Kinetic approach to relativistic dissipation

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Despite a long record of intense effort, the basic mechanisms by which dissipation emerges from the microscopic dynamics of a relativistic fluid still elude complete understanding. In particular, several details must still be finalized in the pathway from kinetic theory to hydrodynamics mainly in the derivation of the values of the transport coefficients. In this paper, we approach the problem by matching data from lattice-kinetic simulations with analytical predictions. Our numerical results provide neat evidence in favor of the Chapman-Enskog [*The Mathematical Theory of Non-Uniform Gases*, 3rd ed. (Cambridge University Press, Cambridge, U.K., 1970)] procedure as suggested by recent theoretical analyses along with qualitative hints at the basic reasons why the Chapman-Enskog expansion might be better suited than Grad's method [Commun. Pure Appl. Math. **2**, 331 (1949)] to capture the emergence of dissipative effects in relativistic fluids.

The basic mechanisms by which dissipative effects emerge from the microscopic dynamics of relativistic fluid remains are still not fully understood in relativistic hydrodynamics. It has been long recognized that the parabolic nature of the Laplace operator is inconsistent with relativistic invariance as it implies superluminal propagation, hence, noncausal and unstable behaviors [1–3]. This can be corrected by resorting to fully hyperbolic formulations of relativistic hydrodynamics whereby space and time come on the same first-order footing, but the exact form of the resulting equations is not fixed uniquely by macroscopic symmetry arguments and thus remains open to debate.

A more fundamental approach is to derive relativistic hydrodynamics from the underlying kinetic theory [4], exploiting the advantages of the bottom-up approach: Irreversibility is encoded within a local $\mathcal{H}$-theorem [5], whereas dissipation results as an emergent manifestation of weak departure from local equilibrium (low Knudsen-number assumption) and the consequent enslaving of the fast modes to the slow hydrodynamic ones, associated with microscopic conservation laws. At no point does this scenario involve second-order derivatives in space, thus, preserving relativistic invariance by construction.

In nonrelativistic regimes, Grad's moments method [6] and the Chapman-Enskog (CE) [7] approach manage to connect kinetic theory and hydrodynamics in a consistent way, i.e., they provide the same transport coefficients. However, the relativistic regime presents a more controversial picture. The Israel [8] and Israel and Stewards (IS) formulation [9], extending Grad's method [6], derives causal and stable equations of motion, at least, for hydrodynamics regimes [10]. Although many earlier works have relied on the IS formulation, recent developments have highlighted theoretical shortcomings [11] and poor agreement with numerical solutions of the Boltzmann equation [12,13].

Recently, several authors have developed new attempts to derive consistent relativistic dissipative hydrodynamics equations. Attempting to circumvent the drawbacks of the IS formulation, Denicol and co-workers [11,14,15] have proposed an extension of the moments methods in which the resulting equations of motion are derived directly from the Boltzmann equation and truncated by a systematic power-counting scheme in the Knudsen number.

