Kardar-Parisi-Zhang modes in d-dimensional directed polymers

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We define a stochastic lattice model for a fluctuating directed polymer in $d \geqslant 2$ dimensions. This model can be alternatively interpreted as a fluctuating random path in two dimensions, or a one-dimensional asymmetric simple exclusion process with $d-1$ conserved species of particles. The deterministic large dynamics of the directed polymer are shown to be given by a system of coupled Kardar-Parisi-Zhang (KPZ) equations and diffusion equations. Using nonlinear fluctuating hydrodynamics and mode coupling theory we argue that stationary fluctuations in any dimension d can only be of KPZ type or diffusive. The modes are pure in the sense that there are only subleading couplings to other modes, thus excluding the occurrence of modified KPZ-fluctuations or Lévy-type fluctuations, which are common for more than one conservation law. The mode-coupling matrices are shown to satisfy the so-called trilinear condition.

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I. INTRODUCTION

The dynamics of one-dimensional many-body systems is presently a topic of intense study. One of the main motivations is to study anomalous transport phenomena, which arise in different contexts and various physical scenarios even when interactions are short-ranged. Specific topics of interest are one-dimensional stochastic equations with local conservations laws [in particular for interface dynamics in the universality class of the one-dimensional noisy Kardar-Parisi-Zhang (KPZ) equation] and stationary spatiotemporal fluctuations in driven diffusion, or anharmonic chains or Hamiltonian fluid dynamics; see, e.g., the collection of articles in Ref. [1] and in the first issue of volume 160 of the *Journal of Statistical Physics* (2015) for recent overviews. In the case of a single locally conserved quantity, the long wave-length fluctuations of the conserved field are generally either diffusive with dynamical exponent $z = 2$ or in the KPZ universality class [2] with dynamical exponent $z = 3/2$.

In this article, we will focus on *coupled* one-dimensional stochastic equations with more than one conservation law. They show a much richer behavior than the single KPZ equation, depending on the details of the models. Fluctuations of the conserved fields can be in a modified KPZ universality class [3] or, more intriguingly, in a discrete family of Lévy universality classes [4] where the dynamical exponents z_i are the Kepler ratios of neighboring Fibonacci numbers and the universal scaling forms of the dynamical structure function are z_i-stable Lévy distributions. The first member in this family is a mode with dynamical exponent $z = 3/2$ as in KPZ, but Lévy scaling function which very recently was proved rigorously for energy fluctuations in a harmonic chain with energy-conserving noise [5]. The second member with dynamical exponent $z = 5/3$ was first firmly established using mode coupling theory for the heat mode in Hamiltonian dynamics for a one-dimensional fluid [6]. Also the limiting value of the Kepler ratios, which is the famous golden mean, can arise [3,4,7].

Here we address the nature of the dynamical structure functions in a higher dimensional setting, viz. for the contour fluctuations in a lattice model for a directed polymer in $d \geqslant 2$ dimensions, somewhat in the spirit of the space-continuous polymer model of [8] for $d = 3$. Our lattice model can be mapped to a fluctuating random path in two dimensions and also to a one-dimensional exclusion process [9–12] generalized to $d-1$ species of particles. We use the latter mapping, taking two different approaches to study the large-scale dynamics and the spatiotemporal fluctuations in the stationary state.

First, focusing on $d = 3$, a dynamical mean-field approach for the particle densities leads to a system of two coupled partial differential equations that each look like a Burgers equation. By introducing a generalized height variable, these equations become coupled KPZ equations. The couplings depend on the rates of the original exclusion process. By varying the rates, one can systematically study the different universality classes. However, two of the entries in the coupling matrices will always remain zero, regardless of the rates in the underlying exchange process.

The second approach is based on nonlinear fluctuating hydrodynamics, which has emerged as a widely applicable and powerful tool for the study of stationary fluctuations of the locally conserved quantities such as energy, momentum, or particle densities [13]. From the exact current-density relation we compute the mode-coupling matrices which allow us to deduce the dynamical universality classes that can occur in the model in any dimension $d \geqslant 2$. We find that only KPZ modes and diffusive modes may occur and that these modes have only subleading couplings between them, which excludes also the occurrence of the modified KPZ universality class. We point out that the mode coupling matrices satisfy the so-called trilinear condition which is relevant for the Gaussian nature of the invariant measure of the associated coarse-grained system of coupled noisy KPZ-equations [14,15].

This paper is organized in the following way. We start by defining in Sec. II the directed polymer model in any dimension

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d that is a generalization of the well-known correspondence between the single–species asymmetric diffusion model and a growing and fluctuating interface in $d = 2$. In Sec. III we focus on $d = 3$ and first derive a system of two coupled nonlinear partial differential equations for a generalized height function from a coarse-graining of the model. Next we study fluctuations via nonlinear fluctuating hydrodynamics. Section IV contains a calculation of the mode-coupling coefficients for an n-component particle exchange process, corresponding to a directed polymer in $d = n + 1$ dimensions. Discussing the case $n = 2$ in detail yields a direct comparison with the height model results. In Sec. V we summarize our results and point to some open problems.

II. DIRECTED POLYMER IN d DIMENSIONS, GENERALIZED HEIGHT FUNCTION, AND THE MULTI-SPECIES ASEP

There is a very nice and well-known mapping between the one-dimensional single-species asymmetric simple exclusion process (ASEP) and a growing and fluctuating interface on a two-dimensional substrate [16,17]. The contour of this interface can equally be interpreted as a model for a directed polymer living on a square lattice in two dimensions. The conformation of the polymer, or equivalently, the height function of the interface, is given by a microstate of the ASEP.

Generalizing to multispecies simple exclusion processes [18], it is natural to search for an analogous construction in higher dimensions. We demonstrate that there is indeed a natural way of defining a directed polymer model in any dimension. This is achieved by identifying the directed polymer with a directed path on a plane perpendicular to the $(1,1,\ldots,1)$-direction of a hypercubic lattice and introducing an associated generalized height function. Below we present the details of this mapping and show that by deriving an equation for the time evolution of the height variable one obtains a set of coupled differential equations that describe either diffusive or KPZ or mixed behavior. The same equations can be derived from the corresponding multispecies simple exclusion process and its master equation dynamics.

A. Details

Consider d species of particles with exclusion, i.e., at most one particle per site, on a one-dimensional chain of L sites, counting a "vacancy" as a species. Particles of different species α and β randomly interchange their positions with rates $g_{\alpha,\beta}$; see Sec. (IV) for a precise definition of this multispecies exclusion process. Then each configuration of the chain can be mapped to a directed path on a d-dimensional hypercubic lattice, which is later projected onto a plane perpendicular to the $(1,1,\ldots,1)$-direction: As you step along the chain, the corresponding steps of the path on the hypercube are given by what species of particle you pass, with each species corresponding to one of the d basis vectors of the hypercube with unit length a. Thus, each step increases the height of the corresponding segment of the directed polymer by $a/\sqrt{d}$ above its anchor point. By convention we take the anchor point to be the origin $\vec{0} = (0,0,\ldots,0)$. We assume no external potential so that in the stationary state each conformation of the directed polymer is equally likely.

