Internal autoresonance in coupled oscillators with slowly decaying frequency

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In this work, we study resonance energy transfer from an impulsively loaded strongly nonlinear oscillator to a weakly coupled linear attachment with a slowly time-decaying stiffness. It is shown that even in the absence of external periodic forcing both oscillators may exhibit the resonance phenomenon, with the permanent response enhancement of the linear oscillator and the corresponding response reduction of the nonlinear actuator. This effect is said to be *internal autoresonance*. The influence of the system parameters on the emergence and stability of autoresonance is investigated both analytically and numerically.

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I. INTRODUCTION

In this work, we study targeted energy transfer (TET) in an oscillator array composed of an impulsively excited strongly nonlinear oscillator with cubic stiffness weakly coupled to a linear oscillator, whose stiffness slowly decays in time. The problem is investigated within the frames of the standard approach to resonance phenomena in nonlinear systems [1], which employs an intrinsic property of a nonlinear oscillator to change both its amplitude and natural frequency when the driving frequency changes. The ability of a nonlinear oscillator to stay captured into resonance due to variance of its structural and/or excitation parameters is known as *autoresonance* (AR).

In a large number of previous works, AR with persistent response enhancement has been identified as one of the most effective methods of excitation of high-energy regimes in a broad range of nonlinear systems. After first studies for the purposes of particle acceleration [2–4] and planetary dynamics [5,6], AR has become a very active field of research; recent advances in this field have been presented and discussed, e.g., in [7–9]. Special attention has been given to *external AR* induced by external forcing with a slowly varying frequency. The particular case of *parametric AR* [10,11] has been interpreted as an extension of classical parametric resonance [12] to the phenomenon of response enhancement in the nonlinear oscillator subjected to parametric excitation with slowly varying frequency.

This work discusses *internal AR* arising due to the slow change of structural parameters in the absence of both external and parametric fast periodic forcing. It will be shown that a strongly nonlinear oscillator having no preferential resonance frequencies may be engaged in resonance due to its intrinsic property to change both its amplitude and natural frequency in accordance with the frequency of the linear attachment. This type of resonance interaction represents *internal AR* in coupled oscillators. It is important to note that internal AR in the system under consideration suggests a reducing response of the nonlinear oscillator such that its frequency diminishes in accordance with the frequency of the linear oscillator. From a practical viewpoint, AR of this type proposes a mechanism of vibration mitigation in a structure subjected to seismic effects (see, e.g., [13]). In the analysis of seismic protection devices, the linear oscillator with a slowly decreasing frequency serves as a model of a one-particle mass-string model with linearly decreasing in time spring stretching, or of an elastic beam subjected to linearly increasing axial compressive loading [14,15].

The rest of this article is organized as follows. The governing equations of the resonant system are introduced in Sec. II. Initial conditions for both oscillators are set equal to zero, except for an initial impulse applied to the nonlinear oscillator. These initial conditions determine the basic limiting phase trajectory (LPT) corresponding to the maximum possible energy transfer from the excited oscillator to the attachment (see, e.g., [16–19], and references therein). Note that the procedures needed to obtain required results remain valid for arbitrary initial conditions but the motion along the LPT is of primary interest because it is associated with the most intense energy transfer. Furthermore, the approaches developed in this work can be extended to the arrays with more complicated nonlinear characteristics provided that the nonlinear oscillator exhibits resonant oscillations with a single dominant frequency close to the natural frequency of the linear oscillator. Nonlinearities of even orders (quadratic, quartic, etc.) do not satisfy this condition.

In Sec. III, asymptotic solutions of the resonant system are derived using the method of multiple scales [20,21]. The obtained solutions confirm the growth of the mean-square amplitude of the linear oscillator together with the simultaneous reduction of the amplitude of the nonlinear oscillator. Furthermore, the asymptotic solutions clearly demonstrate that the backbone curves adequately depict the growth/detuning rate of the mean-square amplitudes, so that the examination of resonance TET can be reduced to the analysis of the corresponding backbone curves. Modifications of LPTs in the systems with different initial conditions are briefly discussed at the end of Sec. III. Numerical results in Sec. IV elucidate the effect of parameters on the intensity of energy transfer. Concluding remarks are collected in Sec. V.

II. RESONANCE DYNAMICS OF COUPLED OSCILLATORS

The equations of motion are given by

$$m_1\frac{d^2U_1}{dt^2}+\gamma U_1^3+c_{12}(U_1-U_2)=0,$$

$$m_2\frac{d^2U_2}{dt^2}+c_2(1-st)U_2+c_{12}(U_2-U_1)=0. \quad (2.1)$$

In (2.1), m_1 and m_2 are the masses of the oscillators, $m_2 \leqslant m_1; c_2$ is the linear spring stiffness constant; γ is the nonlinear spring stiffness; c_{12} is the stiffness parameter of linear coupling; $\omega = (c_2/m_2)^{1/2}$ is the eigenfrequency of the separated linear oscillator in the uncoupled time-invariant system ($c_{12} = 0, s = 0$). The system is assumed to be initially at rest, with a nonzero initial velocity V_0 of the nonlinear oscillator, i.e., $U_1 = U_2 = 0$, $V_1 = \frac{dU_1}{dt} = V_0$, $V_2 = 0$ at $t = 0_+$ (the subscript index "+" will be omitted in further expositions).

