Recurrence relations in one-dimensional Ising models

C. M. Silva da Conceição
Universidade Federal Fluminense, RHS/RCN, 28895-532 Rio das Ostras, Rio de Janeiro, Brazil

R. N. P. Maia
Universidade Federal do Rio de Janeiro, Campus Macaé, 27930-560 Macaé, Rio de Janeiro, Brazil

(Received 30 October 2016; published 13 September 2017)

The exact finite-size partition function for the nonhomogeneous one-dimensional (1D) Ising model is found through an approach using algebra operators. Specifically, in this paper we show that the partition function can be computed through a trace from a linear second-order recurrence relation with nonconstant coefficients in matrix form. A relation between the finite-size partition function and the generalized Lucas polynomials is found for the simple homogeneous model, thus establishing a recursive formula for the partition function. This is an important property and it might indicate the possible existence of recurrence relations in higher-dimensional Ising models. Moreover, assuming quenched disorder for the interactions within the model, the quenched averaged magnetic susceptibility displays a nontrivial behavior due to changes in the ferromagnetic concentration probability.

DOI: 10.1103/PhysRevE.96.032121

I. INTRODUCTION

In the phase transition context, cooperative phenomena are not dependent on details of the intermolecular forces that define microscopic dynamics of the system itself but rather depend on the way the mechanism of propagation of the long-range order occurs. Thus, the Ising model is presented as an important simple model that captures the essence of several cooperative phenomena, thus establishing itself as a tool to understand various aspects of the emergent properties that are shared by numerous macroscopic phenomena to undergo phase transitions. Although Ising model in two and three dimensions are most relevant for describing real systems, the one-dimensional model with random bond might be a first attempt to understand the magnetic properties in quasi-one-dimensional (1D) behavior of some materials [1–4].

It is noteworthy that no magnetic phase transition at finite temperature is possible for the simplest spin $1/2$ chain with nearest-neighbor exchange uniform interaction, i.e., the homogeneous 1D Ising model. This claim follows from Landau's argument, but it also follows from Perron's theorem [5]. Meanwhile, much effort has been done to clarify phase transition phenomena in magnetic systems. Among these works the plane square lattice partition function and magnetic properties were obtained using the standard transfer matrix method. From these works there are results that shows a phase transition in the thermodynamic limit at a well-defined finite temperature [6] for planar lattices with different geometry and uniform exchange interaction.

The existence of linear recurrence relations of second order for the one-dimensional Ising model seems not to have been previously reported. There is an underlying polynomial structure behind the finite-size homogeneous 1D Ising model, obtained when we are looking for a recursive formula for the system's partition function. It is worth noting that the use of a recursion method with a vanishing external magnetic field is not a new idea, since the 1D nonhomogeneous Ising model can be treated with a linear first-order recurrence relation [5]. We show in this paper the existence of a polynomial closed-form solution for the homogeneous model with nonvanishing magnetic field. This solution can be expressed in terms of a polynomial series called generalized Lucas polynomials in the mathematics literature, obtained from the Lucas sequence, which resembles the Fibonacci one [7,8]. We show also how to obtain a recurrence relation for the nonhomogeneous model with nonvanishing magnetic field in matrix form which can be used in order to obtain the thermal magnetic response.

Perfect crystals and uniform magnetic materials usually have spatial symmetries that simplify the theoretical analysis. However, for all real systems that have a certain degree of impurities, the existing symmetries are destroyed, leading to emergence of new symmetries or a complete lack of it at the microscopic level [9]. The quenched average is the proper way to treat systems of this kind, and it consists of an average over all nonhomogeneous parameters in the free energy, i.e., in the logarithm of the partition function [9,10]. Therefore one first needs to perform the thermal average and find the partition function of a non-homogeneous system and then perform another average over the configurational space, which in the nonhomogeneous Ising model involves unequal exchange couplings. There are some works where the replica trick is employed [11,12]. There are some results [13,14] for the one-dimensional random bond Ising model with nearest-neighbor interaction where the authors obtained the energy, entropy, and magnetization in the low-temperature limit with nonvanishing field. Remarkably, they also obtain the magnetic susceptibility for the ground state [13].

The exact quenched average of the magnetic susceptibility is computed in this paper from the quenched averaged free energy for a specific probability distribution in the exchange coupling. We remark that the quenched disorder can introduce a nontrivial behavior of the magnetic susceptibility in a one-dimensional Ising model with short-range interactions. This unusual response is not due to temperature changes but arises from changes in ferromagnetic or antiferromagnetic probability of each bond, with this probability being the control parameter.

Initially, we consider the system and the spin algebra approach in Sec. II, and we show how to obtain recurrence relations for the nonhomogeneous and homogeneous Ising model in Sec. III. Then we analyze a spin chain with quenched disorder in Sec. IV, with some remarks to conclude this paper in Sec. V.