For a hypercube with unit lattice constant $a = 1$ the contour length of the polymer is Ld. The endpoint of the polymer after the L steps of the underlying particle configuration is at height $L/\sqrt{d}$. Its position is determined by the (conserved) number of particles of each species in the chain. In particular, if the number N_α of particles is the same for each species α, i.e., if $N_\alpha = L/d$, then the endpoint of the polymer has coordinates $L/\sqrt{d}(1,1,\ldots,1)$. The projection of the position of the polymer after k steps along the chain onto the hyperplane perpendicular to the $(1,1,\ldots,1)$-direction defines a generalized height variable, which is a $d-1$-dimensional vector.

B. Example in $d = 3$, leading to a path in $d = 2$

For definiteness we discuss in more detail the case $d = 3$, where our generalized height will be shown as the position of the path projected onto a plane perpendicular to the (111) axis of the cube. This path will be in two dimensions. The dynamics of the system is then represented by elementary moves of this path, where one site along the path moves in the only way that is determined by the constraints imposed by the particle exchange dynamics of the exclusion process with three conserved particle species and no vacant sites. Notice that since L is fixed by the dynamics, the particle exchange dynamics correspond to only two genuine conservation laws. This can be seen by identifying one species with vacant sites. We consider periodic boundary conditions for the exclusion process with an equal number of particles of each species which corresponds to periodic boundary conditions for the directed polymer.

To be concrete, we start from a two-species asymmetric exclusion model on a ring with L sites where each site k is either empty or occupied by at most one particle of type A or B. For our purposes it is convenient to think of a vacancy as a further species of particles, denoted by Φ. A microscopic particle configuration is specified by an array of L symbols X_k where $X_k \in \{\Phi, A, B\}$, or, equivalently, by occupation numbers $n_k^X = \delta_{X,X_k}$, which are equal to 1 if the particle at site k is of type X and zero otherwise. It defines a conformation of the directed polymer as described above.

The Markovian stochastic dynamics consists of nearest-neighbor particle exchanges $X_i, X_{i+1} \to X_{i+1}, X_i$ as follows:

Transition	Rate
$A\,B \to B\,A$	r_1
$B\,A \to A\,B$	r_2
$A\,\Phi \to \Phi\,A$	r_3
$\Phi\,A \to A\,\Phi$	r_4
$B\,\Phi \to \Phi\,B$	r_5
$\Phi\,B \to B\,\Phi$	r_6

(1)

To ensure equal equilibrium probabilities for all conformations of the polymer (corresponding to the uniform measure for particle configurations) we impose pairwise balance [19], which yields

$$r_1 + r_4 + r_5 = r_2 + r_3 + r_6. \tag{2}$$

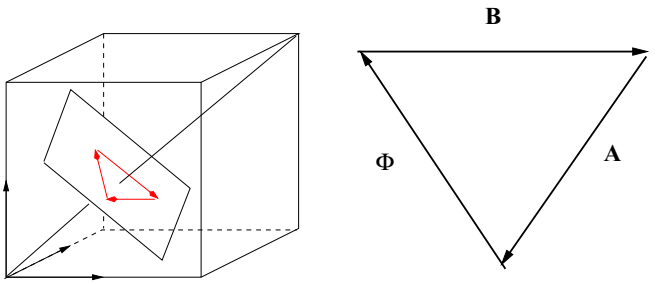

FIG. 1. Projection of the position of the directed polymer onto the plane perpendicular to the (1,1,1) direction (left) and directions for the 2D height function (right).

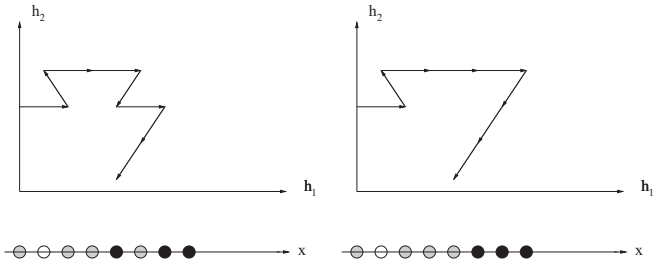

FIG. 2. Two-dimensional random path on the honeycomb lattice and diffusing particles. From the left picture to the right picture, the black particle on lattice site 5 and the gray particle on lattice site 6 have interchanged places and the path has changed accordingly.

The uniform distribution leads to a complete absence of stationary correlations in the thermodynamic limit $L \to \infty$.

The link with the height function and the two-dimensional random path is established as follows. With each of the three species (A, B, or vacancy Φ) we associate one of the three canonical basis vectors $\vec{e}_i$ of the 3D cubic lattice. Thus, starting from the anchor point of the polymer [say, the origin $\vec{0} = (0,0,0)$], the height along the (1,1,1) axis is $k/\sqrt{3}$, where k is the lattice site of the one-dimensional chain of particles. The particle configuration from site 1 to site k on the chain then describes the position of the height vector $\vec{H}_k = \sum_{j=1}^{k} \vec{e}_j^{\,X_j}$ in the plane perpendicular to the (111) direction, reflecting the position of the polymer in three dimensional space at height $k/\sqrt{3}$.

The projection of the three basis vectors onto the plane perpendicular to the (1,1,1) direction is shown in Fig. 1 (left). This results in the following three normalized vectors:

$$\vec{v_\Phi} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{v_A} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \quad \vec{v_B} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Now we can pick basis vectors for the plane perpendicular to the (1,1,1) direction, e.g.,

$$\vec{b_1} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{b_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Expressing the vectors $\vec{v}_i; i = \Phi, A, B$ in terms of the two basis vectors $\vec{b_1}$ and $\vec{b_2}$, they become the two-dimensional unit vectors in the projection plane [see Fig. 1 (right)]:

$$\vec{\Phi} = \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \quad \vec{A} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3}$$

The projected height vector at level k, which is the generalized height function we are after, is then given by

$$\vec{H}_k^{\perp} = \vec{H}_0^{\perp} + \sum_{j=1}^{k} \left(n_k^{\Phi} \vec{\Phi} + n_k^{A} \vec{A} + n_k^{B} \vec{B} \right), \tag{4}$$

where $\vec{H}_0^{\perp}$ is the reference point (taken to be the origin in the description above). This shows that the local occupation numbers give the (discrete) height gradient,

$$\nabla_{(111)} \vec{H}_k^{\perp} := \vec{H}_k^{\perp} - \vec{H}_{k-1}^{\perp} = n_k^{\Phi} \vec{\Phi} + n_k^{A} \vec{A} + n_k^{B} \vec{B}, \tag{5}$$

in (111)-direction. Fluctuations in the height vector are described by nearest neighbor particle swaps as defined above.

Correspondingly the surface path in the plane perpendicular to the (111) direction becomes a planar random path on a honeycomb lattice with unit lattice constant (Fig. 2). A change in the path happens when two particles interchange places.

