Macroscopic violation of the law of heat conduction

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We analyze a model describing an anharmonic macroscopic chain in contact with general reservoirs that follow the Lévy-Itô theorem on the Gaussian-Poissonian decomposition of the measure. We do so by considering a perturbative approach to compute the heat flux and the (canonical) temperature profile when the system reaches the steady state. This approach allows observing a macroscopic violation of the law of the heat conduction equivalent to that found for small ($N = 2$) systems in contact with general reservoirs, which conveys the ascendency of the nature of the reservoirs over the size of the system.

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I. INTRODUCTION

From the ancient Egyptians—to whom the first accounts on the nature of heat are accredited—continuing with pre-Socratic philosophers and later on with eminent figures in the History of Science, it took 3000-off years and Joule's experimental work to reach a proper definition of heat as the amount of energy that is transferred between a system and its surroundings but in the form of whatever kind of work (mechanical, chemical, etc.) [1]. Two decades earlier than "The Mechanical Equivalent of Heat" experiment [2]—and still within the caloric theory—Fourier had established his law stating that the (local) heat flux density, $\vec{h}$, is equal to the product of thermal conductivity, κ, by the negative (local) gradient of the temperature T, $\vec{h} \equiv -\kappa \vec{\nabla} T$, which has been proved thermodynamically correct. With the advent of statistical mechanics—namely, kinetic theory—it was possible to connect mechanical microscopic mechanisms with Fourier's law [3]. In due course, the same microscopical effort was made aiming to figure out the phenomenon of heat transport in crystals in contact with reservoirs at different temperatures, T_C and T_H ($T_C < T_H$) [4,5]. Soon, it was realized that because of the ballistic character of the transmission of the energy, by the harmonic lattice, models of harmonic coupled oscillators are unable to retrieve Fourier's macroscopic behavior; actually, they yield infinite heat conductivity with subsequent dynamically based studies showing that favorable mixing properties assure normal heat transport properties to a system [6] while ergodicity apparently plays a secondary role [7].

Along with the microscopic mechanical features of the system through which heat is transferred, it must be recalled that the thorough characterization of this nonequilibrium problem must take the reservoirs into account. Markovian matters apart, heat reservoirs are assumed as thermal baths—described by either deterministic or stochastic analytical formulations, each presenting its pros and cons [8,9]—yielding Gaussian fluctuations, i.e., presenting a purely continuous Lévy-Itô measure [10] with a single source of stochasticity: the variance. The most typical instances are the Nosé-Hoover thermostat for the former and the Langevin thermostat for the latter. With respect to a stochastic approach to the reservoirs, they allow the employing of a quite useful arsenal of techniques and simplifications in the treatment of Gaussian variables that are provided by both stochastic calculus and probability theory. Nevertheless, the concept of reservoir goes beyond the thermal (heat) classification: in several physical and biological processes we have mechanical(-like) systems in contact with sources of energy which do not abide by the canonical conditions of thermodynamics to be classified as thermal—and thus they are called *athermal* reservoirs—different from other types of sources that act upon the system by performing pure work or exchanging information [11,12].

By reason of their statistical features, athermal reservoirs have been analytically represented by processes other than Gaussian and Brownian. For instance, the shot-noise Poissonian process can be used to represent athermal reservoirs which interact with the system at a rate λ, and effective force of magnitude $\Phi(t)$. When $\langle \Phi(t) \rangle \neq 0$ these reservoirs can be understood as work performing reservoirs, whereas when $\langle \Phi(t) \rangle = 0$ they only change the average energy of the system by stochasticity (variance and higher-order cumulants) and therefore they are viewed as heat sources. The former can be depicted by some types of molecular motors or experimental implementations of ratchets [13,14], whereas the latter can be represented by a (little dense) granular gas [15] or bacterial colonies [16] as well as problems described by generalizations of the Onsager-Machlup fluctuation theory of the second order in time [17]. It is worth mentioning that Poissonian noise has been attracting the attention of the physical community due its applications in a wide set of phenomena, such as (i) solid-state problems wherein shot (singular measure) noise is related to the quantization of the charge [18]; (ii) resistor-inductor-capacitor circuits with injection of power at some rate resembling heat pumps [19]; (iii) surface diffusion and low vibrational motion with adsorbates, e.g., Na/Cu(001) compounds [20]; (iv) biological motors in which shot noise mimics the nonequilibrium stochastic hydrolysis of adenosine triphosphate [21–23]; (v) molecular dynamics when the Andersen thermostat is applied [24]; (vi) the use of detectors based on Josephson junctions in order to probe higher-order cumulants in fluctuating currents [25,26]; and

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(vii) study of shape fluctuations of red blood cell membranes [27]. Concomitantly, other sorts of noises have been considered aiming at better depicting the properties of the source of fluctuations in the system [17,28–33].

Recently, work by two of us, as well as other groups, has put the impact of the nature of the reservoirs on the thermostatistical properties of a mechanical system in the limelight, namely, on the heat conduction of small mechanical systems—actually dimers [34,35]. Explicitly, it was shown that in straightforwardly applying the standard equipartition relation between temperature and mean-square velocity implies awkward results such as having the average heat flux with a "colder" to "hotter" reservoir direction, a clear violation of the law of heat conduction. Within this context, it was introduced [36], and later extended in [37], the concept of temperature of higher order as a way out to amend the inconsistencies between quantitative results and good sense.

Nonetheless, it is well established that the thermodynamical limit plays a crucial role in the results obtained in statistical physics; for example, the physical relevance of the fluctuations vanishes rapidly as the number of elements of the system increases so that we can match microscopical results of statistical mechanics with macroscopical formulas of thermodynamics. It is thus reasonable to ask whether such oddities in the thermostatistical behavior of small systems coupled to athermal reservoirs endure as we increase the size of the system or, contrarily, in the large system limit we end up obtaining standard results. The answer to that question corresponds to the goal of the present article, which is divided as follows: in Sec. II, we introduce the mathematical equations that rule the dynamics of our system: massive particles similar to the β-Fermi-Pasta-Ulam [38,39] problem in contact with thermal and/or athermal reservoirs. In Sec. III, we analytically characterize our method of solution and apply it to the Gaussian bath—for which the results are known—and the quintessential athermal reservoir (the Poissonian case). We present our results on the statistics of heat flux and whether the previous results regarding heat flux inversion persist for macroscopic chains. Finally, in Sec. IV we address further comments on our results and provide some hints about future work.

II. MODEL AND SOLUTION METHOD

As mentioned, the mechanical part of our (non-momentum-conserving) system corresponds to N particles with mass m each, subjected to an anharmonic β-Fermi-Pasta-Ulam [38] interaction potential between them and harmonic pinning to a substrate—with constants k' for the edge and k for the bulk—analytically given by

$$\begin{aligned} m\frac{d^2x_1}{dt^2} &= -\gamma\frac{dx_1}{dt} - k'x_1 - k_1(x_1-x_2) - k_3(x_1-x_2)^3 + \eta_1,\\ m\frac{d^2x_l}{dt^2} &= -kx_l - k_1(2\,x_l - x_{l+1} - x_{l-1}) - k_3(x_l - x_{l-1})^3\\ &\quad - k_3(x_l - x_{l+1})^3,\\ m\frac{d^2x_N}{dt^2} &= -\gamma\frac{dx_N}{dt} - k'x_N - k_1(x_N - x_{N-1})\\ &\quad - k_3(x_N - x_{N-1})^3 + \eta_N, \end{aligned} \tag{1}$$

where the edge particles $l = 1$ and $l = N$ are in contact with the reservoirs, which is represented by (linear) dissipation and a stochastic component η that is the only term that can introduce positive variations of the energy of the system. Recently, the importance of the interaction between the system and the substrate, namely, in the elimination of the cuspidal temperature profile in a chain [40], was presented in [41].