II. THE SPIN OPERATOR ALGEBRA APPROACH

We study a 1D spin-1/2 chain with nearest-neighbor Ising bonds in the presence of an externally applied magnetic field described by the Hamiltonian

$$\hat{H}_N = -\sum_{n=0}^{N-1} J_n \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z - h \sum_{n=0}^{N-1} \hat{\sigma}_n^z ,$$

where $\hat{\sigma}_n^z$ is the z component of the spin operator at site n of the chain and periodic boundary conditions are assumed, i.e., $\hat{\sigma}_N^z = \hat{\sigma}_0^z$. Note that we have used a compact notation $\hat{\sigma}_n \hat{\sigma}_{n+1} \equiv \cdots \hat{1}_{n-1} \otimes \hat{\sigma}_n \otimes \hat{\sigma}_{n+1} \otimes \hat{1}_{n+2} \cdots$ in a Hilbert space of size 2^N since each operator acts in distinct Hilbert subspaces (we also adopt $\hat{\sigma}_n \equiv \cdots \hat{1}_{n-1} \otimes \hat{\sigma}_n \otimes \hat{1}_{n+1} \cdots$). In nonhomogeneous magnetic systems, different exchange interactions must be considered: spin-spin coupling can be ferromagnetic (antiferromagnetic) between sites n and $n+1$ for $J_n > 0$ ($J_n < 0$) with $2|J_n|$ representing the energy cost to break a spin pair bond. In the first case, an aligned spin pair is more likely to occur because in this configuration the energy is minimized, and in the second case an antiparallel pair is favored for the same reason. The external magnetic field h breaks up-down symmetry since each spin also tends to align itself with the field direction, where the spin-field bond energy equals $2h$.

The method is based on spin-1/2 operators algebra that uses as main ingredients the following: (i) all the spin operators commute with each other, $[\hat{\sigma}_n^z, \hat{\sigma}_m^z] = 0\,(0 \leqslant n,m < N-1)$; (ii) all of them are idempotent, $[\hat{\sigma}_n^z]^2 = \hat{1}_n\,(0 \leqslant n < N-1)$; (iii) all of these operators have null trace, $\mathrm{Tr}\,\hat{\sigma}_n = 0\,(0 \leqslant n < N-1)$; and (iv) all traces of any tensor product which contains at least one odd power of a spin operator have null trace because the trace can be calculated in each Hilbert subspace separately. For instance, with this last property we mean $\mathrm{Tr}[\hat{\sigma}_n \otimes \hat{\sigma}_m] = (\mathrm{Tr}\,\hat{\sigma}_n)(\mathrm{Tr}\,\hat{\sigma}_m) = 0\,(0 \leqslant n,m < N-1)$.

The partition function will be calculated through the formal operatorial expansion of the exponential appearing in the canonical ensemble:

$$Z_N(\{K_n\},B) = \mathrm{Tr}\left[\prod_{n=0}^{N-1} e^{K_n \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z} e^{B \hat{\sigma}_n^z}\right], \tag{1}$$

where we have defined the dimensionless parameters $K_n = \beta J_n$ and $B = \beta h$. The property (i) states that all spin operators at different sites commute, which allows us to break the exponential in a product of exponentials indexed by the site location n. From the previous property (ii) it is straightforward to verify that every exponential $e^{K_n \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z}$ and $e^{B \hat{\sigma}_n^z}$ on the right-hand side in the above equation (1) can be rewritten as a linear combination with two operators

$$e^{K_n \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z} = \cosh K_n\, \hat{1}_n \hat{1}_{n+1} + \sinh K_n\, \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z, \tag{2}$$

$$e^{B \hat{\sigma}_n^z} = \cosh B\, \hat{1}_n \hat{1}_{n+1} + \sinh B\, \hat{\sigma}_n^z \hat{1}_{n+1}, \tag{3}$$

where $\hat{1}_n \hat{1}_{n+1} = \cdots \hat{1}_{n-1} \otimes \hat{1}_n \otimes \hat{1}_{n+1} \otimes \hat{1}_{n+2} \cdots$ stands for the identity operator in the full Hilbert space. In this way the result can be obtained if we calculate the trace in Eq. (1):

$$Z_N(\{K_n\},B) = \mathrm{Tr}\Bigg[\prod_{n=0}^{N-1} \left(\cosh K_n\, \hat{1}_n \hat{1}_{n+1} + \sinh K_n\, \hat{\sigma}_n^z \hat{\sigma}_{n+1}^z\right) \times \left(\cosh B\, \hat{1}_n \hat{1}_{n+1} + \sinh B\, \hat{\sigma}_n^z \hat{1}_{n+1}\right)\Bigg]. \tag{4}$$

It is not an easy task to perform the algebric calculation explicitly for an arbitrary large chain size N. However, it suffices to do only the smaller sizes ($N = 2,3,4,5$). We show in Appendix A expressions for the nonhomogeneous Ising model partition function up to $N = 4$ using the previous properties (iii) and (iv) in Eq. (4). These results can be generalized, as will be show in the next section, and can be verified by direct and detailed inspection.

