Killing (absorption) versus survival in random motion

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We address diffusion processes in a bounded domain, while focusing on somewhat unexplored affinities between the presence of absorbing and/or inaccessible boundaries. For the Brownian motion (Lévy-stable cases are briefly mentioned) model-independent features are established of the dynamical law that underlies the short-time behavior of these random paths, whose overall lifetime is predefined to be long. As a by-product, the limiting regime of a permanent trapping in a domain is obtained. We demonstrate that the adopted conditioning method, involving the so-called Bernstein transition function, works properly also in an unbounded domain, for stochastic processes with killing (Feynman-Kac kernels play the role of transition densities), provided the spectrum of the related semigroup operator is discrete. The method is shown to be useful in the case, when the spectrum of the generator goes down to zero and no isolated minimal (ground state) eigenvalue is in existence, like in the problem of the long-term survival on a half-line with a sink at origin.

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I. MOTIVATION

Diffusion processes in a bounded domain (likewise the jump-type Lévy processes) serve as important model systems in the description of varied spatio-temporal phenomena of random origin in nature. When arbitrary domain shapes are considered, one deals with highly sophisticated problems on their own, an object of extensive investigations in the mathematical literature.

A standard physical inventory, in the case of absorbing boundary conditions which are our concern in the present paper, refers mostly to the statistics of exits, e.g., first and mean first exit times, probability of survival, and its asymptotic decay, thence various aspects of the lifetime of the pertinent stochastic process in a bounded domain [1–6]; see also Refs. [7–9].

Typically one interprets the survival probability as the probability that not a single particle may hit the domain boundary before a given time T. The long-time survival is definitely not a property of the free Brownian motion in a domain with absorbing boundaries, where the survival probability is known to decay to zero exponentially with $T \to \infty$ [4,5]. Therefore, the physical conditions that ultimately give rise to a long-living random system, like those considered in Refs. [4,5] (see also Ref. [6]), must result in a specific remodeling (conditioning, deformation, emergent, or "engineered" drift) of the "plain" Brownian motion.

For simple geometries (interval, disk, and the sphere) the exponential decay of the single-particle survival probability has been identified to scale the stationary (most of the time) gas density profile, while that profile and the decay rates stem directly from spectral solutions of the related eigenvalue problem for the Laplacian with the Dirichlet boundary data, respectively on the interval, disk, and sphere; see, e.g., Refs. [1,4,5]. In fact, the square of the lowest eigenfunction, upon normalization, has been found to play the role of the pertinent gas profile density, while the associated lowest eigenvalue of the motion generator determines the decay rate.

These observations have been established within so-called macroscopic fluctuation theory (of particle survival). Effectively, that was also the case in Ref. [6], where a suitable choice of the Monte Carlo updating procedure has resulted in the increase of the survival probability in the diffusion model. That has been paralleled by a temporarily favored motion trend (engineered drift), away from the boundaries, directed towards the midinterval locations.

The long-lifetime regime of a diffusion process in a bounded domain may be comparatively set against that the infinite lifetime (trap, with the inaccessible boundaries); see, e.g., Refs. [10,11] and references therein. The limiting behavior (with respect to the lifetime) of absorbing diffusion processes and symptoms of their convergence towards permanently trapped relatives (never leaving a bounded domain) is worth investigation.

It is useful to mention our earlier analysis of the permanent trapping problem, including a fairly serious question about what is actually meant by the Brownian motion in a trap (interval), with a preliminary discussion of that issue for jump-type processes of Lévy type [11].

A common formal input, both for absorbing and permanently trapped diffusion processes in a bounded domain, is that of spectral problems for Dirichlet Laplacians and Laplacians perturbed by suitable potentials; see Refs. [10–13]. The notion of Markovian semigroup operators and their integral (heat) kernels is here implicit, and a pathwise description by means of the Feynman-Kac formula is feasible.

That entails an exploration of affinities of general killed diffusion processes with diffusions with an infinite lifetime. We point out that the notion of killing stems from a probabilistic interpretation of the Feynman-Kac formula [14–16].

Our discussion departs from much earlier investigations of random processes that either stay forever within a prescribed convex domain or are bound to avoid such a domain while "living" in its exterior [1]. We are strongly motivated by the past mathematical research, whose roots can be traced back to Knight's "taboo processes" [17–19].

We introduce a direct conditioning method that essentially relies on pathwise intuitions underlying the notion of the Feynman-Kac semigroup transition kernel, given the diffusion generator. It is based on the concept of the Bernstein transition function (actually a conditional probability density) [20,22], which in the present paper is explored as a diagnostic tool

for the description of dynamical properties, e.g., an emergent dynamical law, of short-time paths segments in a "bunch," predefined to be long-living, sample trajectories of a stochastic diffusion process with absorption.

The Bernstein function has an appearance of the (x,t)-dependent probability density, associated with paths pinned at two fixed space-time points: (y,s) (initial) and u,T (terminal), $s < t < T$. Our finding is that if $T \gg t$ is large enough, then the Bernstein function is approximated by (and, with $T \to \infty$, ultimately reduces to) the transition probability density $p(x,t|y,s)$ of the diffusion process with an infinite lifetime, where the u dependence is absent. The is the embodiment of the sought-for dynamical law. The corresponding Fokker-Planck equation follows.

Our major playground is Markovian diffusion processes in a bounded domain with absorbing boundaries. Next, we shall demonstrate the validity of extensions of our strategy to absorbing processes in unbounded domains (like a sink on the half-line) and to more general Markovian processes with killing.

We indicate that our conditioning method is not specific to the standard Brownian (diffusions) "territory." One can readily pass from Brownian diffusions to jump-type Lévy stable stochastic processes, whose restriction to a bounded domain, or the case of an unbounded domain with a sink, have been an object of vivid studies in the current literature.

II. INTERVAL WITH ABSORBING ENDS AND TERMS OF SURVIVAL

A. Preliminaries

Diffusion processes in the interval with various boundary conditions have become favored model systems in the statistical physics approach to the Brownian motion and were often exploited in unexpected settings (e.g., prisoner in an expanding cage on the cliff), including the extensions of the formalism to higher dimensions [1,21]; see also Refs. [2,3,7]. In Ref. [1] there is a whole chapter about diffusion in the interval with absorption at its ends. Other inspirations (bounded variation of interest in economy, harmonic and optical traps) can be gained from Refs. [7–9] (see also Ref. [11]) where the permanent trapping problem has been addressed.

