Internal temperature of quantum chaotic systems at the nanoscale

Jiaozi Wang and Wen-ge Wang*

Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China

(Received 22 December 2016; published 6 September 2017)

The extent to which a temperature can be appropriately assigned to a small quantum system, as an internal property but not as a property of any large environment, is still an open problem. In this paper, a method is proposed for solving this problem, by which a studied small system is coupled to a two-level system as a probe, the latter of which can be measured by measurement devices. A main difficulty in the determination of possible temperature of the studied system comes from the back-action of the probe-system coupling to the system. For small quantum chaotic systems, we show that a temperature can be determined, the value of which is sensitive to neither the form, location, and strength of the probe-system coupling, nor the Hamiltonian and initial state of the probe. The temperature thus obtained turns out to have the form of Boltzmann temperature.

DOI: 10.1103/PhysRevE.96.032207

I. INTRODUCTION

Thermal and statistical properties of small quantum systems have been receiving lots of attention in recent years, both theoretical and experimental [1–15]. A key concept in this field, namely, temperature for such systems, has not been fully understood yet [6–8]. In particular, for a small quantum system, which possesses nonweak interactions among its components and is approximately isolated from its environment, the extent to which a temperature can be assigned to it, as an internal property but not as a property of environment, is still an open problem. To solve this problem is a challenge to both theoretical and experimental physics.

On one hand, in the statistical mechanics, temperature can be defined in several ways, which are equivalent in the thermodynamic limit, e.g., that by Boltzmann's entropy [16] and that by Gibbs' entropy [17]. But, there is by far no unique way for extrapolation to small quantum systems [17–24]. Different understandings of this concept may lead to diverse predictions; for example, related to the existence of negative temperature in bounded systems [9–15], debates have been seen [17,25,26]. And different angles of the approach to thermodynamic properties of small systems may lead to different types of definition for temperature [18–21,24,27–30]. To make the situation clarified, a direct consideration of the dynamics at the microscopic level should be unavoidable.

On the other hand, although at the macroscopic scale temperature can be detected in a reliable way by a thermometer, this strategy faces an obstacle when applied to a small system which is coupled to a small probe, the latter of which can be measured by measurement devices. The obstacle comes from the back-action of the system-probe interaction to the studied system. When the back-action can be neglected, the temperature can be studied in a standard way [23,27,28]; however, usually, the back-action is non-negligible due to the smallness of the studied system. A reliable temperature detection can be achieved, only when the influences of the following factors can be suppressed, namely, the form, location, and strength of the system-probe coupling, as well as the Hamiltonian and initial state of the probe.

In this paper, we propose a temperature-detection method, in which the above-discussed back-action can be appropriately taken into account. The method is based on an analysis of the dynamical evolution of the system-probe composite and gives a result insensitive to all the factors discussed above. A close relationship between the statistical mechanics and chaos has been perceived for a long time [31–34]. Hence, we consider possible temperature detection for small quantum chaotic systems. With a two-level system employed as a probe, we show that the above-discussed insensitivity can indeed be achieved in certain situations. Interestingly, it is found that the Boltzmann temperature can appear in a natural way from the dynamical evolution of the composite system.

According to recent progresses achieved in the foundation of quantum statistical mechanics, the so-called typical states within appropriately large energy shells have many properties similar to equilibrium states [35–43]. For this reason, we consider a typical state of the studied system as its initial state, before it is coupled to the probe. It is found that, to accomplish the temperature detection, the system-probe coupling should be appropriately adjusted; specifically, the coupling should be able to induce chaotic motion of the total system, but it should be still weak to ensure narrow eigenfunctions of the total system. The analytical results are tested numerically in an Ising chain in a nonhomogeneous transverse field.

The paper is organized as follows. In Sec. II, we introduce the main setup. In Sec. III, we discuss a reliable method of temperature detection for small quantum chaotic systems and derive an expression for the temperature thus determined. Then, we test the analytical predictions by numerical simulations in Sec. IV. Finally, conclusions are given in Sec. V.

II. THE MAIN SETUP

We use S to denote a considered quantum chaotic system and use $|\varphi_k\rangle$ to denote eigenstates of its Hamiltonian H_S, $H_S|\varphi_k\rangle = E_k|\varphi_k\rangle$. As a quantum chaotic system, its spectrum has no degeneracy. Initially, the system S lies in a (normalized) typical state within an energy shell Γ_0, centered at E_S^0 with a given width δE, namely, $\Gamma_0 = [E_S^0 - \delta E/2, E_S^0 + \delta E/2]$. Explicitly, the typical state is written as

$$|\Phi_0\rangle = \sum_{E_k \in \Gamma_0} D_k|\varphi_k\rangle, \tag{1}$$

*wgwang@ustc.edu.cn

where D_k are Gaussian random numbers with a same variance. We use N_{Γ_0} to indicate the number of energy levels in the energy shell Γ_0.

When a probe is coupled to the system S, the total Hamiltonian is written as

$$H = H_p + \lambda H_I + H_S, \tag{2}$$

with a parameter λ for adjusting the coupling strength. We use $|m\rangle$ of $m = 0,1$ to denote eigenstates of the probe Hamiltonian H_p with eigenvalues e_m, $H_p|m\rangle = e_m|m\rangle$. For brevity, we write unperturbed states of the total system as $|\varphi_k m\rangle$ with energies $E_{km} \equiv E_k + e_m$. Eigenstates of the total Hamiltonian H are denoted by $|\psi_\alpha\rangle$ with energies E_α, $H|\psi_\alpha\rangle = E_\alpha|\psi_\alpha\rangle$, and are expanded as

$$|\psi_\alpha\rangle = \sum_{k,m} C^\alpha_{km}|\varphi_k m\rangle \tag{3}$$

in the unperturbed basis. The initial state of the total system is taken as $|\Psi_0\rangle = |\Phi_0\rangle|m_0\rangle$, undergoing a unitary evolution, $|\Psi(t)\rangle = e^{-iHt}|\Psi_0\rangle$.

For it to be possible to use properties of the probe to detect properties of the system S such as temperature, the motion of the probe should be sufficiently influenced by that of the system S. This requires that the probe-system coupling should not be very weak. Below, we assume that the probe is sufficiently coupled to the system, such that the total system is also a quantum chaotic system. (We revisit this point when discussing numerical results.)

