Cycle-expansion method for the Lyapunov exponent, susceptibility, and higher moments

Patrick Charbonneau,[1,2] Yue (Cathy) Li,[1,3] Henry D. Pfister,[3] and Sho Yaida[1,*]
[1]*Department of Chemistry, Duke University, Durham, North Carolina 27708, USA*
[2]*Department of Physics, Duke University, Durham, North Carolina 27708, USA*
[3]*Department of Electrical and Computer Engineering, Duke University, Durham, North Carolina 27708, USA*
(Received 7 July 2017; published 18 September 2017)

Lyapunov exponents characterize the chaotic nature of dynamical systems by quantifying the growth rate of uncertainty associated with the imperfect measurement of initial conditions. Finite-time estimates of the exponent, however, experience fluctuations due to both the initial condition and the stochastic nature of the dynamical path. The scale of these fluctuations is governed by the Lyapunov susceptibility, the finiteness of which typically provides a sufficient condition for the law of large numbers to apply. Here, we obtain a formally exact expression for this susceptibility in terms of the Ruelle dynamical ζ function for one-dimensional systems. We further show that, for systems governed by sequences of random matrices, the cycle expansion of the ζ function enables systematic computations of the Lyapunov susceptibility and of its higher-moment generalizations. The method is here applied to a class of dynamical models that maps to static disordered spin chains with interactions stretching over a varying distance and is tested against Monte Carlo simulations.

DOI: 10.1103/PhysRevE.96.032129

I. INTRODUCTION

The Lyapunov exponent was initially devised to quantify the rate at which information dissipates in a chaotic dynamical system [1,2]. More concretely, it measures how the distance between two nearby trajectories scales exponentially with time when their initial conditions are infinitesimally close. The quantity has since found a number of other applications. For instance, it gives the free-energy density of one-dimensional spin chains [3] and the entropy rate of stationary hidden Markov models in information theory [4–7]. Interest in Lyapunov exponents continues to spread, as illustrated by the recent study of black-hole scrambling [8,9], which results in the formulation of an upper bound on the Lyapunov exponent for quantum systems [10] (see also Ref. [11]).

When the dynamics of a system can be modeled by a sequence of randomly drawn matrices [12], the Lyapunov exponent is also intimately connected to the rich properties of disordered systems [13]. For the sake of concreteness, consider a sequence of matrices, $\{T_i\}_{i\in\mathbb{N}}$, be they transfer matrices in disordered spin chains or transition-observation matrices in hidden Markov chains [5]. The Lyapunov exponent is then the typical growth rate of the maximum-modulus eigenvalue of the product of these matrices. More formally, we define the finite-sample quantity

$$\lambda_N^{(\alpha)} \equiv \frac{1}{N} \ln \left(\left\| \prod_{i=1}^{N} T_i^{(\alpha)} \right\| \right), \tag{1}$$

where N is the sample size and α denotes the sequence of random matrices drawn independently from some fixed underlying probability distribution. The Lyapunov exponent is then given by the infinite system size limit

$$\lambda \equiv \lim_{N\to\infty} \mathbb{E}[\lambda_N] \equiv \lim_{N\to\infty} \lim_{N_D\to\infty} \frac{1}{N_D} \sum_{\alpha=1}^{N_D} \lambda_N^{(\alpha)}, \tag{2}$$

where the disorder-average $\mathbb{E}[\cdots]$ can be obtained by drawing N_D disorder realizations. Note that the limit does not depend on the choice of matrix norm, $\|\cdots\|$.

Given the ubiquitous appearance of the Lyapunov exponent in products of random matrices [12,14], many methods have been developed for its estimation, including Monte Carlo algorithms [15–17], a perturbative weak-disorder expansion [18,19], a microcanonical method [20], a cycle expansion [21], a Dyson-Schmidt equation [22,23], a scaling method [24,25], an evolution-operator method [26], and an infinite transfer matrix method [27]. Central to all these approaches is the assumption that sample-to-sample fluctuations of λ_N are not so large as to invalidate the law of large numbers. Interestingly, in assessing the applicability of this law, an essential role is played by the second moment of the generalized Lyapunov exponent [28], i.e., the Lyapunov susceptibility,

$$\chi_L \equiv \lim_{N\to\infty} N\big(\mathbb{E}\big[\lambda_N^2\big] - \mathbb{E}[\lambda_N]^2\big).$$

It has indeed been proven under certain conditions on the underlying matrix distribution that the central limit theorem holds if and only if χ_L is finite [29], thus providing a sufficient (though not necessary) condition for the law of large numbers to hold. The susceptibility also appears in rigorous treatments of mean-field spin glasses [30,31] and is related to bond chaos [32–34]. For turbulent flows, a nontrivial susceptibility further signals the existence of intermittency [35,36]. Given the physical and mathematical importance of this quantity [37], it is surprising that it has thus far rarely been explicitly considered.

Here, we develop methods for evaluating the Lyapunov susceptibility and use the results to understand better its behavior. More specifically, we extend the cycle-expansion method [21,38,39], which is based on the Ruelle dynamical ζ function and provides a formally exact expression linking the underlying cycles to the susceptibility. We further find that, when applicable, the cycle-expansion method offers a natural and efficient approach for assessing tails of the Lyapunov-exponent distribution pertaining to the physics of large deviations.

*sho.yaida@duke.edu

The rest of this paper is organized as follows. In Sec. II concrete models that we use to illustrate our methodology are stipulated. Results of Monte Carlo simulations are discussed in Sec. III, and the cycle-expansion method for the Lyapunov susceptibility and its higher-moment generalizations is developed in Sec. IV. In Sec. V results of cycle expansions are compared against those of Monte Carlo simulations. A brief conclusion follows in Sec. VI.

II. MODELS

This section introduces the class of models used in the rest of this work. The models consist of a static one-dimensional chain of N spins with an interaction of range $N_{\rm n}$ captured by transfer matrices, T_a. They thus constitute a generic set of one-dimensional disordered models with finite-range interactions. They can equivalently be viewed as $N_{\rm n}$-neighboring spins that evolve dynamically with a transition matrix T_a hitting at each time step, or as a single spin evolving with finite-time memory. It is worth stressing, however, that these models are chosen mainly for illustrative purposes, and that the methods developed below have a much broader scope of application.