The above equations involve eight independent parameters (including initial impulse). This creates significant computational difficulties, particularly due to the wide range of numerical parameter values. For computational purposes, it is convenient to diminish the number of independent parameters by reducing (2.1) to the dimensionless form. First, we introduce the dimensionless fast and slow time scales $\tau_0 = \omega t$ and $\tau = \varepsilon\tau_0$, respectively. Then, assuming weak coupling, we define the small parameter of the system $\varepsilon = c_{12}/2c_2 \ll 1$; small detuning rate is now defined as $s = 2\varepsilon^2\beta\omega$. The transformed equations of motion are given by the following:

$$\frac{d^2U_1}{d\tau_0^2} + \frac{8}{3}\alpha_1 U_1^3 + 2\varepsilon M(U_1 - U_2) = 0,$$
$$\frac{d^2U_2}{d\tau_0^2} + [1 - 2\varepsilon\zeta_0(\tau)]U_2 + 2\varepsilon(U_2 - U_1) = 0, \quad (2.2)$$

where the relative mass $M = m_2/m_1 \leqslant 1, \zeta_0(\tau) = \beta\tau$, $\alpha_1 = 3\gamma/8m_1\omega^2$. The system is initially at rest with a nonzero initial velocity $v_0 = V_0/\omega$ of the nonlinear oscillator. Introducing the dimensionless variables $u_r = \frac{U_r}{v_0}, v_r = \frac{du_r}{d\tau_0} = \frac{V_r}{v_0}$ $(r = 1,2)$ into (2.2), we reduce the equations of motion to the form

$$\frac{d^2u_1}{d\tau_0^2} + \frac{8}{3}\alpha u_1^3 + 2\varepsilon M(u_1 - u_2) = 0,$$
$$\frac{d^2u_2}{d\tau_0^2} + [1 - 2\varepsilon\zeta_0(\tau)]u_2 + 2\varepsilon(u_2 - u_1) = 0, \quad (2.3)$$

with the coefficient of nonlinearity

$$\alpha = 3V_0^2\gamma/8m_1\omega^4. \quad (2.4)$$

Equations (2.3) include only four independent parameters instead of eight parameters in (2.1). The dimensionless initial conditions

$$u_1(0) = 0, \quad v_1(0) = 1; \quad u_2(0) = 0, \quad v_2(0) = 0 \quad (2.5)$$

determine the LPT of system (2.3) corresponding to the maximum possible energy transfer from the excited oscillator to the attachment. The notion of LPT, first introduced for time-invariant oscillators (e.g., [16,17], and references therein) plays an important role in the analysis of irreversible tunneling in a pair of quasilinear oscillators with slowly time-varying linear spring stiffness [17] as well as in the study of AR in quasilinear chains [9,18,19]. In this work, this notion is extended to strongly nonlinear oscillator arrays. The solution of (2.3) with initial conditions (2.5) is further considered as a basic LPT. Modifications of LPTs in the oscillator arrays with other initial conditions are briefly discussed in Sec. III.

III. ASYMPTOTIC ANALYSIS OF ENERGY TRANSFER

In this work, the existence of 1:1 resonance interactions between the oscillators is assumed. The emergence and stability of this regime is discussed in the Appendix. Under the condition of 1:1 resonance, Eqs. (2.3) can be rewritten as

$$\frac{d^2u_1}{d\tau_0^2} + u_1 + \varepsilon\mu\left(\frac{8}{3}\alpha u_1^3 - u_1\right) + 2\varepsilon M(u_1 - u_2) = 0,$$
$$\frac{d^2u_2}{d\tau_0^2} + [1 - 2\varepsilon\zeta_0(\tau)]u_2 + 2\varepsilon(u_2 - u_1) = 0. \quad (3.1)$$

The scaling parameter $\mu = \varepsilon^{-1}$ is introduced to preserve the formal equivalence of (2.4) and (3.1) under the condition of resonance. We note that overall linear stiffness remains positive and the system is stable in the interval $\tau_0 \in [0,1/(2\varepsilon^2\beta)]$ (Fig. 1). Therefore, the interval I_0: $\tau_0 \in [0,1/(2\varepsilon^2\beta)]$ can be considered as an interval of validity in the study of resonant oscillations.

Asymptotic solutions of (3.1) for small ε are derived using the method of multiple scales [20,21]. The first step for applying this method is to express the system responses in terms of the new complex variables

$$\Psi_r = (v_r + iu_r)e^{-i\tau_0}, \quad \Psi_r^* = (v_r - iu_r)e^{i\tau_0}, \quad r = 1,2, \quad (3.2)$$

where the asterisk denotes complex conjugate. Substituting (3.2) into (3.1) yields the following alternative (still exact)