When the total system is a quantum chaotic system, its energy levels, as well as their spacings, have no degeneracy. It is known that, in this situation, the distance between the reduced density matrix (RDM) of the probe, $\rho(t) = \mathrm{Tr}_S(|\Psi(t)\rangle\langle\Psi(t)|)$, and its long-time average, denoted by $\overline{\rho}$, scales as $N_{\Gamma_0}^{-1/2}$ [42–48]. This implies that, at large N_{Γ_0}, if $\rho(t)$ has a steady state, it should be $\overline{\rho}$.

To derive an expression for $\overline{\rho}$, we note that, when the RDM of the probe is measured experimentally, many realizations of the initial state of the system should be involved. Averaging over these initial states gives $\overline{D_{k_0}D_{l_0}} = \frac{1}{N_{\Gamma_0}}\delta_{k_0 l_0}$. Then, taking the average over a long time period, direct derivation shows that (cf., e.g., Ref. [43])

$$\overline{\rho}_{mm} = \frac{1}{N_{\Gamma_0}} \sum_{E_{k_0}\in\Gamma_0} \sum_{k,\alpha} \left|C^\alpha_{k_0 m_0}\right|^2 \left|C^\alpha_{km}\right|^2. \tag{4}$$

Let us write $\overline{\rho}_{mm}$ as

$$\overline{\rho}_{mm} = \sum_k P^{m_0}_m(E_k), \tag{5}$$

where

$$P^{m_0}_m(E_k) = \frac{1}{N_{\Gamma_0}} \sum_{E_{k_0}\in\Gamma_0} P^{k_0 m_0}_{km}, \tag{6}$$

$$P^{k_0 m_0}_{km} \equiv \sum_\alpha \left|C^\alpha_{k_0 m_0}\right|^2 \left|C^\alpha_{km}\right|^2. \tag{7}$$

The quantity $P^{k_0 m_0}_{km}$ has a simple interpretation; that is, it is the overlap of two local spectral densities of states (LDOS). Specifically, defining a LDOS for an unperturbed state $|\varphi_k m\rangle$ as $\rho^{\mathrm{L}}_{km}(E) = \sum_\alpha |C^\alpha_{km}|^2\delta(E - E_\alpha)$ [49,50], $P^{k_0 m_0}_{km}$ is the overlap of $\rho^{\mathrm{L}}_{km}(E)$ and $\rho^{\mathrm{L}}_{k_0 m_0}(E)$. Although the overlap $P^{k_0 m_0}_{km}$ may show considerable fluctuations with variation of the system's energy E_k, the averaged overlap $P^{m_0}_m(E_k)$ should show a smoother feature for N_{Γ_0} not small.

We note that, for large N_{Γ_0}, off-diagonal elements of $\overline{\rho}$ can be neglected. In fact, applying a result given in Ref. [51] to the system-probe composite we study here with $\mathrm{Tr}_S(H_I) = 0$, one finds that the steady state of the probe should have an approximately diagonal form in the eigenbasis $\{|m\rangle\}$ at λ not small, with off-diagonal elements scaling as $N_{\Gamma_0}^{-1/2}$. In other words, the eigenbasis of the self-Hamiltonian of the probe is a preferred basis [52,53].

III. TEMPERATURE DETECTION

For a probe as a two-level system, which has interacted with the measured system S and has reached a steady state $\overline{\rho}$, one can always get a value of β by fitting the steady state $\overline{\rho}$ to the canonical state $\frac{1}{Z}e^{-\beta H_p}$. This value of β reflects a property of the total system after the interaction. The point is whether it is possible to determine a certain value of β, which reflects a property of the initial state of the system S. For this to be possible, the finally determined value of β should be sensitive to neither the form, location, and strength of the probe-system coupling, nor the Hamiltonian and initial state of the probe.

In this section, we show that the above-discussed goal can be achieved. That is, under appropriate conditions, a value of β can be obtained, which is insensitive to the factors mentioned above.

A. Properties of the function $P^{m_0}_m(E_k)$

In this section, we discuss properties of the function $P^{m_0}_m(E_k)$, which are useful in the study of the steady state $\overline{\rho}$ in Eq. (5). As mentioned previously, the total system is assumed to be a quantum chaotic system, which implies that the eigenfunctions have sufficiently irregular components in the unperturbed basis. This chaotic feature requires that the coupling strength λ is not very small. Meanwhile, we require that λ is not large, such that both eigenfunctions and LDOS are narrow with $w_L \ll \delta E$, where w_L is the averaged width of the LDOS, which is approximately equal to the averaged width of eigenfunctions for λ not large.

We find that, under the conditions discussed above, the function $P^{m_0}_m(E_k)$ has the following three properties: (i) for a fixed value of m_0, this function with $m = 0$ and with $m = 1$ has similar shapes, centered at $(E^S_0 + e_{m_0} - e_m)$; (ii) it has a width approximately equal to δE; and (iii) it is approximately symmetric with respect to its center.

To show the above-discussed properties, let us first consider the sum

$$X_{m_0}(E_\alpha) \equiv \sum_{E_{k_0}\in\Gamma_0} \left|C^\alpha_{k_0 m_0}\right|^2 \tag{8}$$

as a function of the energy E_α. Using this quantity, $P^{m_0}_m(E_k)$ in Eq. (6) can be written as

$$P^{m_0}_m(E_k) = \frac{1}{N_{\Gamma_0}} \sum_\alpha X_{m_0}(E_\alpha)\left|C^\alpha_{km}\right|^2. \tag{9}$$