This, in turn, offers the possibility to include a larger number of moments (with respect to the 14 used in the IS formulation), improving the expressions for the transport coefficients. Starting from similar considerations, Jaiswal *et al.* [16] have included entropic arguments within Grad's method [6] and derived relativistic dissipative hydrodynamics equations which take the same form as the IS formulation although with different expressions for the transport coefficients. When compared to the IS formulation, these developments lead to solutions closer to the Boltzmann equation and, at least, in the ultrarelativistic limit (defined by $\zeta \to 0$, where $\zeta = mc^2/K_B T$ is the ratio of particle rest energy and temperature), they yield transport coefficients in good agreement with those calculated via the CE expansion. Interestingly, the CE method itself remains somewhat less explored [17,18] with relativistic extensions mostly restricted to the relaxation time approximation. More recently, a novel approach, introduced in a series of works by Tsumura and co-workers [19–22], applies renormalization group techniques to the Boltzmann equation. Once again, expressions for bulk (shear) viscosity and heat conductivity coincide with those provided by the CE method. Summing up the present and somewhat not fully conclusive state of affairs is that different theoretical approaches, based on different, if not conflicting assumptions, seem to converge towards the results provided by the CE approach. Conceptual shortcomings of the moments method, recently highlighted also in the nonrelativistic framework [23–26], revolve around the use of second-order spatial derivatives in constitutive hydrodynamical equations [19]. On the other hand, objections to the relativistic Chapman-Enskog expansion point to its link to relativistic Navier-Stokes equations, which suffer from basic
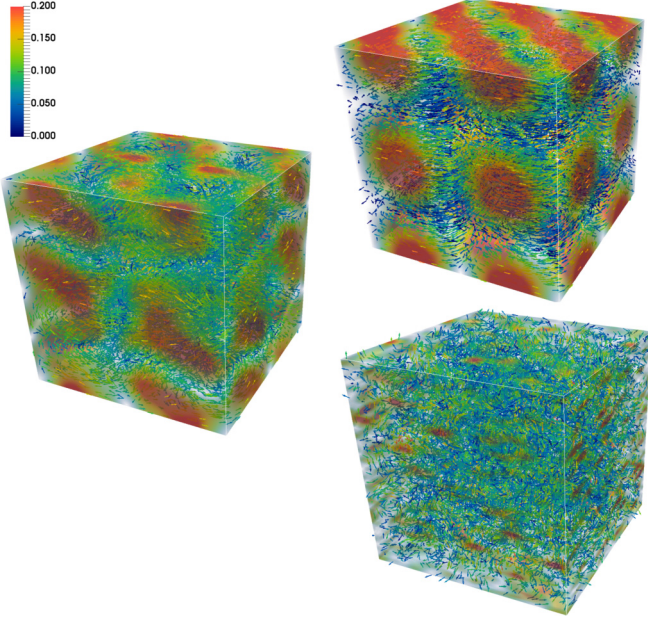
FIG. 1. Three-dimensional Taylor-Green vortex configuration of a viscous relativistic fluid with $\zeta = 0$ and $\tau = 0.51$ (symbols defined in the text). Top: initial configuration; middle: later stage in which the vortex configuration becomes unstable; bottom: final disordered state. The colors code vorticity, and the arrows represent the velocity field.

problems, such as broken causality and resulting instabilities [11,14]. In a less than crystal-clear situation, one would like to validate theory towards experimental data, but a controlled experimental setup is not a viable option at this point in time. Given the circumstances, numerical simulation stands up as a very precious alternative to gain new insight into this problem.

Recent works [27,28] have presented one-dimensional (1D) simulations of the (ultra)relativistic Boltzmann equation in the relaxation time approximation, showing results asymptotically compatible with the CE approach. This paper follows a similar line and reports the results of lattice-kinetic simulations of a relativistic flow in a controlled setup for which an approximate analytical hydrodynamic solution can be derived. We match analytical and numerical results in order to study the dependence of hydrodynamic transport coefficients on parameters defined on the mesoscale. For this purpose, we study the time evolution of a Taylor-Green vortex configuration in two and three spatial dimensions (see Fig. 1) and probe the functional dependence of the transport coefficients on ζ, extending previous work confined to the $\zeta \to 0$ limit. Our main result is a neat indication that CE predictions accurately match numerical data and they do so over a remarkably wide ζ range starting from the ultrarelativistic regime and seamlessly going over to the well-known nonrelativistic case. Our simulations use a recently developed relativistic lattice Boltzmann model (RLBM) [29], able to handle massive particles, providing an analysis of dissipative effects for relativistic but not necessarily ultrarelativistic flows.

In relativistic fluid dynamics, ideal nondegenerate fluids are described by the particle four-flow and energy momentum tensors, which at equilibrium read

$$N_E^\alpha = nU^\alpha, \tag{1}$$

$$T_E^{\alpha\beta} = (P + \epsilon)U^\alpha U^\beta - Pg^{\alpha\beta}, \tag{2}$$

where $U^\alpha = \gamma(1, \boldsymbol{u})$ is the fluid four-velocity, ($\boldsymbol{u}$ is the fluid velocity, $\gamma = 1/\sqrt{1 - u^2}$; we use natural units such that $c = 1$, $K_B = 1$), P is the hydrostatic pressure, and ϵ (n) is the energy (particle) density. We take into account dissipative effects with the Landau-Lifshitz decomposition [5],

$$N^\alpha = N_E^\alpha - \frac{n}{P + \epsilon}q^\alpha, \tag{3}$$

$$T^{\alpha\beta} = T_E^{\alpha\beta} + P^{\langle\alpha\beta\rangle} - \varpi(g^{\alpha\beta} - U^\alpha U^\beta), \tag{4}$$