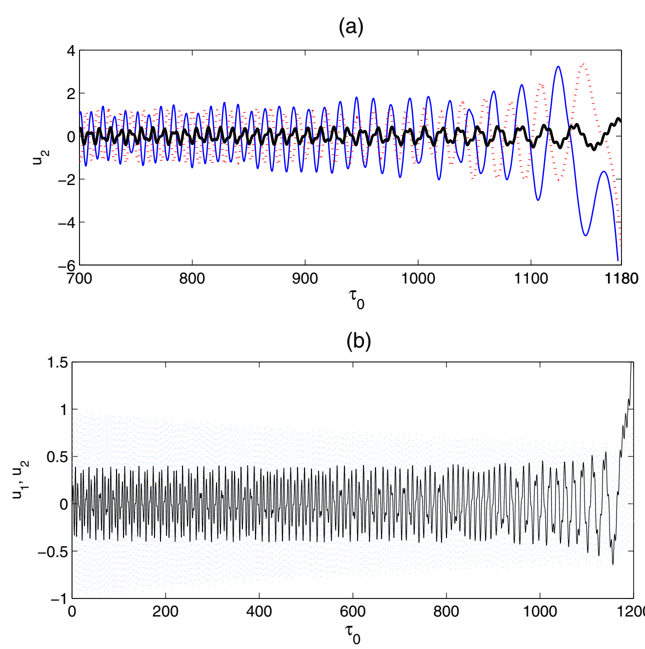

FIG. 1. Responses u_1 and u_2 in system (2.3) with parameters $M = 1$, $\varepsilon = 0.1$, $\beta = 0.05$, and different coefficients of nonlinearity: (a) displacements u_2 of the linear oscillators in system (2.3) with different coefficients of nonlinearity; (b) displacements of both oscillators at nonlinearity $\alpha = 2$.

equations:

$$\frac{d\Psi_1}{d\tau_0} = i\varepsilon\left[\mu\left(\alpha|\Psi_1|^2 - \frac{1}{2}\right)\Psi_1 + M(\Psi_1 - \Psi_2) + G_1\right],$$
$$\Psi_1(0) = 1,$$
$$\frac{d\Psi_2}{d\tau_0} = -i\varepsilon[\zeta_0(\tau)\Psi_2 + (\Psi_1 - \Psi_2) + G_2],$$
$$\Psi_2(0) = 0. \tag{3.3}$$

The terms $G_{1,2}$ include higher harmonics in τ_0 but an explicit expression of these terms is insignificant for further analysis. In the next step, the following asymptotic expansion is introduced:

$$\Psi_r(\tau_0,\tau,\varepsilon) = \psi_r(\tau) + \varepsilon\psi_r^{(1)}(\tau_0,\tau) + O(\varepsilon^2), \quad r = 1,2. \tag{3.4}$$

The equations for the slowly varying envelopes $\psi_r(\tau)$ can be obtained by straightforward averaging of (3.3) with respect to τ_0. The standard procedure [21] yields the following equations for the slow envelopes $\psi_r(\tau)$:

$$\frac{d\psi_1}{d\tau} = i\left[\mu\left(\alpha|\psi_1|^2 - \frac{1}{2}\right)\psi_1 + M(\psi_1 - \psi_2)\right], \quad \psi_1(0) = 1,$$
$$\frac{d\psi_2}{d\tau} = -i[\zeta_0(\tau)\psi_2 + (\psi_1 - \psi_2)], \quad \psi_2(0) = 0. \tag{3.5}$$

Once the solutions $\psi_r(\tau), r = 1,2$, are found, the leading-order approximations to the solutions of the generating system (3.1) can be calculated by (3.2) and (3.4). As a result, we obtain

$$u_{r0}(\tau_0,\tau) = |\psi_r(\tau)|\sin[\tau_0 + \gamma_r(\tau)],$$
$$v_{r0}(\tau_0,\tau) = |\psi_r(\tau)|\cos[\tau_0 + \gamma_r(\tau)],$$
$$r = 1,2, \tag{3.6}$$

where $\gamma_r(\tau_1) = \arg\psi_r(\tau)$. It is well known that the suggested asymptotic procedure leads to an error of approximations of order ε in the time interval $\tau_0 \sim O(1/\varepsilon)$ [1,20,21]. However, a refined analysis has shown that the interval of convergence depends on the properties of approximate solutions (see, e.g., [1,21]). Moreover, relatively large values of ε used in particular problems do not necessarily imply that the derived analytical approximations will be poor at larger times (certain applications can be found, e.g., in [13]). The accuracy of approximations should be additionally verified by numerical simulations.

Note that Eqs. (3.5) are integrable yielding the following integral of motion:

$$|\psi_1(\tau)|^2 + M|\psi_2(\tau)|^2 = |\psi_1(0)|^2 = 1. \tag{3.7}$$

It follows from (3.5) and (3.7) that $|\psi_2(\tau)|^2 \approx \tau^2$ but $|\psi_1(\tau)|^2 \approx 1 - M\tau^2$ in the initial time interval.

The polar representation

$$\psi_1 = ae^{i\delta_1}, \quad \sqrt{M}\psi_2 = be^{i\delta_2}, \quad \Delta = \delta_2 - \delta_1,$$
$$a = \cos\theta, \quad b = \sin\theta \tag{3.8}$$

reduces (3.5) to the following real-valued equations:

$$\frac{d\theta}{d\tau} = -\sqrt{M}\sin\Delta,$$
$$\sin 2\theta\frac{d\Delta}{d\tau} = -\left[\mu\left(\alpha\cos^2\theta - \tfrac{1}{2}\right) + \zeta(\tau)\right]\sin 2\theta - 2\sqrt{M}\cos 2\theta\cos\Delta, \tag{3.9}$$

where $\zeta(\tau) = \zeta_0(\tau) - (1 - M)$. In analogy to quasilinear theory [17], the initial conditions $\theta(0) = 0, \Delta(0) = \pi/2$ determine the LPT of the slow motion as a function of slow time τ.