III. RECURRENCE RELATIONS

A. The nonhomogeneous Ising model

The partition function for the nonhomogeneous Ising model can be extracted from a compact expression and generalized for any spins system of arbitrary size N. Introducing a suitable rank 2 matrix,

$$\mathbb{A}_n(\{K_n\},B) = 2\begin{pmatrix} \alpha\, a_n & \omega\, b_n \\ \omega\, a_n & \alpha\, b_n \end{pmatrix}, \tag{5}$$

where we have defined $a_n = \cosh K_n$, $b_n = \sinh K_n$, $\alpha = \cosh B$, and $\omega = \sinh B$, the exact expression for the partition function for the 1D nonhomogeneous Ising model is given by

$$Z_N = \frac{1}{2}\,\mathrm{Tr}\,\mathbb{W}_N, \quad \mathbb{W}_N = \prod_{n=0}^{N-1} \mathbb{A}_n + \prod_{n=0}^{N-1} \bar{\mathbb{A}}_n, \tag{6}$$

where we have used another matrix obtained by a magnetic field sign change $\bar{\mathbb{A}}_n(\{K_n\},B) = \mathbb{A}_n(\{K_n\},-B)$.

It is possible to establish an alternative approach based in recurrence relation for the matrix $\mathbb{W}_N$, so that this 2×2 matrix depends only on the previous matrices $\mathbb{W}_{N-1}$ and $\mathbb{W}_{N-2}$. The mathematical structure is a nonobvious one because it is a second-order recurrence relation with nonconstant coefficients in matrix form. To show this claim, let us make the following definitions:

$$\mathbb{W}_N = \mathbb{X}_N + \mathbb{Y}_N, \tag{7}$$

where

$$\mathbb{X}_N = \prod_{n=0}^{N-1} \mathbb{A}_n \quad \text{and} \quad \mathbb{Y}_N = \prod_{n=0}^{N-1} \bar{\mathbb{A}}_n.$$

Using Eq. (6), we obtain

$$\mathbb{W}_{N+1} = \mathbb{X}_N \mathbb{A}_N + \mathbb{Y}_N \bar{\mathbb{A}}_N \tag{8}$$

and

$$\mathbb{W}_{N+2} = \mathbb{X}_N \mathbb{A}_N \mathbb{A}_{N+1} + \mathbb{Y}_N \bar{\mathbb{A}}_N \bar{\mathbb{A}}_{N+1}. \tag{9}$$

In order to establish a recurrence relation, let us assume a solution of the type

$$\mathbb{W}_{N+2} = \mathbb{W}_{N+1}\mathbb{P}_{N+1} + \mathbb{W}_N\mathbb{Q}_N \qquad (N \geqslant 0). \tag{10}$$

Once such procedure is employed, it is sufficient to find the values for $\mathbb{P}_{N+1}$ and $\mathbb{Q}_N$. This results in two matrix equations

$$\mathbb{A}_N\mathbb{A}_{N+1} = \mathbb{A}_N\mathbb{P}_{N+1} + \mathbb{Q}_N, \tag{11}$$

$$\bar{\mathbb{A}}_N\bar{\mathbb{A}}_{N+1} = \bar{\mathbb{A}}_N\mathbb{P}_{N+1} + \mathbb{Q}_N, \tag{12}$$

with the following solution:

$$\mathbb{P}_{N+1} = 2\alpha\frac{(a_N + b_N)}{a_N b_N}\begin{pmatrix} b_N a_{N+1} & 0 \\ 0 & a_N b_{N+1} \end{pmatrix} \tag{13}$$

and

$$\mathbb{Q}_N = -4\begin{pmatrix} b_N a_{N+1} & 0 \\ 0 & a_N b_{N+1} \end{pmatrix}. \tag{14}$$

In order to obtain the partition function, it is necessary to choose the suitable initial matrices:

$$\mathbb{W}_0 = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbb{W}_1 = 4\alpha\begin{pmatrix} a_0 & 0 \\ 0 & b_0 \end{pmatrix},$$

obtained on inspection of Eq. (6) and also using the size-dependent coefficient matrices (13) and (14).

In fact, the recursive matrix Eq. (10) represents two independent recurrence relation, since the initial matrices $\mathbb{W}_0$ and $\mathbb{W}_1$ have only nonvanishing elements on the principal diagonal, as well as the coefficient matrices $\mathbb{P}_N$ and $\mathbb{Q}_N$. Thus the exact solution for the nonhomogeneous 1D Ising model is given by the trace $Z_N = (1/2)\,\mathrm{Tr}\,\mathbb{W}_N$ in Eq. (6) that is the solution for the matrix recurrence relation (10).

B. The homogeneous Ising model

A scalar recursive formula with constant coefficients for the finite-size partition function is obtained from the trace of recurrence matrix relation (10) when the homogeneous Ising model is considered, i.e., $K_n = K$ $(0 \leqslant n < N)$. Thus $Z_N(K,B)$ is obtained from $Z_{N-1}(K,B)$ and $Z_{N-2}(K,B)$ only. Most surprising is that the recurrence relation is exactly the one obeyed by the generalized Lucas polynomials [7,8] that are closely related to the generalized Fibonacci polynomials.