The sum $X_{m_0}(E_\alpha)$ can be divided into a smoothly varying part, denoted by $F_{m_0}(E_\alpha)$, and a fluctuating part denoted by R_α,

$$X_{m_0}(E_\alpha) = F_{m_0}(E_\alpha) + R_\alpha. \tag{10}$$

In the case of $\lambda = 0$, there is a one-to-one correspondence between the set $\{|\psi_\alpha\}$ and the set $\{|\varphi_k m\rangle\}$. To indicate this correspondence explicitly, we write the labels k and m as k_α and m_α. It is easy to verify that, at this $\lambda = 0$,

$$X_{m_0}(E_\alpha) = \begin{cases} 1 & \text{if } E_{k_\alpha} \in \Gamma_0 \text{ and } m_\alpha = m_0 \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

where $E_{k_\alpha} = E_\alpha - e_{m_\alpha}$. This implies that

$$F_{m_0}(E_\alpha) = \begin{cases} \frac{\rho_S(E_\alpha - e_{m_0})}{\sum_m \rho_S(E_\alpha - e_m)} & \text{if } E_{k_\alpha} \in \Gamma_0 \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

and

$$R_\alpha = \begin{cases} 1 - \frac{\rho_S(E_\alpha - e_{m_0})}{\sum_m \rho_S(E_\alpha - e_m)} & \text{if } E_{k_\alpha} \in \Gamma_0 \text{ and } m_\alpha = m_0 \\ -\frac{\rho_S(E_\alpha - e_{m_0})}{\sum_m \rho_S(E_\alpha - e_m)} & \text{if } E_{k_\alpha} \in \Gamma_0 \text{ and } m_\alpha \neq m_0 \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

where $\rho_S(E)$ is the (smoothed) density of states of the system. We assume that $\rho_S(E)$ changes slowly in the considered energy region. As a result, variation of $F_{m_0}(E_\alpha) = \frac{\rho_S(E_\alpha - e_{m_0})}{\sum_m \rho_S(E_\alpha - e_m)}$ can be neglected and one has

$$F_{m_0}(E_\alpha) \simeq \begin{cases} c & \text{if } E_{k_\alpha} \in \Gamma_0 \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

where c is some constant. Thus, the function $F_{m_0}(E_\alpha)$ has approximately a rectangular shape, centered at $E_S^0 + e_{m_0}$ with a width δE. In the case that the probe is a single qubit, whose energy scale is much smaller than that of the system S, one has $\rho_S(E_\alpha - e_1) \approx \rho_S(E_\alpha - e_0)$ and $c \approx \frac{1}{2}$.

At small λ, the smoothly varying part of $X_{m_0}(E_\alpha)$, namely, $F_{m_0}(E_\alpha)$, should have a shape with small deviation from that of $\lambda = 0$ discussed above. Specifically, it should have the following properties: (i) being approximately symmetric with respect to a center $(E_S^0 + e_{m_0})$, (ii) having a width close to δE, (iii) varying slowly in the central region of its main body, and (iv) dropping fast at the edges to quite small values. Moreover, the main body of R_α should approximately lie in the same region as that of $F_{m_0}(E_\alpha)$ discussed above.

Since the total system is a quantum chaotic system, which has irregular components in the main bodies of its eigenfunctions, the fluctuating part R_α should fluctuate irregularly. Its contribution to the right-hand side (rhs) of Eq. (9) scales as $1/N_{\Gamma_0}^{1/2}$. Hence, for large N_{Γ_0}, contribution from the fluctuating part can be neglected and one gets

$$P_m^{m_0}(E_k) \simeq \frac{1}{N_{\Gamma_0}} \sum_\alpha F_{m_0}(E_\alpha) \left|C_{km}^\alpha\right|^2. \tag{15}$$

When the coupling is still weak to fulfill the condition $w_L \ll \delta E$, for most of the LDOS $\rho_{km}^L(E_\alpha)$, their main bodies should lie within the slowly varying region of the function $F_{m_0}(E_\alpha)$. For these LDOS, when computing the rhs of Eq. (15), the term $F_{m_0}(E_\alpha)$ can be approximately taken as a constant. Then, noting that $\sum_\alpha |C_{km}^\alpha|^2 = 1$ and the fact that a narrow LDOS $\rho_{km}^L(E_\alpha)$ is approximately centered at $E_\alpha = E_{km}$, from Eq. (15) one finds that

$$P_m^{m_0}(E_k) \simeq \frac{1}{N_{\Gamma_0}} F_{m_0}(E_\alpha)|_{E_\alpha = E_{km}} \tag{16}$$

for most of the energies E_k. The percentage not fulfilling Eq. (16) is proportional to $(w_L/\delta E)$. Thus, for most of the LDOS $\rho_{km}^L(E)$, the function $P_m^{m_0}(E_k)$ has the three properties stated above.

B. Insensitivity to the coupling

In this section, making use of results given in the previous section, we show that a value of β can be determined which is insensitive to the coupling term under the conditions given previously.

Substituting the expression of $P_m^{m_0}(E_k)$ in Eq. (16) into Eq. (5) and making use of the properties of the function $F_{m_0}(E_\alpha)$ discussed above, one finds that, within an error with an upper bound of the order of $(w_L/\delta E)$,

$$\overline{\rho}_{mm} \simeq \frac{1}{N_{\Gamma_0}} \sum_k P_m^{m_0}(E_k) \simeq \frac{1}{N_{\Gamma_0}} \sum_k F_{m_0}(E_{km}).$$

If $\rho_S(E)$ can be approximated by a linear function in the energy shell centered at $E_S^0 + e_{m_0} - e_m$ with a width δE, noting the fact that $F_{m_0}(E + e_m)$ is approximately symmetric within the energy shell with respect to the center, one gets

$$\overline{\rho}_{mm} \simeq G_{\lambda m_0}\, \rho_S\big(E_S^0 + e_{m_0} - e_m\big), \tag{17}$$

where

$$G_{\lambda m_0} = \frac{1}{N_{\Gamma_0}} F_{m_0}\big(E_0^S + e_{m_0}\big)\delta E, \tag{18}$$

being a quantity independent of the label m. The error for the approximation in Eq. (17) scales as $1/N_{\Gamma_0}^{1/2}$ and also as $(w_L/\delta E)$. Equation (17) predicts that

$$\beta \simeq \frac{1}{\Delta_e} \ln \frac{\rho_S\big(E_S^0 + e_{m_0} - e_0\big)}{\rho_S\big(E_S^0 + e_{m_0} - e_1\big)}, \tag{19}$$

where $\Delta_e = e_1 - e_0$. It is clear that the rhs of Eq. (19) is independent of the coupling term λH_I.