Note that the variables $y = a^2, z = b^2$ depict the states of the nonlinear and linear oscillators, respectively. Since the response of the linear oscillator characterizes the emergence and intensity of TET from the excited nonlinear oscillator to the attachment, it is convenient to consider $z(\tau)$ as a new independent variable. Substituting $z = b^2$ into (3.9), we obtain the equations

$$\frac{dz}{d\tau} = -2\sqrt{Mz(1-z)}\sin\Delta,$$
$$\frac{d\Delta}{d\tau} = -\left\{\mu\left[\alpha(1-z) - \frac{1}{2}\right] + \zeta(\tau)\right\} - \sqrt{M}\frac{1-2z}{\sqrt{z(1-z)}}\cos\Delta \tag{3.10}$$

with initial conditions $z(0) = 0, \Delta(0) = \pi/2$. The quasisteady solutions of (3.10) obey the equations

$$\frac{dz}{d\tau} = 0, \quad \frac{d\Delta}{d\tau} = 0. \tag{3.11}$$

The first equation in (3.11) yields $\sin\bar{\Delta} = 0$; the phase $\bar{\Delta} = 0$ is stable (see [17]). In the stable system, the second equation of (3.11) takes the form

$$F(z) = -\mu\left[\alpha(1-z) - \frac{1}{2}\right] - \sqrt{M}\frac{1-2z}{\sqrt{z(1-z)}} = \zeta(\tau). \tag{3.12}$$

If $\zeta_0 \equiv 0, \zeta = -(1 - M)$, then the solution $\bar{z}_0$ of (3.12) determines the stable equilibrium position of the nonlinear oscillator in the conservative system; if $\zeta_0(\tau) \neq 0$, the solution $\bar{z}(\tau)$, which depicts the slow movement of the quasisteady state along the axis $\bar{\Delta} = 0$, determines the *backbone curve* of the oscillator [20]. It is obvious that $\bar{z}(0) = \bar{z}_0$. Substituting $y = 1 - z$ into (3.10)–(3.12), we derive similar equations for the amplitude squared $y = a^2$; the corresponding quasisteady state $\bar{y}(\tau) = 1 - \bar{z}(\tau)$ determines the backbone curve of the nonlinear oscillator.

Since the derivative

$$F'(z) = \mu\alpha + 0.5\sqrt{M}/[z(1-z)]^{3/2} > 0, \quad 0 < z < 1, \tag{3.13}$$

the function $F(z)$ increases with increasing z in the interval of interest, and thus, the root $\bar{z}(\tau)$ of Eq. (3.12) grows with time τ. On the contrary, the solution $\bar{y}(\tau) = 1 - \bar{z}(\tau)$ is decreased as time grows. It follows from (3.13) that these effects hold true for every set of parameters, which ensures resonance interactions between the oscillators. Since the solutions $\bar{y}(\tau)$ and $\bar{z}(\tau)$ are associated with the energy of the nonlinear and linear oscillators, respectively, resonance interactions between the oscillators can be explained and interpreted as TET, where energy is directed from the excited nonlinear oscillator to the linear attachment in a one-way irreversible fashion.

If the initial state of the nonlinear oscillator is changed, e.g., from $u_1(0) = 0, v_1(0) = 1$ to $u_1(0) = 1, v_1(0) = 0$ but $u_2(0) = 0, v_2(0) = 0$, then the system response can be expressed through the complex variables $\tilde{\Psi}_r = (u_r - iv_r)e^{-i\tau_0} = \Psi_r e^{-i\pi/2}$, $\tilde{\Psi}_r^* = (u_r + iv_r)e^{i\tau_0} = \Psi_r^* e^{i\pi/2}$ such that $\tilde{\Psi}_1(0) = \tilde{\Psi}_1^*(0) = 1$, $\tilde{\Psi}_2(0) = \tilde{\Psi}_2^*(0) = 0$. It is easy to deduce that the functions $\tilde{\Psi}_r$ also satisfy Eqs. (3.3) and thus, the above-mentioned conclusions remain valid for the functions $\tilde{\Psi}_r$ and for the corresponding averaged solutions. Hence, system (2.3) with the displaced initial conditions moves along the modified LPT obtained from the basic solution by a simple phase shift. A similar conclusion can be made if the nonlinear oscillator starts at an arbitrary point on the basic LPT.

IV. NUMERICAL SIMULATIONS AND DISCUSSION

In this section, this effect of parameters on the emergence of resonant energy transfer is illustrated by direct numerical simulations. As mentioned earlier in this work, the dimensionless slow system contains four independent parameters but the coefficients depend on all parameters of the original system including the initial impulse. Formula (2.4) indicates similar effects of nonlinearity γ and the squared impulse V_0^2 on the coefficient α. In turn, variations of linear stiffness c_2 and the mass m_2, which determine the frequency $\omega = (c_2/m_2)^{1/2}$, modifies not only the relative mass M and the dimensionless coupling coefficient ε but also the rate $\beta = s/2\varepsilon^2\omega$. This implies that the influence of each original parameter on the system dynamics should be examined separately.

It is mentioned in the Appendix that strongly nonlinear systems with coefficients $\alpha > 1$ may be excluded from consideration because small oscillations of the attachment weakly affect the dynamics of the nonlinear oscillator. In this section, this effect is confirmed by numerical simulations of system (2.3) with parameters $M = 1, \varepsilon = 0.1, \beta = 0.05$, and different coefficients of nonlinearity (Fig. 1).