The generalized Fibonacci polynomials (GFP) in real variables p and q are defined by a second-order linear homogeneous recurrence equation $F_N(p,q) = p\,F_{N-1}(p,q) + q\,F_{N-2}(p,q)\,, N \geqslant 2$ with initial conditions $F_0(p,q) = 0$ and $F_1(p,q) = 1$. This relation, along with the first two polynomials F_0 and F_1, allows the GFP to be generated recursively, whose series expansion is given by [8]:

$$F_N(p,q) = \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} \binom{N-k-1}{k} p^{N-2k-1} q^k.$$

For each term in this polynomial sequence, the higher power in variable p is obtained as $N-1$, i.e., one power lower than the polynomial order. If we start with the same recurrence equation above but choose different initial conditions, then we get another polynomial sequence. In particular, we are interested in the generalized Lucas polynomials (GLP) $L_N(p,q) = p\,L_{N-1}(p,q) + q\,L_{N-2}(p,q)\,, N \geqslant 2$, where the proper initial conditions are $L_0(p,q) = 2$ and $L_1(p,q) = p$, and the series expansion is [8]

$$L_N(p,q) = \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{N}{N-k}\binom{N-k}{k} p^{N-2k} q^k\,.$$

A GLP of order N has exactly N as its higher power in variable p.

From the two roots $\lambda_\pm(p,q) = (p \pm \sqrt{p^2+4q})/2$ of the solution of Fibonacci and Lucas generalized polynomials recurrence relations, it is possible to achieve the general solutions for the linear recurrence relations [8]:

$$L_N(p,q) = \lambda_+^N + \lambda_-^N\,, \quad F_N(p,q) = \frac{\lambda_+^N - \lambda_-^N}{\lambda_+ - \lambda_-}\,. \tag{15}$$

These are Binet form for the GFP recurrence relation and the GLP recurrence relation, respectively, from which many identities can be deduced. Among other important properties of a GLP, one that is shared with the Lucas sequence, is a nonlinear recurrence equation $L_{2N}(p,q) = [L_N(p,q)]^2 - q^N$. Identities involving higher powers than this one can also be easily deduced.

On appropriate choice of parameters $p = 2\alpha(a+b)$ and $q = -4ab$ as proved in Eq. (A3), we obtained through the trace of the recurrence relation in matrix form (10) a recursive formula for the partition function:

$$Z_N(p,q) = p\,Z_{N-1}(p,q) + q\,Z_{N-2}(p,q), \quad N \geqslant 2. \tag{16}$$

For this iteration, we can set the first two terms $Z_0(p,q) = 2$ and $Z_1(p,q) = p$, in order to establish the GLP of order N with parameters $p = p(K,B)$ and $q = q(K,B)$. Hence, we can use p and q to write explicitly the Binet form (15) and recover the well-known finite-size solution of the homogeneous 1D Ising model partition function:

$$Z_N(p,q) \equiv L_N(p,q) = \lambda_+^N + \lambda_-^N. \tag{17}$$

It is important to stress out that the partition function (17) shares all the properties of a GLP (15). Here the discussion reveals an interconnection between the simple homogeneous 1D Ising model and the generalized Fibonacci and Lucas polynomials.

The finite-size Helmholtz free energy is given by $\mathcal{F}_N(p,q) = -k_B T \ln[L_N(p,q)]$ such that the finite-size magnetization per spin equals $M_N/N = 2\,e^{\beta J}\sinh(\beta h)F_N(p,q)/L_N(p,q)$ and vanishes for zero field and finite N for all temperatures. For instance, in the case of the ferromagnetic coupling $(J > 0)$, in the thermodynamic limit this order parameter implies a spontaneous $(h \to 0^\pm)$ ferromagnetic phase transition at $T = 0$ due to a nonanalyticity of $\lim_{N\to\infty} F_N(p,q)/L_N(p,q)$ at this critical point. The finite-size (parallel and isothermal) susceptibility for zero magnetic field is $\chi_N(T) = \beta(1+v)(1-v^N)/[(1+v^N)(1-v)]$, where $v = \tanh(K)$, whose in the thermodynamic limit becomes $\chi(T) = \beta(1+v)/(1-v)$ as described elsewhere [15]. Near criticality the system attains ferromagnetic long-range order revealed by divergent correlation length and magnetic susceptibility. For high temperatures the well-known Curie-Weiss Law is recovered: $\chi_{\mathrm{HT}}(T) = C/(T-\theta)$.

The first-principles calculation that implies Eq. (16) uncover why a numerical investigation [16] has encountered and exploited a nonlinear recurrence identity $Z_{2N}(p,q) = [Z_N(p,q)]^2 - q^N$ for the 2D Ising model partition function, even without realizing this underlying polynomial structure. Next we begin discussing the magnetic properties of the random bond 1D Ising model.

