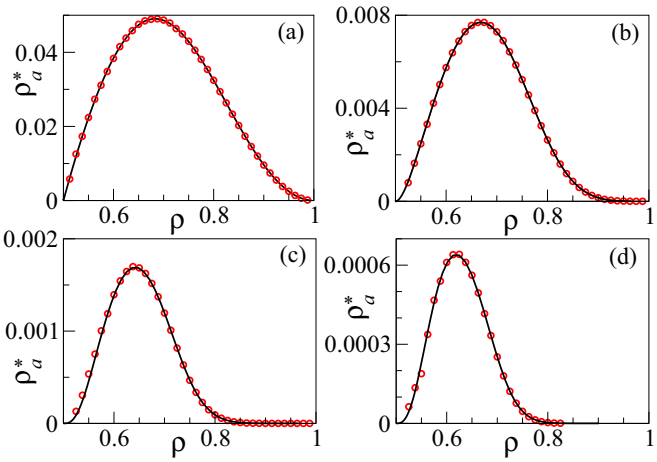

FIG. 4. Parametric plot of $\rho_a^*(z)$, the steady-state average of the active three-rung configuration in (58), as a function of $\rho(z)$. (a), (b), (c), and (d) correspond to even $M = 2$, 4, 6, and 8, respectively. Solid lines are obtained from the transfer-matrix formulation and the symbols correspond to the same obtained from Monte Carlo simulation of the restricted CLG on the $M\times L$ system with $L = 1000$. Clearly, ρ_a^* vanishes at $\rho_c = \frac{1}{2}$ for all M.

$$\lambda(z) = \tfrac{1}{2}(1+3z+\sqrt{1+2z+9z^2}). \tag{61}$$

In the $z\to 0$ limit,

$$\rho(z) = \frac{z}{M}\frac{d}{dz}\ln\lambda_{\max}(z) \simeq \frac{1}{2}+\frac{z}{2}+\mathcal{O}(z^2). \tag{62}$$

Thus the critical density $\rho_c = \frac{1}{2}$ matches with the generic result (57) obtained for even M. Moreover, in the critical regime, we have $z\propto(\rho-\rho_c)$.

The expressions for the eigenvectors are lengthy (omitted here), but the product of the ν_1th element of $|\psi\rangle$ and the $(\nu+1)_1$th element of $\langle\psi|$ is

$$\langle\nu_1|\psi\rangle\langle\psi|(\nu+1)_1\rangle = \frac{1}{2\sqrt{1+2z+9z^2}}. \tag{63}$$

Using this in Eq. (60), in the critical limit we get

$$\rho_a^*(z)\simeq\frac{z^2}{2}+\mathcal{O}(z^3). \tag{64}$$

Thus, in the critical regime, the order parameter for $M=4$ behaves as $\rho_a(z)\sim(\rho-\rho_c)^\beta$, with $\beta=2$. For higher M, extracting the order parameter exponent analytically using Eq. (60) is difficult; for even $M>4$ we proceed to get an estimate from Eq. (59).

In the $z\to 0$ limit,

$$C(z)^L = z^{\nu L}\left[C_\nu^L + z\sum_{j=0}^{L-1}C_\nu^j C_{\nu+1}C_\nu^{L-1-j}+\mathcal{O}(z^2)\right].$$

Here, from Eq. (42) we have

$$C_\nu = \Pi_\nu T = (|\nu_1\rangle\langle\nu_1| + |\nu_2\rangle\langle\nu_2|)T, \tag{65}$$

which has the following properties:

$$\begin{aligned} C_\nu^2 &= |\nu_1\rangle\langle\nu_2|T + |\nu_2\rangle\langle\nu_1|T, \\ C_\nu^{2l} &= C_\nu^2; \quad C_\nu^{2l+1} = C_\nu, \\ \mathrm{Tr}[C_\nu] &= 0; \quad \mathrm{Tr}\big[C_\nu^2\big] = 2, \\ C_\nu|\nu_1\rangle &= |\nu_2\rangle; \quad C_\nu|\nu_2\rangle = |\nu_1\rangle; \end{aligned} \tag{66}$$

where l is a positive integer. The proofs of these relations are straightforward if we use the fact $\langle\nu_k|T|\nu_{k'}\rangle = 1 - \delta_{k,k'}$. We proceed further considering the system size to be $L = 2l$; thus, to leading order in z