Thermally, the solution to this problem is obtained by resorting to the continuous Lévy-Itô measure of the stochastic terms η where

$$\begin{aligned} \langle \eta(t)\rangle_c &= 0,\\ \langle \eta_l(t)\eta_n(t')\rangle_c &= 2\gamma\,T_l\,\delta_{ln}\,\delta(t-t'), \end{aligned} \tag{2}$$

with T_l representing the temperature of each reservoir ($l = 1,N$), which allows the definition of the associated Fokker-Planck equation, whence the calculation of the local average heat flux

$$\langle J_l\rangle \equiv -\left\langle v_l\frac{\partial U(x_l,x_{l+1})}{\partial x_l}\right\rangle \tag{3}$$

$$= \left\langle\left\{[-k_1(x_l - x_{l+1}) - k_3(x_l - x_n)^3]\frac{v_l + v_{l+1}}{2}\right\}\right\rangle \tag{4}$$

is made by employing the eigenvalue method. In the steady state this value is the same for all the particles.

For cases without a full continuous measure, the Lévy-Itô theorem on the decomposition of the measure asserts the singular part of a stochastic variable is written in the form of a Poisson (shot-noise) process

$$\eta(t) = \sum_l \Phi(t)\delta(t - t_l) \tag{5}$$

(l is the sequential order of the lth shot) that physically we bridge with the athermal character of the reservoir [42]. For the sake of simplicity, we shall assume a homogeneous process $\lambda(t) = \lambda$.[1] Despite being possible to consider several distribution functions for $\Phi(t)$, we will restrict our study to the standard family of exponential probability density functions[2]

$$P(\Phi) = \bar{\Phi}^{-1}\exp\left[-\frac{\Phi}{\bar{\Phi}}\right],\quad \overline{\Phi^n} = n!\,\bar{\Phi}^n, \tag{6}$$

namely, its two side extension,[3] $p(\Phi) \sim \exp(-|\Phi|/\bar{\Phi})$, so that there is no contribution from the work to the flux, which is produced when $\langle\eta\rangle = \lambda\langle\Phi\rangle \neq 0$ (see details in Appendix C). In the white-noise cases, the cumulants of the athermal reservoir are defined by the cumulants

$$\langle \eta(t_1)\cdots\eta(t_n)\rangle_c \equiv \lambda\langle\bar{\Phi}^n\rangle\prod_{i=1}^{n-1}\delta(t_{i+1} - t_i). \tag{7}$$

[1]Studies over the statistics of single-particle heterogeneous Poisson systems [42] have shown that they can bring about stochastic resonance phenomena that have not been explored in the case of heat transport yet.

[2]We use the notation $\overline{\cdots}$ for statistics over time and $\langle\cdots\rangle$ for statistics over samples.

[3]Also known as Laplace probability density function.

Heeding the Marcinkiewicz theorem, there must be an infinite number of nonvanishing cumulants, contrarily to the case of the thermal Gaussian case where there must be just one nonvanishing cumulant: its source of stochasticity—the variance that is associated with temperature of reservoir. In the limit $\lambda \to \infty$ and fixing $\langle \eta^2(t) \rangle$ we get standard thermal features. The picture is as follows: as λ increases, the rate of interaction between the system and the bath grows to such an extent that the overall effect is well described taking into account the central limitheorem as in the standard Einstein's Brownian motion theory.

Physically, it is possible to verify equipartition of energy, relating quadratic energy terms (on the degrees of freedom) with the second-order cumulant of $\eta(t)$, as analytically shown in [36,42]. Hence, it is possible to define a canonical temperature

$$T \equiv \frac{1}{2} \frac{\lambda \langle \Phi^2 \rangle}{\gamma}. \tag{8}$$

For athermal cases, represented by Poisson variables, Eq. (5), it is blatantly impossible to turn to continuous measure methods. Recently, a method based on Fourier-Laplace transforming the dynamical equations was able to provide the full statistical characterization of the position and velocity as well as thermostatistical quantities for monomers and dimers [37,43]; in the present case, we explore a multivariate version of that perturbative technique. Explicitly, we note that the typical energy scales of our chain regarding the harmonic and cubic interactions are, respectively,

$$U_{\rm h} = \frac{k_1 x^2}{2}, \quad U_{\rm nl} = \frac{k_3 x^4}{4}. \tag{9}$$

As we want to consider the $U_{\rm h}$ much larger than $U_{\rm nl}$—so that the first-order approximation is already effective—our treatment obeys the condition

$$\frac{U_{\rm nl}}{U_{\rm h}} \ll 1 \to \frac{\frac{k_3 x^4}{4}}{\frac{k_1 x^2}{2}} = \frac{k_3 x^2}{k_1} \sim \frac{k_3 T_N}{k_1^2} \equiv \delta \ll 1, \tag{10}$$

where in the last step of Eq. (10) we have used the equipartition theorem (to which we shall return). Here, it is important to mention that the introduction of nonlinearities in the potential has no effect on the value of $\langle v^2 \rangle$ and hence on the canonical temperature of a system whatever the type of reservoir [36,37]. Under the last inequality of Eq. (10), it is possible to expand all the coordinates in terms of a power series of δ, given by

$$\mathbf{x}_l = \mathbf{x}_l^{(0)} + \delta \mathbf{x}_l^{(1)} + \delta^2 \mathbf{x}_l^{(2)} + \cdots. \tag{11}$$

The equations of motion (1) can be recast in the form

$$m \frac{d^2 \mathbf{x}_l}{dt^2} = \mathcal{D}_{ln} \mathbf{x}_n + \boldsymbol{\eta} + k_3 f_l(\mathbf{x_p}), \tag{12}$$

where $\mathcal{D}(t)$ is a $N \times N$ operator,[4] $\boldsymbol{x}(t) \equiv \{x_1(t), \ldots, x_N(t)\}$ is the vector of the positions, $\boldsymbol{\eta}(t) \equiv \{\eta_1(t), 0, \ldots, 0, \eta_N(t)\}$ represents the multivariate stochastic variable describing the fluctuations introduced by the reservoirs, and $f_i(\mathbf{x_p})$ is an integral operator representing the cubic interparticle interactions in the chain. Plugging Eq. (11) into Eq. (12) and truncating the expansion to first order in δ,

$$m \frac{d^2 \mathbf{x}_l{}^{(0)}}{dt^2} + m\delta \frac{d^2 \mathbf{x}_l{}^{(1)}}{dt^2} = \mathcal{D}_{ln} \mathbf{x}_n^{(0)} + \delta \mathcal{D}_{ln} \mathbf{x}_n^{(1)} + \boldsymbol{\eta} + \delta \frac{k_1^2}{T_N} f_l\big(\mathbf{x_p}^{(0)}\big). \tag{13}$$

The zeroth order in δ corresponds to

$$O(\delta^0) \to m \frac{d^2 \mathbf{x}_l{}^{(0)}}{dt^2} = \mathcal{D}_{ln} \mathbf{x}_n^{(0)} + \boldsymbol{\eta}, \tag{14}$$

and the first order explicitly reads

$$O(\delta^1) \to m \frac{d^2 \mathbf{x}_l{}^{(1)}}{dt^2} = \mathcal{D}_{ln} \mathbf{x}_n^{(1)} + \frac{k_1^2}{T_N} f_l\big(\mathbf{x}_l{}^{(0)}\big). \tag{15}$$

Without loss of generality, assuming the initial condition $x_l(0) = 0$ and $v_l(0) = 0$ for all l, the Fourier-Laplace transforms [44] of the position and velocity read, in the reciprocal space

$$\tilde{x}_l(iq + \varepsilon) \equiv \int_0^\infty x_l(t) e^{-(iq+\varepsilon)t} dt, \tag{16}$$

and

$$\tilde{v}_l(iq + \varepsilon) = (iq + \varepsilon)\tilde{x}_l(iq + \varepsilon), \tag{17}$$

respectively.