Let us consider the free diffusion $\partial_t k = D\Delta_x k$ within an interval $\mathcal{D} = [a,b] \subset R$, with absorbing boundary conditions at its end points a and b. The time and space homogenous transition density, with $x,y \in (a,b), 0 \leqslant s < t$ and $b-a = L$, reads

$$k(x,t|y,s) = \frac{2}{L}\sum_{n=1}^{\infty} \sin\left[\frac{n\pi}{L}(x-a)\right]\sin\left[\frac{n\pi}{L}(y-a)\right] \times \exp\left[-\frac{Dn^2\pi^2}{L^2}(t-s)\right]. \quad (1)$$

Note that $\lim_{t\to s} k(x,t|y,s) \equiv \delta(x-y)$. We point out that in view of the time homogeneity, we can write $k(x,t|y,s) = k_{t-s}(x|y)$. One should keep in mind that k is a symmetric function of x and y, i.e., $k(yt|x,s) = k(y,t|x,s)$.

We deliberately use the notation $k(x,t|y,s)$ if probability is not conserved by the dynamics (it "leaks out" and decays with the growth of time), which is the case for absorbing boundary conditions. If the transition density would conserve probability in the interval (that corresponds to a diffusion never leaving $\mathcal{D}$), the standard notation $p(x,t|y,s)$ will be used.

While setting $a = 0$, $s = 0$, and $y = x_0$ in (1) we arrive at a customary notion of the concentration $k(x,t|x_0,0) = c(x,t)$, typically employed in the literature; see Ref. [1]. Initially one has $\int_0^L c(x,t=0)\,dx = 1$, hence for $t > 0$ the absorption at the boundaries enforces a decay of the probability density, whose time rate may be quantified by $(d/dt)\int_0^L c(x,t)\,dx$ where $t > 0$. The decay of $c(x,t)$ is known to be exponential, and its time rate is determined by the lowest positive eigenvalue of the Laplacian in a bounded domain (the role of so-called eigenfunction expansions [21] needs to be emphasized).

To simplify notation, we note that $x \to x' = (x-a)/L$ transforms the interval $[a,b]$ into $[0,1]$. Another transformation $x \to x' = x - \frac{1}{2}(a+b)$ maps $[a,b]$ into $[-c,c]$, with $c = L/2$, whose special case (set $L = 2$) is the interval $[-1,1]$. From now on we employ the symmetric interval $[-c,c]$ with $L = 2c, c > 0$. We also set $D = 1/2$.

B. (Dirichlet) heat kernel in a bounded domain and its pathwise interpretation

The transition density $k(x,t|y,s)$, as defined by Eq. (1), is known to be an integral (heat) kernel of the semigroup operator $\exp[\frac{1}{2}(t-s)\Delta_{\mathcal{D}}]$, where the notation $\Delta_{\mathcal{D}}$ directly refers to the standard Laplacian in the domain $\mathcal{D}$, with absorbing boundary conditions. The pertinent diffusion process is Markovian and has an explicit semigroup encoding in terms of the motion operator $T_{t-s} = \exp[\frac{1}{2}\Delta_{\mathcal{D}}(t-s)]$.

The semigroup property $T_{t-s}T_{s-r} = T_{t-r}$, with $t > s > r$, implies the validity of the composition rule

$$\int_{\mathcal{D}} k(x,t|y,s)\,k(y,s|z,r)\,dy = k(x,t|z,r).$$

Given a suitable function $f(x)$, its semigroup evolution is defined as

$$f(x) \to f(x,t) = (T_t f)(x) = \exp\left[\frac{1}{2}\Delta_{\mathcal{D}} t\right]$$

$$f(x) = \int_{\mathcal{D}} k(x,t|y,0)\,f(y)\,dy,$$

and $f(x,t)$ is a solution of the "heat" equation in $\mathcal{D}$: $\partial_t f(x,t) = \frac{1}{2}\Delta_{\mathcal{D}} f(x,t)$.

Semigroup kernels admit the pathwise interpretation by means of the Feynman-Kac formula (e.g., the path integral), which has become a classic [14] (actually that is possible in any bounded domain in R^n, an interval being a particular case). Indeed, $k(x,t|y,s)$ is prescribed for a specific "bunch" $\omega \in \Omega_{y,s}^{x,t}$ of sample paths of the Brownian motion: $\omega(\tau), s \leqslant \tau \leqslant t$ [here $\omega(s) = y$ stands for the point of origin while $\omega(t) = x$ is the destination point], all of which survive within $\mathcal{D} \equiv (-c,c)$ up to time $t > s$, e.g., their killing time exceeds t. We have

$$\left\{\exp\left[\frac{1}{2}\Delta_{\mathcal{D}}\,(t-s)\right]\right\}(x,y) = k(x,t|y,s) = \int d\mu_{y,s}^{x,t}(\omega) = \mu_{y,s}^{x,t}(\omega). \quad (2)$$

The notion of the conditional Wiener measure of the set of random paths connecting y with x, in the time interval (s,t) is here implicit [14]. (Note that the term "conditional" means here that two endpoints y and x are fixed, while in case of the Wiener measure only one point, that of the origin of random motion, is fixed.)

Following the traditional lore, we say that $\mu^{x,t}_{y,s}(\omega)$ is a total mass of the pinned paths set $\Omega^{x,t}_{y,s}$. It is the transition kernel $k(x,t|y,s)$ which is the pertinent mass measure.

For clarity of arguments, we presume $y < x$. Let us choose a window $I = [a,b] \subset [y,x]$ with $a < b$ (it is a standard preparatory step in the construction of the conditional Wiener measure). We can assign a mass to a specific subset $\Omega^{x,t}_{y,s}(I)$ of the considered pinned paths set that comprises these sample paths only, which at an intermediate time $r, s < r < t$ reach or cross the window I, before approaching the final destination x at time t. The mass of such subset of sample paths is known to be

$$\mu^{x,t}_{y,s}(I) = \int_I k(x,t|u,r)k(u,r|y,s)\,du < \mu^{x,t}_{y,s}(\mathcal{D}) = k(x,t|y,s). \tag{3}$$

Note that with I replaced by $\mathcal{D}$ in the above integral, the semigroup composition rule yields the mass $k(x,t|y,s)$.