III. COARSE-GRAINED DYNAMICS AND STATIONARY SPATIOTEMPORAL FLUCTUATIONS

A. Coupled KPZ equations for the height function

To study the large-scale behavior of the height function for arbitrary initial states, we define a coarse-grained two-dimensional height variable,

$$\vec{h}(x,t) = \begin{pmatrix} h_1(x,t) \\ h_2(x,t) \end{pmatrix}. \tag{6}$$

Since $\vec{A} + \vec{B} + \vec{\Phi} = 0$, it follows that $\rho_A = \rho_B = \rho_\Phi = 1/3$ for the average particle densities [20]. Therefore, we define coarse-grained local densities $\rho_A(x,t)$, $\rho_B(x,t)$, and $\rho_\Phi(x,t)$ for species A, B, and Φ, respectively, as follows:

$$\begin{aligned} \rho_A(x,t) &= \tfrac{1}{3} + \tfrac{2}{3}\,(\vec{A} \cdot \nabla \vec{h}(x,t)), \\ \rho_B(x,t) &= \tfrac{1}{3} + \tfrac{2}{3}\,(\vec{B} \cdot \nabla \vec{h}(x,t)), \\ \rho_\Phi(x,t) &= \tfrac{1}{3} + \tfrac{2}{3}\,(\vec{\Phi} \cdot \nabla \vec{h}(x,t)). \end{aligned} \tag{7}$$

Here, $\nabla \vec{h}$ denotes the one-dimensional derivative in the direction of the diffusing particles, i.e., along the coarse-grained chain in (111)-direction. Each of the densities fluctuates around its equilibrium value $\frac{1}{3}$ and will be changed proportionally to the change $\nabla \vec{h}$ in the height variable $\vec{h}(x,t)$ projected onto the respective growth direction.

To derive a nonlinear evolution equation for $\vec{h}(x,t)$ we recall that the local particle density describes the gradient of the height vector; see Eq. (5) for the discrete case. To obtain an equivalent continuum description we symmetrize the discrete gradient and expand the ρ_i for $i = A, B, \Phi$ around x to second

order, leading to

$$\rho_A\left(x \pm \tfrac{1}{2}\right) \simeq \tfrac{1}{3} + \tfrac{2}{3}\,(\vec{A}\cdot\nabla\vec{h}) \mp \tfrac{1}{3}(\vec{A}\cdot\Delta\vec{h}) + \cdots \qquad (8)$$

The next step is to consider the time evolution of the height variable $\vec{h}(x,t)$. From the absence of correlations in the stationary distribution and the dynamical rules of the model we find

$$\begin{aligned}\dot{\vec{h}}(x) = {} & (r_3 - r_4)(\vec{\Phi} - \vec{A})\,\rho_\Phi\left(x - \tfrac{1}{2}\right)\,\rho_A\left(x + \tfrac{1}{2}\right) \\ & + (r_1 - r_2)(\vec{B} - \vec{A})\,\rho_B\left(x - \tfrac{1}{2}\right)\,\rho_A\left(x + \tfrac{1}{2}\right) \\ & + (r_5 - r_6)(\vec{\Phi} - \vec{B})\,\rho_\Phi\left(x - \tfrac{1}{2}\right)\,\rho_B\left(x + \tfrac{1}{2}\right). \qquad (9)\end{aligned}$$

This equation describes how the height variable will change after two particles on the lattice will have interchanged places. The increase is proportional to the density of particles and is proportional to the growth direction associated with the interchange process.

We will adopt the following notation:

$$r_1 + r_2 = p, \qquad (10)$$

$$r_5 + r_6 = q, \qquad (11)$$

$$r_3 - r_4 = f_1, \qquad (12)$$

$$r_5 - r_6 = f_2, \qquad (13)$$

$$r_1 - r_2 = f_1 - f_2. \qquad (14)$$

The last equation follows from the pairwise balance requirement Eq. (2). Putting everything into Eq. (9) and denoting transposition of a vector or matrix by a superscript T, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\vec{h}(x) = {} & \begin{pmatrix}\frac{1}{6}(f_1 - 2f_2)\\ \frac{1}{2\sqrt{3}}f_1\end{pmatrix} + \begin{pmatrix}\frac{(f_1-2f_2)}{6} & -\frac{f_1}{2\sqrt{3}}\\ -\frac{f_1}{2\sqrt{3}} & -\frac{(f_1-2f_2)}{6}\end{pmatrix}\nabla\vec{h} \\ & + \begin{pmatrix}\frac{(p+q)}{4} & \frac{(p-q)}{4\sqrt{3}}\\ \frac{(p-q)}{4\sqrt{3}} & \frac{2r_1+2r_5+3(p+q)}{12}\end{pmatrix}\nabla^2\vec{h} \\ & + \begin{pmatrix}(\nabla\vec{h})^T\begin{pmatrix}-\frac{(f_1-2f_2)}{3} & -\frac{f_1}{2\sqrt{3}}\\ -\frac{f_1}{2\sqrt{3}} & 0\end{pmatrix}\nabla\vec{h}\\ (\nabla\vec{h})^T\begin{pmatrix}0 & -\frac{(f_1-2f_2)}{6}\\ -\frac{(f_1-2f_2)}{6} & -\frac{f_1}{\sqrt{3}}\end{pmatrix}\nabla\vec{h}\end{pmatrix} + \cdots \qquad (15)\end{aligned}$$

Taking the gradient on both sides one recognizes two coupled KPZ equations with nonvanishing drift term that mixes the two height components.

We express this system of nonlinear coupled equations in terms of eigenfunctions of matrix multiplying $\nabla\vec{h}$. The eigenvalues are

$$v_{1,2} = \pm\tfrac{1}{3}\sqrt{f_1^2 - f_1 f_2 + f_2^2} =: \pm\tfrac{1}{3}s(f_1,f_2). \qquad (16)$$

The expression under the square root is always positive except for $f_1 = f_2 = 0$ in which case not only $v_1 = v_2 = 0$ but where also the nonlinear term vanishes. This corresponds to the (boring) case of symmetric diffusion which we exclude from our considerations. It is interesting that the two eigenvalues $\lambda_{1,2}$ are then never equal. This implies that the drift term cannot be removed by a Galilei transformation.

When $f_1 = 0$ we do not need to apply a similarity transformation. The result of the transformation to eigenmodes $\vec{\tilde{h}}(x)$ for $f_1 \neq 0$ is

$$\frac{\partial}{\partial t}\vec{\tilde{h}}(x) = \begin{pmatrix}\tilde{v}_1\\ \tilde{v}_2\end{pmatrix} + \begin{pmatrix}-\frac{1}{3}s(f_1,f_2) & 0\\ 0 & \frac{1}{3}s(f_1,f_2)\end{pmatrix}\nabla\vec{\tilde{h}}(x) + \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\nabla^2\vec{\tilde{h}}(x) + \begin{pmatrix}(\nabla\vec{\tilde{h}}(x))^T\begin{pmatrix}N_{11} & N_{12}\\ N_{21} & 0\end{pmatrix}\nabla\vec{\tilde{h}}(x)\\ (\nabla\vec{\tilde{h}}(x))^T\begin{pmatrix}0 & P_{12}\\ P_{21} & P_{22}\end{pmatrix}\nabla\vec{\tilde{h}}(x)\end{pmatrix} + \cdots, \qquad (17)$$

with the average growth velocities

$$\tilde{v}_1 = \frac{f_1(f_1 - 2f_2 + s(f_1,f_2))\left(2f_1^2 + 2f_2(f_2 + s(f_1,f_2)) - f_1(2f_2 + s(f_1,f_2))\right.}{6s(f_1,f_2)|f_1|},$$