Looking to the matrical form of Eq. (1), the Fourier-Laplace transform yields

$$\widetilde{\mathcal{D}}(iq + \varepsilon)\tilde{\boldsymbol{x}}(iq + \varepsilon) = \tilde{\boldsymbol{\eta}}(iq + \varepsilon)$$
$$\tilde{\boldsymbol{x}}(iq + \varepsilon) = \widetilde{\mathcal{A}}(iq + \varepsilon)\tilde{\boldsymbol{\eta}}(iq + \varepsilon), \tag{18}$$

where $\mathcal{A} \equiv \mathcal{D}^{-1}$. From Eq. (18), the position of particle l yields

$$\tilde{x}_l(iq + \varepsilon) = \sum_{n=1,N} \widetilde{\mathcal{A}}(s)_{ln}(iq + \varepsilon)\tilde{\eta}_n(iq + \varepsilon). \tag{19}$$

Similarly, from Eq. (15), we have the column vector for the first-order perturbation of the position

$$\widetilde{\mathbf{x}}_l^{(1)}(iq + \varepsilon) = \frac{k_1^2}{T_N} \widetilde{\mathcal{A}}_{ln}(iq + \varepsilon)\widetilde{f}_n\big(\mathbf{x_p}^{(0)}\big)(iq + \varepsilon). \tag{20}$$

In this steady-state nonequilibrium problem, we can apply the (weak) ergodic property of equivalence between averages over time,

$$\overline{g} \equiv \lim_{\Xi \to \infty} \frac{1}{\Xi} \int_0^\Xi g(t) dt, \tag{21}$$

and over samples, $\langle g \rangle$, where g represents a generic stochastic function. This property allows us to directly benefit from the Fourier-Laplace representation by means of the final-value theorem [44]

$$\overline{g} = \lim_{z \to 0} z \int_0^{+\infty} \exp[-zt] g(t) dt$$
$$= \lim_{z,\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \frac{z}{z - (iq + \varepsilon)} \tilde{g}(iq + \varepsilon), \tag{22}$$

$$\overline{g} = \langle g \rangle = \lim_{z,\varepsilon \to 0} \int_{-\infty}^{+\infty} \frac{dq}{2\pi} \frac{z}{z - (iq + \varepsilon)} \langle \tilde{g}(iq + \varepsilon) \rangle. \tag{23}$$

Because in the stationary state the heat flux is the same between next-nearest-neighbor sites of the chain, we simplify our calculations by restricting our calculations to the heat flux

[4]The form of operators, $\mathcal{A}(s)$ and $\mathcal{D}(s)$, is made explicit in Appendix A.

between particles 1 and 2. Combining Eqs. (4) and (23) we obtain

$$\langle J_1\rangle = \lim_{z,\varepsilon\to 0}\int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{z(iq_2+\varepsilon)}{z-(iq_1+iq_2+2\varepsilon)}\left\{\frac{k_1}{2}\underbrace{\langle[\widetilde{x}_1(iq_1+\varepsilon)-\widetilde{x}_2(iq_1+\varepsilon)][\widetilde{x}_1(iq_2+\varepsilon)+\widetilde{x}_2(iq_2+\varepsilon)]\rangle}_{\text{I}}\right\}$$
$$+\lim_{z,\varepsilon\to 0}\int_{\text{all space}}\frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}\frac{dq_4}{2\pi}\frac{z(iq_4+\varepsilon)}{z-(iq_1+iq_2+iq_3+iq_4+4\varepsilon)}$$
$$\times\left\{\frac{k_3}{2}\underbrace{\lim_{\alpha\to 0}\int_{\text{all space}}\prod_{l=1}^{3}\frac{dq_l}{(2\pi)^3}\langle[\widetilde{x}_1(iq_l+\alpha)-\widetilde{x}_2(iq_l+\alpha)][\widetilde{x}_1(iq_4+\varepsilon)+\widetilde{x}_2(iq_4+\varepsilon)]\rangle}_{\text{II}}\right\}, \tag{24}$$

whose form will depend on the form of the cumulants of $\eta(t)$, as shown in Appendix B.

III. RESULTS

A. The usual case: Gaussian heat reservoirs

This instance corresponds to the traditional case where we have the system put in contact with thermal reservoirs at temperatures T_1 and T_N, each following Eq. (2). Since we work in the Fourier-Laplace space, we must transform Eq. (2) which reads

$$\langle\tilde{\eta}_l(iq_1+\varepsilon)\tilde{\eta}_n(iq_2+\varepsilon)\rangle_c = \frac{2\gamma T_l}{iq_1+iq_2+2\varepsilon}\delta_{ln}, \tag{25}$$

whereas all the other cumulants are equal to zero.

Plugging Eq. (25) into Eq. (24) we obtain the steady-state heat flux with standard thermal reservoirs,

$$\langle J\rangle_{\mathcal{G}} = \gamma^2\frac{\Delta T}{\pi}\int_{-\infty}^{\infty}(iq+\varepsilon)^2\,\widetilde{\mathcal{A}}_{1N}(iq+\varepsilon)\widetilde{\mathcal{A}}_{1N}(-iq-\varepsilon)\,dq + 6\gamma^2 k_1 k_3\left\{\sum_{j=2}^{N}\left[T_1^2\mathcal{I}_4^{[j]}\left(\mathcal{I}_3^{[j,j-1]}+\mathcal{I}_7^{[j,j-1]}\right)\right.\right.$$
$$\left.+T_1T_N\left(\mathcal{I}_3^{[j,j-1]}\mathcal{I}_6^{[j]}+\mathcal{I}_4^{[j]}\mathcal{I}_5^{[j,j-1]}+\mathcal{I}_6^{[j]}\mathcal{I}_7^{[j,j-1]}+\mathcal{I}_4^{[j]}\mathcal{I}_8^{[j,j-1]}\right)+T_N^2\mathcal{I}_6^{[j]}\left(\mathcal{I}_5^{[j,j-1]}+\mathcal{I}_8^{[j,j-1]}\right)\right]+$$
$$\left.-\sum_{j=1}^{N-1}\left[T_1^2\mathcal{I}_4^{[j]}\left(\mathcal{I}_3^{[j,j]}+\mathcal{I}_7^{[j,j]}\right)+T_1T_N\left(\mathcal{I}_3^{[j,j]}\mathcal{I}_6^{[j]}+\mathcal{I}_4^{[j]}\mathcal{I}_5^{[j,j]}+\mathcal{I}_6^{[j]}\mathcal{I}_7^{[j,j]}+\mathcal{I}_4^{[j]}\mathcal{I}_8^{[j,j]}\right)+T_N^2\mathcal{I}_6^{[j]}\left(\mathcal{I}_5^{[j,j]}+\mathcal{I}_8^{[j,j]}\right)\right]\right\}$$
$$+6\gamma^2k_3\left[T_1^2\mathcal{I}_4^{[1]}\mathcal{I}_1+T_1T_N\left(\mathcal{I}_4^{[1]}\mathcal{I}_2+\mathcal{I}_6^{[1]}\mathcal{I}_1\right)+T_N^2\mathcal{I}_6^{[1]}\mathcal{I}_2\right], \tag{26}$$

where the first term $[O(k_3^0)]$ in the limit $N\to\infty$ equals

$$\langle J\rangle_{\mathcal{G}} = \tfrac{\gamma k_1^2}{m\Theta^2}\left(\Pi-\sqrt{\Pi^2-\Theta^2}\right)\Delta T, \tag{27}$$

with

$$\Theta = \frac{2k_1\gamma^2}{m}+2k_1(k_1+k-k'),\quad \Pi = \frac{(2k_1+k)\gamma^2}{m}+(k_1+k)^2+(k_1-k')^2-2k'k,\quad \Delta T\equiv T_N-T_1. \tag{28}$$

The expressions for $\mathcal{I}_n^{[r]}$, $\mathcal{I}_n^{[j,l]}$, $\mathcal{I}_1$, and $\mathcal{I}_2$ are defined in Appendix B. In Fig. 1, we show a comparison between the analytical results given by Eq. (26) and the numerical results obtained by computer simulation of Eq. (1), where it is possible to notice an excellent agreement between the two approaches for the conditions presented in Eq. (10).