The transition density $k(x,t|y,s)$ allows us to define a probability that the process started at y at time s will actually reach an interval $(x, x+\Delta x)$ at time $t > s$. It reads $P(\Omega([y,x]),\Delta x) = k(x,t|y,s)\Delta x$. Likewise we obtain a probability $P(\Omega(I),\Delta x) = \mu^{x,t}_{y,s}(I)\Delta x$ for the I-constrained subset of paths.

The ratio

$$\frac{P(\Omega(I),\Delta x)}{P(\Omega([y,x]),\Delta x)} = \int_I \frac{k(x,t|u,r)\,k(u,r|y,s)}{k(x,t|y,s)}\,du \tag{4}$$

is nothing but a conditional probability quantifying the fraction of mass of the a subset of paths crossing I at time r, while set against the overall mass of all sample paths with origin y at s and destination x, at time t, $s < r < t$.

Under the integral sign in Eq. (4) we encounter a conditional probability density (with respect to u), known as the Bernstein transition function, that has been investigated in a setting quite divorced from the present study contexts [20,22,23]: $B(x,t;u,r;y,s) = k(x,t|u,r)\,k(u,r|y,s)/k(x,t|y,s)$, where $\int_{\mathcal{D}} B(x,t;u,r;y,s)\,du = 1$

C. Conditioning via Bernstein transition function: Dynamical law that underlies long survival

Let us adjust the previous notation for the Bernstein function for the diffusion process (1) to refer to an overall time interval [0,T], whose duration T is arbitrarily assigned, but assumed *a priori* to be large. We focus attention on the transitional (x,t) behavior of the Bernstein function, for times $[s,t] \subset [0,T]$, $T \gg t > s \geqslant 0$), provided we fix two control space-time points: (y,s) for the origin, (u,T) for the target, leaving (x,t) as the unrestricted, "running" one.

Accordingly, we rewrite the Bernstein transition function as

$$B(u,T;x,t;y,s) = \frac{k(u,T|x,t)\,k(x,t|y,s)}{k(u,T|y,s)}, \tag{5}$$

remembering that presently it is a probability density with respect to $x \in [-c,c]$ (e.g., integrates to 1).

Although Eq. (5) explicitly determines the time t evolution of the Bernstein density, given (y,s) and (u,T), the main goal of our subsequent analysis is to deduce the detailed (as yet unspecified) dynamical rule for the Bernstein density, as a function of (x,t) which would have the form of a standard transport equation, appropriate for diffusion processes, like the Fokker-Planck one.

We point out that an analogous problem has been addressed in another context [20,22]. The outcome was the so-called the Bernstein stochastic process, whose Markovianess could be established under suitable (supplementary) conditions.

Since the time interval $(T-s) \gg s$ is large (and likewise $T-t$) if compared to T, the transition density $k(u,T|y,s)$ stands for the mass of all sample paths that have a large overall survival time $\sim T$, while running from y to u within $(-c,c)$. Moreover, the kernel $k(u,T|x,t)$, where $x \in (-c,c)$ may be chosen arbitrarily, refers to paths with a large survival time $\sim T$ as well. To the contrary, the kernel $k(x,t|y,s)$ quantifies the mass of sample paths that run from y to x in a relatively short (compared to T) time interval $t-s$.

Our further reasoning relies on asymptotic (large-time) properties of Feynman-Kac kernels. For a sufficiently large value of T, the dominant terms in the numerator and denumerator of Eq. (5) have a similar form. An exemplary asymptotic of of the transition density $k(u,T|y,s)$) reads

$$k(u,T|x,t) \sim \sin\left[\frac{\pi}{L}(x+c)\right]\sin\left[\frac{\pi}{L}(u+c)\right] \times \exp\left[-\frac{\pi^2}{2L^2}(T-t)\right], \tag{6}$$

and that for $k(u,T|y,s)$ is readily obtained from $k(u,T|x,t))$ by formal replacements $x \to y$, $t \to s$.

Accordingly (remember about $T \gg t$ and $L = 2c$), we arrive at a conditioned transition density where the (familiar in the mathematical literature) Doob-type conditioning of $k(x,t|y,s)$ is nontrivially modified by an emergent time-dependent factor $\exp[+\frac{\pi^2}{8c^2}(t-s)]$. We actually have

$$B(u,T;x,t;y,s) \sim p(x,t|y,s) = k(x,t|y,s)\,\frac{\sin\left[\frac{\pi}{2c}(x+c)\right]}{\sin\left[\frac{\pi}{2c}(y+c)\right]} \times \exp\left[+\frac{\pi^2}{8c^2}(t-s)\right]. \tag{7}$$

Note that $\sin[\frac{\pi}{2c}(x+c)] = \cos(\frac{\pi}{2c}x)$. We recall that the entry $k(x,t|y,s)$ in the above is the transition density of the original process with absorption at the interval endpoints; see Eq. (1).

In the large-T regime both T and u dependence are absent in the approximate formula (7). Thus, the target point u is irrelevant for the description of the dynamical behavior of the Bernstein function for times $\tau \in [s,t]$. There was even no need to execute literally the $T \to \infty$ limit. Once we have $T \gg t$, an approximation (7) is fully legitimate. Moreover the $T \to \infty$ limit actually admits time intervals $[s,t]$ of arbitrary finite duration.

An interesting point is that in Eq. (7), we have arrived at the well-known transition probability density of the unique Markovian diffusion process that never leaves the prescribed

interval (its endpoints are inaccessible from the interior); see, e.g., Refs. [17,18] The transition density (7) of this conditioned diffusion process is the sought-for dynamical law for the time evolution of the Bernstein density (under our premises). The corresponding transport equation is well known as well.