$$\mathrm{Tr}[C(z)^L] = z^{\nu L}\big[\mathrm{Tr}\big[C_\nu^2\big] + \mathcal{O}(z)\big] = z^{\nu L}[2 + \mathcal{O}(z)], \tag{67}$$

and

$$\begin{aligned} \frac{\langle\nu_1|C(z)^{L-2}|(\nu+1)_1\rangle}{z^{\nu(L-2)}} &= \langle\nu_1|C_\nu^{L-2}|(\nu+1)_1\rangle \\ &\quad + z\sum_{j=0}^{L-3}\langle\nu_1|C_\nu^j C_{\nu+1} C_\nu^{L-3-j} \\ &\quad \times |(\nu+1)_1\rangle) + \mathcal{O}(z^2). \end{aligned} \tag{68}$$

Now, $\langle\nu_1|C_\nu^{L-2}|(\nu+1)_1\rangle = \langle\nu_1|C_\nu^2|(\nu+1)_1\rangle = 0$ and we are left with the $\mathcal{O}(z)$ term of Eq. (68). In the sum, all the matrix product terms that end with C_ν would vanish, because $C_\nu|(\nu+1)_1\rangle = \Pi_\nu T|(\nu+1)_1\rangle = 0$ as, for $M > 6$ the rung $(\nu+1)_1$ cannot be a neighbor of any of the rungs in the ν-particle sector. So, the only surviving term in the sum is

$$\langle\nu_1|C_\nu^{L-3}C_{\nu+1}|(\nu+1)_1\rangle = \langle\nu_1|C_\nu C_{\nu+1}|(\nu+1)_1\rangle = 1.$$

Finally, to the lowest order in z, Eq. (68) gives

$$\langle\nu_1|C(z)^{L-2}|(\nu+1)_1\rangle = z^{\nu(L-2)}z. \tag{69}$$

Using this and Eq. (67) in Eq. (59) we obtain

$$\rho_a^* \simeq z^3 \quad \text{for even } M \geqslant 6. \tag{70}$$

To find the order parameter exponent we need to know the behavior of $\rho(z)$ at the critical point. As $z \to 0$,

$$\rho(z) = \frac{\mathrm{Tr}[DC(z)^L]}{\mathrm{Tr}[C(z)^L]} = \frac{1}{2} + \alpha z, \tag{71}$$

where α is a positive constant; a plot of $\rho(z)$ versus z is shown in Fig. 5(a) for $M = 2, 4, 6, 8, 10$. Thus in the critical regime, $z \propto (\rho - \rho_c)$ and ρ_a^* for even $M \geqslant 6$ is proportional to $(\rho - \rho_c)^\beta$ with $\beta = 3$.

To summarize, when the density approaches the critical value $\rho \to \rho_c$, ρ_a^* and thus the order parameter ρ_a behave as

$$\rho_a \simeq \frac{1}{2}\begin{cases} (\rho-\rho_c)^2, & M = 4 \\ (\rho-\rho_c)^3, & M = 6,8,\ldots. \end{cases} \tag{72}$$

Thus, the order parameter exponent for the even M-chain system is

$$\beta = \begin{cases} 1 & \text{for } M = 2(\text{ladder}) \\ 2 & \text{for } M = 4 \\ 3 & \text{for } M = 6,8,\ldots. \end{cases} \tag{73}$$