$$\tilde{v}_2 = \frac{f_1(-f_1 + 2f_2 + s(f_1,f_2))\left(2f_1^2 + 2f_2(f_2 - s(f_1,f_2)) + f_1(-2f_2 + s(f_1,f_2))\right.}{6s(f_1,f_2)|f_1|},$$

of the two projected height variables in normal mode coordinates and the matrix elements

$$M_{11} = \frac{f_2(3(p-q) - (r_1+r_5)) + (f_1 - f_2)(3(p-q) + r_1 + r_5) + 2s(f_1,f_2)(3(p+q) + r_1 + r_5)}{24s(f_1,f_2)},$$

$$M_{22} = \frac{-f_2(3(p-q) + r_1 + r_5) - (f_1 - f_2)(3(p-q) + r_1 + r_5) + 2s(f_1,f_2)(3(p+q) + r_1 + r_5)}{24s(f_1,f_2)},$$

$$M_{12} = -\frac{(-f_1 + 2f_2 + 2s(f_1,f_2))\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 - s(f_1,f_2))}}{24f_1 s(f_1,f_2)\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 + s(f_1,f_2))}}(-2f_1(r_1+r_5) + f_2(-p+q+(r_1+r_5))),$$

$$M_{21} = -\frac{(f_1 - 2f_2 + 2s(f_1,f_2))\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 + s(f_1,f_2))}}{24f_1 s(f_1,f_2)\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 - s(f_1,f_2))}}(-2f_1(r_1 + r_5) + f_2(-p + q + (r_1 + r_5))),$$

$$N_{11} = -\frac{f_1(f_1 - 2f_2 + s(f_1,f_2))\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 + s(f_1,f_2))}}{3s(f_1,f_2)|f_1|},$$

$$P_{22} = -\frac{f_1(-f_1 + 2f_2 + s(f_1,f_2))\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 - s(f_1,f_2))}}{3s(f_1,f_2)|f_1|},$$

$$N_{12} = N_{21} = -\frac{|f_1|((f_1 - f_2)(-(f_1 - f_2)^2 + 5(f_1 - f_2)s(f_1,f_2)) + f_1^2(-f_1 + s(f_1,f_2)))}{6s(f_1,f_2)(f_1)\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 - s(f_1,f_2))}},$$

$$P_{12} = P_{21} = -\frac{|f_1|((f_1 - f_2)((f_1 - f_2)^2 + 5(f_1 - f_2)s(f_1,f_2)) + f_1^2(f_1 + s(f_1,f_2)))}{6s(f_1,f_2)(f_1)\sqrt{6f_1^2 - 6f_1f_2 + f_2(2f_2 + s(f_1,f_2))}},$$

of the phenomenological diffusion matrix. The matrices N and P are the mode coupling matrices which yield the structure of the non-linear part of the coarse-grained evolution equation. We checked that for $f_1 \neq 0$ the expressions under the square roots will always be positive or zero, and that the denominators are not zero. By rewriting these equations in terms of the height gradients $\vec{\rho}(x,t) = \nabla\vec{h}(x,t)$ one gets a system of coupled Burgers equations.

B. Stationary space-time fluctuations

As has become clear from the previous section it is convenient to work with height gradients that map to densities $\rho_\alpha(x,t)$, which are globally conserved, i.e., $\int dx\rho_\alpha(x,t) = L\rho_\alpha$ for a system of length L. A fundamental quantity of interest is the dynamical structure function which are the stationary two-time correlations of the height gradients. For n conserved densities this is an $n \times n$ matrix with the two-point correlations between the (centered) densities $u_\alpha(x,t) = \rho_\alpha(x,t) - \rho_\alpha$ at time t_0 and $t_0 + t$. Because of stationarity t_0 is immaterial and can be set to 0.

1. *Nonlinear fluctuating hydrodynamics*

To study such a system with noisy dynamics on a coarse-grained level we follow the powerful and nonlinear fluctuating hydrodynamics (NLFH) approach [13] whose essence and main insights we briefly summarize.

Consider a system with n conserved densities ρ_α and associated locally conserved currents j_α. On coarse-grained Eulerian scale, where the noise drops out as a result of the law of large numbers, the conservation laws imply that the densities satisfy the nonlinear system of PDEs [9,21],

$$\frac{\partial}{\partial t}\vec{\rho}(x,t) + \frac{\partial}{\partial x}\vec{j}(x,t) = 0, \tag{18}$$

where component $\rho_\alpha(x,t)$ of the vector $\vec{\rho}(x,t)$ is a coarse-grained conserved quantity and the component $j_\alpha(x,t)$ of the current vector $\vec{j}(x,t)$ is the associated locally conserved current. Notice that in our convention $\vec{\rho}$ and $\vec{j}$ are regarded as column vectors.

Because of local stationarity under Eulerian scaling the current is a function of x and t only through its dependence on the local conserved densities. Hence, these equations can be rewritten as

$$\frac{\partial}{\partial t}\vec{\rho}(x,t) + J(x,t)\frac{\partial}{\partial x}\vec{\rho}(x,t) = 0, \tag{19}$$

where $J(x,t)$ is the current Jacobian with matrix elements $J_{\alpha\beta} = \partial j_\alpha/\partial\rho_\beta$, understood as functions of x and t via $\rho_\alpha(x,t)$ via the stationary current-density relation $\vec{j}^*(\vec{\rho})$. In other words, $\vec{j}(x,t) = \vec{j}^*(\vec{\rho}(x,t))$. Obviously, constant densities ρ_α are a (trivial) stationary solution of Eq. (19). Stationary fluctuations of the conserved quantities are captured in the compressibility matrix K that we shall not describe explicitly.

Up to this point the system Eq. (19), and therefore also its expansion in $u_\alpha(x,t)$, is completely deterministic. In the NLFH approach the effect of fluctuations is captured by adding a phenomenological diffusion matrix D and white noise terms ξ_i. This turns Eq. (19) into a system of nonlinear stochastic PDEs. From renormalization group considerations it is known that polynomial nonlinearities of order higher than 4 are irrelevant for the large-scale behavior and order 3 leads at most to logarithmic corrections if the generic quadratic nonlinearity is absent [22]. This justifies an expansion to second order so that the fluctuation fields $u_\alpha(x,t)$ satisfy the system of coupled noisy Burgers equations,

$$\partial_t\vec{u} = -\partial_x\left(J\vec{u} + \tfrac{1}{2}\vec{u}^T\vec{H}\vec{u} - D\partial_x\vec{u} + B\vec{\xi}\right), \tag{20}$$

where $\vec{H}$ is a column vector whose entries $(\vec{H})_\alpha = H^\alpha$ are the Hessians with matrix elements $H^\alpha_{\beta\nu} = \partial^2 j_\alpha/(\partial\rho_\beta\partial\rho_\nu)$. If the quadratic nonlinearity is absent one has diffusive behavior. We stress that the Hessians $\vec{H}$ depend on the stationary densities around which one expands, but not on space and time. Hence they are fixed by the stationary current-density relation $\vec{j}^*(\vec{\rho})$.