At this point, we observe that, despite the fact that the analytical framework we employ is valid for undefined N, for the sake of simplicity and swiftness, we will restrict comparisons between analytical and simulation results to chains of size $N=10$, which can be considered macroscopically large, as visible from Fig. 2.

B. Beyond the standard thermal case: Assuming Poisson reservoirs

Let us now turn to the case where instead of having usual (thermal) reservoirs, we have reservoirs that contain singular measure as well. For the sake of clarity, we consider the simplest case for which all of the measure is singular: (shot-noise) Poissonian reservoirs defined by Eq. (7). In the Fourier-Laplace space they read [42]

$$\langle\eta_\alpha(iq_1+\varepsilon)\ldots\eta_\alpha(iq_n+\varepsilon)\rangle_c \equiv \frac{\lambda_\alpha\langle\bar{\Phi}_\alpha^n\rangle}{\sum_{j=1}^n(iq_j+\varepsilon)}$$
$$\forall_{n\in\mathbb{N}} \text{ and } \alpha = 1,N. \tag{29}$$

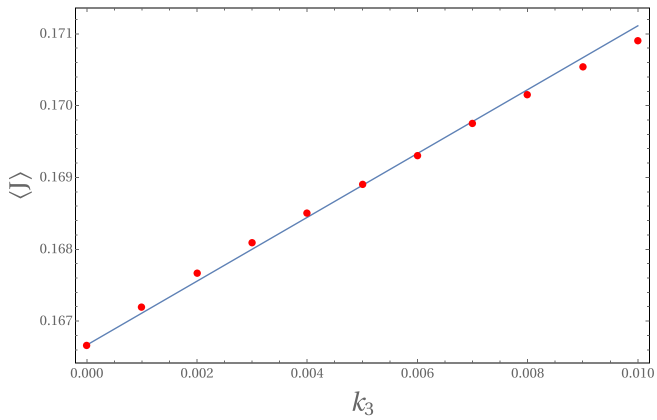

FIG. 1. The heat current in a chain with $N = 10$ particles and $\gamma = m = k' = k_1 = 1$ and $k = \frac{1}{2}$ in contact with thermal reservoirs at $T_C = 1$ and $T_H = 2$. The solid blue line represents the analytical results given by Eq. (26), whereas the points stand for the numerical results obtained by the computer simulation of Eq. (1) following a second-order implementation scheme as presented in [45].

For the general case of two Poissonian reservoirs, we employ definitions Eq. (29) in Eq. (4) and obtain

$$\langle J\rangle_{\mathcal{P}} = \sum_{n=0}^{\infty} k_3^n \sum_{\mathsf{P}} \Delta \prod_{\{l\}\in \mathsf{P}} \langle \Phi^l \rangle \int C_n^{(\mathsf{P})}(\mathbf{q},k_1,k,m,\gamma,\lambda_1,\lambda_N) d\mathbf{q}, \tag{30}$$

where P corresponds to the possible partitions of a block of size n and $\{l\} \in \mathsf{P}$ specifies the exponents in the partition that is equal for both reservoirs ($\Delta\mathcal{A} \equiv \mathcal{A}_1 - \mathcal{A}_N$).

Recalling that these reservoirs allow defining a canonical temperature, Eq. (8), and resorting the definition of cumulants of temperature of nth order [36]

$$\mathcal{T}_n \propto \frac{k_3}{k_1^2}\langle \eta^n\rangle_c = \frac{k_3}{k_1^2}\lambda\langle \Phi^n\rangle, \tag{31}$$

Eq. (30) can be recast to read

$$\langle J\rangle_{\mathcal{P}} = -\kappa_{\text{lin}}\Delta T + \sum_{n=1}^{\infty}\sum_{\mathsf{P}} \kappa_n^{(\mathsf{P})}(\lambda)\cdot\ \Delta\prod_{\{l\}\in\mathsf{P}} \mathcal{T}_l. \tag{32}$$

In first order in k_3 and assuming a symmetrical amplitude of the noise, the flux across the chain is just composed of heat

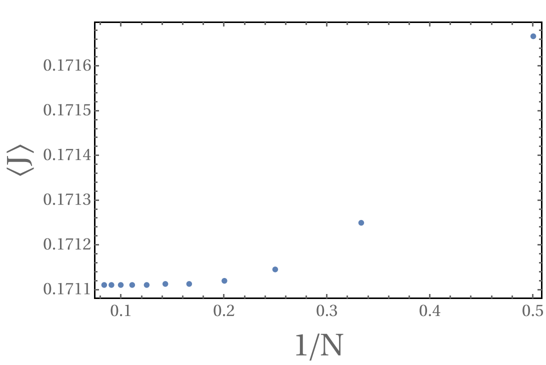

FIG. 2. The heat current in a chain of size N with $\gamma = m = k' = k_1 = 1$ and $k = \frac{1}{2}$ in contact with two reservoirs at temperatures $T_1 = 1$ and $T_N = 2$ vs N^{-1} as given by Eq. (26).

and Eq. (32) expands into

$$\langle J\rangle_{\mathcal{P}} = -\kappa_{\text{lin}}\Delta T + \kappa_1^{(')}(\lambda)\Delta\mathcal{T}_4 + \kappa_1^{('')}(\lambda)\Delta T^2 + \cdots. \tag{33}$$

In Eqs. (32) and (33), κ_{lin} is the conductance for linear systems [4] and we define $\kappa_n^{(\mathsf{P})}(\lambda)$ as the nth-order conductances. Our calculations permit us to grasp that, at each order of k_3 we consider, we call for higher-order cumulants of η_α, i.e., higher sources of stochasticity enter into action. Understanding that the first order of stochasticity—$\langle \eta_\alpha^2\rangle_c$—is related to heat (i.e., a source of energy), higher-order cumulants can be viewed as higher orders of heat. The $\langle v_i^2\rangle$ and $\langle x_i^2\rangle$ profiles show that these sources do not affect the kinetic but affect only potential energy instead.

As k_3 vanishes, we stop activating the higher sources of energy established by the high-order cumulants and every $\kappa_n^{(\mathsf{P})}(\lambda)$ zeros out. On the other hand, in the limit of continuous measure $\lambda \to \infty$, cumulants greater than second order fall off and only the $\kappa_n^{(\mathsf{P})}$ coefficients related to terms ΔT^{n+1} live on so that the sum of the full series yields

$$\lim_{\lambda\to\infty}\langle J\rangle_{\mathcal{P}} = \langle J\rangle_{\mathcal{G}} = -\kappa\Delta T. \tag{34}$$

For the simpler situation in which the athermal Poissonian reservoirs have the same λ, but different typical amplitudes, $\bar{\Phi}_{1,N}$, we have

$$\begin{aligned}\langle J\rangle_{\mathcal{P}} = \langle J\rangle_{\mathcal{G}} + \frac{12\gamma^2 k_1 k_3}{\lambda}\Bigg\{&\sum_{j=2}^{N}\big[T_1^2\big(\mathcal{I}_{11}^{[j,j-1]} + \mathcal{I}_{13}^{[j,j-1]}\big)\\ &+ T_N^2\big(\mathcal{I}_{12}^{[j,j-1]} + \mathcal{I}_{14}^{[j,j-1]}\big)\big] +\\ &- \sum_{j=1}^{N-1}\big[T_1^2\big(\mathcal{I}_{11}^{[j,j]} + \mathcal{I}_{13}^{[j,j]}\big) + T_N^2\big(\mathcal{I}_{12}^{[j,j]} + \mathcal{I}_{14}^{[j,j]}\big)\big]\Bigg\}\\ &+ \frac{12\gamma^2 k_3}{\lambda}\big(T_1^2\mathcal{I}_9^{[1]} + T_N^2\mathcal{I}_{10}^{[1]}\big) + O\big(k_3^2\big).\end{aligned} \tag{35}$$