IV. THE QUENCHED ISING MODEL

Due to impurities randomly distributed in the samples, the free energy of the system must be averaged over an ensemble of samples. This is the proper procedure for systems whose observed relaxation time of disorder is very slow. Here we consider that impurities implies unequal exchange couplings at differents spin pairs.

The quenched average of the free energy for the 1D Ising model with nearest-neighbor random interaction can be calculated with a probability distribution $P(\{J_n\}) = \prod_n P(J_n)$, where for each bond

$$P(J_n) = x\,\delta(J_n - J) + (1-x)\,\delta(J_n + J). \tag{18}$$

Interaction between spins n and $n+1$ has probability x for ferromagnetic interaction and $1-x$ for antiferromagnetic interaction. Therefore the quenched average $[\mathcal{F}_N]_{\rm av} = -\beta^{-1}[\ln Z_N]_{\rm av}$ must be extracted from

$$[\mathcal{F}_N]_{\rm av} = -\beta^{-1}\int \prod_n dJ_n\, P(\{J_n\})\, \ln Z_N(\{J_n\}). \tag{19}$$

Throughout the paper $[\cdots]_{\rm av}$ denotes average over an ensemble of samples. The quenched average $[\ln Z_N]_{\rm av}$ is needed to deal with disorder fluctuations between different samples, while the thermal average was performed to deal with thermal fluctuations of some sample through Eq. (6).

The quenched averaged free energy $[\mathcal{F}_N]_{\rm av}$ is obtained by expanding Eq. (19) in powers of x and $y = 1-x$ (see Appendix B). Ordinarily the quenched averaged magnetic susceptibility per spin is given by derivatives of the quenched averaged free energy with respect to the field

$$[\chi_N]_{\rm av} = -\frac{\beta}{N}\frac{\partial^2}{\partial B^2}[\mathcal{F}_N]_{\rm av}\bigg|_{h=0}. \tag{20}$$

The magnetic field only appears as powers of α, as in Eq. (B1), for example. Derivatives of the quenched averaged free energy therefore demands derivatives of powers of α inside the argument of logarithms. In the zero-field limit the quenched averaged susceptibility simplifies by induction into a helpful expression (Appendix B):

$$[\chi_N]_{\rm av} = \beta\left\{1 + \frac{(2uv)(1-v^2)}{(1-v^{2N})}\,[1+uv+v^2]^{\underline{N-2}}\right\}. \tag{21}$$

Here we defined proper parameters $v = \tanh(\beta J)$ and $u = 2x - 1$, and a suitably binomial-like expansion

$$[r+t]^{\underline{N}} = \sum_{k=0}^{N} r^{\underline{N-k}}\, t^{\underline{k}} \tag{22}$$

for any real numbers r and t. This defines a modified binomial expansion since the binomial coefficient is absent. A finite series expansion can be found using this expansion for the brackets in the quenched averaged susceptibility given by Eq. (21),

$$[1+uv+v^2]^{\underline{N-2}} = \sum_{k=0}^{N-2} v^{2k}\sum_{j=0}^{k}\left(\frac{u}{v}\right)^j, \tag{23}$$

whose closed-form solution can be written from the result of partial sum of geometric sequences:

$$[1+uv+v^2]^{\underline{N-2}} = \frac{v\dfrac{1-(v^2)^{N-1}}{1-v^2} - u\dfrac{1-(uv)^{N-1}}{1-uv}}{v-u} \quad (v<1 \text{ and } v \neq u). \tag{24}$$

The following result for the the finite-size quenched averaged susceptibility (21) is obtained after substituting the closed-form expansion (24):

$$[\chi_N]_{\rm av} = \beta\left\{\frac{v+u}{v-u} - \frac{2u}{v-u}\frac{(1-v^2)}{(1-v^{2N})}\frac{[1-(uv)^N]}{(1-uv)}\right\} \quad (v<1 \text{ and } v\neq u). \tag{25}$$

Note, for probabilities $x = 1$ ($x = 0$), we obtain the magnetic susceptibility of strictly ferromagnetic (antiferromagnetic) interacting spins.

The thermodynamic limit of the quenched susceptibility can be obtained for finite temperature from Eq. (25) with little effort, and it is an analytic function of x and J:

$$[\chi]_{\rm av} = \beta\,\frac{(1+uv)}{(1-uv)} \qquad (v<1). \tag{26}$$

A few remarks on the asymptotic regime follows: the quenched susceptibility goes as $[\chi_{\rm LT}]_{\rm av} \sim \beta x/(1-x)$ for low temperatures (LT), whereas for the antiferromagnetic system ($x = 0$) it vanishes. On the other hand, a Curie-Weiss term $[\chi_{\rm HT}]_{\rm av} \sim C/[T - \theta(x)]$ with a well-defined temperature shift $\theta(x) = 2[J]_{\rm av}/k_B$ for the high-temperature (HT) limit is found. On average, the energy of a spin pair bond is $2[J]_{\rm av}$, which sets an energy scale to the model.