Figure 1(a) illustrates the dynamics of the linear oscillators in system (2.3) with coefficients of nonlinearity $\alpha = 0.5, \alpha = 1, \alpha = 2$. We observe the response enhancement in the systems with $\alpha = 0.5$ and $\alpha = 1$ but small oscillations at $\alpha = 2$. Note that all trajectories become unstable at a point close to $\tau_0 = 1/2\varepsilon^2\beta = 1000$. Figure 1(b) illustrates the responses of the linear and nonlinear oscillators in the system with stronger nonlinearity $\alpha = 2$. It is seen that systems with nonlinearities $\alpha = 0.5$ and $\alpha = 1$ yield similar resonance responses but at $\alpha = 2$ small oscillations of the linear attachment weakly affect the dynamics of the excited nonlinear oscillator.

Taking into account the close similarity of the responses at $\alpha = 1$ and $\alpha = 0.5$, we restrict our focus on the oscillators with $\alpha = 0.5$. Figure 2 illustrates TET in the basic system with parameters $M = 1, \alpha = 0.5, \varepsilon = 0.1 (\mu = 10), \beta = 0.05$. In this case, Eq. (3.12) for the quasisteady solution $\bar{z}(\tau)$ takes the form

$$F(z) = \frac{\mu}{2}z - \frac{1-2z}{\sqrt{z(1-z)}} = \beta\tau. \tag{4.1}$$

The system is stable in the interval $\tau \in [0, 1/(2\varepsilon\beta)]$, or $\tau \in [0,100]$. Figure 2(a) clearly demonstrates that the reduction of the nonlinear response occurs together with an enhanced response of the linear attachment; Fig. 2(b) depicts the amplitude squared $z(\tau)$ of the linear oscillator, the corresponding backbone curve $\bar{z}(\tau)$, and its linear approximation $\bar{z}_l(\tau)$ [see formula (4.2) below] for the basic system. From Fig. 2(b), it is seen that the backbone curve adequately depicts the growth rate of the mean-square amplitude. This means that the study of the resonance effects can be reduced to the analysis of the backbone curves. In general, Eq. (4.1) should be solved numerically. However, Fig. 2(b) demonstrates that the backbone curve $\bar{z}(\tau)$ may be approximated by the function

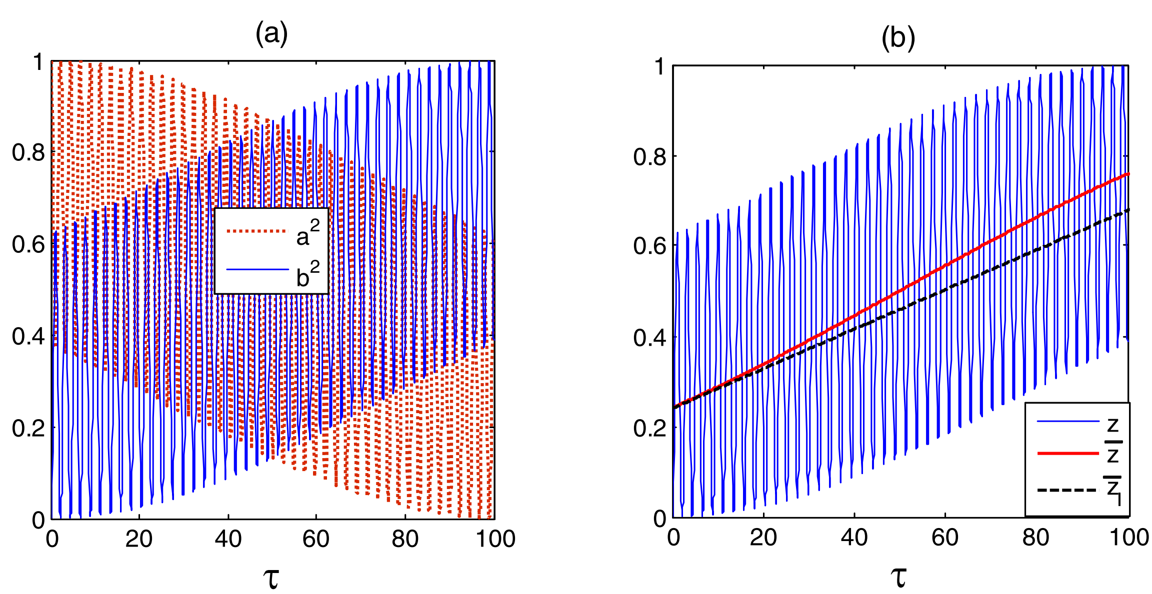

FIG. 2. Responses of the oscillators in the basic system with parameters $M = 1$, $\alpha = 0.5$, $\varepsilon = 0.1$, $\beta = 0.05$: (a) the amplitudes squared a^2 and b^2, (b) the function $z(\tau) = b^2(\tau)$, the backbone curve $\bar{z}(\tau)$ (the solid line), and its linear approximation $\bar{z}_l(\tau)$ (the dashed line) for the linear oscillator.

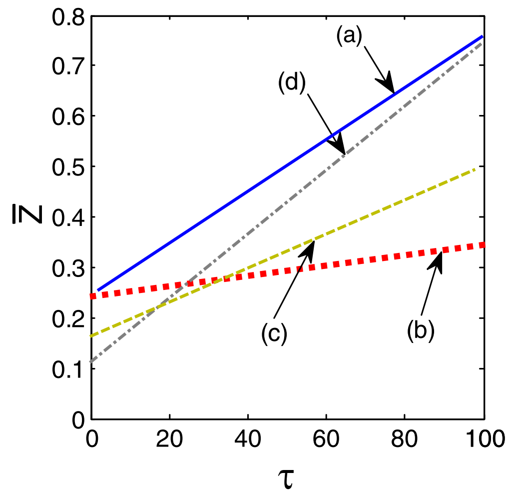

FIG. 3. Backbone curves of the linear oscillator (4.1) with parameters: (a) $\varepsilon = 0.1, M = 1, \beta = 0.05$ (basic); (b) $\varepsilon = 0.1$, $M = 1$, $\beta = 0.01$; (c) $\varepsilon = 0.05$, $M = 1$, $\beta = 0.05$; (d) $\varepsilon = 0.1$, $M = 0.25$, $\beta = 0.05$.

