Reply to "Comment on 'Defocusing complex short-pulse equation and its multi-dark-soliton solution' "

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(Received 13 June 2017; published 11 August 2017)

Our paper [Phys. Rev. E **93**, 052227 (2016)], proposing an integrable model for the propagation of ultrashort pulses, has recently received a Comment by Youssoufa *et al.* [Phys. Rev. E **96**, 026201 (2017)] about a possible flaw in its derivation. We point out that their claim is incorrect since we have stated explicitly that a term is neglected to derive our model equation in our paper. Furthermore, the integrable model is validated by comparing with the normalized Maxwell equation and other known integrable models. Moreover, we show that a similar approximation has to be performed in deriving the same integrable equation as explained in the Comment.

DOI: 10.1103/PhysRevE.96.026202

With the significant progress of laser technology in the past two decades, the study of ultrashort optical pulses has attracted much attention [1]. However, the mathematical description of such ultrashort optical pulses requires a new approach beyond the conventional slowing varying envelope approximation [2]. As a matter of fact, some works have been performed in the literature, and several mathematical models have been proposed and studied [3–7]. Especially, a short-pulse equation was proposed by Schäfer and Wayne [8], which turns out to be integrable and admits Lax pairs [9] and multisoliton solutions [10,11]. Recently, an integrable complex short-pulse equation was proposed in Refs. [12,13] and, separately, was generalized to including the defocusing case [14].

The main criticism by Youssoufa *et al.* in the preceding Comment [15] concerns the approximation used in our papers [14]. To address their criticism, let us first reconfirm our derivation briefly here. Starting from the Maxwell equation and assuming an instantaneous Kerr effect, we obtain the following normalized equation (Eq. (21) in Ref. [14]):

$$E_{zz} - E_{tt} = \pm E + (|E|^2 E)_{tt}. \tag{1}$$

It is noted that the same equation with a positive sign before E has been derived earlier in Refs. [4–6,12]. This equation is considered as a full wave equation to model the ultrashort optical pulses with a few cycles and has been investigated in Refs. [5,6]. In seeking a right-moving wave packet, we assume a multiple scales ansatz,

$$E(z,t) = \epsilon E_0(\tau,z_1,z_2,\ldots) + \epsilon^2 E_1(\tau,z_1,z_2,\ldots) + \cdots, \tag{2}$$

where ϵ is a small parameter and τ and z_n are the scaled variables defined by

$$\tau = \frac{t-z}{\epsilon}, \quad z_n = \epsilon^n z. \tag{3}$$

As a result we obtain the following partial differential equation for E_0 at the order $O(\epsilon)$:

$$-2\frac{\partial^2 E_0}{\partial\tau\,\partial z_1} = \pm E_0 + 2\frac{\partial}{\partial\tau}\left(|E_0|^2\frac{\partial E_0}{\partial\tau}\right). \tag{4}$$

As pointed out in Ref. [14], a term $E_0^2 E_{0,\tau}^*$ (misprinted by $E_0^2 E_{0,\tau}$ in Ref. [14]) is neglected for the purpose of obtaining an integrable nonlinear wave equation. Otherwise, an intermediate equation, which is nonintegrable, turns out to be

$$-2\frac{\partial^2 E_0}{\partial\tau\,\partial z_1} = \pm E_0 + 2\frac{\partial^2}{\partial\tau^2}(|E_0|^2 E_0). \tag{5}$$

Furthermore, by scale transformations,

$$x = \frac{1}{\sqrt{2}}\tau, \quad t = \frac{1}{\sqrt{2}}z_1, \quad q = \sqrt{2}E_0, \tag{6}$$

we arrive at

$$q_{xt} \pm q + \tfrac{1}{2}(|q|^2 q_x)_x = 0, \tag{7}$$

$$q_{xt} \pm q + \tfrac{1}{2}(|q|^2 q)_{xx} = 0, \tag{8}$$

respectively, from (4) and (5).

Thus, we disagree with the claim in the statement [15] that a term (Eq. (17) in Ref. [15]) is missing since we explicitly have stated this neglected term in our paper [14]. Furthermore, to verify the validation of the approximation, in Ref. [14], we first compare the one-soliton solution for the focusing complex short-pulse equation with the ones for the normalized Maxwell equation (1) with the positive sign, the nonlinear Schrödinger (NLS) equation, and the higher-order NLS equation. The results are shown in Fig. 1 in Ref. [14]. For the defocusing case, we also compared the one-soliton solution with the one for the normalized Maxwell equation (1) with the negative sign. The result is Fig. 2 in Ref. [14].

From the above comparisons, it can be seen that the solitary wave solutions for both the focusing and the defocusing complex short-pulse equations is consistent with the ones for the normalized Maxwell equation.

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The physical background underlining the approach by Kuetche *et al.* [13] and the preceding Comment [15] mainly is based on the research performed by Kozlov and co-workers [16–19]. In Refs. [13,15], the nonlinear part of the electric polarization $\mathbf{P}_{\mathrm{NL}}$ is

$$\partial_t \mathbf{P}_{\mathrm{NL}} = \alpha |\mathbf{E}|^2 \mathbf{E}_t + \beta \mathbf{E} \times (\mathbf{E} \times \mathbf{E}_t). \quad (9)$$

As mentioned in Ref. [18], if a cubic nonlinearity $\mathbf{P}_{\mathrm{NL}} = \chi |\mathbf{E}|^2 \mathbf{E}$ is considered due to the Kerr effect, then

$$\partial_t \mathbf{P}_{\mathrm{NL}} = 3\chi |\mathbf{E}|^2 \mathbf{E}_t + 2\chi \mathbf{E} \times (\mathbf{E} \times \mathbf{E}_t). \quad (10)$$

In Refs. [3,19], the nonlinear contribution to the refractive index was studied in more detail. By considering the contribution due to the electronic-vibrational nonlinearity beside the electronic nonlinearity, a more general expression for $\partial_t \mathbf{P}_{\mathrm{NL}}$ takes the form of Eq. (9).

It was pointed out in Ref. [3], $\alpha = \frac{3}{2}\beta$ for purely electronic nonlinearity, and some difference can be brought in a correlation between α and β by the electronic-vibrational mechanism of nonlinearity. However, the case of $\beta = 0$ is too special, which may not be true physically. In other words, an alternative approximation similar to our approach in Ref. [14] is used in deriving an integrable complex short-pulse equation (Eq. (27) in Ref. [15]). Without this approximation, the derivation will lead to a nonintegrable complex short-pulse equation (8).

To summarize, the integrable complex short-pulse equation (7) can be derived to model the propagation of an ultrashort pulse in nonlinear media, either from our approach or from the approach by Youssoufa *et al.* [15] by certain approximations. The integrability of this model equation allows us to obtain more mathematical properties and various exact solutions [20,21]. Meanwhile, a nonintegrable complex short-pulse equation (8) can also be used to describe the ultrashort pulses. To conclude, we argue that the criticism in the Comment [15] is incorrect and Youssoufa *et al.* used an alternative but similar approximation in deriving the same complex short-pulse equation.

The work of B.-F.F. is partially supported by NSF Grant No. 1715991 and the COS Research Enhancement Seed Grants Program at UTRGV. The work of Z.Z. is supported by the National Natural Science Foundation of China under Grant No. 11671255 and by the Ministry of Economy and Competitiveness of Spain under Contract No. MTM2016-80276-P (AEI/FEDER, EU).

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