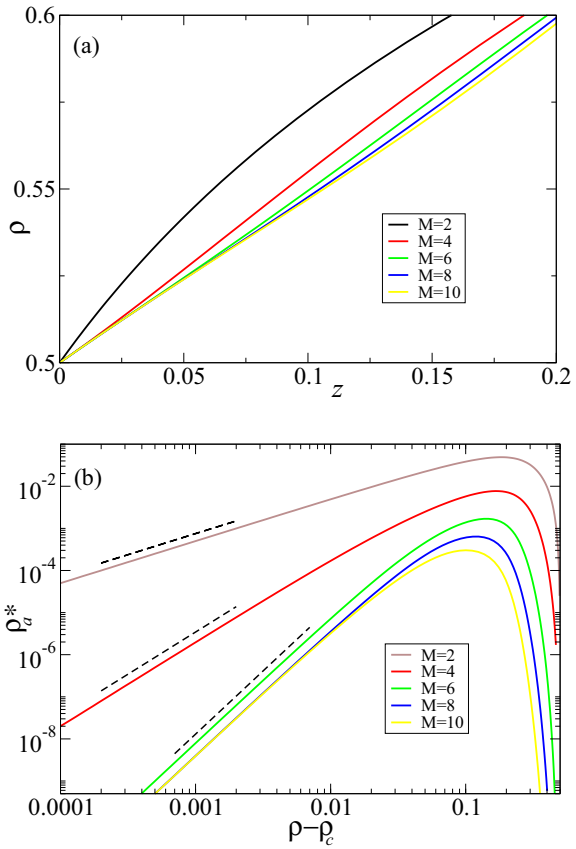

FIG. 5. (a) $\rho(z)$ as a function of z. (b) Log scale plot of ρ_a^* shown in Fig. 4 for $M = 2, 4, 6, 8, 10$ (top to bottom) as a function of $\rho - \rho_c$ shows that $\beta = 1, 2$ for $M = 2, 4$ and $\beta = 3$ for even $M > 4$. Lines with slope 1, 2, 3 are shown in dashed lines for comparison.

In Fig. 4 we have shown the plot of ρ_a^* as a function of ρ for $M = 2, 4, 6, 8$ calculated using the transfer-matrix formulation (solid line) and compared it with the same obtained from the Monte Carlo simulation of the M-chain CLG model with deterministic particle transfer, for chain length $L = 1000$. They clearly indicate that the absorbing transition occurs at $\rho_c = \frac{1}{2}$. In Fig. 5(b) with the same data, ρ_a^* is plotted against $\rho - \rho_c$ in log scale to obtain the order parameter exponent β, which agrees with Eq. (73).

IV. CONCLUSION

In this article, we study the conserved lattice-gas model on a multichain system, where particles having exactly one vacant neighbor are considered active, and they are allowed to hop deterministically to the only vacant neighbor they have. For a single chain, this model reduces to the usual CLG model in 1D, exhibiting a nonequilibrium phase transition from an active to an absorbing state when the density of the system falls below the critical value $\rho_c = \frac{1}{2}$; the critical behavior here is rather trivial, having integer exponents $\beta = 1 = \nu = \eta$. A two-chain conserved lattice-gas model has been studied earlier [21], where particles having at least one occupied neighbor and one vacant neighbor are considered active; absorbing transition in these models turns out to be in the DP universality class,

conjectured as the most robust universality class of absorbing transition. Since the ladder in the thermodynamic limit can be considered as a one-dimensional system, the change of universality class from 1DCLG to DP was rather surprising. A possible reason for the flow to the DP class is the stochasticity: particles having exactly one occupied neighbor must choose one of the other *two* neighbors (which are vacant) as the target site and hop there. If stochastic particle transfer is a relevant perturbation, we should retain 1DCLG universality when this stochasticity is ceased and particle hopping is restricted to be deterministic. Keeping this view in mind we study a restricted CLG dynamics on a ladder (Sec. II) and indeed, the APT turned out to be in the 1DCLG class.

It is natural to expect that this scenario must prevail for any multichain $M \times L$ system, as in the thermodynamic limit $L \to \infty$ (keeping M fixed) the system is effectively one dimensional. This is indeed the case when M is an odd integer and the APT for odd number chains belongs to the 1DCLG universality. The scenario is, however, different when the number of chains is an even $M \geqslant 4$; there, the value of order parameter exponent β depends on the number of chains. For $M = 2$ (ladder) the APT belongs to the 1DCLG universality with $\beta = 1$, whereas for $M = 4$ we get $\beta = 2$, and for any even $M > 4$ the order parameter exponent is $\beta = 3$.

We calculate the critical exponents using a transfer-matrix method, where the steady-state weight can be written as the trace of a matrix string constructed by representing rungs or vertical supports of the M-chain systems as matrices. The number of matrices required for such a matrix product form is the same as the number of periodic binary strings which are devoid of consecutive zeros. This number, and thus the dimension of the transfer matrix grows exponentially as ϕ^M where ϕ is the golden ratio. Along with this, the possible ways a configuration can have local activity also grows quickly and calculation of the order parameter ρ_a, which is the density of active particles, becomes practically impossible as M increases. However, the critical exponent β can be obtained from ρ_a^*, the steady-state average of an active three-rung configuration that contains the minimum number of particles. We substantiate the calculation of ρ_a^* with the numerical values obtained from Monte Carlo simulation.