with

$$q^\alpha = \lambda(\nabla^\alpha T - TU^\alpha \partial_\beta U^\beta),$$
$$P^{\langle\alpha\beta\rangle} = \eta\left(\Delta_\gamma^\alpha \Delta_\delta^\beta + \Delta_\delta^\alpha \Delta_\gamma^\beta - \tfrac{2}{3}\Delta^{\alpha\beta}\Delta_{\gamma\delta}\right)\nabla^\gamma U^\delta,$$
$$\varpi = -\mu\nabla_\alpha U^\alpha;$$

q^α is the heat flux, $P^{\langle\alpha\beta\rangle}$ is the pressure deviator, ϖ is the dynamic pressure, λ is the heat conductivity, and η and μ are the shear and bulk viscosities, respectively. Furthermore, we have

$$\nabla^\alpha = \Delta^{\alpha\beta}\partial_\beta,$$
$$\Delta^{\alpha\beta} = g^{\alpha\beta} - U^\alpha U^\beta,$$
$$\Delta_\beta^\alpha = \Delta^{\alpha\gamma}\Delta_{\gamma\beta}.$$

A kinetic formulation, on the other hand, describes the fluid as a system of interacting particles of rest mass m; the particle distribution function $f(x^\alpha, p^\beta)$ depends on space-time coordinates $x^\alpha = (t, \boldsymbol{x})$ and momenta $p^\alpha = (p^0, \boldsymbol{p}) = (\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p})$; $f(\boldsymbol{x}, t, \boldsymbol{p})d\boldsymbol{x}\,d\boldsymbol{p}$ counts the number of particles in the corresponding volume element in phase space.

The system evolves according to the Boltzmann equation, which, in the absence of external forces, reads as follows:

$$p^\alpha \frac{\partial f}{\partial x^\alpha} = \Omega(f). \tag{5}$$

The collision term $\Omega(f)$ often is replaced by simplified models. For instance, the Anderson-Witting model [30] (a relativistic extension of the well-known Bhatnagar-Gross-Krook [31] formulation), compatible with the Landau-Lifshitz decomposition, reads

$$\Omega = \frac{p^\mu U_\mu}{\tau}(f - f^{\text{eq}}). \tag{6}$$

The equilibrium distribution f^{eq}, following Boltzmann statistics, has been derived many decades ago by Jüttner [32],

$$f^{\text{eq}} \simeq e^{-p^\mu U_\mu/T}. \tag{7}$$

The Anderson-Witting model has just one parameter, the equilibration (proper-)time τ and obeys the conservation equations,

$$\partial_\alpha N^\alpha = 0, \tag{8}$$

$$\partial_\beta T^{\alpha\beta} = 0. \tag{9}$$

As discussed in previous paragraphs, a predictive bridge between kinetic theory and hydrodynamics must provide the

macroscopic transport coefficients λ, μ, η, from the mesoscopic ones (τ in the Anderson-Witting model). Our attempt at contributing further understanding of the issue is based on the following analysis: We: (i) consider a relativistic flow for which we are able to compute an approximate hydrodynamical solution depending on the transport coefficients; (ii) study the same flow numerically with a lattice Boltzmann kinetic algorithm, obtaining a numerical calibration of the functional relation between transport coefficients and τ; (iii) obtain clear-cut evidence that the CE method successfully matches the numerical results and, (iv) double-check our approach using the calibrations obtained in (ii) for a numerical study of a different relativistic flow, successfully comparing with other numerical data obtained by different methods.

We consider Taylor-Green vortices [33], a well-known example of a nonrelativistic decaying flow featuring an exact solution of the Navier-Stokes equations and derive an approximate solution in the mildly relativistic regime. In the nonrelativistic case from the following initial conditions in a two-dimensional periodic domain:

$$u_x(x,y,0) = v_0 \cos(x) \sin(y),$$
$$u_y(x,y,0) = -v_0 \cos(y) \sin(x), \quad x,y \in [0,2\pi], \qquad (10)$$

the solution is given by

$$u_x(x,y,t) = v_0 \cos(x) \sin(y) F(t),$$
$$u_y(x,y,t) = -v_0 \cos(y) \sin(x) F(t), \quad x,y \in [0,2\pi], \quad (11)$$

with

$$F(t) = \exp(-2\nu t), \qquad (12)$$

where ν is the kinematic viscosity of the fluid.