$\bar{z}_l(\tau) = \bar{z}_0 + \xi(\tau)$ satisfying the linearized equation

$$F(\bar{z}_0 + \xi) = F(\bar{z}_0) + F'(\bar{z}_0)\xi = \beta\tau - (1 - M). \quad (4.2)$$

Taking into account that $F(\bar{z}_0) = -(1 - M)$, we obtain the following linear equation for the variation ξ:

$$F'(\bar{z}_0)\xi = \beta\tau, \quad (4.3)$$

In particular, for the basic system we obtain the linearized solution with initial value $\bar{z}_0 = 0.24$, and the terminal value $\bar{z}_{lT} = 0.68$ against the numerical value $\bar{z}_T = 0.76$ [Fig. 2(b)] with the maximum difference between the numerical and approximate results being close to 10%. Direct numerical calculations reveal errors of the same order for all above-mentioned sets of structural parameters. This implies that linear approximations can be used to evaluate the rate of the amplitude growth.

Figure 3 presents the backbone curves for the linear oscillators in the systems with different parameters.

The solid line (a) depicts the backbone curve $\bar{z}(\tau)$ of the basic system. The dotted line (b) corresponds to the backbone curve in the system with coupling stiffness $\varepsilon = 0.1$ and detuning rate $\beta = 0.01$. In this system, the mass and stiffness parameters coincide with the basic parameters but the rate β is decreased due to the decrease of the detuning parameter $s = 2\varepsilon^2\beta\omega$. Note that the interval of stable motion in this system is given by $\tau \in [0,500]$ but, for comparison purposes, motion in the truncated interval $\tau \in [0,100]$ is analyzed. It is seen that the growth rate at $\beta = 0.01$ is small enough, and the resulting process in the interval $\tau \in [0,100]$ cannot be considered as energy localization on the linear oscillator. The dashed line (c) corresponds to the backbone curve $\bar{z}(\tau)$ in the system with the weak-coupling coefficient $\varepsilon = 0.05$. The coefficient $\varepsilon = 0.05$ can be achieved by the reduction of the coupling stiffness c_{12}. As expected, the reduction of the coupling strength diminishes the mean value of the amplitude squared $b^2(\tau)$ compared to the basic system.

Line (d) depicts the enhancement of TET in the system with the relative mass $M = 0.25$. The mass m_1 is assumed to be four times more than in the basic system; the coefficient of nonlinearity γ is also proportionally increased to leave unchanged the coefficient α. The quasisteady solution can now be found from the equation

$$F(z) = \frac{\mu}{2}z - \frac{1 - 2z}{2\sqrt{z(1 - z)}} = \beta\tau - \frac{3}{4}. \quad (4.4)$$

Although the initial value of the backbone curve $\bar{z}_0 = 0.106$ is much less than the corresponding value $\bar{z}_0 = 0.24$ in the basic system, the terminal value $\bar{z}_T = 0.74$ is close to the value $\bar{z}_T = 0.76$ in the system with equal masses. Note that initially the amplitude $|\psi_2| = |b|/\sqrt{M} = 2|b|$ is close to the amplitude of the linear oscillator in the basic system but at large times it becomes twice more than the corresponding amplitude in the

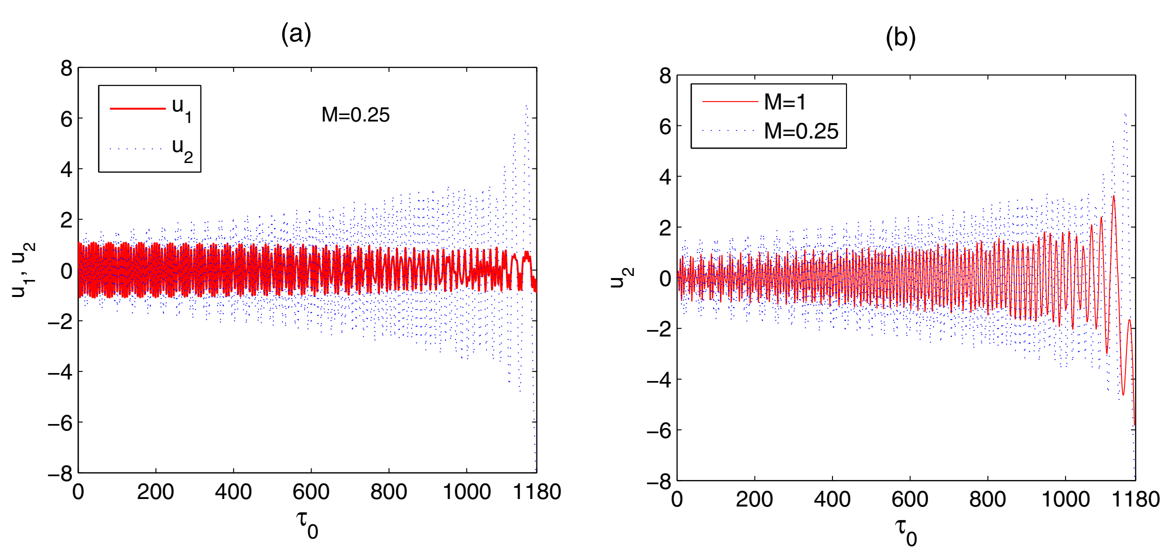

FIG. 4. Responses u_1 and u_2 in the system with parameters $\alpha = 0.5$, $\varepsilon = 0.1$, $\beta = 0.05$, and different values of the relative mass M: (a) responses of both oscillators at $M = m_2/m_1 = 0.25$; (b) responses of the linear oscillators at $M = 0.25$ and $M = 1$ (equal masses).