As the probability x varies between 0 and 1, the average exchange coupling varies in the range $-J \leqslant [J]_{\rm av} \leqslant J$. This temperature $\theta(x)$ should therefore range in the interval $\pm 2[J]_{\rm av}/k_B$, which consequently defines the interactions between spins as predominant ferromagnetic ($[J]_{\rm av},\theta > 0$) or antiferromagnetic ($[J]_{\rm av},\theta < 0$), respectively, as shown in Fig. 1. The demarcation line where one prevailing interaction type stops and the other begins is set when $x = 1/2$, and the system has microscopically interacting spins although it behaves macroscopically like spins solely sensitive to an externally applied magnetic field.

Figure 2 shows a broad maximum in $[\chi]_{\rm av}$ for low probability values x, which is a characteristic of low-dimensional spin systems. This broad peak only emerges below some definite threshold value $x_{\rm th}$ and shifts to higher temperatures with decreasing probability of ferromagnetic bonds. Furthermore, it also shows the magnetic response upturn effect due to impurity in the low-temperature limit.

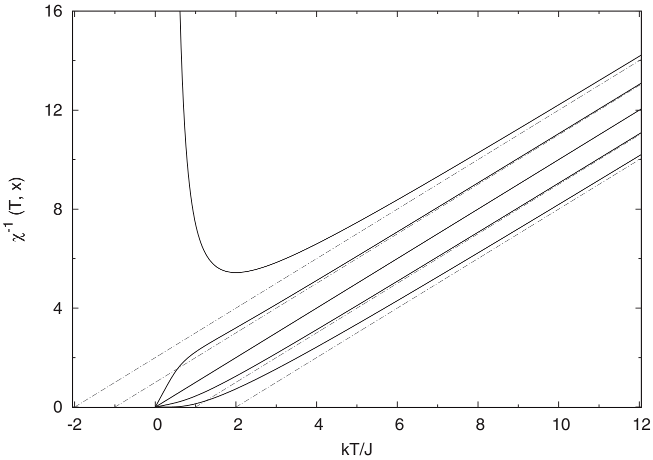

FIG. 1. Temperature dependence of the inverse magnetic susceptibility (solid curves) and its high-temperature assymptotic limit (dashed curves) for several probability values x.

The emergence of minimum and maximum of $[\chi]_{\rm av}$ is revealed by the roots of $\eta = d[\chi]_{\rm av}/dT$. Differentiating Equation (26) with respect to T and looking for the roots, it is found that maximum and minimum temperatures are localized along the dashed curve in Fig. 2 given by

$$\eta(T) = -\frac{v/J}{1-v^2} + \sqrt{\left(\frac{v/J}{1-v^2}\right)^2 + \beta^2}. \tag{27}$$

The maximum of this curve occurs at a threshold value when the condition $\beta_{\rm th}J = \coth(2\beta_{\rm th}J)$ is satisfied. The numerical solution of this transcendental equation is such that $\beta_{\rm th}J \approx 1.033$, where $\chi_{\rm th} \approx 0.2577$ and $x_{\rm th} \approx 0.1125$.

A sharp behavior near the threshold value can be identified in Fig. 2 (inset) which shows the maximum and minimum of the magnetic susceptibility for x values above 0. This indicates a very unusual magnetic behavior. In addition, it has a discontinuous derivative with respect to x at $x \approx 0.1125$.

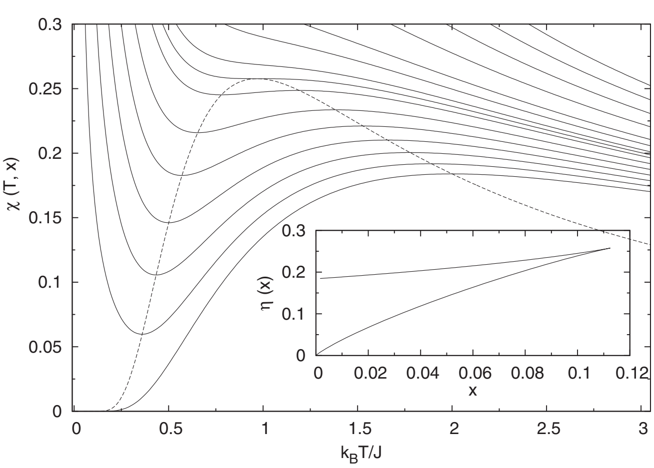

FIG. 2. Temperature dependence of the magnetic susceptibility (solid curve) from lowest $x = 0$ (bottom) to highest $x \approx 0.31$ (top) ferromagnetic probability. Localization of maximum and minimum temperatures (dashed curve) determined by η in Eq. (27). Note the appearance of a threshold temperature whose reduced value is slightly below unity. Inset shows minimum and maximum given by η as a function of the probability x.