In the relativistic case, we need to solve the conservation equations [Eqs. (8) and (9)]. We consider a system with a constant initial particle density and assume that density remains constant. We will verify later this assumption against our numerical results showing that density fluctuations in time are very small. In this case Eq. (8) is satisfied directly, and the expression of the second-order tensor slightly simplifies since $\nabla_\alpha U^\alpha = 0$. Consequently we drop the term depending on bulk viscosity and rewrite the second-order tensor as

$$T^{\alpha\beta} = -P g^{\alpha\beta} + (\epsilon + P) U^\alpha U^\beta + P^{\langle\alpha\beta\rangle}. \qquad (13)$$

We consider the same initial conditions as in Eq. (10) and look for a solution in the form of Eq. (11) with an appropriate function $F_R(t)$ replacing $F(t)$. We plug Eq. (11) in Eq. (13) and derive bulky analytic expressions for the derivatives of the second-order tensor. A linear expansion of these expressions in terms of v_0 yields a much simpler expression for $\partial_\beta T^{\alpha\beta}$, leading to the differential equation,

$$2\eta F_R(t) + (P + \epsilon) F_R'(t) = 0. \qquad (14)$$

Assuming $P + \epsilon$ is constant, for a fixed value of ζ, we derive an explicit solution,

$$F_R(t) = \exp\left(-\frac{2\eta}{P+\epsilon} t\right) F_R(0), \qquad (15)$$

depending on just one transport coefficient, the shear viscosity η. Observe that, although the quantity $P + \epsilon$ exhibits some
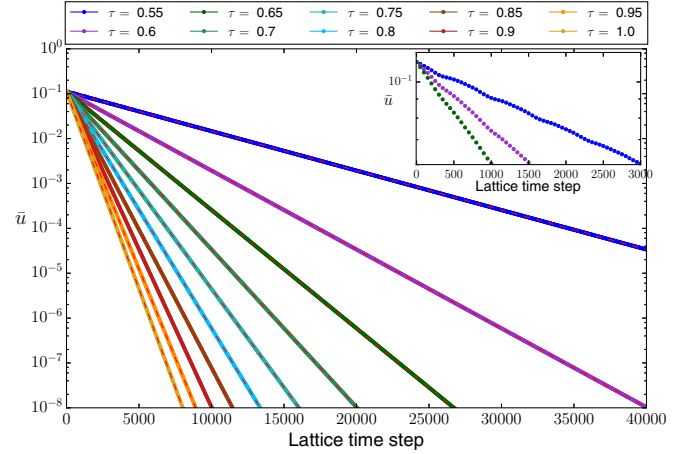

FIG. 2. Simulated time evolution of $\bar{u}$ for selected τ values on a $L = 400$ square lattice ($\zeta = 0$, $v_0 = 0.2$, $n_0 = 1$, $T_0 = 1$). The lines are fits to the exponential decay predicted by Eq. (15). The inset shows nonlinear effects in the early phases of the flow.

time variation (as found in the simulations) due to the evolution of the local temperature, such fluctuations were found to be negligible.

Next, we compare this analytical solution with data obtained via our LB numerical simulation, aiming at linking η to the relaxation time τ. We perform several simulations with different values of the initial speed v_0 and the mesoscopic parameters τ and ζ. We consider small (yet non-negligible) values of u and a very broad range of ζ values smoothly bridging between ultrarelativistic to near nonrelativistic regimes. To this end, it is expedient to introduce the observable $\bar{u}$,

$$\bar{u}^2 = \iint (u_x^2 + u_y^2) dx\, dy \qquad (16)$$

defined to be proportional to $F_R(t)$. Figure 2 gives an example of our numerical results, showing the time evolution of $\bar{u}$, clearly exhibiting an exponential decay. For each set of mesoscopic values, we perform a linear fit of $\ln(\bar{u})$ extracting a corresponding value for η via Eq. (15). We next assume a dependence of η on the mesoscopic parameters, which, on dimensional grounds, reads as

$$\eta = k f(\zeta) P\left(\tau - \tfrac{1}{2}\right), \qquad (17)$$

with $f(\zeta)$ normalized such that $f(0) = 1$. The numerical value of k and the functional form of $f(\zeta)$ contain the physical information on the relation between kinetic and hydrodynamics coefficients. For instance, CE predicts $k = 4/5$ and an expression for $f(\zeta)$ to which we will return shortly; for comparison, Grad's method [6] predicts $k = 2/3$ and a different functional dependence on ζ. We are now able to test that Eq. (17) holds correctly, checking that all measurements of $\eta(\tau)$ at a fixed value of ζ yield a constant value for $k f(\zeta)$. One immediately sees from the second column of Table I that $k = 4/5$ to very high accuracy, consistent with previous results [11,19,27,34]. More interesting is the assessment of the

TABLE I. Fitted values of $kf(\zeta)$ for selected values of τ and ζ. Statistical errors are smaller than 1 in the last displayed digit.