A. Nearest-neighbor (NN) model

The disordered NN Ising model is governed by the Hamiltonian

$$H = -\sum_{i=1}^{N} J_i S_i S_{i+1}, \tag{3}$$

where spins $S_i = \pm 1$ for $i = 1, \ldots, N$ with periodic boundary condition $S_{N+1} = S_1$. The NN interactions $\{J_i\}_{i=1,\ldots,N}$ are randomly drawn to be $\pm J$ with equal probability $1/2$ at each site. The associated transfer matrices, $T_i^{(\alpha)} \in \{T_+, T_-\}$, are then

$$T_\pm \equiv \begin{bmatrix} e^{\pm\beta J} & e^{\mp\beta J} \\ e^{\mp\beta J} & e^{\pm\beta J} \end{bmatrix} = \begin{bmatrix} e^{\pm\tilde\beta} & e^{\mp\tilde\beta} \\ e^{\mp\tilde\beta} & e^{\pm\tilde\beta} \end{bmatrix} \tag{4}$$

for the dimensionless inverse temperature $\tilde\beta \equiv \beta J \equiv \frac{J}{k_{\rm B}T}$. The free-energy density,

$$\beta f_N^{(\alpha)} = -\frac{1}{N} \ln\left[\mathrm{tr}\left(\prod_{i=1}^{N} T_i^{(\alpha)} \right) \right], \tag{5}$$

is thus related to the associated Lyapunov exponent in the thermodynamic limit $N \to \infty$ through Gelfand's formula [40], i.e., $\lim_{N\to\infty} \lambda_N^{(\alpha)} = \lim_{N\to\infty} (-\beta f_N^{(\alpha)})$. For this particular model, each disorder realization can be mapped onto a pure Ising model without disorder by redefining the spins (combined with the possible replacement of periodicity by antiperiodicity at the boundary), and hence is fully solvable, with

$$\lambda = -\beta f = \ln[2\cosh(\tilde\beta)] \tag{6}$$

and $\chi_{\rm L} = 0$.

B. Next-nearest-neighbor (NNN) model

Including NNN interactions is sufficient to make the analysis nontrivial. The Hamiltonian is then

$$H = -\sum_i \left(J_i^{[1]} S_i S_{i+1} + J_i^{[2]} S_i S_{i+2} \right), \tag{7}$$

where $J_i^{[l]} = \pm\frac{J}{\sqrt{2}}$ are independent and identically distributed random variables with amplitude chosen such that the NN scaling of the Lyapunov exponent,

$$\lambda = \ln(2) + \frac{\tilde\beta^2}{2} + O(\tilde\beta^4), \tag{8}$$

is recovered at high temperatures. This model has four possible transfer matrices [41,42],

$$T_{(J^{[1]},J^{[2]})} \equiv \begin{bmatrix} e^{\beta(J^{[1]}+J^{[2]})} & e^{\beta(J^{[1]}-J^{[2]})} & 0 & 0 \\ 0 & 0 & e^{\beta(-J^{[1]}+J^{[2]})} & e^{\beta(-J^{[1]}-J^{[2]})} \\ e^{\beta(-J^{[1]}-J^{[2]})} & e^{\beta(-J^{[1]}+J^{[2]})} & 0 & 0 \\ 0 & 0 & e^{\beta(J^{[1]}-J^{[2]})} & e^{\beta(J^{[1]}+J^{[2]})} \end{bmatrix}, \tag{9}$$

that occur with equal probability, $1/4$. The Lyapunov exponent is here again related to the free-energy density through Gelfand's formula. The model, however, cannot be mapped to a solvable nondisordered model because of the frustration generically introduced by conflicting NN and NNN couplings.

C. Generalized nearest-neighbor models

The generalization of these models to $N_{\rm n}$ nearest neighbors,

$$H = -\sum_i \left(\sum_{l=1}^{N_{\rm n}} J_i^{[l]} S_i S_{i+l} \right), \tag{10}$$

with $J_i^{[l]} = \pm\frac{J}{\sqrt{N_{\rm n}}}$, results in $2^{N_{\rm n}}$ equally probable $2^{N_{\rm n}}$-by-$2^{N_{\rm n}}$ transfer matrices with elements

$$\left[T_{(J^{[1]},\ldots,J^{[N_{\rm n}]})}\right]_{(\sigma_1,\ldots,\sigma_{N_{\rm n}}),(\sigma_2',\ldots,\sigma_{N_{\rm n}+1}')} \equiv \delta_{\sigma_2,\sigma_2'} \cdots \delta_{\sigma_{N_{\rm n}},\sigma_{N_{\rm n}}'} \exp\left[\beta \sum_{l=1}^{N_{\rm n}} J^{[l]} \sigma_1 \sigma_{1+l}' \right], \tag{11}$$

where the dummy spin variables $\sigma_k = \pm 1$ and $\sigma_k' = \pm 1$ span the $2^{N_{\rm n}}$-dimensional vector space. Note that $N_{\rm n} = 1$ recovers the NN model and $N_{\rm n} = 2$ the NNN model, while the limit $N_{\rm n} \to \infty$ corresponds to the Sherrington-Kirkpatrick model with $\pm J/\sqrt{N}$ disorder (which, unlike the model with the canonical Gaussian form [43], has not been solved in the literature). This model therefore offers yet another

