Mathematical and information-geometrical entropy for phenomenological Fourier and non-Fourier heat conduction

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The second law of thermodynamics governs the direction of heat transport, which provides the foundational definition of thermodynamic Clausius entropy. The definitions of entropy are further generalized for the phenomenological heat transport models in the frameworks of classical irreversible thermodynamics and extended irreversible thermodynamics (EIT). In this work, entropic functions from mathematics are combined with phenomenological heat conduction models and connected to several information-geometrical conceptions. The long-time behaviors of these mathematical entropies exhibit a wide diversity and physical pictures in phenomenological heat conductions, including the tendency to thermal equilibrium, and exponential decay of nonequilibrium and asymptotics, which build a bridge between the macroscopic and microscopic modelings. In contrast with the EIT entropies, the mathematical entropies expressed in terms of the internal energy function can avoid singularity paired with nonpositive local absolute temperature caused by non-Fourier heat conduction models.

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I. INTRODUCTION

As a fundamental concept in thermodynamics, entropy is widely discussed and used in macroscopic irreversible phenomena. Strictly speaking, classical thermodynamic entropy is based on equilibrium states and large physical systems. Boltzmann-Gibbs statistical mechanics provides a bridge between the macroscopic quantities and microscopic states, which connects thermodynamic entropy to microscopic distributions. The Boltzmann entropy [1] has a similar structure to the Shannon entropy [2], which is introduced to quantify the information content of data in information theory. Landauer's principle [3–10], which entails the minimum energy loss needed for the erasure operation of one bit of information at a given temperature, further deepens the conceptual relation between thermodynamics and information-theoretic entropies. The principle of entropy increase in thermodynamics provides the tendency for irreversible phenomena, but it is not enough to describe the details of the whole processes. For instance, the Clausius statement [11,12] of the second law governs the direction of heat transfer between two different temperatures, but the transport rate is still unknown. Therefore, supplemental macroscopic phenomenological modelings are needed for complete descriptions and predictions.

Fourier's law is the most classical phenomenological model for heat conduction, which is proved by numerous experiments and widely applied to engineering. As a macroscopic model, Fourier's law models a relation satisfied by local physical quantities,

$$\mathbf{q} + \lambda \nabla T = \mathbf{0}, \tag{1}$$

where $\mathbf{q}$ is the heat flux, λ is the thermal conductivity, and T is the temperature. In statistical mechanics, Fourier's law is derived approximately through some given theoretic assumptions, which also predict possible limitations about this phenomenological model, especially for unsteady problems. From the viewpoint of macroscopic transport theory, Onsager [13] pointed out that Fourier's law neglects the time needed for acceleration of the heat flow, which has been verified by further theoretical analyses and experiments [14–18]. The non-Fourier effects in macroscopic heat conduction are usually called "second sound" or "heat wave," and to handle the wavelike transport, several macroscopic phenomenological models were developed. The Cattaneo-Vernotte (CV) model [19,20] is the most typical one, whose hyperbolic governing equation predicts a finite wave velocity of heat propagation. The CV model is often generalized for non-Fourier mathematical modeling, i.e., the Jeffrey model [21,22], a linear superposition of the CV and Fourier heat conductions. The single-phase-lagging (SPL) model [23] is another "natural extension" [24,25] of the CV model, which will reduce to the CV model by making a first-order Taylor series approximation. With the further extension of the SPL model, the dual-phase-lagging (DPL) model [26] introduces the lagging influence of the temperature gradient. Besides the lagging types, the CV model was recently generalized to fractional heat conduction [27–29], which is usually applied to thermoelasticity [30] with original memory behaviors. Besides extending the CV model, there are other different methods for non-Fourier mathematical modeling. The Guyer-Krumhansl (GK) model [31–34] is a well-known model derived from the linearized phonon Boltzmann equation. The two-temperature (TT) model [35] considers the coupling processes of different mechanisms in heat conduction. To analyze the local details predicted by these nonequilibrium heat conductions, classical thermodynamic entropy is directly extended as local-equilibrium entropy in the framework of classical irreversible thermodynamics (CIT) [36]. For Fourier's law, the positive thermal conductivity guarantees the validity of the Clausius statement in local areas, guaranteeing a non-negative form of the CIT entropy production rate $\mathbf{q} \cdot \nabla(\frac{1}{T})$. However, it is found that the local Clausius statement could be violated by non-Fourier models, i.e., the CV model [37,38], resulting in negative local CIT

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entropy generation. Extended irreversible thermodynamics (EIT) [36] is then developed to overcome this defect by introducing nonequilibrium intrinsic variables to the classical definition of entropy. For a given heat conduction model, if the intrinsic variables and their constitutive relations with corresponding entropies are assumed appropriately, the EIT entropy production rate will have a non-negative form. The physical meanings of these EIT entropies are not specific and mathematically, non-Fourier models typically have the potential to predict nonpositive local absolute temperature [39,40], which will generate singularity in the EIT entropies because of their mathematical expressions containing $\ln T$.