	$kf(\zeta)$						
τ	$\zeta = 0$	$\zeta = 1.6$	$\zeta = 2$	$\zeta = 3$	$\zeta = 4$	$\zeta = 5$	$\zeta = 10$
0.600	0.8003	0.8319	0.8448	0.8587	0.8892	0.8994	0.9311
0.700	0.8002	0.8318	0.8447	0.8584	0.8888	0.8990	0.9302
0.800	0.8002	0.8318	0.8447	0.8583	0.8887	0.8989	0.9300
0.900	0.8002	0.8318	0.8447	0.8583	0.8887	0.8988	0.9299
1.000	0.8002	0.8317	0.8446	0.8582	0.8887	0.8988	0.9299

functional behavior of $f(\zeta)$. The CE expansion predicts [5]

$$f(\zeta) = \frac{\zeta^3}{12}\left(\frac{3}{\zeta^2}\frac{K_3(\zeta)}{K_2(\zeta)} - \frac{1}{\zeta} + \frac{K_1(\zeta)}{K_2(\zeta)} - \frac{Ki_1}{K_2(\zeta)}\right), \quad (18)$$

with $Ki_1 = \int_0^\infty e^{-\zeta \cosh(t)}/\cosh(t)dt$.

Our numerical findings for $kf(\zeta)$ are shown in Fig. 3; for some ζ values we have used several different quadratures for our LB method (see Ref. [29]), the corresponding results differing from each other by approximately 1%; we consider this an estimate of our systematic errors. Figure 3 also shows the CE prediction [Eq. (18)] that almost perfectly matches our results (we remark that *no* free parameters are involved in this comparison) and nicely goes over to the well-known nonrelativistic limit for large values of ζ. For a more quantitative appreciation of the significance of our result, we also plot the predictions of Grad's method [6], which obey the following equation:

$$f(\zeta) = \frac{3}{2}\frac{K_3^2(\zeta)}{K_2(\zeta)K_4(\zeta)}. \quad (19)$$

Comparison of the two curves allows concluding that our level of resolution is adequate to discriminate between the two options. We performed the same procedure for fully three dimensional simulations, and the corresponding results hold
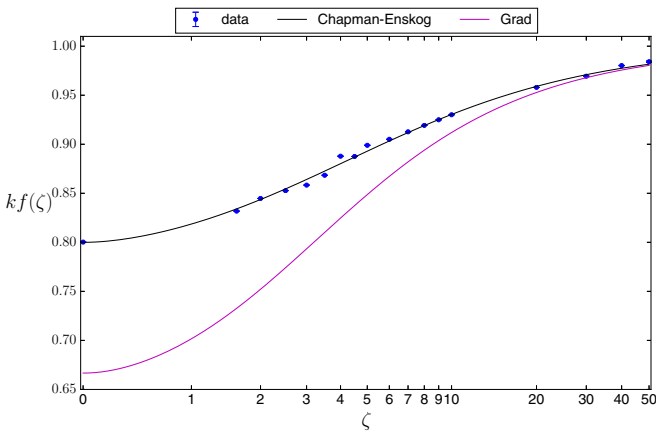
FIG. 3. Measured value $kf(\zeta)$ as a function of ζ. The black (magenta) lines are analytic results of the Chapman-Enskog (Grad's [6]) methods for the relativistic Boltzmann equation. To improve resolution at small ζ values, we map $\zeta \to \ln(\zeta + \sqrt{1 + \zeta^2})$ on the x axis.
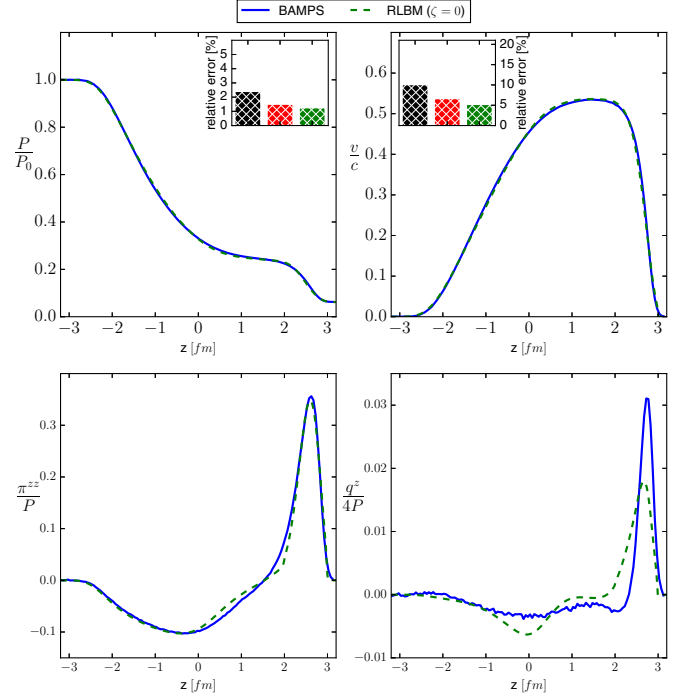
FIG. 4. Comparison of BAMPS and our RLBM for the Riemann problem at $t = 3.2$ fm ($\zeta = 0$, $\eta/s = 0.1$). Top: (left) pressure profile, (right) velocity profile. The error bars are the $L2$ difference between BAMPS and RLBM using Grad [6] $L = 1600$ (black), Chapman-Enskog $L = 1600$ (red), and Chapman-Enskog $L = 12\,800$ (green). Bottom: (left) viscous pressure tensor, (right) heat flux profile.