To proceed further it is convenient to transform into normal modes $\vec{\phi} = R\vec{u}$, where $RJR^{-1} = \mathrm{diag}(v_\alpha)$ and the transformation matrix R. The eigenvalues v_α of J play the role of characteristic speeds that on microscopic scale describe the speed of local perturbations [23]. One thus arrives at

$$\partial_t\phi_\alpha = -\partial_x[v_\alpha\phi_\alpha + \vec{\phi}^T G^\alpha\vec{\phi} - \partial_x(\tilde{D}\vec{\phi})_\alpha + (\tilde{B}\vec{\xi})_\alpha], \tag{21}$$

with $\tilde{D} = RDR^{-1}$ and $\tilde{B} = RB$. The matrices

$$G^{\alpha} = \frac{1}{2}\sum_{\gamma} R_{\alpha\gamma}(R^{-1})^{T} H^{\gamma} R^{-1} \tag{22}$$

are the mode coupling matrices with the mode-coupling coefficients $G^{\alpha}_{\beta\gamma} = G^{\alpha}_{\gamma\beta}$, which are, by construction, symmetric. They are said to satisfy the trilinear condition if they satisfy also the symmetry $G^{\alpha}_{\beta\gamma} = G^{\beta}_{\alpha\gamma}$ [14,15].

The main quantities of interest are then dynamical structure functions,

$$S^{\alpha\beta}(x,t) = \langle \phi^{\alpha}(x,t)\phi^{\beta}(0,0)\rangle, \tag{23}$$

which describe the stationary space-time fluctuations of the normal modes. They satisfy the normalization

$$\int_{-\infty}^{\infty} dx\, S^{\alpha\beta}(x,t) = \delta_{\alpha,\beta}, \tag{24}$$

which arises from the conservation law and the normalization condition $RKR^{T} = \mathbb{1}$. It is important to note that in the absence of long-range order and long-range jumps generally the product JK of the Jacobian with the compressibility matrix K is symmetric, which can be proved mathematically rigorously [24]. This guarantees that on macroscopic scale the full nonlinear system Eq. (19) is hyperbolic [25], i.e., characteristic velocities v_{α} are real.

When the characteristic velocities are all different, i.e., in the strictly hyperbolic case, the off-diagonal terms $S^{\alpha\beta}$ decay quickly and for long times and large distances one is left with the diagonal elements $S^{\alpha\alpha}(x,t)$ which are asymptotically universal functions $S^{\alpha\alpha}(x,t) \sim t^{-1/z_{\alpha}} f(u_{\alpha})$ with the scaling variable $u_{\alpha} = (x - v_{\alpha}t)^{z_{\alpha}}/t$. Here z_{α} is the dynamical exponent.

These scaling functions can be evaluated using mode coupling theory [13,26]. As pointed out in the introduction, in systems with short-range interactions there is an infinite discrete family of universality classes with dynamical exponents z_{α} that are the Kepler ratios of neighboring Fibonacci numbers $F_{\alpha+2}/F_{\alpha+1}$ [4], beginning with $z_1 = 2 = F_3/F_2$ corresponding to diffusion and Gaussian scaling function f, followed by $z_{\alpha} = 3/2,5/3,8/5,\ldots$. Also the limit value of this sequence, which is the golden mean $\phi = (1+\sqrt{5})/2$, can arise.

Which dynamical universality classes appear depends on which diagonal elements of the mode coupling matrix vanish. A full classification for $n = 2$ is given in Refs. [3,7] and for general n in Ref. [26]. For $n = 2$ one can have diffusion with $z = 2$, and also exponents $z = 3/2,5/3,\phi$. The dynamical exponent $z = 3/2$ can describe the KPZ universality class [2] (in which case the scaling function f is the celebrated Prähofer-Spohn function [27]), or a modified KPZ universality class with unknown scaling function [3], or a Lévy universality class [3,5,26,28]. The $z = 5/3$ Lévy class characterizes the heat mode in anharmonic chains [29,30] and one-dimensional fluids obeying Hamiltonian dynamics [6].[1] Experimental evidence for anomalous heat conduction has been found in single multiwalled carbon and boron-nitride nanotubes at room temperature [34].

The upshot of the mode coupling treatment of NLFH is that *the dynamical universality classes can be directly inferred from the structure of the mode coupling matrices, which in turn is fully determined by the stationary current-density relation* $\vec{j}^{*}(\vec{\rho})$ *for the conserved densities* $\vec{\rho}$ *of the system.*

The theory of nonlinear fluctuating hydrodynamics combined with mode-coupling theory is rather robust. It relies fundamentally on the presence long-lived long wave-length modes which arise from the conservation laws. Excluded are (i) systems that exhibit long-range order in the stationary state, in which case complex characteristic velocities indicative of phase separation [35–37] may arise. (ii) In systems with long-range interactions other discrete dynamical exponents may appear, e.g., the ballistic universality class with $z = 1$ in nearest-neighbor hopping with long-range dependence of the hopping rate [38–40], or in models with long-range jumps such as the raise-and-peel model [41], or the Oslo rice pile model for which a numerically determined dynamical exponent is conjectured to take the rational value $z = 10/7$ [42]. (iii) Also integrable models with non-local conservation laws might conceivably exhibit dynamical exponents that are not Kepler ratios. However, so far there is no evidence for such an anomaly [43].

The family of height models considered here falls into neither of these three long-range categories (i)–(iii) and therefore one expects all dynamical exponents to be the Kepler ratios derived in Ref. [4]. They appear in combinations that can be derived from the mode coupling matrices for a general number of conservation laws following Ref. [26] and specifically for $n = 2$ from the earlier work [3,7]. In the following we compute the mode coupling matrices for the directed polymer model first for $n = 2$ (corresponding to $d = 3$) and then for general n to work out the dynamical universality classes of the n generalized height functions.

2. *Fluctuations in d = 3*

In the following, we apply the approach based on NLFH that we have outlined above to the directed polymer model in three dimensions, with the aim of identifying its universal classes through analysis of the mode-coupling matrix.

When $f_1 = 0$ the matrices appearing in the quadratic term in the right-hand side of Eq. (15) are the mode coupling matrices Eq. (22) introduced above. One sees that the height variable h_2 has neither a quadratic self-coupling nor a nonlinear coupling to h_1. On the other hand, h_1 has a nonvanishing quadratic nonlinearity, but no quadratic coupling to the diffusive mode. Hence according to [3,7] the evolution of h_2 is diffusive and mode 1 is KPZ.

In the matrices N and P one recognizes the mode coupling matrices G^{α} Eq. (22) arising from NLFH. Thus the universality classes can be identified. Since both mode coupling matrices have generically nonvanishing self-coupling coefficients N_{11} and P_{22} we arrive at the conclusion that generically the two-component height model has two KPZ modes whose centers of mass drift away from each other with speeds Eq. (16). Similar

[1]The dynamical exponent $z = 5/3$ has also been reported for heat transport in hard-point particle gases [31], but universality for this system has been challenged recently [32,33].

models were studied by Kim and den Nijs [44] and Ferrari, Sasamoto, and Spohn [14].