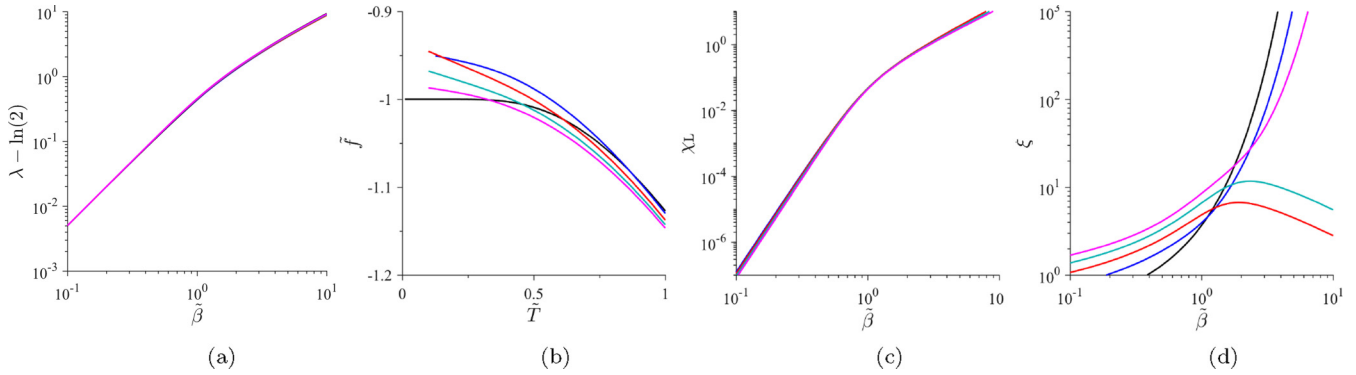

FIG. 1. Temperature dependence of the Lyapunov observables obtained by Monte Carlo simulations for $N_n = 2$ (blue), 3 (red), 4 (cyan), and 5 (magenta), along with the exact results for $N_n = 1$ (black). (a) Subtracting the high-temperature limit, ln(2), from λ shows it to scale as $\tilde{\beta}^2$ for $\tilde{\beta} \ll 1$ and as $\tilde{\beta}$ for $\tilde{\beta} \gg 1$. (b) The normalized free-energy density $\tilde{f} \equiv f/J$ as a function of the normalized temperature $\tilde{T} \equiv T/J$. For $N_n \geqslant 3$, the normalized free-energy monotonically varies as the interaction range increases. It is expected to asymptote to the mean-field Sherrington-Kirkpatrick limit with $\pm J/\sqrt{N}$ disorder distribution, but such a disordered model has not yet been solved. (c) The Lyapunov susceptibility, χ_L, also shows two regimes. Its growth as $\tilde{\beta}^2$ for $\tilde{\beta} \gg 1$ reveals the finiteness of $\lim_{N\to\infty} N(\mathbb{E}[f_N^2] - \mathbb{E}[f_N]^2)$, which governs sample-to-sample fluctuations in the thermal free energy. (d) The correlation length, ξ, diverges exponentially toward $T = 0$ for $N_n = 1$, 2, and 5, but is maximal at $\tilde{T} \sim 0.5$ for $N_n = 3$ and 4, thus demonstrating the nonuniversality of the appropriate order parameter across models of disordered spin chains. Note that for all the temperatures considered, $\xi < N_{equil} = 0.1N$. Note also that the error on the various quantities is of order $\sqrt{\chi_L/(N_D N)}$, and is thus much smaller than the width of the lines in this figure.

way of interpolating between finite- and infinite-dimensional systems [44–48].

III. MONTE CARLO SIMULATIONS

The Lyapunov exponent [Eq. (1)] and its susceptibility [Eq. (3)] for the above models can be directly evaluated by computing the largest eigenvalue for the product of each sequence of matrices. Because such computation for a large number of matrices results in numerical inaccuracy, any reasonable implementation cannot apply this scheme directly, but instead keeps track of the growth rate of the vector magnitude [15,16,49]. More specifically, we here randomly pick an initial normalized 2^{N_n}-dimensional vector, $\mathbf{v}(i = 1)$, evaluate the magnification factor $m_i^{(\alpha)} = \|T_i^{(\alpha)}\mathbf{v}(i)\|$ after each transfer matrix multiplication, and then define a new normalized vector, $\mathbf{v}(i+1) \equiv T_i^{(\alpha)}\mathbf{v}(i)/m_i^{(\alpha)}$. In order to lose memory of the arbitrarily chosen initial vector, the first N_{equi} equilibration steps are discarded; hence, the estimate for the sample Lyapunov exponent is

$$\lambda_N^{(\alpha)} = \frac{1}{N} \sum_{i=N_{equi}+1}^{N+N_{equi}} \ln\left(m_i^{(\alpha)}\right). \tag{12}$$

Averaging over N_D samples provides an estimate of the Lyapunov exponent, while computing the sample variance yields the Lyapunov susceptibility upon proper normalization with N. This scheme can be further generalized to extract the second-largest eigenvalue of the product of random matrices, λ_{sub}, through Gram-Schmidt orthogonalization [16]. This second eigenvalue encodes the correlation length (or correlation time from a dynamical viewpoint), $\xi \equiv 1/(\lambda - \lambda_{sub})$. We here obtain results with $N_{equi} = 10^5$, $N = 10^6$, and $N_D = 10^5$. In particular, the equilibration time N_{equi} is chosen to be much longer than the correlation length/time at all temperatures considered.

Results for λ and χ_L are given in Figs. 1(a) and 1(c), respectively, for $N_n = 1, \ldots, 5$. They all show the same qualitative trend. At high temperatures, the Lyapunov exponent is well described by Eq. (8), with $\lambda(T \to \infty) = \ln(2)$, the entropy density of noninteracting spins; at low temperatures, $\lambda \sim \tilde{\beta}$, which is consistent with the free energy approaching a constant at $T = 0$ [Fig. 1(b)]. In that same limit, the susceptibility scales as $\chi_L \sim \tilde{\beta}^2$, which suggests that sample-to-sample fluctuations in the free-energy density are $O(N^{-1/2})$. Hence, with our choice of N and N_D, the estimates of the Lyapunov exponent have an accuracy roughly of one part in 100 000, which is much smaller than the thickness of the lines in Fig. 1.

The correlation length, ξ, is reported in Fig. 1(d). For $N_n = 2$ and $N_n = 5$, the length grows exponentially toward $T = 0$, just as in the pure Ising model with $N_n = 1$. For $N_n = 3$ and 4, by contrast, ξ initially grows upon cooling but then decays, reaching a maximum around $T/J \sim 0.5$. This result may seem surprising at first, but in fact reflects the subtlety of defining order parameters. The relevant quantity depends on the details of the microscopic interactions and is thus nonuniversal across models [50–53]. As an illustration, consider the Sherrington-Kirkpatrick $N_n \to \infty$ limit, for which ordering is of a completely different (amorphous) nature. In order to capture amorphous ordering at low temperatures, correlation functions have to be appropriately modified. It should therefore not be surprising that $\xi = 1/(\lambda - \lambda_{sub})$ associated with a particular N_n-spin correlation function does not exhibit a low-temperature divergence for some of the models intermediate between $N_n = 1$ and $N_n = \infty$.

IV. REPLICA TRICK

In this section, we first review the use of the replica trick to average over disorder, and then develop the cycle-expansion method. We show below how the latter is closely related to the former, but surmounts some of its implementation difficulties.