The extra terms besides $\langle J\rangle_{\mathcal{G}}$ do not vanish in the limit $N \to \infty$; only in the limit $\lambda \to \infty$—i.e., when the measure of the noise approaches continuity and physically the athermal reservoirs turn into effective thermal reservoirs [e.g., the (granular) gases in the reservoirs increase their densities and become liquids]—the second and fourth terms on the right-hand side vanish and the results of Eq. (26) are recovered. In other words, in making the mechanical system—through which heat is transferred between the two reservoirs—converge on the thermodynamic limit, we do not quell the impact of the athermal properties of the reservoirs. The nonlinear mechanical features of the system and the athermal properties—i.e., the existence of higher-order cumulants (stochasticity)—of the reservoirs continue coupling and we are still able to compute a singular contribution to the heat flux.

Violating the law of heat conduction in a macroscopic system. The finding that the nonlinear nature of the mechanical system activates the higher-order stochasticity (nth cumulants with $n > 3$) of athermal systems in the form of sources of energy opened the door to microscopic violations of the law of heat conduction [34] as well as the zeroth law of thermodynamics [35] using an overdamped $N = 2$ system where the heat flows on average from the colder to the hotter

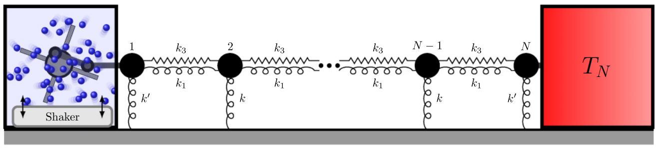

FIG. 3. Schematic representation of a pinned anharmonic chain of arbitrary size $N \gg 1$. In contact with the particle "1" we put the colder reservoir with temperature T_1 corresponding to a granular gas which hits a paddle with a rate λ leading to a stochastic force $\eta_1(t)$ with a characteristic scale, $\bar{\Phi}$, so that $T_1 \propto \lambda\, \bar{\Phi}^2$; on the other edge, the particle "N" is in contact with a thermal reservoir at temperature $T_N > T_1$. (Courtesy of Lucianno A.C.A. e Defaveri.)

reservoir. The results we have obtained in the previous section prove that macroscopic systems keep on activating the higher-order cumulants of η and having higher-order reservoirs acting upon it. This means that in the case of general reservoirs—analytically represented by a composition of Gaussian and Poissonian noises according to the Lévy-Itô theorem—we can still have inversion of the heat flux direction depending on the balance between all the cumulants of η_1 and η_N.

To illustrate this situation, we resort to the utmost situation where we have a chain connecting the two extreme cases with respect to the nature of the reservoir: an athermal (symmetrical) Poissonian reservoir—with canonical temperature T_1—e.g., in the form of a granular gas and a thermal reservoir with $T_N > T_1$, as depicted in Fig. 3. This particular choice on the reservoirs is made in furtherance of clearness only and the assumption of more intricate cases would confirm our conclusions.

Within this setup, and following our calculations, namely, Eq. (4), the heat flux reads up to first order in k_3,

$$\langle J\rangle_{\mathcal{P}} = \langle J\rangle_{\mathcal{G}} + \Bigg\{ \sum_{j=2}^{N} \big[T_1^2\big(\mathcal{I}_{12}^{[j,j-1]} + \mathcal{I}_{14}^{[j,j-1]}\big)\big] - \sum_{j=1}^{N-1} \big[T_1^2\big(\mathcal{I}_{12}^{[j,j]} + \mathcal{I}_{14}^{[j,j]}\big)\big] \Bigg\} \frac{12\gamma^2 k_1 k_3}{\lambda} + \frac{12\gamma^2 k_3}{\lambda} T_1^2 \mathcal{I}_{10}^{[1]}, \qquad (36)$$

and a comparison between that equation and numerical simulation obtained by integrating the equations of motion Eq. (1) is presented in Fig. 4.

At first glance, the behavior shown in Fig. 4 seems physically wrong. However, this reckoning is strongly based on standard assumption that the temperature of a reservoir is only related to the second-order cumulant of the noise. For thermal reservoirs, this is correct because that is the unique source of stochasticity, but a simplistic definition for athermal cases, though. In that case, we have additional term stochasticity, which physically translate into extra sources of energy. This new term can actually excel the standard form of heat and switch the direction of the flux, as our calculations and simulations show.

Considering our fully nonequilibrium system, its physics—i.e., the role of higher-order cumulants as sources of heat flux is uncovered by relying on perturbation theory and the

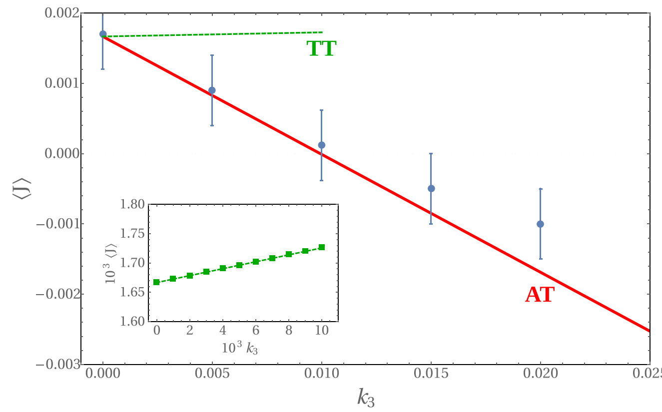

FIG. 4. The heat current in a chain with $N = 10$ particles in the conditions depicted in Fig. 3 with the following parameters with the following values: $T_1 = 1.99$, $T_N = 2$, $\lambda = 4$, $\gamma = m = k' = k_1 = 1$, and $k = \frac{1}{2}$. The solid red line (AT) is computed considering one athermal and one thermal reservoir and the dashed green line is for the case of two thermal reservoirs (TT) under the same mechanical and temperature conditions (the inset is a zoom), whereas the blue dots are obtained by numerical simulation using the approach presented in the Appendix of [42] to cope with the athermal reservoir. In both cases, we use the first-order approximation, Eqs. (36) and (26). The error bars computed at $\pm 5 \times 10^{-4}$ correspond to the error expected from sampling over 10^8 measurements (the same would apply to the symbols of the numerical simulations in the inset).

developing of Eq. (24)—becomes easier to understand when we analyze the contribution from the higher-order cumulants to the heat flux, namely, the second (negative) term inside the curly brackets, that is added up with the canonical temperature heat current.

IV. CONCLUDING REMARKS

In this article, we have cast light on the problem of heat conduction between athermal reservoirs through macroscopic mechanical systems. Those general reservoirs are described by stochastic variables with a singular contribution to their Lévy-Itô measures and therefore cannot be characterized by solely establishing value of the variance, which is associated to the temperature of the reservoir, but considering the entire set of cumulants instead that are naturally understood as sources of stochasticity.