Besides the extensions of thermodynamic entropy, there are also "mathematical entropies" for partial differential equations. These entropic functions usually have similar structures to Boltzmann's H function, and like the H function, they are nonincreasing in isolated systems which is analogous to the increase of thermodynamic entropy. In this work, we introduce these mathematical entropic functions to phenomenological heat conductions. The entropic functions are connected to several information-geometrical conceptions with specific long-time behaviors relying on the governing equations. These long-time behaviors exhibit a wide diversity and physical pictures in phenomenological heat conductions, including the tendency to thermal equilibrium, and exponential decay of nonequilibrium and asymptotics, which build a bridge between the macroscopic and microscopic modelings. In contrast with the EIT entropies, the mathematical entropies expressed in terms of the energy functions can avoid singularity paired with nonpositive local absolute temperature caused by non-Fourier heat conduction models. Some aspects of this problem could also be avoided by limiting the upper bound of the admissible heat flux, which is determined by the finite propagation speed of heat pulses and internal energy density [41–43].

II. FOURIER'S HEAT CONDUCTION

We start from the case of constant properties, in which λ, the mass density ρ, and the specific heat c_V are constants. The governing equation of this case is the linear parabolic equation, whose commonly used mathematical entropy defined in $\mathbb{R}^n$ is [44,45]

$$H_\infty(t) = \int_{\mathbb{R}^n} T(\mathbf{x},t) \ln \frac{T(\mathbf{x},t)}{T_\infty(x)} dV, \tag{2}$$

where $T_\infty(x) = \lim_{t\to+\infty} T(x,t)$, $t \in [0,+\infty)$, $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$, and $dV = dx_1 dx_2 \cdots dx_n$. In a finite n-dimensional torus D, a convex and non-negative entropic functional [45] is written as (non-negativity is guaranteed by convexity)

$$H_F(t) = \frac{1}{\text{meas}(D)} \int_D \frac{T}{T_0} \ln \frac{T}{T_0} dV, \tag{3}$$

where meas(D) is the measure on D and T_0 is the usual average of T, $T_0 = \frac{1}{\text{meas}(D)} \int_D T dV$. Consider the distribution $f_T = \frac{T}{T_0}$, which satisfies the normalization condition $\frac{1}{\text{meas}(D)} \int_D f_T dV = 1$, and we will have that $H_F(t) = H_F[f_T(.,t)] = H_F(f_T)$ has a similar structure to Boltzmann's H function. For each distribution f_T with temperature average T_0, $f_E = \frac{T_0}{T_0} = 1$ is the corresponding equilibrium distribution with a homogeneous distribution of thermal energy (note that the equilibrium distribution is determined by the value of thermal energy rather than the asymptotic state predicted by a given determinate problem). Then, $H_F(t)$ can be considered as the relative entropy or Kullback-Leibler (KL) divergence [46] between the two distributions,

$$H_F(t) = \frac{1}{\text{meas}(D)} \int_D f_T \ln \frac{f_T}{f_E} dV = \text{D}_{\text{KL}}(f_T \| f_E). \tag{4}$$

The KL divergence is an important information-geometrical concept measuring the difference between two distributions. Thus $H_F(t)$ reflects the deviation between a heat conduction process and its corresponding equilibrium state with the same total internal energy, or the nonequilibrium degree of heat conduction. Under periodic or adiabatic boundary conditions [45], we have $\frac{dH_F(t)}{dt} \leqslant 0$, showing the tendency to equilibrium in Fourier heat conduction. In a convex D, $\frac{d}{dt}|\frac{dH_F(t)}{dt}| = -\frac{d^2 H_F(t)}{dt^2} \leqslant 0$ also holds, and hence the rate of tending to equilibrium is dissipative. Besides the distance between two distributions, the KL divergence also has a meaning of "information gain," which is often applied to feature selection. For heat conduction with a fixed thermal energy, the equilibrium state without temperature difference is the state of minimum "quality," and the deviation from the equilibrium state can be considered as a "gain" because the quality of thermal energy is improved by temperature difference. Thus $H_F(t)$ is the information gain of a heat conduction process with respect to its equilibrium state with the same thermal energy. Different from other distance functions, i.e., Euclidean distance, the KL divergence is only a premetric quantity because of its asymmetry $D_{\text{KL}}(f_T \| f_E) \neq D_{\text{KL}}(f_E \| f_T)$. A natural symmetrization for this information gain of heat conduction is $\text{D}_S(f_T \| f_E) = \frac{1}{2} D_{\text{KL}}(f_T \| f_E) + \frac{1}{2} D_{\text{KL}}(f_E \| f_T)$. The Jensen-Shannon (JS) divergence [47] is another commonly used symmetrization, $D_{\text{JS}}(f_T \| f_E) = \frac{1}{2} D_{\text{KL}}(f_T \| \frac{f_E + f_T}{2}) + \frac{1}{2} D_{\text{KL}}(f_E \| \frac{f_E + f_T}{2})$. The two symmetrizations are semimetrics because the triangle inequality needed by metrics is not satisfied.