In the case that the eigenfunctions of the total system have on average a Lorentz shape [54], one can derive an explicit expression for the function $P_m^{m_0}(E_k)$ (see Appendix A),

$$P_m^{m_0}(E_k) \approx \frac{(\theta_+ - \theta_-)\rho_S\big(E_S^0\big)}{\pi \rho_T\big(E_S^0 + e_{m_0}\big)}, \tag{20}$$

where $\theta_\pm = \arctan \frac{2x_0 \pm \delta E}{2w_L}$, with $x_0 = E_k + e_m - E_S^0 - e_{m_0}$, and ρ_T is the density of states of the total system. It is not difficult to verify that the rhs of Eq. (20) has the three properties discussed above for $P_m^{m_0}(E_k)$.

C. Insensitivity to the probe

The value of β given in Eq. (19) depends on both the initial state and the Hamiltonian of the probe. In this section, we determined a value of β which is independent of the these two factors.

With the dependence on m_0 written explicitly, β_{m_0} in Eq. (19) has the following explicit expressions:

$$\beta_0 \simeq \frac{1}{\Delta_e} \ln \frac{\rho_S(E_S^0)}{\rho_S(E_S^0 - \Delta_e)}, \quad \beta_1 \simeq \frac{1}{\Delta_e} \ln \frac{\rho_S(E_S^0 + \Delta_e)}{\rho_S(E_S^0)}. \tag{21}$$

It is seen that the average $\overline{\beta} = \frac{1}{2}(\beta_0 + \beta_1)$ satisfies the relation

$$\overline{\beta} \simeq \beta_{\rm sm}, \tag{22}$$

where $\beta_{\rm sm}$ is a Boltzmann temperature, given in statistical mechanics for macroscopic systems from Boltzmann's entropy [16],

$$\beta_{\rm sm} = \left.\frac{\partial \ln \rho_S(E)}{\partial E}\right|_{E=E_S^0}, \tag{23}$$

which is clearly independent of the probe.

Furthermore, Eq. (22) can be obtained under a more generic initial condition of the probe, namely, for $|\psi_0\rangle = \sum_m c_m|m\rangle$ with a random relative phase between c_0 and c_1. In fact, in this case, within the second-order expansion of $\ln \rho_S$ with respect to Δ_e, one can show that (see Appendix B)

$$\beta \simeq \sum_{m_0} |c_{m_0}|^2 \beta_{m_0}. \tag{24}$$

Then, taking the average over all possible values of $|c_{m_0}|^2$, one gets the same averaged value of β as in Eq. (22).

To summarize, when the following conditions are satisfied, a temperature $\overline{\beta} \simeq \beta_{\rm sm}$ can be assigned to a quantum chaotic system S, which can be detected by a probe qubit: (i) N_{Γ_0} for the initial state of S is sufficiently large; (ii) the total system is a quantum chaotic system, whose eigenfunctions have sufficiently irregular coefficients in the unperturbed basis; (iii) $w_L \ll \delta E$; and (iv) δE is sufficiently small for linear approximation of $\rho_S(E)$ within related energy shells.

IV. NUMERICAL TESTS

In this section, we test the results given above, by numerical simulations performed in an Ising chain composed of N $\frac{1}{2}$-spins in a nonhomogeneous transverse field. The Hamiltonian of the system is written as

$$H_S = \mu_x \sum_{i=1}^{N} \sigma_x^i + \mu_1 \sigma_z^1 + \mu_4 \sigma_z^4 + \mu_z \sum_{i=1}^{N-1} \sigma_z^i \sigma_z^{i+1}, \tag{25}$$

where $\sigma_{x,z}$ indicate Pauli matrices. The probe, with a Hamiltonian $H_p = \omega_p \sigma_x^p$, is coupled to the ith spin of the Ising chain, with an interaction Hamiltonian,

$$\lambda H_I = \lambda \sigma_z^p \otimes \sigma_z^i. \tag{26}$$

The energy shell for the initial state is chosen to be narrow but contains a large number of levels. For $N = 14$, N_{Γ_0} is about 500.

The parameters μ_x, μ_z, μ_1, and μ_2 are adjusted, such that the system S is in a quantum chaotic regime, in which the nearest-level-spacing distribution $P(s)$ is close to the Wigner distribution $P_W(s) = \frac{\pi}{2} s \exp(-\frac{\pi}{4}s^2)$, the latter of which is almost identical to the prediction of the random matrix theory (RMT) [32,55]. In order to determine the

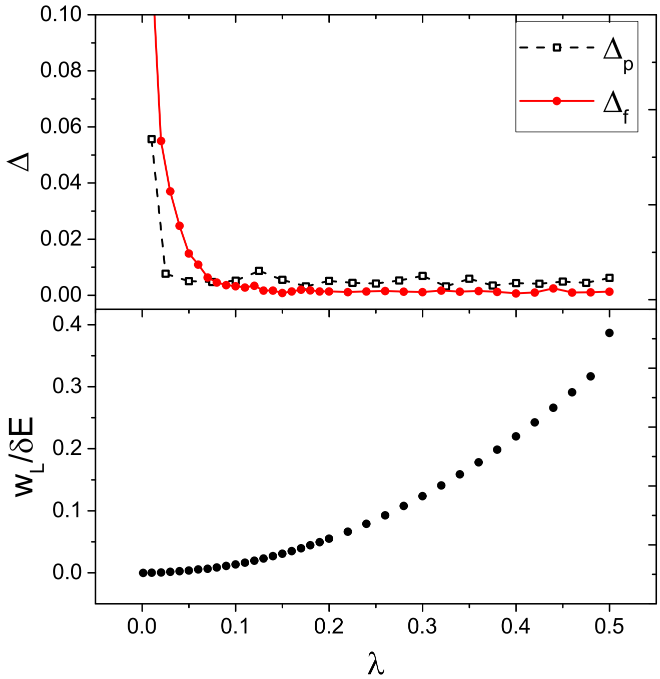

FIG. 1. Top: "Distances" to quantum chaos for the total system versus the coupling strength λ for $N = 14$. The distance Δ_p (see the text; open squares connected by dashed line) indicates a measure given by the statistics of the spectrum, and Δ_f (solid circles connected by solid line) is for the statistics of eigenfunctions. Bottom: The ratio $w_L/\delta E$ versus λ.

quantum chaotic regime of the coupling strength λ, we have studied the distance between $P(s)$ and $P_W(s)$, measured by $\Delta_p = \int |I(s) - I_W(s)|\, ds$. Here, $I(s)$ indicates the cumulative distribution of $P(s)$, $I(s) = \int_0^s P(s')\, ds'$, and $I_W(s)$ is the cumulative Wigner distribution, $I_W(s) = \int_0^s P_W(s')\, ds'$. As seen in the upper panel of Fig. 1, Δ_p drops quite fast, reaching a quite small value at $\lambda \approx 0.025$.