Notice, however, that $s(f,f) = s(f,0) = f$. Therefore, when $f_1 = f_2 =: f \neq 0$ one has $s(f,f) = f$ and, therefore, $N_{11} = 0$, $P_{22} \neq 0$. In this case, mode 1 is diffusive while mode 2, which has no coupling to the diffusive mode, is KPZ. On the other hand, when $f_1 = f \neq 0$ and $f_2 = 0$ one gets $N_{11} \neq 0$, $P_{22} = 0$, which is the same scenario with the role of two modes interchanged. Therefore, also mixed dynamics may occur. In the trivial case where $f_1 = f_2 = 0$ both modes are diffusive.

IV. THE n-COMPONENT PARTICLE EXCHANGE PROCESS

As discussed above the mapping between the height model and exclusion can be applied to any dimension $d \geqslant 2$. Here we define the corresponding multispecies exclusion process and discuss it in detail in the hopping rates for which the stationary distribution factorizes. We shall call this process the n-component particle exchange process (PEP). For more general exclusion processes with nearest-neighbor particle exchange and nonfactorized stationary distributions we refer to Refs. [45,46] and, for the present context, to Ref. [14]. We derive the exact mode coupling matrices in explicit form and thus identify the possible universality classes for arbitrary dimension d.

A. Definition and stationary properties

In the n-component PEP an exclusion particle of type $\alpha \in \{0,1,\ldots,M\}$ on site k exchanges with type β on site $k+1$ with rate $g_{\alpha,\beta}$, symbolically

$$A_\alpha A_\beta \to A_\beta A_\alpha \quad \text{with rate } g_{\alpha,\beta}.$$

Type 0 is called vacancy and we speak of M distinct conserved species of particles. The total number of particles of each species in the system is denoted N_α. We consider L sites with periodic boundary conditions. It is convenient to decompose the rates into a symmetric part $w_{\alpha,\beta} = w_{\beta,\alpha} > 0$ for $\alpha \neq \beta$ and an antisymmetric part $f_{\alpha,\beta} = -f_{\beta,\alpha}$ in the form

$$g_{\alpha,\beta} = \tfrac{1}{2}(w_{\alpha,\beta} + f_{\alpha,\beta}). \tag{25}$$

Positivity of the rates implies $w_{\alpha,\beta} \geqslant |f_{\alpha,\beta}|$. For convenience we define $w_{\alpha,\alpha} = f_{\alpha,\alpha} = 0$ and denote the vacuum driving fields for particles with vacant neighbors by

$$f_\alpha := f_{\alpha,0}, \tag{26}$$

which implies, by definition, $f_0 = 0$. If for some α one has $w_{\alpha,0} = |f_\alpha|$, the vacuum motion of species α is totally asymmetric.

From pairwise balance [19] we find that the canonical stationary distribution with N_α particles is uniform, provided that the condition

$$f_{\alpha,\beta} = f_\alpha - f_\beta \tag{27}$$

is satisfied with driving fields in the physical domain $|f_\alpha| \leqslant w_{\alpha,0}$. It follows that the grandcanonical stationary ensemble with fluctuating particle numbers is a product measure defined by fugacities μ_α, or equivalently, particle densities

$$\rho_\alpha := \langle N_\alpha \rangle / L = \frac{e^{\mu_\alpha}}{\sum_{\alpha'=0}^{M} e^{\mu_{\alpha'}}}. \tag{28}$$

The product structure leads to the covariances (generalized compressibilities)

$$\begin{aligned}\kappa_{\alpha\beta} := \frac{\partial \rho_\alpha}{\partial \mu_\beta} &= 1/L\langle (N_\alpha - \langle N_\alpha \rangle)(N_\beta - \langle N_\beta \rangle)\rangle \\ &= \rho_\alpha(\delta_{\alpha,\beta} - \rho_\beta).\end{aligned} \tag{29}$$

We denote the compressibility matrix with matrix elements $\kappa_{\alpha\beta}$ by K. By construction, $K = K^T$ is symmetric.

Consider the local density $\rho_k^\alpha := \langle n_k^\alpha \rangle$, i.e., the expectation of the local particle number $n_k^\alpha \in \{0,1\}$. Particle number conservation implies the discrete continuity equation

$$\frac{d}{dt}\rho_k^\alpha = j_{k-1}^\alpha - j_k^\alpha, \tag{30}$$

where, by definition of the process, the expected local current of species α is given by

$$j_k^\alpha = \sum_{\beta=0}^{M} g_{\alpha,\beta}\langle n_k^\alpha n_{k+1}^\beta \rangle - g_{\beta,\alpha}\langle n_k^\beta n_{k+1}^\alpha \rangle. \tag{31}$$

In the grandcanonical stationary distribution one has

$$j_\alpha = \rho_\alpha \left(f_\alpha - \sum_{\beta=1}^{M} f_\beta \rho_\beta \right). \tag{32}$$

This follows from the factorization property of the grandcanonical stationary distribution.

B. Collective velocities

As discussed above one expects in the hydrodynamic limit on Euler scale the system of conservation laws Eq. (19) where J is the flux Jacobian with matrix elements,

$$J_{\alpha\beta} = \frac{\partial j_\alpha}{\partial \rho_\beta} = \left[f_\alpha - \sum_{\gamma=1}^{M} f_\gamma \rho_\gamma \right] \delta_{\alpha,\beta} - f_\beta \rho_\alpha. \tag{33}$$

To derive the normal modes for nonzero densities and nonzero driving fields we introduce the diagonal matrices $\hat{\rho} := \mathrm{diag}(\rho_\alpha)$ and $\hat{f} := \mathrm{diag}(f_\alpha)$ with the densities and driving fields resp. on the diagonal. Then we can write

$$J = D^{-1} B D, \tag{34}$$

where $D = \sqrt{\hat{f}/\hat{\rho}}$ and $B = B^T$. The nondiagonal matrix elements of B are $B_{\alpha\beta} = -\sqrt{f_\alpha \rho_\alpha f_\beta \rho_\beta}$. This implies that J can be diagonalized with the help of D and an orthogonal matrix $\mathcal{O}$. With $\hat{J} := \mathrm{diag}(v_i)$, one can write

$$R J R^{-1} = \hat{J}, \tag{35}$$

where $R = Q^{-1}\mathcal{O}D$ and $R^{-1} = D^{-1}\mathcal{O}^T Q$ with an invertible diagonal matrix $Q = \mathrm{diag}(q_\alpha)$. Notice also that $R^T = D\mathcal{O}^T Q^{-1}$ and $(R^{-1})^T = Q\mathcal{O}D^{-1}$. Choosing Q such that

$$R K R^T = \mathbb{1}, \tag{36}$$

one obtains an orthonormal basis of the modes. To compute the matrix Q we observe that

$$J = KD^2 - \sum_\alpha f_\alpha \rho_\alpha \mathbb{1}. \tag{37}$$

Thus, $RKR^T = Q^{-1}\mathcal{O}DJD^{-1}\mathcal{O}^T Q^{-1} + \sum_\alpha f_\alpha \rho_\alpha Q^{-1}\mathcal{O}\mathcal{O}^T Q^{-1} = (RAR^{-1} + \sum_\alpha f_\alpha \rho_\alpha \mathbb{1})Q^{-2}$. This yields

$$q_\alpha^2 = v_\alpha + \sum_\alpha f_\alpha \rho_\alpha. \tag{38}$$

We remark that decomposing J into a traceless part and the trace yields

$$J = \tilde{J} + \frac{1}{M}\sum_\alpha f_\alpha (1 - (M+1)\rho_\alpha)\mathbb{1}, \tag{39}$$

which can be written in the form $J = D^{-1}\tilde{B}D + V\mathbb{1}$ with traceless and symmetric $\tilde{B}$. For the completely symmetric state with $\rho_\alpha = 1/(M+1)$ as for the generalized height model one has $V = 0$ and the collective velocities are the eigenvalues of $\tilde{B}$. On the other hand, for equal driving fields $f_\alpha = f$ one has $V = f(1 - \sum_{\alpha=1}^M \rho_\alpha)$, which vanishes only for the completely filled lattice. This then is the multispecies simple exclusion process.