V. CONCLUDING REMARKS

We study the 1D Ising model with nearest-neighbor interaction in the presence of an external magnetic field. Considering nonhomogeneous interaction, we showed how to compute the partition function from an algebraic method whose main feature encompass recurrence relations. In particular, for the homogenous 1D Ising model, we describe a direct connection between the generalized Lucas polynomials and the partition function. It automatically establishes a recurrence relation for the partition function.

The presented analytical method is able to introduce recursive formulas for the homogeneous and nonhomogeneous 1D Ising model, which is a nonobvious result if one is restricted to the standard transfer matrix method. In a numerical approach [16] a nonlinear recurrence relation was investigated for the finite-size homogeneous 2D Ising model without external magnetic field, identical to the one obeyed by the generalized Lucas polynomials $Z_{2N}(p,q) = [Z_N(p,q)]^2 - q^N$. The analytical results presented in this paper for the 1D Ising model gives a mathematical background approach to the physical problem, and we assert that future research can clarify if the finite-size partition function of a 2D Ising model also obeys a linear recurrence equation.

Furthermore, in order to study the presence of impurities in magnetic samples, we consider the quenched disorder to deal with systems whose relaxation time of exchange couplings is very slow. We perform the quenched average of the free energy and obtained the magnetic properties of this disordered systems and found a nontrivial behavior for the magnetic susceptibility for certain values of probabilities x or $1 - x$ of a ferromagnetic or antiferromagnetic interaction between nearest-neighbor spins. We compute the threshold value $x_{\rm th} \approx 0.1125$ along with the threshold temperature and the threshold magnetic susceptibility solving a transcendental equation. We emphasize that singularities due to thermal fluctuations in homogeneous systems are completely different from nontrivial behavior due to samples fluctuations in disordered systems.

Finally, we stress that our result for the quenched average of the magnetic susceptibility (26) is not defined for the ground state of the system. It was obtained for a specific order of limits, i.e., first we take the zero field limit, and then we take the thermodynamic limit. In a different approach [13], the asymptotic limit near zero temperature was taken first, and then the other limits were performed, where the authors found a finite magnetic susceptibility at zero temperature. Therefore our results complement the quenched averaged magnetic susceptibility for finite temperatures not yet reported.

ACKNOWLEDGMENT

The authors acknowledge R. G. Amorim and O. Lourenço for interesting comments regarding the initial manuscript.

APPENDIX A: PARTITION FUNCTIONS

Using Eq. (4) and tensor properties mentioned in Sec. II, we obtained the following expressions for the partition functions for low N values:

$$Z_2 = 2^2[(a_0a_1 + b_0b_1)\alpha^2 + (a_0b_1 + b_0a_1)\omega^2]$$

$$\begin{aligned} Z_3 = &\, 2^3[a_0a_1a_2 + b_0b_1b_2]\alpha^3 \\ &+ 2^3[a_0a_1b_2 + b_0b_1a_2]\alpha\omega^2 \\ &+ 2^3[a_0b_1a_2 + b_0a_1b_2]\alpha\omega^2 \\ &+ 2^3[a_0b_1b_2 + b_0a_1a_2]\alpha\omega^2 \end{aligned}$$

$$\begin{aligned} Z_4 = &\, 2^4[a_0a_1a_2a_3 + b_0b_1b_2b_3]\alpha^4 \\ &+ 2^4[a_0a_1a_2b_3 + b_0b_1b_2a_3]\alpha^2\omega^2 \\ &+ 2^4[a_0a_1b_2a_3 + b_0b_1a_2b_3]\alpha^2\omega^2 \\ &+ 2^4[a_0a_1b_2b_3 + b_0b_1a_2a_3]\alpha^2\omega^2 \\ &+ 2^4[a_0b_1a_2a_3 + b_0a_1b_2b_3]\alpha^2\omega^2 \\ &+ 2^4[a_0b_1b_2a_3 + b_0a_1a_2b_3]\alpha^2\omega^2 \\ &+ 2^4[a_0b_1b_2b_3 + b_0a_1a_2a_3]\alpha^2\omega^2 \\ &+ 2^4[a_0b_1a_2b_3 + b_0a_1b_2a_3]\omega^4. \end{aligned}$$

All these partition functions are summarized in the general formula (6). These partitions functions could be obtained from the recurrence relation in matrix form (10). Considering the homogeneous case, the coefficients matrices (13) and (14) simplifies each in a product of a scalar with the identity matrix

$$\mathbb{P}_{N+1} = 2\alpha(a+b)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A1}$$

and

$$\mathbb{Q}_N = -4ab\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A2}$$

These scalars are $p = 2\alpha(a+b)$ and $q = -4ab$. Thus the recurrence relation for the partition function of a homogeneous system are given by

$$Z_{N+2}(p,q) = p\, Z_{N+1}(p,q) + q\, Z_N(p,q), \quad N \geqslant 0. \tag{A3}$$