Previous results on heat transport in small systems have proven that the interaction between them and athermal reservoirs leads to a heat flux formula consisting of additional terms that open the door to the violation of the law of heat conductance. The key to unravel that paradox is the understanding that the extra stochasticity provided by the high-order nonvanishing cumulants defining athermal reservoirs act upon the system as (higher-order) sources of energy, more precisely, heat. These sources are called up when the system presents nonlinearities and remain silent otherwise. In the latter linear case, the heat flux will be controlled by the difference between the temperature (originated from the stochasticity of first order, the variance) of the reservoirs, whereas in the former the tagging as "hotter" and "colder" of each of the reservoirs depends on the cumulant structure and the level of nonlinearity in the system; explicitly, as Fig. 4 shows, the distinction between hotter and colder when at least one of the reservoirs is athermal depends on the level of nonlinearity in the system because a given reservoir can be the hotter—in the sense that on average the heat flows outward—up to some threshold value k_3^* and colder—with the heat flowing inward on the average—after that (and vice versa). Moreover, an athermal reservoir with the same canonical temperature T as that of a thermal reservoir is always the "hotter" one in a nonlinear situation since it will be possible to measure a heat flux athermal $\rightarrow$ thermal set forth by the high-order cumulants. Equivalently, we can consider two athermal reservoirs which show the same canonical temperature, but a different cumulant structure [46]; in that case, the heat flux is still equal to zero when $k_3 = 0$ and depends on the cumulant structures, which rule the high-order sources of energy of the reservoirs, for $k_3 \neq 0$.

In respect to heat transport, our present work shows the nature of the reservoirs outclasses the scale of a mechanical system through which heat flows, in the sense that for a macroscopic ($N \gg 1$) nonlinear system in contact with different athermal reservoirs we can still obtain the same sort of paradoxical results as for small ($N = 2$) systems. That provides further significance to the results of [34,35,37].

Bearing in mind the standard framework and the fact that a "reverse" heat flow implies a decrease in the entropy, the present result cues us in to a possible overall macroscopic violation of the second law of the thermodynamics. At this point, we must recall that athermal reservoirs are naturally out of equilibrium: this means their existence involves a continuous production of entropy which sets off the entropy decrease that is taking place in the reservoirs plus chain subsystem and hence the second law of thermodynamics is still verified overall (a reasoning related to Haff's law [47] on this matter is introduced in Appendix D).

Last, we deem that the present result opens the door to brand new studies within the problem of heat and energy transport in low-dimensional nonequilibrium systems—this time subjected to athermal reservoirs—and for which the signal relation between the value of the average flux and the difference between the canonical temperatures, as depicted by Fourier law, can be modified. Having shown that, the natural move is the assessment of scaling relations. From the decades of studies on systems in contact with thermal reservoirs we have learned that such relations are quite sensitive to the mechanical details such as the existence of on-site interactions (pinning) which break momentum conservation in the system [4,41], the nature and magnitude of the nonlinearity of the potentials [48,49]—either on the interactions between the elements of the chain or the on-site potential [50–52]—or mass dispersion among others [53]. With respect to theory, future work on this type of systems should explore the relations between heat flux and the specificities of the chain, namely, its scaling ones. With respect to experiment, we expect that the development of realistic apparatuses (similar to Fig. 3) will allow measuring the predicted flux reversion.

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APPENDIX A: MATRIX FOR $\mathcal{D}(s)$ AND $\widetilde{\mathcal{A}}(s)$

The matrix of dynamics, $\mathcal{D}(s)$, in Laplace space is written as

$$\mathcal{D}(s) = \begin{pmatrix} ms^2+\gamma s+k_1+k' & -k_1 & \cdots & \cdots & \cdots & 0 \\ -k_1 & ms^2+2k_1+k & -k_1 & \cdots & \cdots & 0 \\ 0 & -k_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & -k_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & \cdots & 0 & -k_1 & ms^2+2k_1+k & -k1 \\ 0 & \cdots & 0 & 0 & -k_1 & ms^2+\gamma s+k_1+k' \end{pmatrix},$$

and its inverse is

$$\widetilde{\mathcal{A}}(s) = \begin{pmatrix} \widetilde{A}_{11}(s) & \widetilde{A}_{12}(s) & \widetilde{A}_{13}(s) & \cdots & \cdots & \widetilde{A}_{1N}(s) \\ \widetilde{A}_{21}(s) & \widetilde{A}_{22}(s) & \widetilde{A}_{32}(s) & \cdots & \cdots & \widetilde{A}_{2N}(s) \\ \vdots & \vdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ \widetilde{A}_{N1}(s) & \cdots & \cdots & \cdots & \cdots & \widetilde{A}_{NN}(s) \end{pmatrix}.$$

The elements $\widetilde{\mathcal{A}}(s)_{ij}$ have the structure

$$\widetilde{\mathcal{A}}(s)_{ij} = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_{2n-3}s^{2n-3} + a_{2n-2}s^{2n-2}}{Det[\mathcal{D}(s)]},$$

where n represents the dimension of the chain, and the a_k are different constants for each $\widetilde{\mathcal{A}}(s)_{ij}$.

Taking the particular case $N = 4$ and using the parameters $k' = k = k_1 = \gamma = m = 1$, $\mathcal{D}(s)$ and $\widetilde{\mathcal{A}}(s)$ are written, respectively, as

$$\widetilde{\mathcal{D}}(s)=\begin{pmatrix} s^2+s+2 & -1 & 0 & 0\\ -1 & s^2+3 & -1 & 0\\ 0 & -1 & s^2+3 & -1\\ 0 & 0 & -1 & s^2+s+2\end{pmatrix},$$

$$\widetilde{\mathcal{A}}(s)=\frac{1}{\mathrm{Det}[\widetilde{\mathcal{D}}(s)]}\begin{pmatrix}\widetilde{\mathcal{A}}(s)_{11} & \widetilde{\mathcal{A}}(s)_{12} & \widetilde{\mathcal{A}}(s)_{13} & \widetilde{\mathcal{A}}(s)_{14}\\ \widetilde{\mathcal{A}}(s)_{21} & \widetilde{\mathcal{A}}(s)_{22} & \widetilde{\mathcal{A}}(s)_{23} & \widetilde{\mathcal{A}}(s)_{24}\\ \widetilde{\mathcal{A}}(s)_{31} & \widetilde{\mathcal{A}}(s)_{32} & \widetilde{\mathcal{A}}(s)_{33} & \widetilde{\mathcal{A}}(s)_{34}\\ \widetilde{\mathcal{A}}(s)_{41} & \widetilde{\mathcal{A}}(s)_{42} & \widetilde{\mathcal{A}}(s)_{43} & \widetilde{\mathcal{A}}(s)_{44}\end{pmatrix},$$

where $\mathrm{Det}[\widetilde{\mathcal{A}}(s)]$ is

$$\mathrm{Det}[\widetilde{\mathcal{D}}(s)]=s^8+2s^7+11s^6+16s^5+40s^4+38s^3+54s^2+26s+21,$$

and for entries we have

$$\widetilde{\mathcal{A}}(s)_{11}=\widetilde{\mathcal{A}}(s)_{44}=s^6+s^5+8s^4+6s^3+19s^2+8s+13,$$
$$\widetilde{\mathcal{A}}(s)_{12}=\widetilde{\mathcal{A}}(s)_{43}=s^4+s^3+5s^2+3s+5,$$
$$\widetilde{\mathcal{A}}(s)_{13}=\widetilde{\mathcal{A}}(s)_{42}=2+s+s^2,$$
$$\widetilde{\mathcal{A}}(s)_{14}=\widetilde{\mathcal{A}}(s)_{41}=1,$$
$$\widetilde{\mathcal{A}}(s)_{21}=\widetilde{\mathcal{A}}(s)_{34}=s^4+s^3+5s^2+3s+5,$$
$$\widetilde{\mathcal{A}}(s)_{22}=\widetilde{\mathcal{A}}(s)_{33}=s^6+s^5+8s^4+6s^3+19s^2+8s+13,$$
$$\widetilde{\mathcal{A}}(s)_{23}=\widetilde{\mathcal{A}}(s)_{32}=s^4+2s^3+5s^2+4s+4,$$
$$\widetilde{\mathcal{A}}(s)_{24}=\widetilde{\mathcal{A}}(s)_{31}=2+s+s^2.$$