Like any other absorbing phase transition into multiple absorbing configurations, Monte Carlo simulation of these models also suffer from the choice of initial condition [31]; it is presumed that the critical steady states of these systems are hyperuniform [32] and the system takes an unusually long time to relax and achieve that. One must carefully choose initial conditions which preserve the natural correlations of the stationary state. Again unlike the 1DCLG model ($M = 1$) where all supercritical configurations are active, for $M \geqslant 2$ chains the supercritical states have (i) absorbing configurations in the supercritical region and (ii) active configurations which are dynamically inaccessible. For example, when M is even, there are only two configurations at ρ_c which are devoid of CZs; since one of the sublattices is completely occupied, in this configuration each particle has exactly four vacant neighbors and one can create absorbing configurations with higher density by adding additional particles, keeping two neighbors of every particle vacant. Also, the dynamically inaccessible active configurations are not so uncommon; some examples $M = 2,3$ are

$$\begin{pmatrix} \ldots 01\hat{1}010\ldots \\ \ldots 101101\ldots \end{pmatrix}; \quad \begin{pmatrix} \ldots 10111\ldots \\ \ldots 01\hat{1}01\ldots \\ \ldots 11110\ldots \end{pmatrix}. \tag{74}$$

To avoid both kinds of configurations in numerical simulations we start with an initial state that contains the rung $|M_1\rangle$ which is fully occupied and the rungs which have minimum number $\nu = \lfloor \frac{M+1}{2} \rfloor$; of course, care must be taken so that the initial configuration is devoid of CZs. The conserved density of the system $\rho = \zeta + (1-\zeta)\frac{\nu}{M}$ can be tuned by changing the number of $|M_1\rangle$ rungs ζL. In Fig. 6 we plot $\rho_a(t)$ as a function of t for density $\rho = 0.53$ and $L = 10^4$ starting from a random IC (where ρL particles are placed at randomly chosen sites, avoiding multiple occupancies) and natural IC, created from the rungs $\{\nu_k\}$ of the ν-particle sector and the rung $\{M_1\}$. Clearly, the random IC takes a long time to relax and produce undershooting, whereas the natural IC relaxes very quickly.

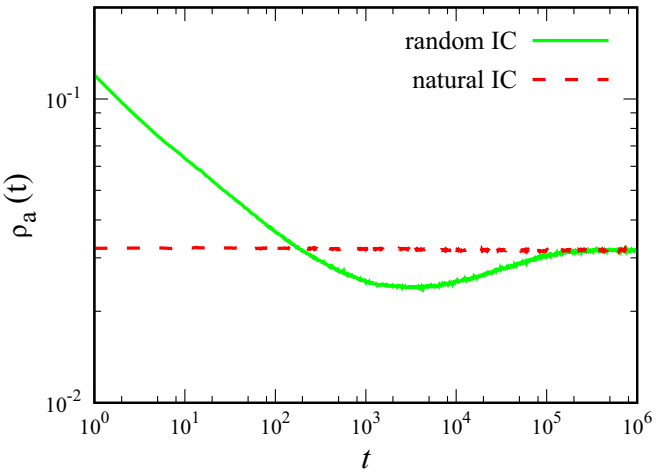

FIG. 6. The density of active particles $\rho_a(t)$ as a function t for $M = 2$. The evolution from a random initial condition (IC) exhibits undershooting and long relaxation to the stationary state; both these ill effects are avoided if we use the natural IC. Here $L = 10^4$ and $\rho = 0.53$.

In the calculation of the partition function, however, we have summed over all configurations which are devoid of CZs, without avoiding (i) absorbing configurations with $\rho > \rho_c$ and (ii) the dynamically inaccessible active configurations. We presume that at any supercritical density, the fraction of such configurations in comparison to the total number of configurations devoid of CZs vanishes in the thermodynamic limit. This assumption must be true as, for any M, as ρ_a^* obtained from the numerical simulations match with the analytical results obtained using a transfer matrix and the partition function; a proof, though desirable, is missing.

It is rather surprising that the critical exponents of this class of models depend on the geometry of the lattice. For even M, the system with two or four chains for which we

get $\beta = 1,2$ respectively, may be considered as the finite size effect, although unusual. The most surprising point is the large M limit, where β explicitly depends on whether M is odd or even; in this case the $M \to \infty$ limit is nontrivial. It remains to see what the critical behavior of the restricted CLG model is in two dimensions.

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[30] This is certainly true when $\rho > \frac{1}{2}$; in that case there might be consecutive 0's in the initial configuration, but eventually the activity reaches those regions and destroys them. However, for $\rho < \frac{1}{2}$, there are no configurations which are devoid of consecutive 0's.
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