As seen in the analytical derivation of temperature given in the previous section, the property, which has been really used, is certain irregular behavior of the eigenfunctions. Such a property of eigenfunctions is not necessarily guaranteed by properties of the spectrum. Hence, a direct study of statistical properties of the eigenfunctions is needed. Numerical simulations in several models, including the Ising chain studied here, show that the following quantity Δ_f is useful for this purpose [56]: $\Delta_f = \int |f(x) - f_{\rm RMT}(x)|\, dx$. Here, $f(x)$ indicates the cumulative distribution of rescaled components in main bodies of the eigenfunctions, with $x = C_{km}^{\alpha}/\sqrt{\Pi_m(\varepsilon)}$, where $\Pi_m(\varepsilon) = \langle |C_{km}^{\alpha}|^2\rangle$ indicates the average shape of the eigenfunctions and $f_{\rm RMT}(x)$ is a cumulative Gaussian distribution predicted by the RMT [55]. As seen in the upper panel of Fig. 1, Δ_f reaches its lowest-value region at $\lambda \approx 0.1$. Thus, for $\lambda \gtrsim 0.1$, the eigenfunctions should have the needed irregular behaviors.

The lower panel of Fig. 1 shows that w_L reaches 10% of δE at $\lambda \approx 0.25$. Thus, the averaged overlap $P_m^{m_0}(E_k)$ is expected to have the three properties stated previously for λ above 0.1 and somewhat below 0.25. Indeed, we found that $P_m^{m_0}(E_k)$ are close to the prediction in Eq. (20) and possess the three

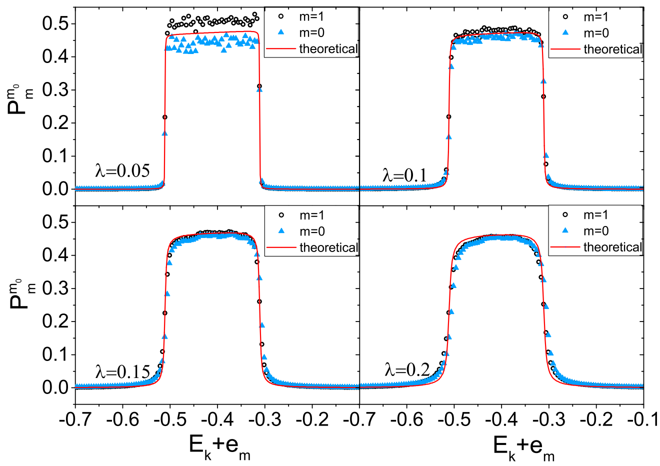

FIG. 2. Shapes of $P_m^{m_0}(E_k)$ for $m = 0$ (open circles) and $m = 1$ (triangles), with $\delta E = 0.2$ and $\Delta_e = 0.6$, plotted as a function of E_{km} for clearness in comparison. The solid curves represent the analytical prediction in Eq. (20).

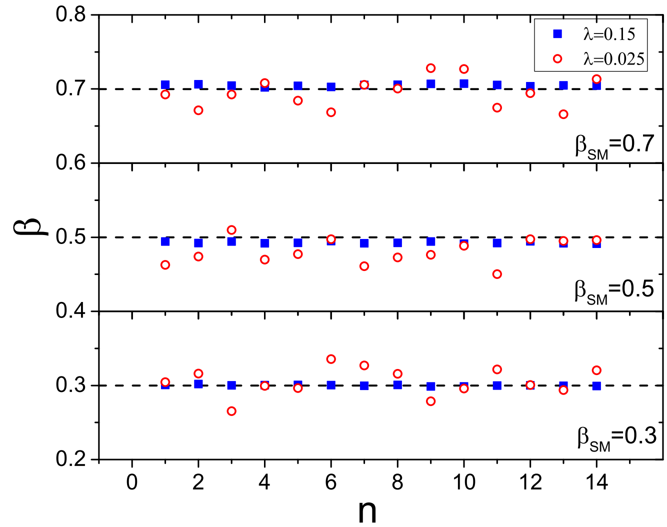

FIG. 4. Values of $\overline{\beta}$, when the probe is coupled to the nth spin of the chain, for $\lambda = 0.15$ (solid squares) and for $\lambda = 0.025$ (open circles).

properties in this intermediate regime of λ (as illustrated in Fig. 2). Consistently, $\overline{\beta} = \frac{\beta_1+\beta_0}{2}$ has been found quite close to $\beta_{\rm sm}$ in this regime of λ for $N = 14$ (Fig. 3).

Figure 3 shows that, decreasing the value of N and, thus, decreasing the number N_{Γ_0} for energy levels in the initial energy shell, the fluctuation of $\overline{\beta}$ becomes stronger. In fact, for $N = 8$, the fluctuations are quite strong, such that no reliable temperature detection can be done by the probe. Furthermore, at quite small λ, even for large N ($N = 14$), the fluctuation of $\overline{\beta}$ is also quite large, such that there is no reliable temperature detection. In fact, in this case, the two systems are not sufficiently coupled; as a result, one cannot get the temperature of the system S from properties of the probe. (This point is obvious in the extreme case of zero coupling.)