APPENDIX B: INTEGRALS OF HEAT FLUX FOR GAUSSIAN AND POISSONIAN BATHS

1. Integrals related to Gaussian bath

In order to clarify a little bit more of the analytical computation, we show here the functions utilized to determine the heat current in our work. As the expressions become so lengthy, it is useful to introduce here some auxiliary functions given by

$$\chi^{\pm}_{\{j,j+1\}}(iq_m+\alpha)=\widetilde{\mathcal{A}}_{\{j,1\}}(iq_m+\alpha)\pm\widetilde{\mathcal{A}}_{\{j+1,1\}}(iq_m+\alpha),$$
$$\Omega^{\pm}_{\{j,j+1\}}(iq_m+\alpha)=\widetilde{\mathcal{A}}_{\{j,N\}}(iq_m+\alpha)\pm\widetilde{\mathcal{A}}_{\{j+1,N\}}(iq_m+\alpha). \tag{B1}$$

We are now able to write down some integrals related to term II. Considering the heat flux between the first and second particles, one gets

$$\mathcal{I}_1=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\chi^{+}_{\{1,2\}}(iq_1+\varepsilon)\chi^{-}_{\{1,2\}}(-iq_1-\varepsilon), \tag{B2}$$
$$\mathcal{I}_2=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\Omega^{+}_{\{1,2\}}(iq_1+\varepsilon)\Omega^{-}_{\{1,2\}}(-iq_1-\varepsilon). \tag{B3}$$

We now define a couple of variables which are useful to term I:

$$\Gamma_j(iq_1+\varepsilon)=\widetilde{\mathcal{A}}_{\{1,j\}}(iq_1+\alpha)-\widetilde{\mathcal{A}}_{\{2,j\}}(iq_1+\alpha),$$
$$\Lambda_j(iq_2+\varepsilon)=\widetilde{\mathcal{A}}_{\{1,j\}}(iq_2+\alpha)+\widetilde{\mathcal{A}}_{\{2,j\}}(iq_2+\alpha), \tag{B4}$$

allowing us to write the integrals which compose I and II as

$$\mathcal{I}_3^{[l,j]}=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\Lambda_l(iq_1+\varepsilon)\chi^{-}_{\{j,j+1\}}(iq_1+\varepsilon)\chi^{-}_{\{1,2\}}(-iq_1-\varepsilon), \tag{B5}$$
$$\mathcal{I}_4^{[j]}=\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}\chi^{-}_{\{j,j+1\}}(iq_2+\alpha)\chi^{-}_{\{j,j+1\}}(-iq_2-\alpha), \tag{B6}$$
$$\mathcal{I}_5^{[l,j]}=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\Lambda_l(iq_1+\varepsilon)\Omega^{-}_{\{j,j+1\}}(iq_1+\varepsilon)\Omega^{-}_{\{1,2\}}(-iq_1-\varepsilon), \tag{B7}$$
$$\mathcal{I}_6^{[j]}=\int_{-\infty}^{\infty}\frac{dq_2}{2\pi}\Omega^{-}_{\{j,j+1\}}(iq_2+\alpha)\Omega^{-}_{\{j,j+1\}}(-iq_2-\alpha), \tag{B8}$$
$$\mathcal{I}_7^{[l,j]}=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\Gamma_l(-iq_1-\varepsilon)\chi^{-}_{\{j,j+1\}}(-iq_1-\varepsilon)\chi^{+}_{\{1,2\}}(iq_1+\varepsilon), \tag{B9}$$
$$\mathcal{I}_8^{[l,j]}=\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}(iq_1+\varepsilon)\Gamma_l(-iq_1-\varepsilon)\Omega^{-}_{\{j,j+1\}}(-iq_1-\varepsilon)\Omega^{+}_{\{1,2\}}(iq_1+\varepsilon). \tag{B10}$$

2. Integrals related to Poissonian bath

The main difference between the computation involving the Gaussian and Poissonian noises at first order in δ, comes from terms related to averages over cumulants of fourth order. Besides the use of Wick theorem, the evaluation of this kind of term makes use of Eq. (29) and consequently adds some extra integrals in the expression of the Gaussian case. The new contributions to the heat flux when the Poissonian is injecting energy into the chain are

$$\mathcal{I}_9^{[j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}\chi^-_{\{j,j+1\}}(iq_1+\varepsilon)\chi^-_{\{j,j+1\}}(iq_2+\varepsilon)\chi^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon)\chi^+_{\{1,2\}}(iq_3+\varepsilon), \tag{B11}$$

$$\mathcal{I}_{10}^{[j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}\Omega^-_{\{j,j+1\}}(iq_1+\varepsilon)\Omega^-_{\{j,j+1\}}(iq_2+\varepsilon)\Omega^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon)\Omega^+_{\{1,2\}}(iq_3+\varepsilon), \tag{B12}$$

$$\mathcal{I}_{11}^{[l,j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}(-iq_1-\varepsilon)\chi^-_{\{1,2\}}(iq_1+\varepsilon)\Lambda_l(-iq_1-\varepsilon)\chi^-_{\{j,j+1\}}(iq_2+\varepsilon) \times \chi^-_{\{j,j+1\}}(iq_3+\varepsilon)\chi^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon), \tag{B13}$$

$$\mathcal{I}_{12}^{[l,j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}(-iq_1-\varepsilon)\Omega^-_{\{1,2\}}(iq_1+\varepsilon)\Lambda_l(-iq_1-\varepsilon)\Omega^-_{\{j,j+1\}}(iq_2+\varepsilon) \times \Omega^-_{\{j,j+1\}}(iq_3+\varepsilon)\Omega^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon), \tag{B14}$$

$$\mathcal{I}_{13}^{[l,j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}(iq_1+\varepsilon)\chi^+_{\{1,2\}}(iq_1+\varepsilon)\Gamma_l(-iq_1-\varepsilon)\chi^-_{\{j,j+1\}}(iq_2+\varepsilon) \times \chi^-_{\{j,j+1\}}(iq_3+\varepsilon)\chi^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon), \tag{B15}$$

$$\mathcal{I}_{14}^{[l,j]} = \int_{\text{all space}} \frac{dq_1}{2\pi}\frac{dq_2}{2\pi}\frac{dq_3}{2\pi}(iq_1+\varepsilon)\Omega^+_{\{1,2\}}(iq_1+\varepsilon)\Gamma_l(-iq_1-\varepsilon)\Omega^-_{\{j,j+1\}}(iq_2+\varepsilon) \times \Omega^-_{\{j,j+1\}}(iq_3+\varepsilon)\Omega^-_{\{j,j+1\}}(-iq_1-iq_2-iq_3-3\varepsilon). \tag{B16}$$

APPENDIX C: THE WORKING NATURE OF A POISSON RESERVOIR WITH NONZERO AVERAGE AMPLITUDE

In order to show that the work component of a Poisson reservoir is removed we assume a *reductio ad absurdum*–like approach by considering the equation of motion of a particle in contact with a Poissonian (shot-noise) bath, attached to a linear spring and subject to gravity

$$m\ddot{y} = -ky^2 - mg - \gamma\dot{x} + \eta, \tag{C1}$$

where η is a Poisson white noise.

Analytically that noise is given by

$$\langle \eta(t_1)\cdots\eta(t_n)\rangle_c = \lambda\phi_n\delta(t_1-t_2)\delta(t-t')\cdots\delta(t_{n-1}-t_n). \tag{C2}$$

The initial condition is $y(0) = \dot{y}(0) = 0$.