As a comparison, in the framework of classical irreversible thermodynamics [36], the definition of entropy in heat conduction is

$$S_{\text{CIT}} = \int_D \rho c_V \ln \frac{T}{T_C} dV, \tag{5}$$

where T_C is the reference temperature. By setting $T_C = T_0$, another convex and non-negative entropic function can be given from the CIT entropy,

$$H_{\text{CIT}}(t) = -\frac{1}{\text{meas}(D)} \frac{S_{\text{CIT}}}{\rho c_V} = -\frac{1}{\text{meas}(D)} \int_D \ln \frac{T}{T_0} dV. \tag{6}$$

This dimensionless entropic functional is also dissipative, $\frac{dH_{\text{CIT}}(t)}{dt} \leqslant 0$, for isolated Fourier heat conduction, but its second-order derivative $\frac{d}{dt}|\frac{dH_{\text{CIT}}(t)}{dt}|$ can be positive in high-dimensional problems ($n \geqslant 2$) [48]. It also means that the CIT entropy production rate is not necessarily dissipative. Therefore, both the CIT entropy increase and dissipation of $H_F(t)$ can reflect the tendency to equilibrium, but the dissipative rate of tending to equilibrium exists only for $H_F(t)$. Accordingly, in Fourier heat conduction, $H_F(t)$ provides more physical pictures about the dissipative behaviors tending to equilibrium than the CIT entropy. $H_F(t)$ also has other long-time behaviors, i.e., exponential decay [45],

$$H_F(t) \leqslant H_F(0)\exp\left(-\frac{t}{\tau_F}\right), \tag{7}$$

where τ_F is a positive constant. Taking into account the meaning of the KL divergence, this behavior shows the exponential decay of the deviation from equilibrium. In transport theory, the relaxation time approximation [49–51], which is a widely used linearization method for the Boltzmann equation, will also lead to an exponential decay about the deviation between the distribution function f and its equilibrium state f_0,

$$(f-f_0) = (f-f_0)|_{t=0}\exp\left(-\frac{t}{\tau_B}\right), \tag{8}$$

where τ_B is the relaxation time. The exponential decay in Eq. (8) is established under the condition of $\nabla f = 0$ in every local element, since the exponential decay of $D_{\mathrm{KL}}(f_T\|f_E)$ holds when the system is adiabatic, which is a global condition. Equation (7) is a global estimation of the exponential decay expressed by macroscopic temperature or thermal energy. For this reason, a "macroscopic global thermal relaxation time" for the macroscopic phenomenological description of heat conduction can be defined as $\tau_{GF} = \inf \tau_F$. In contrast, the exponential decay in Eq. (8) is a local approximation of microscopic distribution functions. Similarly, we can also define a "microscopic global thermal relaxation time" $\tau_{GB} = \inf \tau_G$, where $\tau_G > 0$ satisfies the exponential decay for the global integral of the nonequilibrium degree,

$$\int_{D^n} D_{\mathrm{KL}}(f\|f_0)dV \leqslant \exp\left(-\frac{t}{\tau_G}\right)\int_{D^n}[D_{\mathrm{KL}}(f\|f_0)|_{t=0}]dV. \tag{9}$$

If the local distribution function is close enough to its equilibrium state $(\frac{f-f_0}{f_0}) = o(1)$, we have

$$\int f\ln\frac{f}{f_0} \cong \int \frac{(f-f_0)^2}{2f_0}. \tag{10}$$

From Eq. (10), it is found that τ_{GB} can be understood as a global average of $2\tau_B$. Two global thermal relaxation times are so far proposed: τ_{GF} is a macroscopic estimation according to the phenomenological model, and τ_{GB} is for microscopic distribution function on the basis of the Boltzmann equation with the relaxation time approximation. The theoretical basics of the two relaxation times are radically different, but if their corresponding mathematical modelings of heat conduction are coincident enough, a global averaged estimation expressed by macroscopic temperature can be provided for the relaxation time of microscopic distribution function $\tau_{GF} \cong \tau_{GB} \cong 2\tau_B$.