We have also tested the insensitivity of the measured value $\overline{\beta}$ to the location of coupling, for λ lying in the intermediate regime discussed above, as illustrated in Fig. 4 for $\lambda = 0.15$. On the other hand, the figure shows that, for quite small values of λ, say, for $\lambda = 0.025$, consistent with the results shown in Fig. 3, the value of $\overline{\beta}$ is sensitive to the location of coupling. Furthermore, we have studied dependence of the difference $|\beta_1 - \beta_0|$ on the spin number N. The density of states, ρ_S, has approximately a Gaussian shape [57], $\rho_S(E) \approx A\exp(-\alpha E^2)$. This gives $|\beta_1 - \beta_0| \simeq 2\alpha\Delta_e$. Numerically we found that $\alpha \propto \frac{1}{N+c}$ with $c \sim O(1)$; hence, $|\beta_1 - \beta_0| \propto \frac{\Delta_e}{N+c}$, approaching zero in the limit $N \to \infty$.

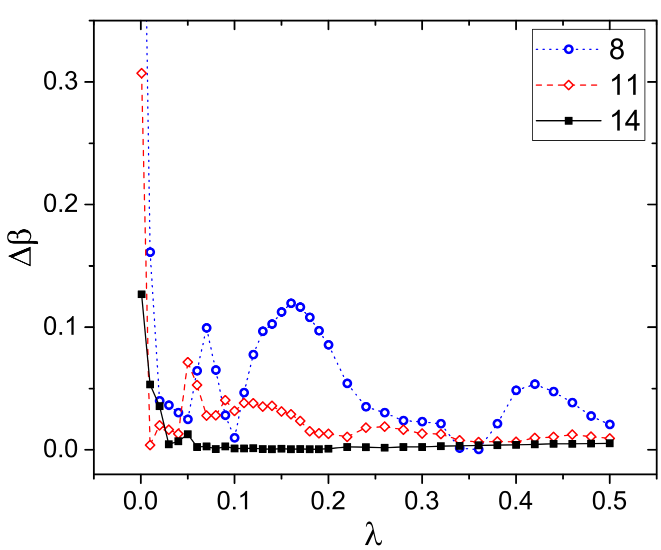

FIG. 3. The difference $\Delta\beta = |\overline{\beta} - \beta_{\rm sm}|$ versus λ. The value of E_S^0 for the initial state corresponds to $\beta_{\rm sm} = 0.3$.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, a method is proposed by which the temperature of a small quantum chaotic system can be detected by a probe qubit, which is appropriately coupled to the studied system. This method appropriately takes into account the back-action of the probe to the state of the studied system. The obtained temperature is determined by the derivative of the logarithm of the density of states of the studied system, in the same manner as a Boltzmann temperature for macroscopic systems. The extent to which a temperature can be assigned to the system has also been studied. Due to the smallness of the studied system, fluctuations around the obtained results are not negligible, scaling as $N_{\Gamma_0}^{-1/2}$ and also as $(w_L/\delta E)$, where N_{Γ_0} is the number of states in the initial energy shell with a width δE and w_L is the average width of the LDOS. Finally, we note that the proposed method should be feasible for experimental study of temperature under today's technology.

The system studied above initially lies in a pure state as a typical state in an energy shell, which may be effectively related to a microcanonical state [35]. In principle, the method proposed in this paper can also be used to study other types of initial states, for example, initially lying in a canonical state or in a nonequilibrium state [30,58,59]. It would be of interest to study temperature properties for such initial states in the future.

ACKNOWLEDGMENTS

The authors are grateful to J. Gong, G. Casati, and G. Benenti for valuable discussions and suggestions. This work was partially supported by the Natural Science Foundation of China under Grants No. 11275179 and No. 11535011, and the National Key Basic Research Program of China under Grant No. 2013CB921800.

APPENDIX A: DERIVATION OF EQ. (20)

In this Appendix, we derive Eq. (20), when eigenfunctions of the total system have on average a Lorentz shape with a width w_L. In this case, one has

$$\overline{|C^{\alpha}_{km}|^2} \approx \frac{w_L}{\pi\rho_T(E_\alpha)} \frac{1}{(E_\alpha - (E_k + e_m))^2 + w_L^2}, \tag{A1}$$

where the average is taken over neighboring levels [54].

Noting Eqs. (8) and (10), the smoothly varying part $F_{m_0}(E_\alpha)$ can be written as

$$F_{m_0}(E_\alpha) = \sum_{E_{k_0}\in\Gamma_0} \overline{|C^{\alpha}_{k_0 m_0}|^2}. \tag{A2}$$

When N_{Γ_0} is large, the summation in Eq. (A2) can be approximated by an integration over the energy of the system S, with $\int dE \rho_S(E)$. Substituting Eq. (A1) into the obtained integration, one gets

$$F_{m_0}(E_\alpha) \simeq \frac{\rho_S(E_S^0) w_E}{\rho_T(E_\alpha)\pi} \int_{E_\alpha - E_S^0 - e_{m_0} - \frac{\delta E}{2}}^{E_\alpha - E_S^0 - e_{m_0} + \frac{\delta E}{2}} \frac{1}{x^2 + w_E^2}\, dx, \tag{A3}$$

where $x = E_\alpha - (E + e_{m_0})$. Then, noting Eq. (16) and the fact that $E_{km} = E_k + e_m$, straightforward derivation shows that

$$P_m^{m_0}(E_k) \simeq \frac{\rho_S(E_S^0)}{\pi\rho_T(E_T^0)} \times \left(\arctan\frac{2x_0 + \delta E}{2w_E} - \arctan\frac{2x_0 - \delta E}{2w_E}\right), \tag{A4}$$

where $x_0 = E_k + e_m - E_S^0 - e_{m_0}$.

APPENDIX B: DERIVATION OF EQ. (24) FOR A GENERIC INITIAL STATE OF THE PROBE

In this Appendix, we show that Eq. (24) holds, within the second-order approximation with respect to $\Delta_e = e_1 - e_0$, under a generic initial condition of the probe, $|\psi_0\rangle = \sum_m c_m|m\rangle$ with a random relative phase between c_0 and c_1. Below, for brevity, in this Appendix we omit the overline of $\overline{\rho}$.