By general principles we deduce [17,18] the forward drift of the conditioned diffusion process in question

$$b(x) = \nabla \ln \cos\left(\frac{\pi}{2c}x\right) = -\frac{\pi}{2c}\tan\left(\frac{\pi}{2c}x\right) \tag{8}$$

and the transport equation in the Fokker-Planck form (partial derivatives are executed with respect to x)

$$\partial_t \rho = \tfrac{1}{2}\Delta\rho - \nabla(b\rho). \tag{9}$$

The transition density $p(x,t|y,s)$ [Eq. (7)] is its solution. We have also the probability transport rule valid for any probability density $\rho(x,t) = \int_{-c}^{c} \rho(y)p(x,t|y,0)\,dy$ with $\rho_0(y)$ considered as the initial data for the F-P evolution.

Accordingly, the dynamics of the Bernstein function (5), on the $s,t,t-s$ time scales that are small relative to T, is well approximated by that of a transition probability density (7) of the diffusion process never leaving the interval. The larger the presumed survival time T of sample paths in question, the larger may be (ultimately arbitrary, for $T \to \infty$) the time duration of the "small-scale" $[s,t]$ process.

The dynamics (7) is thus a generic (albeit approximate for finite T) property of all long-living trajectories of random motion in the interval with absorbing ends. The requirement of an infinite survival time (permanent trapping) leaves us with the diffusion process (7), with a guarantee that no trajectory may reach the interval endpoints.

D. Feynman-Kac kernel in the interval and eigenfunction expansions

We point out an obvious link of the previous analysis with the standard (quantum mechanical by provenance) spectral problem for the operator $-\frac{1}{2}\Delta$ in an infinite well, supported on the interval $(-c,c)$, denoting $\lambda_n, n = 1,2,\ldots$ the eigenvalues and $\phi_n, n = 1,2,\ldots$ the orthonormal basis system composed of the eigenfunctions ϕ_n.

In particular a "miraculous" emergence of the time-dependent factor $\exp[+\frac{\pi^2}{2L^2}(t-s)]$ might seem to be annoying at the first glance. However, this factor secures the existence of an asymptotic invariant probability density. Since $\pi^2/2L^2 = \lambda_1$ is the lowest eigenvalue of $-\frac{1}{2}\Delta_{\mathcal{D}}$, we effectively encounter here a standard additive "renormalization" $-\frac{1}{2}\Delta_{\mathcal{D}} - \lambda_1$ in the semigroup generator definition. Then the strictly positive operator (bottom of the spectrum is λ_1) becomes replaced by the non-negative operator with the bottom of the spectrum at zero; see, e.g., Refs. [14,15].

The spectral decomposition of $(1/2)\Delta_{\mathcal{D}}$ allows us to rewrite $k(x,t|y,s)$ in a handy form:

$$k(x,t|y,s) = \sum_{n=1}^{\infty} e^{-\lambda_n(t-s)}\,\phi_n(x)\phi_n(y). \tag{10}$$

It is clear that under suitable regularity assumptions concerning the long-time T behavior of Feynman-Kac kernels, specifically that we have $k(u,T|y,s) \sim \phi_1(y)\phi_1(u)\exp[-\lambda_1(T-s)]$, one arrives at the legitimate $T \to \infty$ expression:

$$p(x,t|y,s) = k(x,t|y,s)\frac{\phi_1(x)}{\phi_1(y)}e^{+\lambda_1(t-s)} \tag{11}$$

as anticipated previously [set, e.g., $\lambda_1 = \frac{\pi^2}{2L^2}$ and $\phi_1(x) = \sqrt{\frac{2}{L}}\cos(\pi x/L)$].

Given the limiting transition probability density (11), we can now assume that $t \gg s$, By resorting to the large time behavior of the kernel (6), we readily deduce the relaxation pattern of the random motion, namely, and we get

$$p(x,t|y,s) \sim [\phi_1(x)]^2 = \rho(x) = \frac{2}{L}\cos^2(\pi x/L). \tag{12}$$

While keeping in memory the $L^2(-c,c)$ normalization of $\phi_1(x)$ we have thus identified the stationary invariant probability density $\rho(x)$ of the diffusion process, conditioned never to leave the interval. In Fig. 1 we have depicted the stationary probability density (12), while set against the forward drift (8).

E. Time rates to equilibrium

Collecting the generic features (typicalities) of the diffusion in the interval with absorbing ends, i.e., (1) $k(x,t|y,0) = \sum_j \exp(-\lambda_j t)\,\phi_j(y)\phi_j(x)$, (2) the transition density of the conditioned process in the universal form $p(x,t|y,0) = k(x,t|y,0)\,e^{+\lambda_1 t}\,\phi_1(x)/\phi_1(y)$, and (3) the relaxation asymptotic $\lim_{t\to\infty} p(x,t|y,0) = \rho(x) = \phi_1^2(x)$, we can address an issue of the time rates towards equilibrium in the conditioned random motion (11).

Let us denote $\tilde{k}(x,t|y,0) = e^{\lambda_1 t}k(x,t|y,0)/\phi_1(x)\phi_1(y)$. We employ an estimate (known to be valid for $t > 1$) [25] (see also Ref. [26]):

$$|\tilde{k}(x,t|y,0) - 1| \leqslant C\,e^{-(\lambda_2-\lambda_1)t}, \tag{13}$$

where C is a suitable constant. That yields immediately multiply both sides by $[\rho(x) = \phi_1^2(x)]$ the general formula for the time rate to equilibrium, provided we name the invariant stationary density $\rho(x)$ the equilibrium density of the process:

$$|p(x,t|y,0) - \rho(x)| \leqslant C\,e^{-(\lambda_2-\lambda_1)t}\,\rho(x). \tag{14}$$

We have the exponentially fast approach toward equilibrium (realized via the invariant density shape rescaling by en exponential factor), whose speed depends on the spectral gap between two lowest eignvalues.

In the interval $(-1,1)$ we have [compare, e.g., Eq. (1)] $\lambda_1 = \pi^2/8$ and $\lambda_2 = \pi^2/2$, hence $\lambda_2 - \lambda_1 = 3\pi^2/8$. We have also $\rho(x) = \cos^2(\pi x/2)$.

The above rate formula has much broader significance, since it is generally valid for diffusions in convex domains (irrespective of space dimensions) [25].

F. Semigroup transcript of the Fokker-Planck dynamics

Let us employ the standard textbook notation [21] with dimensional constants kept explicit. Scalings leading to dimensionless notations, used throughout the present paper, are obvious.