Another frequently used behavior of $H_F(t)$ is the long-time asymptotics [45]. For instance, consider the relative entropies between the distributions $(f_{T_1}, f_{T_2}, \ldots, f_{T_n}) = (\frac{T_1}{T_0}, \frac{T_2}{T_0}, \ldots, \frac{T_n}{T_0})$ with the same boundary temperature and thermal energy,

$$H_R\left(f_{T_i}\|f_{T_j}\right) = \frac{1}{\mathrm{meas}(D)}\int_D \frac{T_i}{T_0}\ln\frac{T_i}{T_j}dV. \tag{11}$$

$H_R(f_{T_i}\|f_{T_j})$, which reflects the discrimination between two temperature distributions, is decreasing,

$$\frac{dH_R\left(f_{T_i}\|f_{T_j}\right)}{dt} = \frac{1}{\mathrm{meas}(D)}\int_D -\frac{\lambda T_j^2}{\rho c_V T_0 T_i}\left|\nabla\left(\frac{T_i}{T_j}\right)\right|^2 dV \leqslant 0. \tag{12}$$

For an arbitrary boundary, because of the nonzero boundary heat flux, the systems may not tend to equilibrium but the discriminations between the systems are still dissipative. Therefore, all systems will tend asymptotically to an asymptotical distribution, which means the effects caused by different initial conditions are being eliminated. One well-known consequence of the long-time asymptotics is that for a time-independent boundary, the temperature distributions with different initial values will finally tend to an identical steady solution. From the viewpoint of information theory, the long-time asymptotics shows that the initial information gain caused by initial nonequilibrium is lost in the "thermal information" transmission. For this special case with time-independent boundary, the long-time asymptotics does not require the same total internal energy because the time-independent boundary has already given an asymptotic value of the total internal energy. As another nonincreasing mathematical entropy, $H_{\mathrm{CIT}}(t)$ is not suitable to be a measure of the discrimination between temperature distributions. That is because the discrimination $H_{\mathrm{CIT}}(f_{T_i}\|f_{T_j}) = -\frac{1}{\mathrm{meas}(D)T_0}\int_D \ln\frac{T_i}{T_j}dV$ is not non-negative and consequently, its derivative cannot determine whether this discrimination is dissipative in time. Although $|H_{\mathrm{CIT}}(f_{T_i}\|f_{T_j})|$ can be applied, the continuity will be broken which makes the calculation of $\frac{d|H_{\mathrm{CIT}}(f_{T_i}\|f_{T_j})|}{dt}$ insignificant (even impossible) for general problems. In summary, $H_F(t)$ can provide richer mathematical behaviors and physical pictures than the CIT entropy. Actually, the CIT entropy is a direct extension of thermodynamic entropy relying on local equilibrium, which is independent of the heat conduction law, since $H_F(t)$ is directly for the governing equation of heat conduction.

Furthermore, nonextensive types of entropy families [45] or entropic indices [52] can also be introduced as follows:

$$H^{(k)}(t) = \frac{1}{\mathrm{meas}(D)k(k-1)}\int_D\left[\left(\frac{T}{T_0}\right)^k - 1\right]dV, \tag{13}$$

$$R^{(\alpha)}(t) = \frac{1}{\alpha-1}\ln\left[\frac{1}{\mathrm{meas}(D)}\int_D\left(\frac{T}{T_0}\right)^\alpha dV\right], \tag{14}$$

Where $k \neq 0,1$ and $\alpha > 1$. The two above-mentioned entropies are the zero-order $[H_{\mathrm{CIT}}(t) = \lim_{k\to 0} H^{(k)}(t)]$ and first-order $[H_F(t) = \lim_{k\to 1} H^{(k)}(t)]$ limits of $H^{(k)}(t)$. $k = 2$ is also a special case, in which $H^{(k)}(t) = H^{(2)}(t)$ is usually called the "energy integral" with the "energy function" $E_F = \frac{1}{2}(\frac{T}{T_0})^2$ (or $\frac{1}{2}T^2$). In mathematics, $H^{(2)}(t)$ is used for discussing the well-posedness of second-order parabolic equations. $R^{(\alpha)}(t)$ has a similar form to Renyi entropy [52] with the distribution f_T, and it can be considered as the Renyi divergence between f_T and its equilibrium state with the same thermal energy,

$$D_R^{(\alpha)}(f_T\|f_E) = \frac{1}{\alpha-1}\ln\left[\frac{1}{\mathrm{meas}(D)}\int_D \frac{f_T^\alpha}{f_E^{\alpha-1}}dV\right] = R_F^{(\alpha)}(t). \tag{15}$$