Taking the average over the initial states $|\psi_0\rangle$, due to the random relative phase between c_0 and c_1, one gets

$$\rho_{mm} = \sum_{m_0} |c_{m_0}|^2 \rho_{mm}^{(m_0)}, \tag{B1}$$

where $\rho_{mm}^{(m_0)}$ indicates the rhs of Eq. (4), with the dependence on m_0 written explicitly. As discussed in the main text, the averaged RDM has an approximately diagonal form in the eigenbasis of the self-Hamiltonian H_p. In this basis, the parameter β in the canonical state $\frac{1}{Z}\exp(-\beta H_p)$ can written as

$$\beta = -\frac{1}{\Delta_e} \ln\frac{\rho_{11}}{\rho_{00}}. \tag{B2}$$

Substituting Eq. (B1) into the above expression of β, one gets

$$\beta = -\frac{1}{\Delta_e} \ln\frac{|c_1|^2\rho_{11}^{(1)} + |c_0|^2\rho_{11}^{(0)}}{|c_1|^2\rho_{00}^{(1)} + |c_0|^2\rho_{00}^{(0)}}. \tag{B3}$$

Making use of the expression of $\rho_{mm}^{(m_0)}$ in Eq. (17), it is not difficult to find that

$$\rho_{mm}^{(m_0)} \simeq \frac{\rho_S(E_S^0 + e_{m_0} - e_m)}{\sum_{m'} \rho_S(E_S^0 + e_{m_0} - e_{m'})}. \tag{B4}$$

For example, for $m = m_0 = 1$, one has

$$\rho_{11}^{(1)} \simeq \frac{\rho_S(E_S^0)}{\rho_S(E_S^0) + \rho_S(E_S^0 + \Delta_e)}. \tag{B5}$$

Expanding $\ln\rho_S(E_S^0 + \Delta_e)$ in the Taylor expansion and keeping the second-order term, one finds that

$$\rho_S(E_S^0 + \Delta_e) \simeq \rho_S(E_S^0)\exp\left(\beta_{\rm sm}\Delta_e + \beta'_{\rm sm}\Delta_e^2/2\right), \tag{B6}$$

where $\beta_{\rm sm}$ is defined in Eq. (23): $\beta_{\rm sm} = \frac{\partial \ln\rho_S}{\partial E}|_{E=E_S^0}$. Substituting Eq. (B6) into Eq. (B5), one gets

$$\rho_{11}^{(1)} \simeq \frac{1}{1 + \exp\left(\beta_{\rm sm}\Delta_e + \beta'_{\rm sm}\Delta_e^2/2\right)}. \tag{B7}$$

Similarly, one can compute other elements $\rho_{mm}^{(m_0)}$.

To simplify the notation, we introduce two quantities χ_+ and χ_-:

$$\chi_+ = \exp\left(\beta_{\rm sm}\Delta_e + \beta'_{sm}\Delta_e^2/2\right), \tag{B8}$$

$$\chi_- = \exp\left(\beta_{\rm sm}\Delta_e - \beta'_{sm}\Delta_e^2/2\right). \tag{B9}$$

It is not difficult to find that

$$\rho_{11}^{(1)} \simeq \frac{1}{1+\chi_+}, \quad \rho_{00}^{(1)} \simeq \frac{\chi_+}{1+\chi_+},$$
$$\rho_{11}^{(0)} \simeq \frac{1}{1+\chi_-}, \quad \rho_{00}^{(0)} \simeq \frac{\chi_-}{1+\chi_-}. \tag{B10}$$

Substituting these expressions into Eq. (B3), after simple algebra, we get

$$\beta \simeq -\frac{1}{\Delta_e}\ln\frac{|c_1|^2(1+\chi_-) + |c_0|^2(1+\chi_+)}{|c_1|^2\chi_+(1+\chi_-) + |c_0|^2\chi_-(1+\chi_+)}$$
$$\simeq -\frac{1}{\Delta_e}\ln\frac{1 + |c_1|^2\chi_- + |c_0|^2\chi_+}{|c_1|^2\chi_+ + |c_0|^2\chi_- + \exp(2\beta_{\rm sm}\Delta_e)}. \tag{B11}$$

When $(\beta'_{\rm sm}\Delta_e^2)$ is small, one can write

$$\exp\left(\beta'_{\rm sm}\Delta_e^2/2\right) \simeq 1 + \beta'_{\rm sm}\Delta_e^2/2. \tag{B12}$$

Using this approximation, Eq. (B11) can be further written as

$$\beta \simeq \beta_{\rm sm} - \frac{1}{\Delta_e}\ln\frac{1 + \exp(\beta_{\rm sm}\Delta_e)\left[1 - (|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2\right]}{1 + (|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2 + \exp(\beta_{\rm sm}\Delta_e)}. \tag{B13}$$

Then, using the approximation that

$$1 \pm (|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2 \simeq \exp\left(\pm(|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2\right), \tag{B14}$$

straightforward derivation gives

$$\beta \simeq \beta_{\rm sm} - \frac{1}{\Delta_e}\ln\frac{1+\exp(\beta_{\rm sm}\Delta_e)\exp\left(-(|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2\right)}{\exp\left((|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e^2/2\right)+\exp(\beta_{\rm sm}\Delta_e)} = \beta_{\rm sm} + (|c_1|^2 - |c_0|^2)\beta'_{sm}\Delta_e/2. \tag{B15}$$

Finally, noting that $|c_1|^2 + |c_0|^2 = 1$ and using the expressions of β_0 and β_1 in Eq. (21), one gets

$$\beta \simeq |c_1|^2(\beta_{\rm sm} + \beta'_{sm}\Delta_e/2) + |c_0|^2(\beta_{\rm sm} - \beta'_{sm}\Delta_e/2) \simeq |c_1|^2\beta_1 + |c_0|^2\beta_0, \tag{B16}$$

which gives Eq. (24).