C. Mode-coupling coefficients

The Hessians

$$H^\gamma_{\alpha\beta} := \frac{\partial^2 j^\gamma}{\partial\rho_\alpha \partial\rho_\beta} = \partial_\alpha J_{\gamma\beta} \tag{40}$$

are constants

$$H^\gamma_{\alpha\beta} = -(f_\alpha \delta_{\beta,\gamma} + f_\beta \delta_{\alpha,\gamma}). \tag{41}$$

This simple form allows us to compute explicitly the mode-coupling coefficients

$$G^\gamma_{\alpha\beta} := \frac{1}{2}\sum_\lambda R_{\gamma\lambda}[(R^{-1})^T H^\lambda R^{-1}]_{\alpha\beta}. \tag{42}$$

According to the definitions given above, we have

$$D_{\alpha\beta} = \sqrt{\frac{f_\alpha}{\rho_\alpha}}\delta_{\alpha,\beta} \tag{43}$$

and

$$R_{\alpha\beta} = q_\alpha^{-1}\sqrt{\frac{f_\beta}{\rho_\beta}}\mathcal{O}_{\alpha\beta},$$

$$(R^{-1})_{\alpha\beta} = q_\beta\sqrt{\frac{\rho_\alpha}{f_\alpha}}\mathcal{O}_{\beta\alpha} = q_\beta^2\frac{\rho_\alpha}{f_\alpha}(R^T)_{\alpha\beta},$$

$$(R^T)_{\alpha\beta} = q_\beta^{-1}\sqrt{\frac{f_\alpha}{\rho_\alpha}}\mathcal{O}_{\beta\alpha},$$

$$((R^{-1})^T)_{\alpha\beta} = q_\alpha\sqrt{\frac{\rho_\beta}{f_\beta}}\mathcal{O}_{\alpha\beta} = q_\alpha^2\frac{\rho_\beta}{f_\beta}R_{\alpha\beta}. \tag{44}$$

Hence, by straightforward computation

$$\begin{aligned}
G^\gamma_{\alpha\beta} &= \frac{1}{2}\sum_\lambda\sum_\mu\sum_\nu R_{\gamma\lambda}((R^{-1})^T)_{\alpha\mu}H^\lambda_{\mu\nu}(R^{-1})_{\nu\beta}\\
&= -\frac{1}{2}\sum_\mu\sum_\nu[f_\mu R_{\gamma\nu}((R^{-1})^T)_{\alpha\mu}(R^{-1})_{\nu\beta}\\
&\quad + f_\nu R_{\gamma\mu}((R^{-1})^T)_{\alpha\mu}(R^{-1})_{\nu\beta}]\\
&= -\frac{1}{2}\sum_\mu[f_\mu\,((R^{-1})^T)_{\alpha\mu}\,\delta_{\beta,\gamma} + f_\mu\,(R^{-1})_{\mu\beta}\,\delta_{\alpha,\gamma}]\\
&= -\frac{1}{2}\left[q_\alpha^2\sum_\mu R_{\alpha\mu}\rho_\mu\,\delta_{\beta,\gamma} + q_\beta^2\sum_\mu R_{\beta\mu}\rho_\mu\,\delta_{\alpha,\gamma}\right]\\
&= -\frac{1}{2}\big[q_\alpha^2(R\vec{\rho})_\alpha\,\delta_{\beta,\gamma} + q_\beta^2(R\vec{\rho})_\beta\,\delta_{\alpha,\gamma}\big].
\end{aligned} \tag{45}$$

We point out the nontrivial trilinear property $G^\gamma_{\alpha\beta} = G^\alpha_{\gamma\beta}$, which one expects for systems where the driving force does not change the stationary distribution [14,15].

For the diagonal elements, one has

$$G^\gamma_{\alpha\alpha} = -q_\alpha^2\,(R\vec{\rho})_\alpha\,\delta_{\alpha,\gamma}. \tag{46}$$

Hence, generically all modes are KPZ and there are only subleading corrections since $G^\gamma_{\alpha\alpha} = 0$ for $\alpha \neq \gamma$. If one of the coefficients $q_\alpha^2(R\vec{\rho})_\alpha$ vanishes, then this mode is diffusive and all other modes evolve independently of this mode.

D. Details for two conservation laws

We return to the case $n = 2$ and at least one driving field nonzero and present the diagonalization explicitly and in detail for arbitrary densities. For the two-component PEP with arbitrary densities ρ_α, we have

$$j_1 = f_1\rho_1(1-\rho_1) - f_2\rho_1\rho_2, \quad j_2 = f_2\rho_2(1-\rho_2) - f_1\rho_1\rho_2, \tag{47}$$

and the compressibility matrix is given by

$$K = \begin{pmatrix}\rho_1(1-\rho_1) & -\rho_1\rho_2\\ -\rho_1\rho_2 & \rho_2(1-\rho_2)\end{pmatrix}. \tag{48}$$

We find

$$\begin{aligned}
J &= \begin{pmatrix} f_1(1-2\rho_1) - f_2\rho_2 & -f_2\rho_1\\ -f_1\rho_2 & f_2(1-2\rho_2) - f_1\rho_1\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{2}(f_1(1-\rho_1) - f_2(1-\rho_2)) & -f_2\rho_1\\ -f_1\rho_2 & -\frac{1}{2}(f_1(1-\rho_1) - f_2(1-\rho_2))\end{pmatrix} + \frac{1}{2}\sum_{\alpha=1}^2 f_\alpha(1-3\rho_\alpha)\mathbb{1}.
\end{aligned} \tag{49}$$

To compute the eigenvalues of J we use Eq. (39). For $n = 2$ this yields as eigenvalues of $\tilde{B}$ the quantities $\pm\sqrt{\det \tilde{B}}$ and, therefore,

$$v_{1,2} = \frac{1}{2}\Bigg[\sum_{\alpha=1}^{2} f_\alpha(1-3\rho_\alpha) \pm\sqrt{[f_1(1-\rho_1)-f_2(1-\rho_2)]^2+4f_1\rho_1 f_2\rho_2}\Bigg]. \quad (50)$$

In the domain of interest $0 < \rho_1 + \rho_2 < 1$ one has $\det \tilde{B} > 0$. Hence, the corresponding system of conservation laws is strictly hyperbolic. For the special case $\rho_1 = \rho_3 = 1/3$ we recover Eq. (16).