APPENDIX B: QUENCHED AVERAGES

The quenched average of $\ln Z_N$ was calculated using the thermal average (6) and the probability distribution (18) and (19) for low N values:

$$\begin{aligned} [\ln Z_2]_{\text{av}} = &\, x^2 \ln[\alpha^2(a+b)^2 - 2ab] \\ &+ 2xy \ln[\alpha^2(a+b)(a-b)] \\ &+ y^2 \ln[\alpha^2(a-b)^2 + 2ab] \end{aligned}$$

$$\begin{aligned} [\ln Z_3]_{\text{av}} = &\, x^3 \ln[\alpha(a+b)(\alpha^2(a+b)^2 - 3ab)] \\ &+ 3x^2y \ln[\alpha(a-b)(\alpha^2(a+b)^2 - ab)] \\ &+ 3xy^2 \ln[\alpha(a+b)(\alpha^2(a-b)^2 + ab)] \\ &+ y^3 \ln[\alpha(a-b)(\alpha^2(a-b)^2 + 3ab)] \end{aligned}$$

$$\begin{aligned} [\ln Z_4] = &\, x^4 \ln[\alpha^4(a+b)^4 - 4\,ab\alpha^2(a+b)^2 + 2\,a^2b^2] \\ &+ 4x^3y \ln[\alpha^2(a^2-b^2)(\alpha^2(a+b)^2 - 2ab)] \\ &+ 4x^2y^2 \ln[\alpha^4(a+b)^2(a-b)^2 + 4a^2b^2\alpha^2 - 2a^2b^2] \\ &+ 2x^2y^2 \ln[\alpha^4(a+b)^2(a-b)^2 + 2a^2b^2] \\ &+ 4xy^3 \ln[\alpha^2(a^2-b^2)\,(\alpha^2(a-b)^2 + 2ab)] \\ &+ y^4 \ln[\alpha^4(a-b)^4 + 4\,ab\alpha^2(a-b)^2 + 2\,a^2b^2]. \end{aligned}$$

We defined parameters $a = \cosh(\beta J)$, $b = \sinh(\beta J)$, and $\alpha = \cosh(\beta h)$ and calculated this average up to $N = 5$.

The quenched average of the magnetic susceptibility with vanishing field

$$[\chi_N]_{\text{av}} = \beta\left\{1 + \frac{2}{N}\, S_N\right\} \tag{B1}$$

was found from the following polynomials in parameters u and v:

$$S_2 = \frac{2uv}{1+v^2}, \tag{B2}$$

$$S_3 = \frac{3uv\,[1 + uv + v^2]}{1 + v^2 + v^4}, \tag{B3}$$

$$S_4 = \frac{4uv\,[1 + uv + v^2 + uv^3 + u^2v^2 + v^4]}{1 + v^2 + v^4 + v^6}. \tag{B4}$$

In the numerators all brackets follows a simple rule with increasing N, a certain powerlike expansion of $1 + uv + v^2$ with binomials coefficients suppressed given by Eq. (22). In all denominators a finite series with a common ratio v^2 whose a summation index ranges from 0 to $N-1$ has appeared. Thus it induced the result

$$S_N = Nuv\,\frac{(1-v^2)}{(1-v^{2N})}\,[1 + uv + v^2]^{\underline{N-2}}. \tag{B5}$$

The induction procedure described in this appendix allows one to obtain Eq. (21).

[1] D. W. Aldous *et al.*, Inorg. Chem. **46**, 1277 (2007).
[2] F. H. Aidoudi *et al.*, Dalton Trans. **43**, 568 (2014).
[3] E. E. Kaul, H. Rosner, V. Yushankhai, J. Sichelschmidt, R. V. Shpanchenko, and C. Geibel, Phys. Rev. B **67**, 174417 (2003).
[4] Y. Savina, O. Bludov, V. Pashchenko, S. L. Gnatchenko, P. Lemmens, and H. Berger, Phys. Rev. B **84**, 104447 (2011).
[5] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992).

[6] K. Huang, *Statistical Mechanics* (Wiley, New York, 1987).
[7] G.-S. Cheon, H. Kim, and L. W. Shapiro, Discrete Appl. Math. **157**, 920 (2009).
[8] T. Amdeberhan, X. Chen, V. H. Moll, and B. E. Sagan, Ann. Comb. **18**, 541 (2014).
[9] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[10] S. F. Edwards and P. W. J. Anderson, J. Phys. F: Met. Phys. **5**, 965 (1975).
[11] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975).
[12] M. Mezard, G. Parisi, and M. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[13] B. Derrida, J. Vannimenus, and Y. Pomeau, J. Phys. C **11**, 4749 (1978).
[14] J. F. Fernandez, Phys. Rev. B **16**, 5125 (1977).
[15] T. Antal, M. Droz, and Z. Rácz, J. Phys. A: Math. Gen. **37**, 1465 (2004).
[16] G. Nandhini and M. V. Sangaranarayanan, J. Chem. Sci. **121**, 595 (2009).