A. Replica trick

For systems with quenched disorder, the Lyapunov exponent in Eq. (1) involves averaging over a logarithm, which is often analytically intractable. The replica trick sidesteps this problem by looking instead at [13,54,55]

$$L(n) \equiv \lim_{N\to\infty} L_N(n) \equiv \lim_{N\to\infty} \frac{1}{N} \ln\{\mathbb{E}[e^{nN\lambda_N}]\}$$
$$= \lim_{N\to\infty} \frac{1}{N} \ln\left\{\mathbb{E}\left[\left\|\prod_{i=1}^{N} T_i\right\|^n\right]\right\}, \tag{13}$$

which, for integer n, can be regarded as the logarithm of the average over n replicated samples [56–58]. This quantity is both analytically and computationally more tractable. The Lyapunov exponent can then be obtained as

$$\left.\frac{dL(n)}{dn}\right|_{n=0} = \lim_{n\to 0}\lim_{N\to\infty} \frac{dL_N(n)}{dn}$$
$$= \lim_{n\to 0}\lim_{N\to\infty} \frac{\mathbb{E}[e^{nN\lambda_N}\lambda_N]}{\mathbb{E}[e^{nN\lambda_N}]}$$
$$= \lim_{N\to\infty} \mathbb{E}[\lambda_N] = \lambda, \tag{14}$$

assuming that the order of the limits over N and n can be swapped, which is not (yet) a mathematically rigorous step [59]. It is possible, however, to rigorously establish that both limits exist and that $L'(0) \geqslant \lambda$ (see Appendix A). A similar computation and set of assumptions yield the Lyapunov susceptibility [60]

$$\chi_{\mathrm{L}} = \left.\frac{d^2L(n)}{dn^2}\right|_{n=0}. \tag{15}$$

We next assume that the generalized Lyapunov exponent $L(n)$ is an analytic function for $n \in [0,\infty)$. Although once again not rigorous, this hypothesis is physically reasonable. No transition—including a replica-symmetry-breaking transition—can indeed occur at finite temperature in one-dimensional systems with short-range interactions [61].

Based on these results and assumptions, one might expect the Lyapunov exponent and susceptibility to be obtained by extrapolating the slope and curvature, respectively, of $L(n)$ computed at positive integer n to $n = 0$ [57]. Specifically, the assumed analyticity permits a Taylor expansion,

$$L(n) = L(0) + L'(0)n + \tfrac{1}{2}L''(0)n^2 + \tfrac{1}{6}L'''(0)n^3 + \cdots, \tag{16}$$

with $L(0) = 0$ (by definition), $L'(0) = \lambda$, and $L''(0) = \chi_{\mathrm{L}}$. Figure 2, however, makes clear the technical difficulty of such an extrapolation. As discussed in Sec. III, models with $N_{\mathrm{n}} \geqslant 2$ in the low-temperature regime, $\tilde{\beta} \gg 1$, have $\chi_L \propto \tilde{\beta}^2$, while $\lambda \propto \tilde{\beta}$. As a result, $\frac{L(n)}{n}$ dips quickly as n approaches the origin. This rapid curbing prevents the reliable extrapolation of the intercept from function evaluations at positive integer n, even if these evaluations are obtained with machine precision and elaborate extrapolation schemes, such as Padé approximants, are used. In practice, at low temperatures such a scheme simply fails.

In passing, we note that the replica trick can also be used to recover an exact integral equation for the Lyapunov exponent

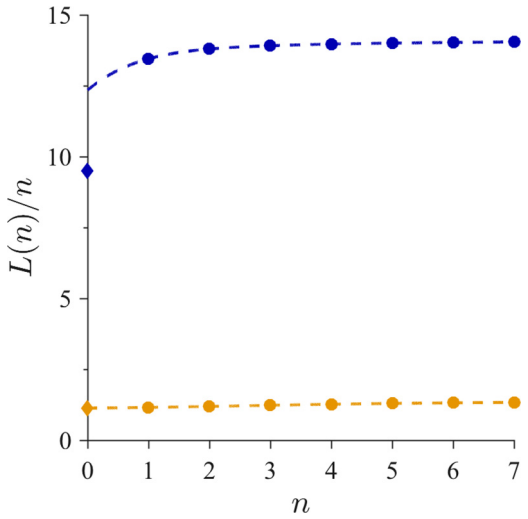

FIG. 2. Generalized Lyapunov exponents evaluated at integer values of n for the NNN model at $\tilde{\beta} = 1$ (orange) and $\tilde{\beta} = 10$ (navy blue). The diamonds at $n = 0$ denote the values for $\lim_{n\to 0} \frac{L(n)}{n} = \lambda$ obtained by Monte Carlo simulations. The values of the generalized Lyapunov exponent are obtained through the tensorial replication technique described in Refs. [12,57] for $n = 1,\ldots,7$. A sextic polynomial fit (dashed lines) gets more or less the correct result for $\tilde{\beta} = 1$, but fails dramatically for $\tilde{\beta} = 10$.

[22,23]. In order to attain the accuracy of order ϵ through such a scheme, however, the computational cost scales as $(1/\epsilon)^{m-1}$ for m-by-m transfer matrices due to the need for discretizing the interval of length m into steps of size ϵ. Thus, for $m > 3$ (i.e., for $N_{\mathrm{n}} > 1$) this approach quickly becomes outperformed by the Monte Carlo algorithm, which has a computational cost that scales as $(1/\epsilon)^2$.

B. Cycle expansion

In order to avoid the numerical challenge of a direct replica extrapolation, we instead consider the cycle-expansion method [21,38,39]. This computational scheme begins by constructing the Ruelle dynamical ζ function [62,63],

$$\zeta^{-1}(z,n) \equiv \exp\left\{-\sum_{N=1}^{\infty} \frac{1}{N}[ze^{L_N(n)}]^N\right\}, \tag{17}$$

which can be evaluated systematically by cycle expansion. Recall that $L_N(0) = 0$; hence, for $n = 0$ the series that appears in the argument of the exponential can be explicitly summed to yield $\zeta^{-1}(z,0) = 1 - z$, which has a zero at $z = 1$. In general, given the thermodynamic limit $\lim_{N\to\infty} L_N(n) = L(n)$,

$$\zeta^{-1}(e^{-L(n)},n) = 0\,. \tag{18}$$

Differentiating the above relation with respect to n then yields [64]

$$\lambda = L'(0) = -\partial_n\zeta^{-1}(1,0) \quad \text{and}$$
$$\chi_{\mathrm{L}} = L''(0) = \lambda^2 - \partial_n^2\zeta^{-1}(1,0) + 2\lambda\partial_z\partial_n\zeta^{-1}(1,0).$$

Given the formal expression for the ζ function from the cycle expansion [as detailed in Appendix B, $P^\star$ denotes the set of pseudocycles $\mathcal{G}$, with the sign $\mathcal{M}(\mathcal{G})$, the probability $p(\mathcal{G})$, the

length $\ell(\mathcal{G})$, and the spectral radius $\rho(\mathcal{G})$, i.e., (the product of) the largest eigenvalue(s)],

$$\zeta^{-1}(z,n) = \sum_{\mathcal{G}\in P^\star} \mathcal{M}(\mathcal{G}) p(\mathcal{G}) z^{\ell(\mathcal{G})} e^{n \ln \rho(\mathcal{G})}, \tag{19}$$

we first recover the expression for the Lyapunov exponent [14,21],

$$\lambda = -\sum_{\mathcal{G}\in P^\star} \mathcal{M}(\mathcal{G}) p(\mathcal{G}) \ln \rho(\mathcal{G}), \tag{20}$$

and then obtain the Lyapunov susceptibility,

$$\chi_{\mathrm{L}} = \lambda^2 + \sum_{\mathcal{G}\in P^\star} \mathcal{M}(\mathcal{G}) p(\mathcal{G}) \ln \rho(\mathcal{G}) [2\lambda \ell(\mathcal{G}) - \ln \rho(\mathcal{G})]. \tag{21}$$