Laplace-Fourier transforming the equation of motion we get

$$\tilde{y}(s) = -\frac{mg}{sR(s)} + \frac{\tilde{\eta}(s)}{R(s)},$$

where $R(s) = ms^2 + \gamma s + k = m(s^2 + \theta s + \omega^2) = m(s - \kappa_+)(s-\kappa_-)$, where $\kappa_\pm = (-\theta \pm i\sqrt{4\omega^2-\theta^2})/2$.

Considering that

$$y(t) = \int_0^\infty dt\,\delta(t-t_1)\,y(t_1) = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi}\,e^{(iq_1+\epsilon)t}\,\tilde{y}(iq_1+\epsilon) = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi} \times e^{(iq_1+\epsilon)t}\left(-\frac{mg}{(iq_1+\epsilon)R(iq_1+\epsilon)} + \frac{\tilde{\eta}(iq_1+\epsilon)}{R(iq_1+\epsilon)}\right),$$

the first integral reads

$$I_1 = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi}e^{(iq_1+\epsilon)t}\left(-\frac{mg}{(iq_1+\epsilon)\,R(iq_1+\epsilon)}\right) = -\frac{g}{\kappa_1\kappa_2} + \frac{g(e^{\kappa_2 t}\kappa_1 - e^{\kappa_1 t}\kappa_2)}{\kappa_2(-\kappa_2+\kappa_1)\kappa_1}.$$

Note that

$$I_1(t=0) = 0$$

and

$$\lim_{t\to\infty} I_1 = -\frac{g}{\kappa_1\kappa_2} = -\frac{g}{\omega^2}.$$

The second integral reads

$$I_2 = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi}e^{(iq_1+\epsilon)t}\left(\frac{\tilde{\eta}(iq_1+\epsilon)}{R(iq_1+\epsilon)}\right),$$

and is a function of the noise.

Let us, for instance, study $\langle y(t)\rangle$. We need to take the average $\langle I_2\rangle$

$$\langle I_2\rangle = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi}e^{(iq_1+\epsilon)t}\left(\frac{\langle\tilde{\eta}(iq_1+\epsilon)\rangle}{R(iq_1+\epsilon)}\right) = \lim_{\epsilon\to 0}\int_{-\infty}^\infty \frac{dq_1}{2\pi}e^{(iq_1+\epsilon)t}\left(\frac{\lambda\,\phi_1}{(iq_1+\epsilon)R(iq_1+\epsilon)}\right) = \frac{\lambda\phi_1}{\kappa_1\kappa_2} - \frac{\lambda\phi_1(e^{\kappa_2 t}\kappa_1 - e^{\kappa_1 t}\kappa_2)}{\kappa_2(-\kappa_2+\kappa_1)\kappa_1}.$$

In the limit $t \to \infty$, the stable position is

$$y_f \equiv \lim_{t\to\infty} y(t) = \frac{\lambda\phi_1 - g}{\omega^2} = \frac{m(\lambda\phi_1 - g)}{k}.$$

Observe that for $\lambda\phi_1 = g$, the final average position is the initial one. However, whenever the inequality holds, say $\lambda\phi_1 - g = \delta > 0$, we have

$$y_f = \frac{m\delta}{k},$$

which corresponds to the elongation of the spring due to the force difference $m(\lambda\phi_1 - g)$ between the gravity and the Poisson "wind." We shall take the limits $\delta, k \to 0$ with $\delta/k = \Delta \gg 1$.

The average potential energy of the system will be

$$U = mgy_f + \frac{1}{2}ky_f^2 = m^2 g\frac{\delta}{k} + \frac{km^2\delta^2}{2k^2}.$$

We can obtain the instantaneous velocity by taking the time derivative of the position

$$v(t) = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}e^{(iq_1+\epsilon)t} \times \left(-\frac{mg}{R(iq_1+\epsilon)} + \frac{(iq_1+\epsilon)\,\tilde{\eta}(iq_1+\epsilon)}{R(iq_1+\epsilon)}\right).$$

The first and second integrals now give us an average of zero at the limit of large times.

The second cumulant will give the local canonical temperature

$$\langle v^2(t)\rangle_c = \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{dq_2}{2\pi}e^{(iq_1+iq_2+2\epsilon)t} \times \frac{(iq_1+\epsilon)(iq_2+\epsilon)\langle\tilde{\eta}(iq_1+\epsilon)\,\tilde{\eta}(iq_1+\epsilon)\rangle_c}{R(iq_1+\epsilon)R(iq_2+\epsilon)}$$

$$= \lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{dq_1}{2\pi}\frac{dq_2}{2\pi}e^{(iq_1+iq_2+2\epsilon)t} \times \frac{(iq_1+\epsilon)(iq_2+\epsilon)\lambda\phi_2}{(iq_1+iq_2+2\epsilon)R(iq_1+\epsilon)R(iq_2+\epsilon)}.$$

The limit of infinite time gives us

$$\langle v^2(\infty)\rangle_c = \frac{\lambda\phi_2}{2\theta}.$$

Using the exponential probability density function for the Poisson kicks, which naturally has an average value different to zero, we write

$$\phi_1 = \int_0^{\infty} dy\frac{e^{-y/\phi_0}}{\phi_0}y = \phi_0,$$

$$\phi_2 = \int_0^{\infty} dy\frac{e^{-y/\phi_0}}{\phi_0}y^2 = 2\phi_0^2.$$

Let us compare the thermodynamic temperature and the work done as time becomes large. We have

$$m\langle v^2(\infty)\rangle_c = T = m\frac{\lambda\phi_2}{2\theta} = \frac{\lambda\phi_0^2 m^2}{\gamma}.$$

We can now compare the potential energy, accumulated due to the work done by the Poisson wind, and the heat absorbed that shows in the temperature value. Since $\lambda\phi_0 \approx g$, we have

$$\frac{T/2}{U} = \frac{\frac{\lambda\phi_0^2 m^2}{2\gamma}}{U} = \frac{g^2m^2}{2\gamma\lambda U}.$$

By taking the limits $\phi_0 \to 0$, and $\lambda \to \infty$, keeping $\lambda\phi_0 \approx g$, we see that $T \ll U$. All noise cumulants vanish, except for the first one in this soft gas wind model. The athermal reservoir gives mainly work to the particle.

That said, if the Poissonian shot noise exhibits a vanishing value for its average, that the work produced is equal to zero and there is no contribution from work to the energy flux which turns into heat flux alone.

APPENDIX D: ENTROPY PRODUCTION OF A GRANULAR GAS

Concerning the production of entropy rate of an athermal reservoir, it was demonstrated in [15] that a granular gas (reservoir) can produce non-Gaussian noise upon a motor (system). While the work produced scales with the granular temperature—according to [47]—the rate of energy loss by the granular gas (which has to be constantly replaced so the gas might reach a steady state) in the reservoir is proportional to $(1-e)\rho V T_G^{3/2}$, where e is the coefficient of restitution, ρ is the density of grains, V is the volume, and T_G is the granular temperature.

The work extracted (from the reservoir), per unit time, will be a small fraction of the grains total kinetic energy $\epsilon\rho V T_G$. On the one hand, the system will reach a temperature that is of the order of T_G at the steady state. Furthermore, from the system's point of view, the entropy reduction rate will be, at most, of the order of

$$\frac{\epsilon\rho V T_G}{T_G} \propto \epsilon\rho V.$$

On the other hand, in order to sustain the steady-state regime of the reservoir, the energy dissipation rate goes as $(1-e)\,\rho V T_G^{3/2}$, yielding an entropy production rate of the order of

$$\frac{(1-e)\rho V T_G^{3/2}}{T} \propto \sqrt{T_G}\frac{T_G}{T},$$

where T is the thermodynamical temperature of the grains. Usually, $T_G/T \approx 10^{10}$–10^{12}. Thus, we can say the entropy production rate is orders of magnitude larger than the anomalous entropy reduction due to the non-Gaussian character of the athermal reservoir noise function.

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