[1] N. Linden, S. Popescu, and P. Skrzypczyk, Phys. Rev. Lett. **105**, 130401 (2010).
[2] O. Fialko and D. W. Hallwood, Phys. Rev. Lett. **108**, 085303 (2012).
[3] M. Serra-Garcia, A. Foehr, M. Molerón, J. Lydon, C. Chong, and C. Daraio, Phys. Rev. Lett. **117**, 010602 (2016).
[4] H. T. Quan, Y.-x. Liu, C. P. Sun, and F. Nori, Phys. Rev. E **76**, 031105 (2007).
[5] J. Eisert, M. Friesdorf, and C. Gogolin, Nat. Phys. **11**, 124 (2015).
[6] M. Hartmann, G. Mahler, and O. Hess, Phys. Rev. Lett. **93**, 080402 (2004).
[7] M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, and J. Eisert, Phys. Rev. X **4**, 031019 (2014).
[8] A. De Pasquale, D. Rossini, R. Fazio, and V. Giovannetti, Nat. Commun. **7**, 12782 (2016).
[9] N. F. Ramsey, Phys. Rev. **103**, 20 (1956).
[10] M. J. Klein, Phys. Rev. **104**, 589 (1956).
[11] A. S. Oja and O. V. Lounasmaa, Rev. Mod. Phys. **69**, 1 (1997).
[12] A. Rapp, S. Mandt, and A. Rosch, Phys. Rev. Lett. **105**, 220405 (2010).
[13] P. Medley, D. M. Weld, H. Miyake, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **106**, 195301 (2011).
[14] S. Braun *et al.*, Science **339**, 52 (2013).
[15] S. Mandt, A. E. Feiguin, and S. R. Manmana, Phys. Rev. A **88**, 043643 (2013).
[16] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958).
[17] S. Hilbert, P. Hänggi, and J. Dunkel, Phys. Rev. E **90**, 062116 (2014).
[18] V. V. Flambaum and F. M. Izrailev, Phys. Rev. E **55**, R13 (1997).
[19] G. Benenti, G. Casati, and D. L. Shepelyansky, Eur. Phys. J. D **17**, 265 (2001).
[20] G. Casati, B. V. Chirikov, I. Guarneri, and F. M. Izrailev, Phys. Lett. A **247**, 140 (1998).
[21] C. Mejía-Monasterio, T. Prosen, and G. Casati, Europhys. Lett. **72**, 520 (2005).
[22] S. Genway, A. F. Ho, and D. K. K. Lee, Phys. Rev. A **86**, 023609 (2012).
[23] O. Fialko, Phys. Rev. E **92**, 022104 (2015).
[24] F. Borgonovi, F. Mattiotti, and F. M. Izrailev, Phys. Rev. E **95**, 042135 (2017).
[25] J. Dunkel and S. Hilbert, Nat. Phys. **10**, 67 (2014).
[26] J. Poulter, Phys. Rev. E **93**, 032149 (2016).
[27] V. L. Berdichevsky, *Thermodynamics of Chaos and Order* (Addison Wesley Longman, Essex, UK, 1997).
[28] D. H. E. Gross, *Microcanonical Thermodynamics: Phase Transitions in Small Systems*, Lecture Notes in Physics Vol. 66 (World Scientific, Singapore, 2001).
[29] M. Bander, arXiv:cond-mat/9609218.
[30] J. Casas-Vázquez and D. Jou, Rep. Prog. Phys. **66**, 1937 (2003).
[31] A. J. Lichtenberg and M. A. Liebermann, *Regular and Chaotic Dynamics*, 2nd ed. (Springer-Verlag, New York, 1992).
[32] *Quantum Choas: Between Order and Disorder*, edited by G. Casati and B. V. Chirikov (Cambridge University Press, Cambridge, U.K., 1994).
[33] V. L. Berdichevsky and M. V. Alberti, Phys. Rev. A **44**, 858 (1991).
[34] V. M. Bannur, P. K. Kaw, and J. C. Parikh, Phys. Rev. E **55**, 2525 (1997).
[35] S. Popescu, A. J. Short, and A. Winter, Nat. Phys. **2**, 754 (2006).
[36] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett. **96**, 050403 (2006).
[37] A. Riera, C. Gogolin, and J. Eisert, Phys. Rev. Lett. **108**, 080402 (2012).
[38] W.-g. Wang, Phys. Rev. E **86**, 011115 (2012).
[39] A. Altland and F. Haake, Phys. Rev. Lett. **108**, 073601 (2012).
[40] L. F. Santos, F. Borgonovi, and F. M. Izrailev, Phys. Rev. Lett. **108**, 094102 (2012).
[41] C. Gogolin and J. Eisert, Rep. Prog. Phys. **79**, 056001 (2016).
[42] P. Reimann, Phys. Rev. Lett. **101**, 190403 (2008).
[43] N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. E **79**, 061103 (2009).
[44] J. Von Neumann, Z. Phys. **57**, 30 (1929).
[45] A. J. Short, New J. Phys. **13**, 053009 (2011).
[46] P. Reimann, Phys. Scr. **86**, 058512 (2012).
[47] A. J. Short and T. C. Farrelly, New J. Phys. **14**, 013063 (2012).
[48] P. Reimann, Nat. Commun. **7**, 10821 (2016).
[49] V. V. Flambaum, A. A. Gribakina, G. F. Gribakin, and M. G. Kozlov, Phys. Rev. A **50**, 267 (1994).
[50] W.-g. Wang, F. M. Izrailev, and G. Casati, Phys. Rev. E **57**, 323 (1998).
[51] L. He and W.-g. Wang, Phys. Rev. E **89**, 022125 (2014).
[52] J. P. Paz and W. H. Zurek, Phys. Rev. Lett. **82**, 5181 (1999).

[53] W.-g. Wang, J. Gong, G. Casati, and B. Li, Phys. Rev. A **77**, 012108 (2008); includes derivations of Eqs. (17) and (20), and discussion about the validity of Eq. (22) under random initial states of the probe.
[54] V. V. Flambaum and F. M. Izrailev, Phys. Rev. E **61**, 2539 (2000).
[55] F. Haake, *Quantum Signatures of Chaos*, 3rd ed. (Springer-Verlag, Berlin, 2010).
[56] J. Wang and W.-g. Wang, Chaos Solitons Fractals **91**, 291 (2016).
[57] Y. Y. Atas and E. Bogomolny, J. Phys. A **47**, 335201 (2014).
[58] D. Pagel, P. Nalbach, A. Alvermann, H. Fehske, and M. Thorwart, New J. Phys. **15**, 105008 (2013).
[59] D. Pagel, A. Alvermann, and H. Fehske, Phys. Rev. E **87**, 012127 (2013).