Higher moments of the distribution for λ_N can also be obtained by further differentiating with respect to n. For example, the third derivative is proportional to the skewness,

$$\begin{aligned} L'''(0) &= \lim_{N\to\infty} N^2 \{\mathbb{E}[(\lambda_N - \mathbb{E}[\lambda_N])^3]\} \\ &= -\lambda^3 + 3\chi_{\mathrm{L}}\lambda - \partial_n^3 \zeta^{-1}(1,0) + 3\lambda \partial_z \partial_n^2 \zeta^{-1}(1,0) \\ &\quad - 3\lambda^2 \partial_z^2 \partial_n \zeta^{-1}(1,0) + 3(\chi_{\mathrm{L}} - \lambda^2) \partial_z \partial_n \zeta^{-1}(1,0), \end{aligned} \tag{22}$$

and the fourth derivative is proportional to the kurtosis,

$$\begin{aligned} L''''(0) = \lim_{N\to\infty} N^3 \{&\mathbb{E}[(\lambda_N - \mathbb{E}[\lambda_N])^4] \\ &- 3\mathbb{E}[(\lambda_N - \mathbb{E}[\lambda_N])^2]^2\}. \end{aligned} \tag{23}$$

Because each differentiation brings down an overall factor of N, higher-order derivatives are associated with ever-refined information about the distribution.

We can further generalize the cycle expansion to glean information about the whole Lyapunov characteristic exponent spectrum and, in particular, about the second-largest eigenvalue that controls the correlation length. The derivation of the cycle-expansion expression for the ζ function indeed only depends on the positivity and cyclic nature of the weight. Hence, the above formulas also provide the magnitude of the subleading eigenvalues via a straightforward replacement of the spectral radius, $\rho(\mathcal{G})$, by the magnitude of the corresponding rank eigenvalues.

V. COMPARISON

In this section we contrast the strengths and weaknesses of the Monte Carlo treatment and of the cycle expansion, starting with their computational efficiencies. Generically, given N_{TM} transfer-transition matrices to draw from, the number of terms to be evaluated asymptotically grows as N_{TM}^k at the kth order of the cycle expansion. Hence, while the cycle-expansion method provides a computationally efficient method when N_{TM} is of order one, the attainable numerical accuracy quickly deteriorates with increasing N_{TM}, as previously noted (see, e.g., Ref. [17]). Although a careful comparison of computational costs depends on implementation details, we empirically find that the cycle-expansion method converges much faster than the Monte Carlo algorithm for $N_{\mathrm{TM}} = 2$, while its efficiency already lags for $N_{\mathrm{TM}} = 4$, at least as far as the Lyapunov exponent is concerned. For the models at hand, the computational cost of the cycle expansion can be curtailed by setting $J^{[1]} > 0$ through spin redefinitions, which reduces $N_{\mathrm{TM}} = 2^{N_{\mathrm{n}}}$ by a factor of two. With this trick, we have carried out the cycle expansion to order $k = 11$ for the NNN ($N_{\mathrm{n}} = 2$) and $N_{\mathrm{n}} = 3$ models, which suffices to recover Monte Carlo results within their accuracy (see Fig. 3). Comparable accuracy is, however, harder to achieve for $N_{\mathrm{n}} \geqslant 4$. As far as the attainable numerical accuracy of the Lyapunov exponent is concerned, the Monte Carlo algorithm thus almost always outperforms the cycle-expansion method.

A different balance is, however, reached for the Lyapunov susceptibility and higher moments. With a naive implementation of the Monte Carlo algorithm, it becomes increasingly challenging to assess quantities related to the large-deviation scaling, such as skewness and kurtosis (see, however, Ref. [17] for an efficient resampling method). By contrast, cycle expansions do not encounter such difficulty and

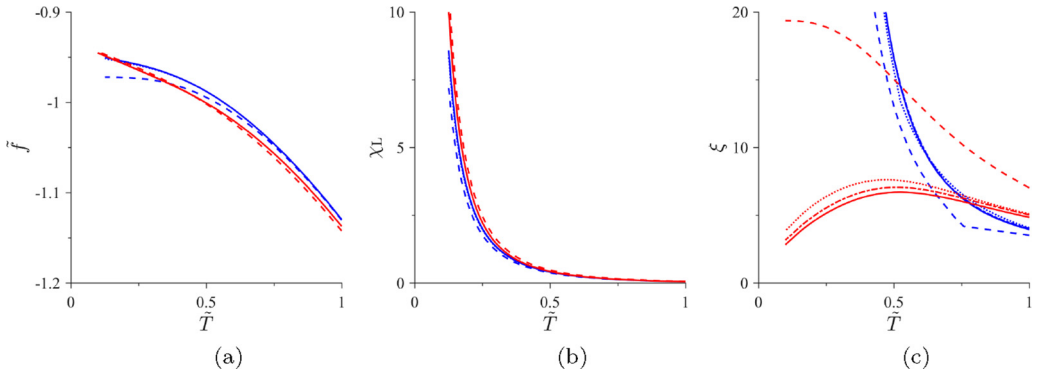

FIG. 3. Lyapunov observables obtained through the cycle-expansion method for $N_{\mathrm{n}} = 2$ (blue) and 3 (red) at order $k = 3$ (dashed), 7 (dotted), and 11 (dash-dotted), along with the Monte Carlo simulations results (solid). (a) The normalized free-energy $\tilde{f}$ as a function of the normalized temperature $\tilde{T}$. (b) The Lyapunov susceptibility, χ_{L}, as a function of $\tilde{T}$. (c) The correlation length, ξ, as a function of $\tilde{T}$. Convergence of the cycle expansion for the subleading eigenvalue is slower than for the leading eigenvalue, but nonetheless suffices to recover qualitative features including the nonmonotonic temperature evolution of $\xi(T)$ for $N_{\mathrm{n}} = 3$.