Based on the entropic functions, we can provide a nonequilibrium entropy family by introducing the effects of the deviation from equilibrium $S_{HT} = S_{HT}[S_{eq}, D(f_T \| f_E)]$, in which S_{eq} is equilibrium entropy and $D(f_T \| f_E)$ is $D_{\rm KL}(f_T \| f_E)$ or $D_R^{(\alpha)}(f_T \| f_E)$. A linearization of S_{HT} is

$$S_{HT} = (1-w)S_{eq} - wk_B D(f_T \| f_E), \tag{16}$$

where k_B is the Boltzmann's constant and w is the weight coefficient of nonequilibrium. In contrast with CIT entropy, the nonequilibrium degree $D(f_T \| f_E)$ is considered as an intrinsic variable of S_{HT}. To make this entropic functional of phenomenological macroscopic heat conduction agree with the microscopic expression, we can select proper weight coefficients to approximate the Boltzmann-Gibbs entropy, which is independent of the postulate of local equilibrium. As the nonextensive types of entropy families are introduced, not only Boltzmann-Gibbs statistical mechanics but also nonextensive statistical mechanics [53–56] could be approximated.

The above entropic functions can also be applied to nonlinear cases, where the thermal conductivity is expressed as $\lambda(T) = \lambda_0 T^m$ (λ_0 is a positive constant and $m > -1$). $H^{(k)}(t)$ (including $k \to 0,1$) are still non-negative and convex. Under periodic or adiabatic boundary conditions, the dissipation rates of $H^{(k)}(t)$ are as follows:

$$\frac{dH_F^{(k)}(t)}{dt} = \begin{cases} -\frac{4\lambda_0}{\text{meas}(D)\rho c_V (k+m)^2}\int_D \left|\nabla\left(\frac{T}{T_0}\right)^{\frac{k+m}{2}}\right|^2 dV, & k+m \neq 0 \\ -\frac{\lambda_0}{\text{meas}(D)\rho c_V}\int_D \left|\nabla\left(\ln\frac{T}{T_0}\right)\right|^2 dV, & k+m = 0 \end{cases}. \tag{17}$$

It should be emphasized that $m > -1$ is necessary, not only for the convexity but also for the existence of non-negative solutions. When $m \leqslant -1$, the governing equation becomes the "singular diffusion equation" or "fast diffusion equation" [57–59], which is usually paired with the ill-posedness of non-negative solutions (the existence of solutions even depends on dimensions). However, in the famous linear phenomenological heat transfer law [60,61],

$$\mathbf{q} = L_q \nabla\left(\frac{1}{T}\right) = -\frac{L_q}{T^2}\nabla T, \tag{18}$$

where L_q is the phenomenological coefficient, corresponding to $m = -2$. This constitutive assumption aims at the equality between the CIT entropy production rate and $L_q^{-1}\mathbf{q}\cdot\mathbf{q}$, but in mathematics, astonishingly, entropylike functions are used to prove the nonexistence of non-negative solutions submitting this type [57]. For more complicated cases $\lambda = \lambda(T) > 0$ [$\lim_{T\to 0^+} |\lambda(T)| < +\infty$], the entropic function is written as [62]

$$H_{NL}(t) = \frac{1}{\text{meas}(D)}\int_D \left[\int_0^T \frac{d\xi}{\lambda(\xi)T_0}\int_{T_0}^{\xi}\frac{\lambda(\varsigma)}{\varsigma}d\varsigma\right] dV, \tag{19}$$

with a dissipation rate in periodic or adiabatic boundary conditions as follows:

$$\frac{dH_{NL}(t)}{dt} = -\frac{1}{\text{meas}(D)}\int_D \frac{T}{g(T)T_0}\left|\frac{g(T)\nabla T}{T}\right|^2 dV. \tag{20}$$

Phonon heat conduction in the low-temperature limit ($T \to 0$) is taken as an example [63–66], in which $g_{T\to 0}(T) = AT^3$ and $\rho c_V = BT^3$ (A and B are positive constants). The governing equation is written as $BT^3\frac{\partial T}{\partial t} = \nabla(AT^3\nabla T)$, which can be transformed into a linear equation $\frac{\partial v}{\partial t} = \nabla(\frac{A}{B}\nabla v)$ by setting $v = T^4$. $H^{(k)}(t)$ can subsequently be applied by replacing T with v. For other more elaborate physical property modeling, if $\forall T > 0$ the internal energy function $u(T)$ satisfies $\frac{du(T)}{dT} > 0$; the governing equation can be transformed into $\frac{\partial u}{\partial t} = \nabla\{g[T(u)]\frac{dT(u)}{du}\nabla u\}$ and $H_{NL}(t)$ can then be applied. Additionally, the condition $\lim_{u\to 0^+} |g[T(u)]\frac{dT(u)}{du}| < +\infty$ should be satisfied to avoid the singularity and nonexistence of solutions.