a similar degree of accuracy; details will be presented in an expanded version of this paper.

Finally, in order to provide a further test of the robustness of our calibration procedure, we consider a significantly different problem, we simulate a 1D shock tube problem in the ultrarelativistic regime ($\zeta = 0$) comparing with BAMPS [35], a Monte Carlo numerical solver for the full Boltzmann equation. This simulation uses a $1 \times 1 \times LZ$ lattice and keeps the ratio $\eta/s = 0.1$ fixed (s is the entropy density). The initial conditions for the temperature are $T_A = 400$ MeV for $z < 0$ and $T_B = 200$ MeV for $z \geqslant 0$. The initial values for the pressure step are $P_A = 5.43$ and $P_B = 0.339$ GeV/fm^3.

Figure 4 shows that our results are in excellent agreement with those of BAMPS. The error bars show the improvement obtained adopting the CE method for the transport coefficients (red bars) over previous results [36] using Grad's method [6] of moments (black bars). In Fig. 4 we also present the profile of the π^{zz} component of the pressure viscous tensor and of the q^z component of the heat flux, showing good agreement with results produced by BAMPS for the former quantity, whereas non-negligible differences arise for the latter. The reason is that, since the Anderson-Witting model only provides a free parameter τ, a fine description of several transport coefficients would require extending it to a multirelaxation time collisional operator.

Summarizing, we have investigated the kinetic pathway to dissipative relativistic hydrodynamics by comparing lattice-kinetic simulations with analytical results based on the Chapman-Enskog method. We find very neat evidence

supporting recent theoretical findings in favor of the Chapman-Enskog procedure, which we tentatively interpret as the failure of Grad's method [6] to secure positive definiteness of Boltzmann's distribution function. Since violations of positive definiteness are most likely to occur in the high-energy tails of the distribution, it is natural to speculate that they should be of particular relevance to the relativistic hydrodynamic regime in which tails are significantly more populated than in the nonrelativistic case. These results are potentially relevant to the study of a wide host of dissipative relativistic hydrodynamic problems, such as electron flows in graphene and quark-gluon plasmas [37,38]. A further intriguing question pertains to the relevance of this analysis to strongly interacting holographic fluids obeying the anti-de Sitter-conformal field theory (AdS-CFT) bound [39]. Indeed, although such fluids are believed to lack a kinetic description altogether since quasiparticles are too short lived to carry any physical relevance, they are still amenable to a *lattice* kinetic description reaching down to values of η/s well below the AdS-CFT bound [40,41]. Work to explore the significance of the AdS-CFT bounds in lattice fluids is currently underway.

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