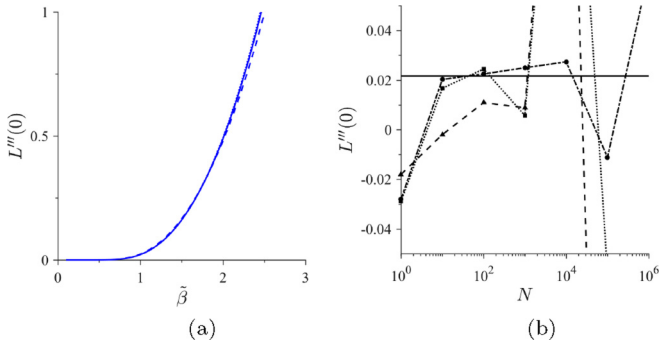

FIG. 4. (a) The skewness for the NNN model obtained by the cycle expansion at order $k = 3$ (dashed), 7 (dotted), 11 (dash-dotted), and 15 (solid) quickly converges with increasing k. (b) The skewness at $\tilde{\beta} = 1$ evaluated directly through Monte Carlo simulations as a function of the number of spins N with a fixed number of disorder realization $N_D = 10^1$ (dashed line with triangles), 10^3 (dotted line with squares), and 10^5 (dash-dotted line with circles) fluctuates around the cycle-expansion result for $k = 15$ (solid line). The skewness and higher moments are associated with fine features of the distribution that otherwise approaches a sharply peaked Gaussian distribution for large N, making their reliable estimates by simple Monte Carlo numerically challenging.

quickly converge (see Fig. 4). The cycle-expansion method thus offers a reliable computational tool for assessing higher moments pertaining to the large-order behavior, at least when the number of possible transfer matrices is small or when the symmetry of the problem reduces the computational cost associated with evaluating the spectral radius of cycles.

VI. CONCLUSION

We have developed the cycle-expansion method to compute observables pertaining to the distribution of Lyapunov exponents in systems with disorder. The cycle expansion, when its computation can be feasibly carried out to tenth order or so, reproduces the Lyapunov exponent and susceptibility results from Monte Carlo simulations, and yields far more accurate estimates of higher-order moments, such as the skewness. The derivation of these cycle-expansion expressions, however, crucially relies on the analyticity of the generalized Lyapunov exponent. While such analyticity appears physically reasonable in the absence of replica symmetry breaking, a formal proof is still lacking. It would also be interesting to develop a method that could capture the large-order behavior of higher-dimensional systems and systems with continuous distributions of quenched randomness, for which intricate dynamical effects, such as glassiness, are expected.

Data associated with this work are available from the Duke Digital Repository [65].

ACKNOWLEDGMENTS

We thank M. Lian, J. C. Mattingly, and F. Zamponi for stimulating discussions and suggestions. We also acknowledge the Duke Compute Cluster for computational times. This work was supported by a grant from the Simons Foundation (Grant No. 454937, Patrick Charbonneau).

APPENDIX A: PROPERTIES OF THE GENERALIZED LYAPUNOV EXPONENT

Though the results contained in this appendix are likely known to experts in the field, we here provide their succinct derivations.

In order to prove the existence of the limit defining the generalized Lyapunov exponent, we observe that

$$\begin{aligned}(N+M)L_{N+M}(n) &= \ln \mathbb{E}\left[\left\|\prod_{i=1}^{N+M} T_i\right\|^n\right]\\ &\leqslant \ln \mathbb{E}\left[\left(\left\|\prod_{i=1}^{N} T_i\right\|^n \left\|\prod_{i=N+1}^{N+M} T_i\right\|^n\right)\right]\\ &\leqslant \ln \mathbb{E}\left[\left\|\prod_{i=1}^{N} T_i\right\|^n\right] + \ln \mathbb{E}\left[\left\|\prod_{i=N+1}^{N+M} T_i\right\|^n\right]\\ &= NL_N(n) + ML_M(n).\end{aligned}$$

The last two steps hold because T_i is an independent and identically distributed sequence, although the argument can also be extended to handle random matrices generated by a finite-memory Markov process. The result shows that $NL_N(n)$ is subadditive in N and implies that

$$L(n) \equiv \lim_{N\to\infty} \frac{1}{N} L_N(n) = \inf_{N\geqslant 1} \frac{1}{N} L_N(n)$$

exists for $n \in [0,\infty)$. Similarly, the derivative

$$L'_N(n) = \frac{\mathbb{E}\left[\left\|\prod_{i=1}^N T_i\right\|^n \ln\left\|\prod_{i=1}^N T_i\right\|\right]}{\mathbb{E}\left[\left\|\prod_{i=1}^N T_i\right\|^n\right]}$$

exists for all $n \in [0,\infty)$ and $L'_N(0) = \mathbb{E}[\ln\|\prod_{i=1}^N T_i\|]$.

The convexity of $L_N(n)$ can be shown as

$$\begin{aligned}L_N[\alpha n_0 + (1-\alpha)n_1] &= \frac{1}{N}\ln \mathbb{E}\left[\left\|\prod_{i=1}^{N} T_i\right\|^{\alpha n_0+(1-\alpha)n_1}\right]\\ &\leqslant \frac{1}{N}\ln\left[\left(\mathbb{E}\left\|\prod_{i=1}^{N} T_i\right\|^{n_0}\right)^{\alpha}\left(\mathbb{E}\left\|\prod_{i=1}^{N} T_i\right\|^{n_1}\right)^{1-\alpha}\right]\\ &= \alpha L_N(n_0) + (1-\alpha)L_N(n_1),\end{aligned}$$

where Hölder's inequality is used in the second step. Because $L_N(n)$ is convex and differentiable, it follows that $L(n)$ is convex and $L'_N(n) \to L'(n)$ at all points where $L'(n)$ exists [66]. This further implies that

$$L'(0^+) \geqslant \lim_{N\to\infty} L'_N(0) = \lambda.$$

APPENDIX B: NOTATIONS FOR CYCLE-EXPANSION EXPRESSIONS

A product of ℓ transfer matrices, $G = T_{a_1}T_{a_2}\cdots T_{a_\ell}$, specifies a cycle of length $\ell(G)$. The probability of G appearing among all the cycles of the same length ℓ is denoted as $p(G)$, which in our models uniformly equals $(\frac{1}{2^{N_n}})^{\ell(G)}$. A cycle G is prime if there is no cycle G' of length $\ell(G') < \ell(G)$