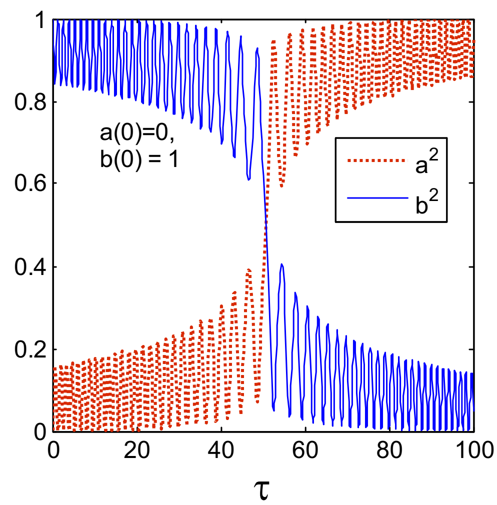

FIG. 5. Tunneling generated by the unit initial impulse imposed on the linear oscillator: the solid line corresponds to the excited linear oscillator; the dotted line depicts the response of the nonlinear oscillator.

system with equal masses. This result can be understood with the help of approximations (3.13), (4.2), and (4.3). It follows from (3.13) and (4.2) that the initial state $\bar{z}_0$ is decreased with decreasing M. On the other hand, expression (3.13) shows that the derivative $F'(\bar{z}_0)$ grows when the parameters M and $\bar{z}_0$ are diminished, thus compensating the gap between the initial points of the backbone curves at different values of the relative mass M.

The obtained approximations are confirmed by the results of numerical simulations for system (2.3) with parameters $\alpha = 0.5, \varepsilon = 0.1, \beta = 0.05$, and different coefficients M (Fig. 4).

Finally, we note that an application of an initial impulse to the linear oscillator while the nonlinear one is initially at rest transforms self-sustained resonance into adiabatic tunneling with irreversible energy transfer from the excited linear oscillator to the nonlinear attachment. An example for the basic system with parameters $\varepsilon = 0.1, M = 1, \beta = 0.05, \alpha = 0.5$ is presented in Fig. 5.

The earlier work [22] described a similar process in a system with the potential of the nonlinear oscillator $U_1 = \frac{1}{2}c_1u_1^2 + \frac{1}{4}\gamma u_1^4$ such that $c_1/m_1 = c_2/m_2$. It was shown that, although the system is initially engaged in resonance, energy transfer from the excited linear oscillator gives rise to adiabatic tunneling, in which energy of the excited oscillator falls due to escape from resonance, together with a simultaneous rapid increase of energy of the nonlinear oscillator. In contrast to tunneling, the above-discussed decreasing resonance response of the excited nonlinear oscillator is caused by capture of the oscillator into resonance with slowly decaying frequency.

V. CONCLUSIONS

In this article, we have studied energy transfer from an impulsively excited strongly nonlinear oscillator to a linear attachment with a slowly time-decaying stiffness. It is shown that even in the absence of external or parametric periodic forcing the system may exhibit *internal* AR, with permanent growth of the mean-square amplitude of the linear attachment and the simultaneously decreasing amplitude of the nonlinear oscillator. In the first part of the article, the governing equations of the slow dynamics are derived under the condition of 1:1 resonance interactions between the oscillators. The conditions of capture into resonance are collected in the Appendix. The numerical results obtained in the second part of the work motivate the investigations of the effect of parameters on the emergence of AR. In particular, it is shown that AR may occur in the array with moderate nonlinearity but the growth of nonlinearity results in failure of resonance capture and localization of energy on the excited oscillator. A similar effect is observed in the system with diminishing relative mass.

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APPENDIX: CONDITIONS OF 1:1 INTERNAL RESONANCE

In this work, resonance energy transfer under the condition of 1:1 resonance between the oscillators is investigated. It is important to note that, in contrast to the linear oscillator the frequency of a strongly nonlinear oscillator is not a pregiven constant but directly depends on the energy of oscillations determined by both nonlinearity and initial conditions. This implies that 1:1 resonance in the time-invariant analog of the coupled system (2.3) may occur if the natural frequency of the separated nonlinear oscillator

$$\frac{d^2u_1}{d\tau_0^2} + \frac{8}{3}\alpha u_1^3 = 0 \quad \text{(A1)}$$

with initial conditions $u_1(0) = 0$, $v_1(0) = 1$ is close to unity. The frequency of nonlinear oscillations can be approximately computed by the harmonic balance method [20], i.e., by seeking the time-periodic response in the form

$$u_1(\tau_0) = A\,\sin[\Omega(A)\tau_0], \quad \frac{du_1}{d\tau_0} = \Omega(A)A\,\cos[\Omega(A)\tau_0] \quad \text{(A2)}$$