Let us consider the Langevin equation $dX_t = b(X_t)\,dt + \sqrt{2\nu}B_t$, where the drift velocity is a gradient field, e.g.,

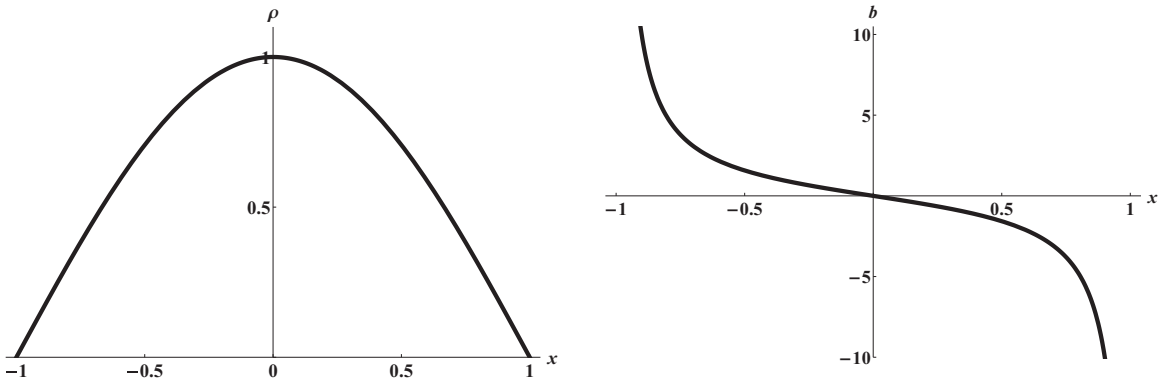

FIG. 1. Conditioned Brownian equilibrium in the interval with inaccessible endpoints: the permanent trapping enclosure. Left: the probability density $\rho(x) = \cos^2(\pi x/2)$; right: the forward drift $b(x) = -(\pi/2)\tan(\pi x/2)$ of the Fokker-Planck equation in a trap $(-1,1) \subset R$, [24].

$b = -\nabla\mathcal{V}$. The corresponding Fokker-Planck equation takes the form $\partial_t \rho = \nu\Delta\rho - \nabla(b\rho)$.

We assume that asymptotically, the Fokker-Planck dynamics sets down at the equilibrium (stationary) solution $\rho_*(x)$, e.g., $\lim_{t\to\infty} \rho(x,t) = \rho_*(x)$. Since the drift $b(x)$ does not depend on time, the Fokker-Planck equation implies that $b = \nu\nabla \ln \rho_*$, i.e., the stationary solution ρ_* fixes b and in reverse. Accordingly, we arrive at the Boltzmann-Gibbs form of $\rho_*(x) = \exp[-\mathcal{V}(x)/\nu]$. That is consistent with the primary definition of $b = -\nabla\mathcal{V}$.

Following tradition [21] let us introduce a multiplicative decomposition of $\rho(x,t)$:

$$\rho(x,t) = \Phi(x,t)\rho_*^{1/2}(x). \tag{15}$$

The introduced positive function $\Phi(x,t)$ satisfies the generalized diffusion equation

$$\partial_t \Phi = \nu\Delta\Phi - V\Phi, \tag{16}$$

where the potential field $V = V(x)$ is given (up to an additive constant allowing to make positive any bounded from below V):

$$V = \frac{1}{2}\left(\frac{b^2}{2\nu} + \nabla b\right). \tag{17}$$

Equation (15) actually derives from the semigroup dynamics $\exp(-t\hat{H}/\nu)$ describing a stochastic process with killing, if $V(x)$ is positive-valued (that is, trivial for potentials which are bounded from below, since we can always add a suitable constant to make the potential positive-definite).

If the spectral solution for $\hat{h} = -\nu\Delta + V$ allows for an isolated eigenvalue at the bottom of the spectrum, denoted λ_1, we can always introduce $\hat{H} - \lambda_1$ as the semigroup generator. Then (preserving only the discrete part of the spectral decomposition), $\Phi(x,t) = \exp(+\lambda t)\sum_{n=1} c_n \exp(-\lambda_n t)\phi_n(x) \to \phi_1(x) = \rho_*^{1/2}(x)$ In that case the dynamics of $\Phi(x,t)$ asymptotically sets down at $\rho_*^{1/2}(x)$.

Remark. It is an amusing exercise to check that by setting $\nu = 1/2$ and inserting the drift field expression (8), we actually obtain $V(x) = 0$ identically in the open interval $(-c,c)$. Note also that $b = \nabla \ln \rho_*$, where $\rho_* = (\phi_1)^2$; see also Ref. [11].

III. DISK WITH AN ABSORBING BOUNDARY AND THE CONDITIONING

To make clear the links with the past and current mathematical literature on a similar subject, let us, e.g., quote the major result of Ref. [27] (see also Refs. [28] and [26,29]), which has been actually formulated for bounded planar domains.

We adopt the original notation of Ref. [27] to that used in the present paper and stress that the order of variables x and y is here interchanged if compared with that in Ref. [27], formula (1.1). Namely, let Ω stand for a planar domain of finite area, λ is the first positive eigenvalue of half the Laplacian $(1/2)\Delta$ in Ω, and ϕ is the first L^2 normalized eigenfunction $\int_\Omega \phi^2 = 1$.

Let $k(x,t|y,0)$ be the fundamental solution of the heat equation with Dirichlet boundary conditions. Then for any $x \in \Omega$, there holds

$$\lim_{t\to\infty} \frac{e^{\lambda t}\, k(x,t|y,0)}{\phi(y)\phi(x)} = 1 \tag{18}$$

uniformly in $y \in \Omega$. Equation (18) clearly is instrumental in defining the Brownian motion conditioned to stay forever in Ω. Let us recall then the long-time behavior of the transition kernel as reported in Eq. (7). The same asymptotic in the present notation would read $k(x,t|y,0) \sim \phi(y)\phi(x)\exp(-\lambda t)$.

As a simple example of the bounded (and convex) planar domain, we take a domain in the regular disk shape, i.e., bounded by the circle of a fixed radius R. The spectral solution for such two-dimensional domain is clearly in reach, albeit somewhat murky from the casual (user-friendly) point of view.

The spectral decomposition formula (8) for the transition density of the process in a disk with absorbing boundary is valid here, with suitable amendments that reflect the two-dimensional setting.