III. NON-FOURIER HEAT CONDUCTION

The CV model is the most typical model predicting wavelike heat transport, but it has the potential to violate the Clausius statement in local areas because the local CIT entropy production rate $q\cdot\nabla(\frac{1}{T})$ might be negative. To guarantee a non-negative form of the local entropy generation [36], heat flux is introduced as an intrinsic variable in the EIT entropy with a constitutive relation assumed as $S_{\rm CV} = S_{eq} - \frac{\tau}{2\lambda T^2}\mathbf{q}\cdot\mathbf{q}$. Formally speaking, $S_{\rm CV}$ has a non-negative local entropy production rate $\dot{S}_{\rm CV} = \frac{1}{\lambda T^2}\mathbf{q}\cdot\mathbf{q}$. However, it is also revealed that the CV model can predict nonpositive local absolute temperature [39,40], which will obviously generate singularity in $S_{\rm CV}$ and $\dot{S}_{\rm CV}$. Thus, strictly speaking, this EIT entropy with a non-negative form of the local entropy generation could still be infeasible. The negative values of absolute temperature can be avoided by imposing initial conditions which can limit the upper bound of the propagation speed in heat conduction [41–43], which is a natural requirement but we here use a different method. To avoid the singularity caused by nonpositive local absolute temperature, we introduces an energy function for the CV model,

$$E_{\rm CV} = w\left[\frac{\lambda\tau}{\rho c_V}|\nabla T|^2 + \tau^2\left(\frac{\partial T}{\partial t}\right)^2\right]. \tag{21}$$

Compared with $D_{\rm KL}(f_T \| f_E)$, which reflects the global deviation from equilibrium, $E_{\rm CV}$ represents the local nonequilibrium degree in wavelike heat transport. Similar to Fourier's law, the global nonequilibrium degree of the CV model is also dissipative in adiabatic problems,

$$\int_D \frac{\partial E_{\rm CV}}{\partial t}dV = -\int_D 2\tau\left(\frac{\partial T}{\partial t}\right)^2 dV \leqslant 0. \tag{22}$$

A non-negative and nonincreasing entropic functional is then given by

$$R_{\rm CV}(t) = \ln\left[\frac{1}{\text{meas}(D)}\int_D\left(1 + \frac{E_{\rm CV}}{T_0^2|_{t=0}}\right)dV\right]. \tag{23}$$

If a heat conduction problem can guarantee a unique positive equilibrium temperature $T_{EQ} > 0$ (this requirement is obvious in physics, but might not be satisfied by the CV model), $T_0|_{t=0}$ can also be replaced by T_{EQ}. Although the local Clausius statement is not satisfied by the CV model, the dissipation of the global nonequilibrium degree still shows the tendency to equilibrium. Nevertheless, Eq. (22) indicates that the dissipation of the global nonequilibrium degree will stop when $\frac{\partial T}{\partial t} = 0$. It means that $\int_D |\nabla T|^2 dV$ is conserved in steady adiabatic problems, and hence the temperature difference can exist steadily, which is impossible in Fourier heat conduction. For more sophisticated hyperbolic phenomenological models, including the Jeffrey model, GK model, TT model, and first-order Taylor series expansion of the DPL model, the governing equations take the following form:

$$C_1 \frac{\partial T}{\partial t} + C_2 \frac{\partial^2 T}{\partial t^2} = \nabla^2 T + C_3 \frac{\partial}{\partial t}(\nabla^2 T), \tag{24}$$

where C_i $(i = 1,2,3)$ are positive coefficients. The energy function reflecting the local nonequilibrium degree is given by the following form:

$$E_W = w\left[\frac{C_2}{C_1^2}|\nabla T|^2 + \left(\frac{C_2}{C_1}\right)^2\left(\frac{\partial T}{\partial t}\right)^2\right]. \tag{25}$$

The corresponding dissipative rate and entropic function are

$$\int_D \frac{\partial E_W}{\partial t} dV = -\frac{2C_2}{C_1^2}\int_D \left[C_1\left(\frac{\partial T}{\partial t}\right)^2 + C_3\left(\frac{\partial \nabla T}{\partial t}\right)^2\right] dV, \tag{26}$$

$$R_W(t) = \ln\left[\frac{1}{\text{meas}(D)}\int_D \left(1 + \frac{E_W}{T_0^2|_{t=0}}\right) dV\right]. \tag{27}$$