with $G = (G')^{\ell(G)/\ell(G')}$ [21]. For example, $T_1T_2T_2T_1$ is prime, but $T_1T_2T_1T_2$ is not. Prime cycles are further grouped into equivalence classes, in which two products are identified if they are related by a cyclic permutation, such as $T_1T_2T_4T_5$ and $T_4T_5T_1T_2$. The set of all such equivalent classes is denoted P. A size-h subset, $\mathcal{G} = \{G_1, G_2, \ldots, G_h\}$, is known as a pseudocycle, where $G_\mu \in P$ for $\mu = 1, \ldots, h$ and $G_\mu \neq G_\nu$ for $\mu \neq \nu$ [14]. In particular, note that $\mathcal{G} = \{T_1T_2T_3, T_2T_3, T_1T_2\}$ is a pseudocycle, while $\{T_1T_2T_3, T_1T_2, T_1T_2\}$ is not because the element T_1T_2 is repeated. The set of all pseudocycles, which is the set of all subsets of equivalent classes of prime cycles, is denoted $P^\star$. Finally, various quantities are naturally defined as (i) the length $\ell(\mathcal{G}) = \sum_{\mu=1}^{h} \ell(G_\mu)$, (ii) the probability function $p(\mathcal{G}) = \prod_{\mu=1}^{h} p(G_\mu)$, (iii) the Möbius function $\mathcal{M}(\mathcal{G}) = (-1)^h$, and (iv) the weight $\rho(\mathcal{G}) = \prod_{\mu=1}^{h} \rho(G_\mu)$, where $\rho(G_\mu)$ is the spectral radius of the matrix G_μ.

Cycle expansions are truncated at kth order by summing over all the pseudocycles of length $\ell(\mathcal{G}) \leqslant k$, where the same maximum length k should be used in the cycle-expansion expressions of $\partial_z^{s_1}\partial_n^{s_2}\zeta^{-1}(1,0)$ for all s_1 and s_2. With this truncation scheme, dilatation symmetry is preserved. That is, uniformly multiplying transfer matrices by c, $T_a \to cT_a$, makes the Lyapunov exponent $\lambda \to \lambda + \ln(c)$, while the susceptibility χ_{L} and higher-moments remain invariant. To confirm this symmetry, it is useful to use the identity $\sum_{\mathcal{G}\in P^\star;\ell(\mathcal{G})=\ell_0} \mathcal{M}(\mathcal{G})p(\mathcal{G}) = \delta_{\ell_0,0} - \delta_{\ell_0,1}$ that follows from Eq. (19) evaluated at $n = 0$, where, in particular, $\zeta^{-1}(z,0) = 1 - z$.

[1] A. M. Lyapunov, The general problem of the stability of motion, Int. J. Control **55**, 531 (1992).

[2] A. Pikovsky and A. Politi, *Lyapunov Exponents: A Tool to Explore Complex Dynamics* (Cambridge University Press, Cambridge, UK, 2016).

[3] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part I, Phys. Rev. **60**, 252 (1941).

[4] D. Blackwell, The entropy of functions of finite-state Markov chains, in *Trans. First Prague Conf. Information Thoery, Statistical Decision Functions, Random Processes* (Czechoslovak Academy of Sciences, Prague, Czechoslovakia, 1957), pp. 13–20.

[5] Henry D. Pfister, On the capacity of finite state channels and the analysis of convolutional accumulate-*m* codes, Ph.D. thesis, University of California, San Diego, CA, 2003.

[6] T. Holliday, A. Goldsmith, and P. Glynn, Capacity of finite state channels based on Lyapunov exponents of random matrices, IEEE Trans. Inf. Theory **52**, 3509 (2006).

[7] P. Jacquet, G. Seroussi, and W. Szpankowski, On the entropy of a hidden Markov process, Theor. Comput. Sci. **395**, 203 (2008).

[8] P. Hayden and J. Preskill, Black holes as mirrors: quantum information in random subsystems, J. High Energy Phys. 09 (2007) 120.

[9] Y. Sekino and L. Susskind, Fast scramblers, J. High Energy Phys. 10 (2008) 065.

[10] J. Maldacena, S. H. Shenker, and D. Stanford, A bound on chaos, J. High Energy Phys. 08 (2016) 106.

[11] J. Kurchan, Quantum bound to chaos and the semiclassical limit, arXiv:1612.01278.

[12] A. Crisanti, G. Paladin, and A. Vulpiani, Products of random matrices, in *Statistical Physics* (Springer Science & Business Media, Berlin, 2012), Vol. 104.

[13] M. Mézard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).

[14] J. L. Nielsen, Lyapunov exponent for products of random matrices, in *Chaos: Classical and Quantum—Projects*, edited by P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, G. Vattay, N. Whelan, and A. Wirzba (1997) at http://chaosbook.org/projects/index.shtml, last accessed on 2017-06-27.

[15] G. Benettin, L. Galgani, and J. M. Strelcyn, Kolmogorov entropy and numerical experiments, Phys. Rev. A **14**, 2338 (1976).

[16] G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, Lyapunov characteristic exponents for smooth dynamical systems and for Hamiltonian systems: A method for computing all of them. Part 1. Theory, Meccanica **15**, 9 (1980).

[17] J. Vanneste, Estimating generalized Lyapunov exponents for products of random matrices, Phys. Rev. E **81**, 036701 (2010).

[18] E. Gardner, C. J. Itzykson, and B. Derrida, The Laplacian on a random one-dimensional lattice, J. Phys. A: Math. Gen. **17**, 1093 (1984).

[19] B. Derrida and E. Gardner, Lyapounov exponent of the one dimensional Anderson model: Weak disorder expansions, J. Phys. (France) **45**, 1283 (1984).

[20] J. M. Deutsch and G. Paladin, Product of Random Matrices in a Microcanonical Ensemble, Phys. Rev. Lett. **62**, 695 (1989).

[21] R. Mainieri, Zeta Function for the Lyapunov Exponent of a Product of Random Matrices, Phys. Rev. Lett. **68**, 1965 (1992).

[22] B. Derrida and H. J. Hilhorst, Singular behaviour of certain infinite products of random 2×2 matrices, J. Phys. A: Math. Gen. **16**, 2641 (1983).

[23] M. Weigt and R. Monasson, Replica structure of one-dimensional disordered Ising models, Europhys. Lett. **36**, 209 (1996).

[24] G. Paladin and M. Serva, Analytic Solution of the Random Ising Model in One Dimension, Phys. Rev. Lett. **69**, 706 (1992).

[25] P. S. Davids, Analytic structure of the one-dimensional random-bond Ising model, J. Phys. A: Math. Gen. **27**, 6703 (1994).

[26] Z.-Q. Bai, On the cycle expansion for the Lyapunov exponent of a product of random matrices, J. Phys. A: Math. Theor. **40**, 8315 (2007).

[27] Z.-Q. Bai, An infinite transfer matrix approach to the product of random 2×2 positive matrices, J. Phys. A: Math. Theor. **42**, 015003 (2009).