Since we are interested in the long-T duration of the conditioned process, in view of our previous discussion in Sec. II, we need basically the knowledge of the stationary density and the forward drift. In the present case these read respectively:

$$\rho(\mathbf{r}) \sim \frac{j_0^2\left(\frac{z_1 r}{R}\right)}{j_1^2(z_1)} = \phi_1^2(\mathbf{r}), \tag{19}$$

where $z_1 = 2.4048\ldots$ is the first positive zero of the Bessel function $j_0(r)$, and $\mathbf{r} = (x,y)$ below we shall use

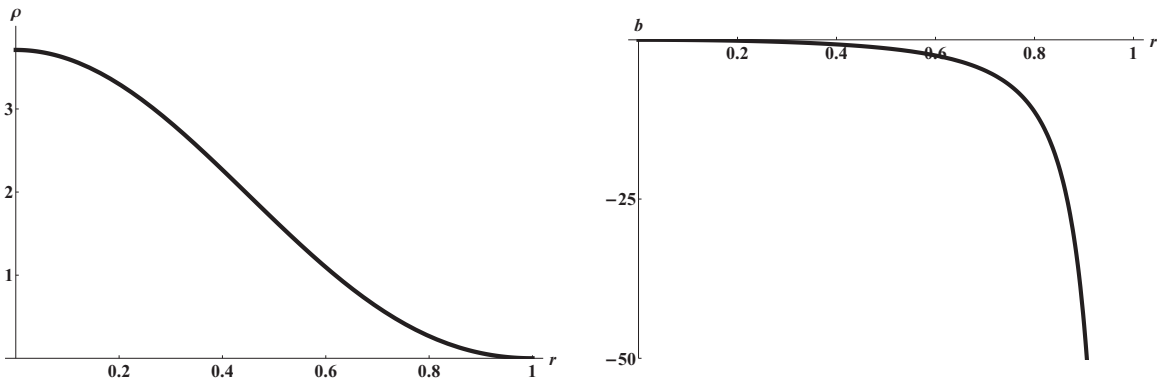

FIG. 2. Conditioned Brownian equilibrium in the unit disk with an inaccessible boundary: the permanent trapping enclosure. Left: the probability density $\rho(x)$; right: the forward drift $b(x)$ of the Fokker-Planck equation in a trapping disk [24].

$[\hat{\mathbf{r}} = (1/r)(x,y)]$. The density is purely radial, hence only the radial component of $\mathbf{b}(\mathbf{r}) = b(\mathbf{r})\,\hat{\mathbf{r}}$ is different from zero with

$$b(\mathbf{r}) = \frac{\partial}{\partial r}\,\ln\phi_1(\mathbf{r}) = -\frac{z_1 j_1\left(\frac{z_1 r}{R}\right)}{R j_0(z_1 r)}. \tag{20}$$

The stationary probability denity (19) and the forward drift (20) are comparatively depicted in Fig. 2. We have here a fully fledged two-dimensional Brownian motion, with a drift $\mathbf{b}(\mathbf{r})$ that is a purely radial vector field. The Fokker-Planck equation is two-dimensional: $\partial_t\rho(\mathbf{r},t) = (1/2)\Delta\rho - \nabla(\mathbf{b}\rho)$. For completeness, we shall reproduce this equation in the polar coordinates, with the radial form of the drift being explicit:

$$\partial_t\rho = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\rho - \left[\frac{1}{r}\frac{\partial(rb\rho)}{\partial_r}\right]. \tag{21}$$

We point out that the time rate formulas become more complicated in dimensions exceeding $D > 1$, since if we admit the general (not exclusively radial) random motion the disk, the Laplacian spectrum becomes degenerate (except for the ground state) and the eigenvalues (increasing) order is not set merely by $n = 1,2,\ldots$, but needs to account for the angular label l [30–32]. For example, we know the the first positive zero of j_0 ($l = 0$ sector) approximately equals 2.4048, the second zero equals 5.5201, while the first zero of j_1 (l=1 sector) reads 3.8137, and the first zero of j_2 equals 5.1356.

Remark. The survival problem on the disk can be reformulated in a pictorial way by referring directly to the front-cover picture of Redner's monograph [1] and taking some inspiration from Refs. [2,3], even though the problems addressed there refer to absorbing boundary conditions and first passage time issues. Take literally a (somewhat drunken) Brownian wanderer on top of the island (disk) which is surrounded by a cliff (plus predators in the ocean). Add a psychologically motivated fear component to the wanderer's behavior, when one is close to the island boundary. Then we readily arrive at the following problem: How should that fearful Brownian agent move to increase his chances for survival? Our disk solution, establishing the regime of the inaccessible boundary, gives the optimal stationary probability density (transition density likewise, albeit not in a handy closed form) and the drift field. The random wanderer obeying the associated Fokker-Planck equation and following the underlying Langevin) dynamics would live indefinitely on the island (disk), while moving safely away from the inaccessible boundaries with no chance to reach them.

As a useful supplement, let us add that the three-dimensional spherical well problem, with absorption at the boundary, can be addressed in the very same way. The probability density reads [here $\mathbf{r} = (x,y,z)$ and $\hat{\mathbf{r}}$ stand s for a unit vector]

$$\rho(\mathbf{r}) = \frac{1}{2\pi R}\,\frac{\sin^2(\pi r/R)}{r^2} = \phi_1^2(\mathbf{r}), \tag{22}$$

where R stands for the sphere radius. The forward drift of the pertinent stochastic process, that never leaves the sphere interior, reads

$$\mathbf{b}(\mathbf{r}) = \frac{\partial_r\phi_1(\mathbf{r})}{\phi_1(\mathbf{r})}\,\hat{\mathbf{r}} = \left[\frac{\pi}{R}\cot\left(\frac{\pi r}{R}\right) - \frac{1}{r}\right]\hat{\mathbf{r}} = b(\mathbf{r})\hat{\mathbf{r}}. \tag{23}$$

We have here a fully fledged three-dimensional Brownian motion, with a drift $\mathbf{b}(\mathbf{r})$. The Fokker-Planck equation is now three-dimensional, and its radial form is reproduced for completeness:

$$\partial_t\rho = \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\rho - \left[\frac{1}{r}^2\frac{\partial(r^2 b\rho)}{\partial_r}\right]. \tag{24}$$