By defining $\frac{E_W}{T_{EQ}^2}\ln\frac{E_W}{T_{EQ}^2}|_{E_W=0} = \lim_{E_W\to 0^+}(\frac{E_W}{T_{EQ}^2}\ln\frac{E_W}{T_{EQ}^2}) = 0$, $f_W = \frac{IE_W}{T_{EQ}^2}$, and $f_{WE} = \frac{T_{EQ}^2}{T_{EQ}^2} = 1$, the KL divergence showing the nonequilibrium degree of heat wave can be proposed,

$$D_{\text{KL}}(f_W \| f_{WE}) = \frac{1}{\text{meas}(D)}\int_D \frac{IE_W}{T_{EQ}^2}\ln\frac{IE_W}{T_{EQ}^2} dV, \tag{28}$$

where I is the normalized coefficient satisfying $\frac{1}{\text{meas}(D)}\int_D \frac{IE_W}{T_{EQ}^2} dV = 1$. More generally, the energy functions of wavelike transport with the wave velocity V_h can be written as $E = E(\varphi)$, whose intrinsic variable is $\varphi = |\nabla T|^2 + \frac{1}{V_h^2}(\frac{\partial T}{\partial t})^2 = (\frac{\partial T}{\partial |\mathbf{x}|})^2 + \frac{1}{V_h^2}(\frac{\partial T}{\partial t})^2$. This selection method of the intrinsic variable is to guarantee the positions of $|\mathbf{x}|$ and $V_h t$ are equal in the energy functions because of their equal positions in the traveling wave solutions $T(\mathbf{x},t) = T(|\mathbf{x}| \pm V_h t)$. For the dissipation and convexity of the corresponding nonequilibrium degrees, $E(\varphi)$ should also satisfy $E(0) = 0$, $\frac{dE(\varphi)}{d\varphi} > 0$, and $\frac{d^2E(\varphi)}{d\varphi^2} \leqslant 0$.

Phase lagging is another type of phenomenological non-Fourier heat conduction. The DPL model [26] is a typical example,

$$\mathbf{q}(\mathbf{x},t+\tau_q) + \lambda \nabla T(\mathbf{x},t+\tau_T) = 0, \tag{29}$$

where τ_q is the relaxation time of the heat flux and τ_T is the relaxation time of temperature gradient. The Taylor series expansions of the DPL model are often connected to the lattice Boltzmann method [34] and used to give hyperbolic heat conductions. Like heat wave models, phase-lagging models can also violate the Clausius statement in local areas. The violation will be shown by one-dimensional (1D) problems in $[0,l] \times [0,+\infty)$, which obeys the DPL model and satisfies $\frac{\lambda\pi(\tau_q - \tau_T)}{\rho c_V l^2} = \frac{1}{2}$. If the initial condition is taken $T|_{t=0} = T_0(1 + \alpha \sin\frac{\pi x_1}{l})$ and the boundary conditions are taken $T|_{x_1=0,l} = T_0$ (T_0 and α are positive constants), one solution of this problem is given by

$$T_1(x_1,t) = T_0\left[1 + \alpha\cos\frac{\pi t}{2(\tau_q - \tau_T)}\sin\frac{\pi x_1}{l}\right]. \tag{30}$$

The local CIT entropy production rate of this solution is

$$\mathbf{q}\cdot\nabla\left(\frac{1}{T}\right) = -\frac{\lambda\alpha^2\pi^2T_0^2}{2l^2T_1^2}\cos^2\frac{\pi x_1}{l}\sin\frac{\pi t}{\tau_q - \tau_T}, \tag{31}$$

which is not non-negative. Similar to heat wave models, an EIT entropy can be provided for the DPL model,

$$S_D(x_1,t) = \int_0^t \frac{\rho c_V}{T(x_1,\varepsilon+\tau_T)}\frac{\partial T(x_1,\varepsilon+\tau_q)}{\partial \varepsilon} d\varepsilon, \tag{32}$$

with a non-negative local entropy production rate $\frac{\lambda|\nabla T(x_1,t+\tau_T)|^2}{T^2(x_1,t+\tau_T)}$. Compared with the local-equilibrium entropy, whose local entropy flux is $\mathbf{J_E}(\mathbf{x},t) = \frac{q(\mathbf{x},t)}{T(\mathbf{x},t)}$, this EIT entropy reflects the effects of phase-lagging heat conduction in its local entropy flux, which is modified as $\mathbf{J_D}(\mathbf{x},t) = \frac{q(\mathbf{x},t+\tau_q)}{T(\mathbf{x},t+\tau_T)}$. If we adopt the equilibrium initial condition $T|_{t=0} = T_0$ and adiabatic boundary conditions $q|_{x_1=0,l} = 0$, there is a series of periodic solutions with an arbitrary constant C_1,

$$T_2(x_1,t) = C_1\cos\frac{n\pi x_1}{l}\sin\frac{\pi t}{2(\tau_q - \tau_T)} + T_0. \tag{33}$$

The periodic solutions demonstrate that the phase-lagging heat conduction in an adiabatic system might neither reach nor tend to thermal equilibrium, and initial thermal equilibrium can even be broken by heat transport. However, the second law of thermodynamics requires the spontaneous tendency to thermal equilibrium. Thus, although we can provide appropriate EIT entropies with non-negative generations for phase-lagging models, violations of the second law still exist. It is concluded that besides non-negative local entropy production rates, the generalized entropic functions should also provide physical pictures of the tendency to equilibrium.