To compute R we define the orthgonal matrix

$$\mathcal{O} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \quad (51)$$

Straightforward computation shows that J is diagonalized with the choice

$$\tan(2\phi) = \frac{2\sqrt{f_1\rho_1 f_2\rho_2}}{f_1(1-\rho_1)-f_2(1-\rho_2)}. \quad (52)$$

This yields, together with Eq. (36),

$$R = \begin{pmatrix} q_1^{-1}\sqrt{\frac{f_1}{\rho_1}}\cos\phi & -q_1^{-1}\sqrt{\frac{f_2}{\rho_2}}\sin\phi \\ q_2^{-1}\sqrt{\frac{f_1}{\rho_1}}\sin\phi & q_2^{-1}\sqrt{\frac{f_2}{\rho_2}}\cos\phi \end{pmatrix},$$
$$R^{-1} = \begin{pmatrix} q_1\sqrt{\frac{\rho_1}{f_1}}\cos\phi & q_2\sqrt{\frac{\rho_1}{f_1}}\sin\phi \\ -q_1\sqrt{\frac{\rho_2}{f_2}}\sin\phi & q_2\sqrt{\frac{\rho_2}{f_2}}\cos\phi \end{pmatrix}, \quad (53)$$

where

$$q_i^2 = v_i + f_1\rho_1 + f_2\rho_2. \quad (54)$$

The Hessians are

$$H^1 = -\begin{pmatrix} 2f_1 & f_2 \\ f_2 & 0 \end{pmatrix}, \quad H^2 = -\begin{pmatrix} 0 & f_1 \\ f_1 & 2f_2 \end{pmatrix}, \quad (55)$$

and Eq. (42) yields

$$G^1 = -\frac{1}{2}\begin{pmatrix} 2g_1 & g_2 \\ g_2 & 0 \end{pmatrix}, \quad G^2 = -\frac{1}{2}\begin{pmatrix} 0 & g_1 \\ g_1 & 2g_2 \end{pmatrix}, \quad (56)$$

with coupling constants

$$g_1 = q_1(\sqrt{f_1\rho_1}\cos\phi - \sqrt{f_2\rho_2}\sin\phi),$$
$$g_2 = q_2(\sqrt{f_1\rho_1}\sin\phi - \sqrt{f_2\rho_2}\cos\phi). \quad (57)$$

As expected, generically both modes are KPZ with subleading corrections.

Care has to be taken if $f_1 = 0$ and $f_2 = f \neq 0$. Then,

$$j_1 = -f\rho_1\rho_2, \quad j_2 = f\rho_2(1-\rho_2) \quad (58)$$

and

$$J = \begin{pmatrix} -f\rho_2 & -f\rho_1 \\ 0 & f(1-2\rho_2) \end{pmatrix}. \quad (59)$$

This yields the collective velocities

$$v_1 = -f\rho_2, \quad v_2 = f(1-2\rho_2) \quad (60)$$

and

$$R = \begin{pmatrix} \sqrt{\frac{1-\rho_2}{\rho_1(1-\rho_1-\rho_2)}} & \sqrt{\frac{\rho_1}{(1-\rho_1-\rho_2)(1-\rho_2)}} \\ 0 & \frac{1}{\sqrt{\rho_2(1-\rho_2)}} \end{pmatrix},$$
$$R^{-1} = \begin{pmatrix} \sqrt{\frac{\rho_1(1-\rho_1-\rho_2)}{1-\rho_2}} & -\rho_1\sqrt{\frac{\rho_2}{1-\rho_2}} \\ 0 & \sqrt{\rho_2(1-\rho_2)} \end{pmatrix}. \quad (61)$$

For the Hessians, one has

$$H^1 = -f\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H^2 = -2f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (62)$$

Then Eq. (42) yields

$$G^1 = -\frac{f}{2}\sqrt{\rho_2(1-\rho_2)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$G^2 = -f\sqrt{\rho_2(1-\rho_2)}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (63)$$

(We remind the reader that the labels at G and H are upper indices, not powers.) Hence, mode 1 is diffusive and mode 2 is KPZ.

The case $f_2 = 0$ and $f_1 = f \neq 0$ follows by symmetry. We also consider $f_1 = f_2 = f$. In this case, Eq. (52) yields $\tan\phi = \sqrt{\rho_1/\rho_2}$, in which case Eq. (57) gives $g_1 = 0$, or $\tan\phi = -\sqrt{\rho_2/\rho_1}$ in which case $g_2 = 0$. Hence, one of the modes is diffusive, as first argued in Ref. [47].

V. CONCLUSIONS

We have treated coupled nonlinear stochastic PDE equations of KPZ type in two different contexts: A model for directed polymers in $d = 3$ where we derived from a dynamical mean field approach a system of two coupled partial differential equations, and from nonlinear fluctuating hydrodynamics theory where the same equations are shown to follow from conservation laws for the densities and the presence of noise. These equations can then be treated in mode coupling theory. Both approaches lead to the same structure of the mode coupling terms.

Next, we generalized the lattice gas approach to an arbitrary number n of conserved particle species, corresponding to a model for directed polymers in $d = n + 1$ dimensions. Thus, we give a direct physical link between fluctuations in the conformations of the polymer and the underlying particle exchange processes on the lattice. This allows in particular to understand and access different cases of the general classification given in Refs. [3,7] for two conservation laws and for an arbitrary number of conservation laws in Refs. [4,26]. It turns out that stationary spatiotemporal fluctuations in the polymer model are generally either diffusive or in the universality class of the one-dimensional KPZ equation.

This behavior of the coupled one-dimensional KPZ equations is in contrast to the two-dimensional KPZ equation which was investigated mostly numerically over the last years [48–52]. A major difference to the one-dimensional KPZ

universality class shows up already at the level of critical exponents. The 1D static critical KPZ exponent χ, which in lattice gas language reflects the fluctuations of the particle number in a large but finite segment of the 1D lattice, takes the exact value $1/2$, while in two dimensions one finds numerically quite accurate values ranging from 0.363 to 0.393, depending on the choice of microscopic model (see Ref. [52] for an overview that includes also earlier numerical results). For the scaling exponent $\beta = \chi/z$ Halpin-Healy finds in two dimensions numerical values between 0.235 and 0.248 [49] rather than the exact value $1/3$ for one dimension. On the microscopic level the difference between the fluctuating polymer considered here and the three-dimensional directed polymer model of Ref. [49] is that in our case each polymer conformation has the same energy, while in Ref. [49] the directed polymer is a directed walk through a 3D lattice of random energy sites such that the total path energy is the sum of the site energies visited along the way. It thus appears that two coupled one-dimensional KPZ equations behave fundamentally differently from a two-dimensional single KPZ equation.

There is an interesting open problem: The totally asymmetric two-component model ($w_{10} = f_1$, $w_{20} = f_2$ $w_{12} = f_1 - f_2$) is integrable [53]. Can one use the integrability to obtain directly the exact scaling form of the dynamical structure function? From the results of the present work one expects this to be the Praehofer-Spohn scaling function [27] for each mode separately.

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