IV. FEYNMAN-KAC KERNELS FOR KILLED STOCHASTIC PROCESSES AND THE KILLING REMOVAL

It is well known that operators of the form $\hat{H} = -(1/2)\Delta + V \geqslant 0$ with $V \geqslant 0$ give rise to transition kernels of diffusion-type Markovian processes with killing (absorption), whose rate is determined by the value of $V(x)$ at $x \in R$. That interpretation stems from the celebrated Feynman-Kac (path integration) formula, which assigns to $\exp(-\hat{H}t)$ the positive integral kernel:

$$[\exp(-(t-s)\left(-\frac{1}{2}\Delta + V\right)](y,x)$$
$$= \int \exp\left\{-\int_s^t V[\omega(\tau)]\,d\tau\right\} d\mu_{s,y,x,t}(\omega). \tag{25}$$

In terms of Wiener paths that kernel is constructed as a path integral over paths that get killed at a point $X_t = x$ with an extinction probability $V(x)\,dt$, in the time interval$(t,t+dt)$ (note that physical dimensions of V before scaling them out

were J/s, that is, usually secured by a factor $1/2mD$ or $1/\hbar$). The killed path is henceforth removed from the ensemble of ongoing Wiener paths.

Given a discrete spectral solution for $\hat{H} = -(1/2)\Delta + V$ with $V(x) \geqslant 0$, comprising the monotonically growing series of nondegenerate positive eignevalues, with real $L^2(R)$ eigenfunctions, the integral kernel of $\exp(-t\hat{H})$ has the time-homogeneous form

$$k(y,x,t) = k(x,y,t) = \sum_j \exp(-\epsilon_j t)\,\phi_j(y)\phi_j(x). \tag{26}$$

Consider the harmonic oscillator problem with $\hat{H} = (1/2)(-\Delta + x^2)$. The integral kernel of $\exp(-t\hat{H})$ is given by a classic Mehler formula:

$$\begin{aligned} k(x,y,t) &= [\exp(-t\hat{H})](y,x) = \frac{1}{\sqrt{\pi}}\exp[-(x^2+y^2)/2] \\ &\quad\times \sum_{n=0}^{\infty}\frac{1}{2^n n!}H_n(y)H_n(x)\exp(-\epsilon_n t) \\ &= \exp(-t/2)\,(\pi[1-\exp(-2t)])^{-1/2} \\ &\quad\times \exp\left[\frac{1}{2}(x^2-y^2) - \frac{(x-e^{-t}y)^2}{(1-e^{-2t})}\right] \\ &= \frac{1}{(2\pi\sinh t)^{1/2}}\exp\left[-\frac{(x^2+y^2)\cosh t - 2xy}{2\sinh t}\right], \end{aligned} \tag{27}$$

where $\epsilon_n = n + \frac{1}{2}$, $\phi_n(x) = [4^n(n!)^2\pi]^{-1/4}\exp(-x^2/2)\,H_n(x)$ is the $L^2(R)$ normalized Hermite (eigen)function, while $H_n(x)$ is the nth Hermite polynomial $H_n(x) = (-1)^n(\exp x^2)\frac{d^n}{dx^n}\exp(-x^2)$. Note a conspicuous presence of the time-dependent factor $\exp(-t/2)$.

Let us replace t by $T-t$ and accordingly consider $k(x,T|y,s) = k(T-t,x,y)$. Now we pass to the conditional transition density (5) and investigate the large-T (eventually $T\to\infty$) regime. Since for large-value T we have (compare, e.g., also Ref. [14])

$$k(x,t|u,T) \sim \frac{1}{\sqrt{\pi}e^{(T-t)/2}}\,e^{-\frac{1}{2}(x^2+u^2)}, \tag{28}$$

by repeating the conditioning procedure of Sec. II, we readily arrive at the approximation [cf. (7)] of the Bernstein transition function by a transition probability density $p(x,t|y,s) = p_{t-s}(x|y)$ of the familiar Ornstein-Uhlenbeck process

$$\begin{aligned} &B(u,T;x,t;y,s) \to p(x,t|y,s) \\ &= k(x,t,y,s)\,\frac{\exp(-x^2/2)}{\exp(-y^2/2)}\,e^{(t-s)/2} \\ &= k(y,x,t)\,\frac{\phi_1(x)}{\phi_1(y)}\,e^{+\epsilon_1(t-s)} = [\pi(1-e^{-2(t-s)})]^{-1/2} \\ &\quad\times\exp\left[-\frac{(x-e^{-(t-s)}y)^2}{(1-e^{-2(t-s)})}\right], \end{aligned} \tag{29}$$

where $\phi_1(x) = \pi^{-1/2}\exp(-x^2/2)$ and $\epsilon_1 = 1/2$ have been accounted for.

Here the Fokker=Planck operator takes the form $L_{FP} = (1/2)\Delta - \nabla[b(x)\cdot]$ and $b(x) = -x$. Clearly $b(x) = \nabla\ln\phi_1(x)$, as should be the case. The asymptotic (invariant, stationary) probability density of the pertinent process reads $\rho(x) = \phi_1^2(x) = (1/\pi)\exp(-x^2)$. For earlier considerations on how to transform the Feynman-Kac averages for processes with killing into those that refer to processes without any killing, see, e.g., Ref. [33].

V. HANDLING THE SINK: SURVIVAL ON THE HALF-LINE AND BESSEL PROCESSES

The conditioning procedure up to now seems to rely heavily on contracting semigroups, whose generators have purely discrete spectral solutions with the bottom eigenvalue being well separated from the rest of the spectrum.

The above outlined conditioning procedure will surely work for potentials that are bounded from below continuous functions, since we can always redefine potentials with a bounded from below negative part, by adding to $V(x)$ a modulus of its minimal value $|V(x_{\min})|$ or a modulus of any of its multiple (identical) local minima: $V(x)\to V(x)+|V(x_{\min})|$ so arriving at $V(x)\geqslant 0$. The redefined potential is nonnegative and gives rise to the diffusion-type process with killing, whose transition density $k(x,t|y,s)$ is given by the Feynman-Kac formula [14,15].