[28] H. Fujisaka, Statistical dynamics generated by fluctuations of local Lyapunov exponents, Prog. Theor. Phys. **70**, 1264 (1983).

[29] H. Ishitani, A central limit theorem for the subadditive process and its application to products of random matrices, Publ. Res. Inst. Math. Sci. **12**, 565 (1977).

[30] M. Aizenman, J. L. Lebowitz, and D. Ruelle, Some rigorous results on the Sherrington-Kirkpatrick spin glass model, Commun. Math. Phys. **112**, 3 (1987).

[31] J. Baik and J. O. Lee, Fluctuations of the free energy of the spherical Sherrington-Kirkpatrick model, J. Stat. Phys. **165**, 185 (2016).

[32] A. J. Bray and M. A. Moore, Chaotic Nature of the Spin-Glass Phase, Phys. Rev. Lett. **58**, 57 (1987).
[33] J.-P. Bouchaud, F. Krzakala, and O. C. Martin, Energy exponents and corrections to scaling in Ising spin glasses, Phys. Rev. B **68**, 224404 (2003).
[34] T. Aspelmeier, Free-Energy Fluctuations and Chaos in the Sherrington-Kirkpatrick Model, Phys. Rev. Lett. **100**, 117205 (2008).
[35] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, Characterisation of intermittency in chaotic systems, J. Phys. A: Math. Gen. **18**, 2157 (1985).
[36] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, UK, 1995).
[37] F. Cecconi, M. Cencini, A. Puglisi, D. Vergni, and A. Vulpiani, From the law of large numbers to large deviation theory in statistical physics: An introduction, in *Large Deviations in Physics: The Legacy of the Law of Large Numbers*, edited by A. Vulpiani, F. Cecconi, M. Cencini, A. Puglisi, and D. Vergni (Springer, Berlin, Heidelberg, 2014), pp. 1–27.
[38] P. Cvitanović, Invariant Measurement of Strange Sets in Terms of Cycles, Phys. Rev. Lett. **61**, 2729 (1988).
[39] R. Artuso, E. Aurell, and P. Cvitanović, Recycling of strange sets. I. Cycle expansions, Nonlinearity **3**, 325 (1990).
[40] I. Gelfand, Zur Theorie der Charaktere der Abelschen topologischen Gruppen, Rec. Math. [Mat. Sbornik] N.S. **9**, 49 (1941).
[41] W. Selke and M. E. Fisher, Monte Carlo study of the spatially modulated phase in an Ising model, Phys. Rev. B **20**, 257 (1979).
[42] The theorem in Ref. [29] stipulates that the entries of transfer matrices must all be positive while each transfer matrix in our generalized nearest-neighbor models has zeros for $N_n \geqslant 2$. In order to apply the theorem, we can instead look at products of N_n such matrices, which have all positive entries and are also independently and identically distributed.
[43] D. Sherrington and S. Kirkpatrick, Solvable Model of A Spin-Glass, Phys. Rev. Lett. **35**, 1792 (1975).
[44] G. Kotliar, P. W. Anderson, and D. L. Stein, One-dimensional spin-glass model with long-range random interactions, Phys. Rev. B **27**, 602(R) (1983).
[45] T. Aspelmeier, W. Wang, M. A. Moore, and H. G. Katzgraber, Interface free-energy exponent in the one-dimensional Ising spin glass with long-range interactions in both the droplet and broken replica symmetry regions, Phys. Rev. E **94**, 022116 (2016).
[46] S. Franz, T. Jörg, and G. Parisi, Overlap interfaces in hierarchical spin-glass models, J. Stat. Mech. (2009) P02002.
[47] M. Castellana and G. Parisi, Renormalization group computation of the critical exponents of hierarchical spin glasses, Phys. Rev. E **82**, 040105(R) (2010).
[48] P. Charbonneau and S. Yaida, Nontrivial Critical Fixed Point for Replica-Symmetry-Breaking Transitions, Phys. Rev. Lett. **118**, 215701 (2017).
[49] Ch. Skokos, The Lyapunov characteristic exponents and their computation, Lect. Notes Phys. **790**, 63 (2010).
[50] R. J. Baxter, Eight-Vertex Model in Lattice Statistics, Phys. Rev. Lett. **26**, 832 (1971).
[51] R. J. Baxter and F. Y. Wu, Ising model on a triangular lattice with three-spin interactions. I. The eigenvalue equation, Aust. J. Phys. **27**, 357 (1974).
[52] S. Krinsky and D. Mukamel, Ising models with $n >\sim 2$-component order parameters, Phys. Rev. B **16**, 2313 (1977).
[53] M. N. Barber, Non-universality in the Ising model with nearest and next-nearest neighbor interactions, J. Phys. A: Math. Gen. **12**, 679 (1979).
[54] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge University Press, Cambridge, UK, 1934).
[55] S. F. Edwards and P. W. Anderson, Theory of spin glasses, J. Phys. F **5**, 965 (1975).
[56] J. B. Pendry, 1D localisation and the symmetric group, J. Phys. C **15**, 4821 (1982).
[57] J.-P. Bouchard, A. Georges, D. Hansel, P. Le Doussal, and J. M. Maillard, Rigorous bounds and the replica method for products of random matrices, J. Phys. A: Math. Gen. **19**, L1145 (1986).
[58] M. J. de Oliveira and A. Petri, Generalized Lyapunov exponents for products of correlated random matrices, Phys. Rev. E **53**, 2960 (1996).
[59] J. L. van Hemmen and R. G. Palmer, The replica method and solvable spin glass model, J. Phys. A: Math. Gen. **12**, 563 (1979).
[60] Note that in the dynamical system literature the function $L(n)$ is known as the generalized Lyapunov exponent [28,35], which is related through a Legendre transform to the Cramér function governing the large-deviation behavior of the Lyapunov exponent λ_N [37].
[61] D. Ruelle, Analycity properties of the characteristic exponents of random matrix products, Adv. Math. **32**, 68 (1979).
[62] D. Ruelle, *Thermodynamic Formalism* (Addison-Wesley, Boston, MA, 1978).
[63] D. Ruelle, Dynamical zeta functions and transfer operators, Not. Am. Math. Soc. **49**, 887 (2002).
[64] A similar expression for the Lyapunov susceptibility has been obtained within the evolution-operator formalism in Ref. [27].
[65] Duke Digital Repository at https://doi.org/10.7924/G88050N6.
[66] Ivan B. Lacković, On the behaviour of sequences of left and right derivatives of a convergent sequence of convex functions, Publ. Elektroteh. Fak., Univ. Beogradu, Ser.: Mat. Fiz. **735–762**, 19 (1982).