The above energy functions have a simple form for common non-Fourier models, but for nonlinear non-Fourier conductions [67–69], this method will also become more sophisticated. Generally speaking, most phenomenological heat conduction models could reduce to Fourier's law or the CV model by neglecting certain behaviors, which predict dissipative physical behaviors. Mathematically, their governing equations [70] (mainly including damped wave equations and semilinear parabolic equations) can define forward regularizing flows in certain adequate phase spaces containing absorbing sets. The equations with these physical and mathematical characteristics can be summarized as "dissipative evolution equations." "ε entropy" [71], proposed by Kolmogorov, is another important entropy to study the long-time behaviors of the dissipative evolution equations. For a compact or precompact set $\mathbf{S}$ in a metric space $\mathbf{N}$, its ε entropy [70–73] is written as $H_\varepsilon(\mathbf{S},\mathbf{N}) = \log_2 B_\varepsilon(\mathbf{S},\mathbf{N})$, where $B_\varepsilon(\mathbf{S},\mathbf{N})$ is the minimal number of ε balls covering $\mathbf{S}$. The corresponding fractal dimension is then defined as

$$\dim_F(\mathbf{S},\mathbf{N}) = \lim_{\varepsilon\to 0}\left[\sup \frac{H_\varepsilon(\mathbf{S},\mathbf{N})}{\log_2 \frac{1}{\varepsilon}}\right]. \tag{34}$$

The fractal dimension is a widely used concept for studying the attractors and solution operators of the dissipative evolution equations, which can provide long-time behaviors including the asymptotics and exponential decay of attractors [70,72–74]. A semilinear parabolic system with a nonlinear source term $R(U)$ is taken as a simple example,

$$\frac{\partial U}{\partial t} + L(U) = R(U) \tag{35}$$

where U is a scalar or vector and $L(U)$ is a positive, self-adjoint and linear operator, i.e., $L(U) = \nabla^2 U$. In this case, ε entropy can be used to provide an exponential decay for the solution operator $\mathcal{S}$ and the exponential fractal attractor $\mathcal{M}$ [73,74],

$$\text{dist}_H(\mathcal{S}\text{U},\mathcal{M}) = \text{dist}_H(\mathcal{S}\text{U},\mathcal{M})|_{t=0}\exp\left(-\frac{t}{\tau_A}\right), \tag{36}$$

where dist_H is the standard asymmetric Hausdorff pseudodistance and τ_A is a positive constant. The relaxation behavior of phenomenological heat conductions is subsequently provided.

IV. CONCLUSIONS

In the present work, mathematical entropic functions are proposed for phenomenological heat conduction models, which are understood through information-geometrical conceptions and thermal equilibrium. It should be mentioned that information-theoretical concepts based on microscopic statistical mechanics have been applied to the framework of extended irreversible thermodynamics [75–78], while the information-theoretical concepts in this work are expressed by macroscopic and phenomenological quantities. The long-time behaviors of these entropies can exhibit more abundant physical pictures than the CIT and EIT entropies, including the tendency to thermal equilibrium, and exponential decay of nonequilibrium and asymptotics. The mathematical behaviors and physical pictures also make a connection between macroscopic and microscopic mathematical modelings. A global averaged estimation expressed by local macroscopic temperature is provided for the relaxation time of microscopic distribution function $\tau_{GF} \cong \tau_{GB} \cong 2\tau_B$. In addition, the entropic indices connect phenomenological heat conduction to nonextensive thermodynamics, which provides a perspective beyond Boltzmann-Gibbs statistical mechanics.

Although the EIT entropies can guarantee non-negative forms for the local entropy production rates, they have the potential to cause singularity because of the nonpositive local absolute temperature caused by non-Fourier heat conductions. By introducing non-negative energy functions, generalized entropies can overcome this defect, meanwhile guaranteeing the tendency to thermal equilibrium. Incidentally, ε entropy which is used to study the long-time behaviors of dissipative evolution equations (including damped wave equations and semilinear parabolic equations) demonstrates its possibility for studying more sophisticated nonlinear models which can reduce to the CV model or Fourier's law.

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