It is our aim to demonstrate that we need not have a discrete spectral solution at hand. The employed conditioning procedure appears to work properly, also when the spectrum of the involved semigroup generator is continuous. This is, e.g., the case for Brownian motion on a half-line with an absorbing barrier (sink).

We set the sink at 0 and consider the Brownian motion as being restricted to the positive semiaxis ($x\in R^+$). The pertinent transition density is obtained via the method of images, by employing the standard Brownian transition probability density induced by $[(1/2)\Delta]$

$$p(x,t|y,s) = [2\pi(t-s)]^{-1/2}\,\exp[-(x-y)^2/2\pi(t-s)], \tag{30}$$

namely,

$$\begin{aligned} k(x,t|y,s) &= p(x.t|y,s) - p(x,t|-y,s) = \frac{2}{\sqrt{2\pi(t-s)}} \\ &\quad\times\exp\left[-\frac{x^2+y^2}{2(t-s)}\right]\sinh\left(\frac{xy}{t-s}\right). \end{aligned} \tag{31}$$

The large-T behavior of $k(u,T|y,s)$ is easily inferred to imply (the situation is less straightforward than in the previous examples, since the time label T persists in the exponent; note that we replace $T-s$ and $T-t$ by T)

$$k(u,T|y,s) \sim \frac{2}{\sqrt{2\pi T}}\exp\left[-\frac{y^2+u^2}{2T}\right]\frac{uy}{T}. \tag{32}$$

Accordingly, in the large-T regime, the Bernstein function is approximated by an intriguing functional expression:

$$B(u,T;x,t;y,s) \sim p(x,t|y,s)\exp\left[\frac{y^2-x^2}{2T}\right], \tag{33}$$

where we single out, as a leading factor, an immediately recognizable transition probability function of the Bessel

process

$$p(x,t|y,s) = k(x,t|y,s)\,\frac{x}{y}. \tag{34}$$

We have fixed the (y,s) and (u,T) data and there is no u-dependence in the asymptotic expression (33). Consequently, $\exp(y^2/2T)$ is irrelevant as far as the large-T regime is concerned. What matters, however, is the remaining term $\exp(-x^2/2T)$, which is not quite innocent for x that are comparable in size with $T^{1/2}$. Surely, for large finite T the exponential term may be regarded to very be close to one for not too large values of x, such as $x < T^{1/2}/100$, since $\exp(-1/20\,000) \sim 1\text{–}5\times10^{-5}$ is sufficiently close to 1.

Pushing $T \to \infty$ refers to an eternal survival and involves the pointwise convergence of $B(u,T;x,t;y,s)$ to $p(x,t|y,s)$ in Eq. (33), ultimately leaving us with a transition probability density for the Bessel process.

The forward drift of this process is known [17,18] to be equal to $b(x) = \nabla \ln x = 1/x$. The Fokker-Planck generator takes the familiar (Bessel process) form $L_{FP} = (1/2)\Delta - [b(x)\cdot]$. We note that the point 0 is presently inaccessible for the process.

Told otherwise, the one-dimensional Brownian motion starting from $y > 0$, conditioned to remain positive up to time T, converges as $T \to \infty$ to the radial process of the three-dimensional Brownian motion, known as the Bessel process.

Remark. In the one-parameter family of Bessel processes, with drifts of the form $b(x) = (1+2a)/2x$, in the case of $a \geqslant 0$, the point $x = 0$ is never reached from any $y > 0$ with probability 1. To the contrary, for $a < 0$, the barrier at $x = 0$ is absorbing (sink).

Let us recall the backward generator of the process: $(1/2)\Delta + b(x)\nabla$ with $b(x) = (1+2a)/2x$. The one-parameter family of pertinent transition densities reads

$$k_a(x,t|y,0) = \frac{y^{2a+1}}{t(xy)^a}\,\exp\left[-\frac{x^2+y^2}{2t}\right] I_{|a|}\left(\frac{xy}{2t}\right). \tag{35}$$

Let us consider the special case of $a = \pm(1/2)$ for which the modified Bessel function takes a handy form:

$$I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,\sinh z. \tag{36}$$

It is easy to verify that $k_{-1/2}(x,t|y,0)$ coincides with the transition density of the Brownian motion constrained to stay on $R+$, with a sink at 0. The generator simply is $(1/2)\Delta$ on R^+, with absorbing boundary at 0.

On the other hand $k_{+1/2}(x,t|y,0)$ is a transition probability density of the Bessel process with $b(x) = 1/x$ (e.g., the Brownian motion conditioned to never reach 0, if started from any $y > 0$). Its F-P generator reads $(1/2)\Delta - \nabla[b(x)\cdot]$, with $b(x) = \nabla \ln x$.

VI. PROSPECTS

Our conditioning strategy involving the Bernstein transition probability densities heavily relies on the large-time asymptotic properties of transition densities for processes with absorption (killing). Once the kernels $k(x,t|y,s)$ are given the spectral resolution form (e.g., the eigenfunction expansions) we can expect that our considerations should be safely extended to Lévy-stable processes, additively perturbed by suitable potential functions. That refers to Lévy motions in energy landscapes of Refs. [26,29] and references therein.

In the case of a discrete spectral resolution of the random motion generator, transition kernels are in principle computable and amenable to asymptotic procedures of Secs. II to V. Basically, the kernels have no known explicit analytic forms. If we know the lowest eigenvalue and the corresponding eigenfunction, the conditioning itself can be imposed in exactly the same way as before (via Bernstein transition functions). A link of the killed stochastic process and its "eternally living" partner can surely be established for jump-type processes as well.

For deceivingly simple problems (albeit in reality, technically quite involved) of the Lévy-stable finite and infinite well or its spherical well analog, numerically accurate and approximate analytic formulas are known for the ground states. We have in hand their shapes (hence the resultant stationary probability densities of the conditioned process) together with corresponding lowest eigenvalues [35–38]. Analogous results were established for some Lévy stable oscillators [34,35,39,40]

The solution for the half-line Lévy-stable problem with an absorbing barrier (rather involved and available in terms of an approximate analytic expression) is also in existence [41] and may be used to deduce the process living eternally on the half-line, following our conditioning method. All that needs more elaborate analysis, which we relegate to the future.

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