with frequency $\Omega(A) = \alpha^{1/2}A$. Taking into account initial conditions, we obtain

$$\Omega(A)A = \alpha^{1/2}A^2 = 1, \quad A = \alpha^{-1/4}, \quad \Omega(A) = \alpha^{1/4} \quad \text{(A3)}$$

and, respectively, $\Omega = 1$ if $\alpha = 1$. Note that the approximate analysis allows considering a more general set of parameters $\alpha \sim 1$. Indeed, the parameters $\alpha = 2$ and $\alpha = 0.5$ correspond, respectively, to the frequencies $\Omega = 1.18$ and $\Omega = 0.84$, which are close to the required resonance frequency $\Omega = 1$ but formula (A3) shows that the amplitude A decreases with increasing α. This effect becomes even more enhanced for the coupled oscillators (Fig. 1).

Prerequisites for the emergence of TET in the oscillator array can be formally obtained in the same way as for quasilinear systems [17]. However, the dependence of the

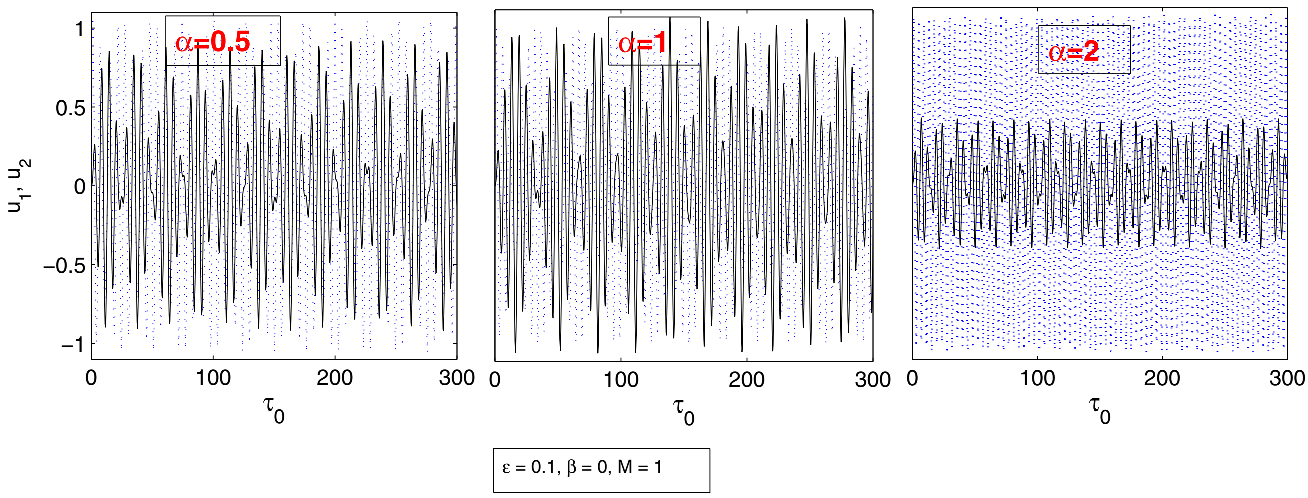

FIG. 6. Responses u_1 and u_2 in the time-invariant coupled oscillators with parameters $M = 1$, $\varepsilon = 0.1$, $\beta = 0$, and different coefficients of nonlinearity. u_1: dashed line; u_2: solid line.

coefficients on a number of parameters complicates the theoretical analysis. We simplify the problem by studying the dynamics of the attachment under the condition $|z| \ll 1$. Substituting $b = \sqrt{z}$ into (3.10) and assuming $|b| \ll 1, \zeta_0 = 0$, we reduce (3.10) to the following equations:

$$\frac{db}{d\tau} = -\sqrt{M}\sin\Delta,$$
$$b\frac{d\Delta}{d\tau} = -\left\{\mu\left[\alpha(1-b^2) - \frac{1}{2}\right] - (1-M)\right\}b - \sqrt{M}\cos\Delta, \quad \text{(A4)}$$

which agree with the averaged equations for the amplitude and the phase of the Duffing oscillator. Using the results earlier obtained for the Duffing oscillator [16], we conclude that the linear attachment performs small oscillations if the condition

$$27\mu\alpha M < [\mu\alpha - 0.5\mu - (1-M)]^3 \quad \text{(A5)}$$

holds true. Let $M = 1, \mu = 10(\varepsilon = 0.1)$. In this case, condition (A5) holds at $\alpha > 1.2$; otherwise, $27\mu\alpha > (\mu\alpha - 0.5\mu)^3$. In other words, at $\alpha < 1.2$ the system exhibits energy exchange manifested as large oscillations of both oscillators but at $\alpha > 1.2$ energy remains localized on the excited nonlinear oscillator (Fig. 6). A similar result for system (2.3) with slowly decreasing in time linear stiffness is demonstrated in Fig. 1. Also, in the particular case of $M = \frac{1}{4}$ we find that the energy remains localized on the excited nonlinear oscillator at $\alpha > 0.98$.

The obtained results have a clear physical meaning: energy transfer and exchange may occur for moderate coefficients of nonlinearity satisfying conditions (A3) and (A5); a further increase of nonlinearity leads to localization of energy on the excited nonlinear oscillator. At the same time, the reduction of the relative mass M is equivalent to the reduction of the coupling strength, thus diminishing energy transferring to the attachment and reducing its response.

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